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CONTENTS

No. 1 JULY

Observables in the Extreme Relativistic Representation of the Dirac Equation ..... P. M. Mathews and A. Sankaranarayanan 1

On the Quantization of Physical Space-Time Operators. II ..... Ichie Watanabe 7

Nuclear Interactions in the $p_{1/2}$-$g_{9/2}$ Configurations ..... V. K. Thankappan, Y. R. Waghmare and S. P. Pandya 22

Self-Trapping of an Electron by the Acoustical Mode of Lattice Vibration. I ..... Yutaka Toyozawa 29

The Pair Correlation Function of an Imperfect Electron Gas in High Densities ..... Seizo Ueda 45

Space-Time Correlation Function in the Theory of Electrical Conductivity ..... Isao Mannari 51

Particle-Hole Bound States in the Many Fermion System ..... Tetsuo Gotô 84

Some Converging Examples of the Perturbation Series in the Quantum Field Theory ..... Yusuke Kato 99

Magnetohydrostatic Equilibrium for Fluid Enclosed in an Axisymmetric Toroidal Surface ..... Yusuke Kato 123

On a Convergent Model of Quantum Field Theory with Indefinite Metric ..... Kan-ichi Yokoyama 131

Letters to the Editor:

Operator Equations in Two Field Theory Models ..... P. G. Federbush 148

A New Model for the High Energy Nuclear Interaction ..... S. Hasegawa 150

Hard Core and Shape Parameter in the Effective Range Expansion ..... T. Hamada and I. D. Johnston 153

Errata:

Dynamical Theory of Ultrasonic Attenuation in Metals (Vol. 25, p. 327) ..... N. Takimoto 155

Contents of Journal of the Phys. Soc. of Japan, Vol. 16, No. 6

No. 2 AUGUST

Fixation of Physical Space-Time Coordinates and Equation of Motion of Two-Body Problem ..... Toshiei Kimura 157

Generalized Theory of Condensing Systems ..... Kazuyosi Ikeda 173

Nuclear Finite Size Effects on the Internal Conversion in the μ-Mesonic He^3 Atom.....Hajime Narumi and Shigeru Matsuo 221

Symmetry of the Strong Interaction in the Four-Dimensional Isotopic-Spin Space and in the Three-Dimensional κ-Space .....
..... Seitaro Nakamura and Shigeo Sato 231

Propagation of Shock Waves in Inhomogeneous Medium. IV
——Second Order Approximation——.....Noboru Ohyama 251

Stability of Axially Asymmetric Nuclear Deformation .....
.....Jun Hiura, Shōta Suekane and Hajime Tanaka 263

Letters to the Editor:

Nambu Model for Elementary Particles and the Composite Model for Pion....C. Ihara 270

Another Possible Model of Quantum Field Theory with Indefinite Metric..K. Yokoyama 272

Test of the Intermediary Boson Theory ..... S. Nakamura and K. Itami 274

Hypothesis of the Isotopic Spin Independence in Extremely High Energy Region
.....M. Kawaguchi 277

K^--p Interactions at 1.15 Bev/c.....S. Minami 278

Field Theory of Nucleon-Nucleon Scattering .....I. Sato 281

Effect of the Bond Polarization on the I-I-coupling in Hydrocarbons.....E. Hiroike 283

No. 3 SEPTEMBER

Single Particle versus Collective Excitation in the Direct Interactions
.....Taro Tamura and Tokuo Terasawa 285

The Singularity of Propagators in Field Theory and the Structure of Space-Time .....Yoshiro Takano 304

The Polarization of Deuteron.....Yasutaro Takahashi 315

Nuclear Matter at Low Density with Attractive and Singular Interactions.....Yasushi Wada 321

Integral Representations for Scattering Amplitudes in Perturbation Theory ..... Noboru Nakanishi 337

***K* Meson-Pion Interaction and $K^-+p\rightarrow\bar{K}^0+\pi^-+p$ Reaction**.....
..... Shigeo Minami 356

Elastic π^--p Scattering at 1.4 Bev..... Shigeo Minami 375

Remarks on the Lowlying Baryon Isobars.....
..... R. E. Marshak and S. Okubo 381

A Simple Model for Two-Channel Reaction in Field Theory and Its Application to the Y^* Problem ..... Yasunori Fujii 391

Letters to the Editor:

Isotopic Abundance of Carbon in Primary Cosmic Rays ..... H. Hasegawa and K. Ito 418
The Shielding of a Fixed Charge in a Metal ..... S. Shinohara 420
Remarks on Interacting Many-Fermion System ..... T. Ishihara and M. Yasuno 422
On the Validity of the Adiabatic Approximation in the Theory of Electron Broadening in a Plasma ..... S. Matsuo and H. Narumi 424
Effet de la polarisation chargée sur l'interaction entre ions lourds .....K. Nakamura 426

Errata

On the Kinetic Equation for Electrons in Metals at Low Temperatures....Tohru Morita 429

Contents of Journal of the Phys. Soc. of Japan, Vol. 16, No. 7 & No. 8
Contents of Supplement of the Prog. Theor. Phys., No. 17—No. 20

No. 4 OCTOBER

On a Model of Nucleon Structure and Meson-Nucleon Interaction. II ..... Hiroshi Nakamura 431

Renormalization of the Partition Function in the Relativistic Statistical Mechanics of Electron Gas ..... Keiji Watanabe 453

On Vertical Representation of Bethe-Salpeter Amplitudes..... ..... Masakuni Ida and Kazumi Maki 470

Macroscopic Causality and Lower Limit for the Energy Derivative of the Scattering Phase Shift ——*Relativistic Case*—— ....Takesi Ogimoto 483

The Contribution of Multiple Scattering within the Nucleus to the Elastic Scattering of Nucleons by Nuclei ..... Alvin M. Saperstein 489

Observables of a Dirac Particle..... ..... P. M. Mathews and A. Sankaranarayanan 499

On a Radiative Decay of Σ^+ .....Makoto Sawamura 505

Effects of the γ-3π Interaction on the Electromagnetic Properties of Deuteron ..... Yasunori Fujii and Masaaki Kawaguchi 519

Methods for Detecting High Energy Photons in Primary Cosmic Rays ..... Takashi Murayama 541

***K*π Resonance Effect to the Form Factors in the K_{l3} Decays**..... .....Jugoro Iizuka 554

Letters to the Editor:

Some Notes on the Effects of Vector Meson on the Electromagnetic Structure of Nucleon.....R. Kubo and H. Nakamura 565
Evidence for Quadratic Spin-Orbit Interaction .....N. R. Yoder and P. Signell 567

Motion of an Impurity Atom in One-Dimentional Lattice ..... S. Kashiwamura 568
Note on K^--p Interactions at 400 Mev/c ..... S. Minami 569
On the Generalized Nilsson Diagram ..... T. Inoue 572

No. 5 NOVEMBER

Approximate Constancy of Adiabatic Invariants ..... L. M. Garrido 577
Theory of Ultrasonic Attenuation in Metals ..... Yosuke Nagaoka 589
Subtraction in the Chew-Mandelstam Equation and Determination of the Low-Energy Scattering Amplitude ..... Takeshi Kanki 611
Dispersion Relations in Nucleon-Nucleon Scattering and Resonances in Many-Pion System ..... Yasuo Hara 627
On $\pi+N\rightarrow Y+\pi+K$ Reaction ..... Shigeo Minami 647
Dynamical Theory of Ultrasonic Attenuation. II ——*Note on the Weinreich Relation*—— ..... Noboru Takimoto 659
Alpha-Particle Reduced Widths in Heavy Nuclei .... Kichinosuke Harada 667
Nonstatic One-Boson-Exchange Potentials ..... Norio Hoshizaki, I Lin and Shigeru Machida 680
Magnetic Ordering in the Rare-Earth Metals with More than Half-Filled 4*f* Shells ..... Hirosi Miwa and Kei Yosida 693
Correlation between Measurements in Quantum Theory ..... Henry Margenau and Robert Nyden Hill 722
Theory of Impurity Band Conduction in Semiconductors ——*An Approach to Random Lattice Problem*—— ..... Takeo Matsubara and Yutaka Toyozawa 739
Non-Static Effects in the Nucleon-Nucleon Interaction ..... Philip Connors and Peter Signell 757
Persistent Ring Currents in an Ideal Bose Gas ..... J. M. Blatt 761

Letters to the Editor:

On a Fixation of Gauge and True Observables of Yang-Mills Field ..... T. Kimura 790
A Note on the Non-leptonic Decay Modes of the Ξ-Hyperon ..... A. P. Balachandran and K. Venkatesan 792
Phonon-Phonon Interaction in Liquid Helium II ..... K. Kawasaki 793
Sound Attenuation in Liquid Helium II at Very Low Temperatures ..... K. Kawasaki 795
Analysis of $(n, n'\gamma)$ Cross Section for Nuclei $A\sim100$ ..... A. Sugie, S. Kikuchi, S. Momiyama and T. Suzuki 797
A Note on the Dielectric Constant and Causality ..... A. Morita and Y. Osaka 799
Three-Pion Resonance and High Energy Phenomena ..... S. Minami 801
Proton-Antiproton Collision and Isobaric States ..... S. Minami 803

Errata :

Parametric Integral Formulas and Analytic Properties in Perturbation Theory (Suppl. No. 18, p. 1) ..... N. Nakanishi 806

Contents of Supplement of the Prog. Theor. Phys., No. 17–No. 20

Contents of Journal of the Phys. Soc. of Japan, Vol. 16, No. 9 & No. 10

No. 6 DECEMBER

Partial Equilibrium Model for Nuclear Reactions ..... Ko Izumo 807

Relativistic Quantum Field Theory of Unstable Particle ..... Mitsuru Hama and Sho Tanaka 829

Equilibrium Abundances of Stable Isotopes ..... Hira Lal Duorah 844

On the Plasma Oscillations in Metal Foils ..... Hideo Kanazawa 851

Energy Spectrum and Determinant Method in Quantized Field Theory ..... Shûkô Azuma 861

On the Quaternary Intermediate Boson in the Weak Interaction ..... Yasuhisa Katayama 879

Nuclear Radiation from Solids ..... Tetsuo Eguchi 900

Spin and Statistics ..... Lowell Brown and Julian Schwinger 917

Integral Representations for Scattering Amplitudes in Perturbation Theory. II ..... Noboru Nakanishi 927

Internal Structure of Very Massive Stars ..... Shiro Sakashita and Chushiro Hayashi 942

Nature of the First Excited States of the Vibrational Even-Even Nuclei ..... Tara Tamura and Takeshi Udagawa 947

Viscosity of Fermi Particle Systems ..... Hisashi Nishimura and Hazime Mori 967

Stellar Synthesis of the Proton-Rich Heavy Elements ..... Kensai Ito 990

Letters to the Editor :

Associated Production in Pion-Nucleon Collisions and Effects of Resonant States ..... T. Tsuchida, T. Sakuma and S. Furui 1005

On the Interaction of the Pion and Cascade Particle ..... S. Iwao 1008

Imaginary Part of the Optical Potential for Nuclei $A \sim 60$ and $A \sim 100$ ..... A. Sugie, K. Harada and H. Horie 1010

Effect of Pion-Pion Scattering Resonance on Nucleon Structure ..... A. Takahashi 1012

Proof of Partial-Wave Dispersion Relations for Pion-Pion and Nucleon-Nucleon Scatterings in Perturbation Theory ..... K. Yamamoto 1014

Hypothesis of the Isotopic Spin Independence. II ..... M. Kawaguchi 1016

Contents and Author Index to Vol. 26 ..... i–ix

AUTHOR INDEX TO VOLUME 26

Azuma-S. Energy Spectrum and Determinant Method in Quantized Field Theory. 861

Balachandran-A. P. & Venkatesan-K. A Note on the Non-leptonic Decay Modes of the Ξ-Hyperon..... 792(L)
Blatt-J. M. Persistent Ring Currents in an Ideal Bose Gas 761
Brown-L. & Schwinger-J. Spin and Statistics..... 917

Connors-P. & Signell-P. Non-Static Effects in the Nucleon-Nucleon Interaction 757

Duorah-H. L. Equilibrium Abundances of Stable Isotopes 844

Eguchi-T. Nuclear Radiation from Solids 900

Federbush-P. G. Operator Equations in Two Field Theory Models..... 148(L)
Fujii-Y. A Simple Model for Two-Channel Reaction in Field Theory and Its Application to the Y^* Problem 391
—— **& Kawaguchi-M.** Effects of the γ-3π Interaction on the Electromagnetic Properties of Deuteron..... 519
Furui-S. (See Tsuchida-T.) ..... 1005(L)

Garrido-L. M. Approximate Constancy of Adiabatic Invariants 577
Gotô-T. Particle-Hole Bound States in the Many Fermion System 84

Hama-M. & Tanaka-S. Relativistic Quantum Field Theory of Unstable Particle 829
Hamada-T. & Johnston-I. D. Hard Core and Shape Parameter in the Effective Range Expansion ..... 153(L)
Hara-Y. Dispersion Relations in Nucleon-Nucleon Scattering and Resonances in Many-Pion System..... 627
Harada-K. Alpha-Particle Reduced Widths in Heavy Nuclei..... 667
——(See Sugie-A.) .....1010(L)
Hasegawa-H. & Ito-K. Isotopic Abundance of Carbon in Primary Cosmic Rays.. 418(L)
Hasegawa-S. A New Model for the High Energy Nuclear Interaction..... 150(L)
Hayashi-C. (See Sakashita-S.) 942
Hill-R. N. (See Margenau-H.) 722
Hiroike-E. Effect of the Bond Polarization on the I-I-coupling in Hydrocarbons..283(L)
Hiura-J., Suekane-S. & Tanaka-H. Stability of Axially Asymmetric Nuclear Deformation 263
Horie-H. (See Sugie-A.) .....1010(L)
Hoshizaki-N., Lin-I & Machida-S. Non-static One-Boson-Exchange Potentials.... 680

Ida-M. & Maki-K. On Vertical Representation of Bethe-Salpeter Amplitudes..... 470
Ihara-C. Nambu Model for Elementary Particles and the Composite Model for Pion ..... 270(L)
Iizuka-J. $K\pi$ Resonance Effect to the Form Factors in the K_{l3} Decays 554
Ikeda-K. Generalized Theory of Condensing Systems..... 173
Inoue-T. On the Generalized Nilsson Diagram..... 572(L)
Ishihara-T. & Yasuno-M. Remarks on Interacting Many-Fermion System ..... 422(L)
Itami-K. (See Nakamura-S.) ..... 274(L)
Ito-K. (See Hasegawa-H.) ..... 418(L)
—— Stellar Synthesis of the Proton-Rich Heavy Elements..... 990
Iwao-S. On the Interaction of the Pion and Cascade Particle .....1008(L)
Izumo-K. Partial Equilibrium Model for Nuclear Reactions..... 807

Johnston-I. D. (See Hamada-T.) 153

Kanazawa-H. On the Plasma Oscillations in Metal Foils..... 851
Kanki-T. Subtraction in the Chew-Mandelstam Equation and Determination of the Low-Energy Scattering Amplitude 611
Kashiwamura-S. Motion of an Impurity Atom in One-Dimensional Lattice.... 568(L)
Katayama-Y. On the Quaternary Intermediate Boson in the Weak Interaction..... 879
Kato-Y. Some Converging Examples of the Perturbation Series in the Quantum Field

Theory 99
—— Magnetohydrostatic Equilibrium for Fluid Enclosed in an Axisymmetric Toroidal Surface 123
Kawaguchi-M. Hypothesis of the Isotopic Spin Independence in Extremely High Energy Region..... 277(L)
—— (See Fujii-Y.) 519
—— Hypothesis of the Isotopic Spin Independence. II .....1016(L)
Kawasaki-K. Phonon-Phonon Interaction in Liquid Helium II..... 793(L)
—— Sound Attenuation in Liquid Helium II at Very Low Temperatures ..... 795(L)
Kikuchi-S. (See Sugie-A.) ..... 797(L)
Kimura-T. Fixation of Physical Space-Time Coordinates and Equation of Motion of Two-Body Problem..... 157
—— On a Fixation of Gauge and True Observables of Yang-Mills Field .... 790(L)
Kubo-R. & Nakamura-H. Some Notes on the Effects of Vector Meson on the Electromagnetic Structure of Nucleon..... 565(L)

Lin-I (See Hoshizaki-N.) 680

Machida-S. (See Hoshizaki-N.) 680
Maki-K. (See Ida-M.) 470
Mannari-I. Space-Time Correlation Function in the Theory of Electrical Conductivity 51
Margenau-H. & Hill-R. N. Correlation between Measurements in Quantum Theory 722
Marshak-R. E. & Okubo-S. Remarks on the Lowlying Baryon Isobars 381
Mathews-P.M. & Sankaranarayanan-A. Observables in the Extreme Relativistic Representation of the Dirac Equation 1
—— **& Sankaranarayanan-A.** Observables of a Dirac Particle 499
Matsubara-T. & Toyozawa-Y. Theory of Impurity Band Conduction in Semiconductors ——*An Approach to Random Lattice Problem*—— 739
Matsuo-S. (See Narumi-H.) 221
—— **& Narumi-H.** On the Validity of the Adiabatic Approximation in the Theory of Electron Broadening in a Plasma.... 424(L)
Minami-S. K^--p Interactions at 1.15 Bev/c ..... 278(L)
—— K Meson-Pion Interaction and $K^-+p\rightarrow\bar{K}^0+\pi^-+p$ Reaction..... 356
—— Elastic π^--p Scattering at 1.4 Bev.... 375
—— Note on K^--p Interactions at 400 Bev/c ..... 569(L)
—— On $\pi+N\rightarrow Y+\pi+K$ Reaction..... 647
—— Three-Pion Resonance and High Energy Phenomena ..... 801(L)
—— Proton-Antiproton Collision and Isobaric States ..... 803(L)
Miwa-H. & Yosida-K. Magnetic Ordering in the Rare-Earth Metals with More than Half-Filled 4f Shells..... 693
Momiyama-S. (See Sugie-A.)..... 797(L)
Mori-H. (See Nishimura-H.)..... 967
Morita-A. & Osaka-Y. A Note on the Dielectric Constant and Causality..... 799(L)
Murayama-T. Methods for Detecting High Energy Photons in Primary Cosmic Rays.. 541

Nagaoka-Y. Theory of Ultrasonic Attenuation in Metals 589
Nakamura-H. On a Model of Nucleon Structure and Meson-Nucleon Interaction. II..... 431
—— (See Kubo-R.) ..... 565(L)
Nakamura-K. Effet de la polarisation chargée sur l'interaction entre ions lourds .. 426(L)
Nakamura-S. & Itami-K. Test of the Intermediary Boson Theory ..... 274(L)
—— **& Sato-S.** Symmetry of the Strong Interaction in the Four-Dimensional Isotopic-Spin Space and in the Three-Dimensional κ-Space..... 231
Nakanishi-N. Integral Representations for Scattering Amplitudes in Perturbation Theory 337
—— Integral Representations for Scattering Amplitudes in Perturbation Theory. II.. 927
Narumi-H. & Matsuo-S. Nuclear Finite Size Effects on the Internal Conversion in the μ-Mesonic He^3 Atom..... 221
—— (See Matsuo-S.) ..... 424(L)
Nishimura-H. & Mori-H. Viscosity of Fermi Particle Systems..... 967

Ogimoto-T. Macroscopic Causality and Lower Limit for the Energy Derivative of the Scattering Phase Shift ——*Relativistic Case*—— 483
Ohyama-N. Propagation of Shock Waves in Inhomogeneous Medium. IV ——*Second*

Order Approximation—— 251
Okubo-S. (See Marshak-R. E.) 381
Osaka-Y. (See Morita-A.) ..... 799(L)

Pandya-S. P. (See Thankappan-V. K.) .. 22

Sakashita-S. & Hayashi-C. Internal Structure of Very Massive Stars 942
Sakuma-T. (See Tsuchida-T.) .....1005(L)
Sankaranarayanan-A. (See Mathews-P. M.) 1
—— (See Mathews-P. M.) 499
Saperstein-A. M. The Contribution of Multiple Scattering within the Nucleus to the Elastic Scattering of Nucleons by Nuclei 489
Sato-I. Field Theory of Nucleon-Nucleon Scattering ..... 281(L)
Sato-S. (See Nakamura-S.) 231
Sawamura-M. On a Radiative Decay of Σ^+ 505
Schwinger-J. (See Brown-L.) 917
Shinohara-S. The Shielding of a Fixed Charge in a Metal ..... 420(L)
Signell-P. (See Yoder-N. R.) ..... 567(L)
—— (See Connors-P.) 757
Suekane-S. (See Hiura-J.) 263
Sugie-A., Kikuchi-S., Momiyama-S. & Suzuki-T. Analysis of $(n, n'\gamma)$ Cross Section for Nuclei $A\sim100$ ..... 797(L)
——, **Harada-K. & Horie-H.** Imaginary Part of the Optical Potential for Nuclei $A\sim60$ and $A\sim100$ .....1010(L)
Suzuki-T. (See Sugie-A.) ..... 797(L)

Takahashi-A. Effect of Pion-Pion Scattering Resonance on Nucleon Structure ....1012(L)
Takahashi-Y. The Polarization of Deuteron 315
Takano-Y. The Singularity of Propagators in Field Theory and the Structure of Space-Time 304
Takimoto-N. Dynamical Theory of Ultrasonic Attenuation. II ——*Note on the Weinreich Relation*—— 659
Tamura-T. & Terasawa-T. Single Particle versus Collective Excitation in the Direct Interactions 285
—— **& Udagawa-T.** Nature of the First Excited States of the Vibrational Even-Even Nuclei 947
Tanaka-H. (See Hiura-J.) 263
Tanaka-S. (See Hama-M.) 829
Terasawa-T. (See Tamura-T.) 285
Thankappan-V.K., Waghmare-Y. R. & Pandya-S. P. Nuclear Interactions in the $p_{1/2}$-$g_{9/2}$ Configurations 22
Toyozawa-Y. Self-Trapping of an Electron by the Acoustical Mode of Lattice Vibration. I 29
——(See Matsubara-T.) 739
Tsuchida-T., Sakuma-T. & Furui-S. Associated Production in Pion-Nucleon Collisions and Effects of Resonant States. 1005(L)

Udagawa-T. (See Tamura-T.) 947
Ueda-S. The Pair Correlation Function of an Imperfect Electron Gas in High Densities 45

Venkatesan-K. (See Balachandran-A. P.) ..... 792(L)

Wada-Y. Nuclear Matter at Low Density with Attractive and Singular Interactions. 321
Waghmare-Y. R. (See Thankappan-V. K.) 22
Watanabe-I. On the Quantization of Physical Space-Time Operators. II 7
Watanabe-K. Renormalization of the Partition Function in the Relativistic Statistical Mechanics of Electron Gas 453

Yamamoto-K. Proof of Partial-Wave Dispersion Relations for Pion-Pion and Nucleon-Nucleon Scatterings in Perturbation Theory .....1014(L)
Yasuno-M. (See Ishihara-T.) ..... 422(L)
Yoder-N. R. & Signell-P. Evidence for Quadratic Spin-Orbit Interaction .... 567(L)
Yokoyama-K. On a Convergent Model of Quantum Field Theory with Indefinite Metric 131
—— Another Possible Model of Quantum Field Theory with Indefinite Metric.. 272(L)
Yosida-K. (See Miwa-H.) 693

Errata

Dynamical Theory of Ultrasonic Attenuation in Metals (Vol. 25, p. 327) ..... N. Takimoto 155
On the Kinetic Equation for Electrons in

Metals at Low Temperatures (Vol. 25, p. 845) ..... T. Morita 429
Parametric Integral Formulas and Analytic Properties in Perturbation (Suppl. No. 18, p. 1) ..... N. Nakanishi

Contents of Journal of the Phys. Soc. of Japan

Vol. 16, No. 6 ..... No. 1
Vol. 16, No. 7 & No. 8 ..... No. 3
Vol. 16, No. 9 & No. 10 ..... No. 5

Contents of Suppl., Prog. Theor. Phys.

No. 17—No. 20 ..... No. 3

Progress of Theoretical Physics, Vol. 26, No. 1, July 1961

Observables in the Extreme Relativistic Representation of the Dirac Equation

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A new choice of operators is given to represent dynamical variables of a Dirac particle in the extreme relativistic representation.

§ 1. Introduction

Considerable attention has recently been paid to the construction of special representations of the Dirac equation which emphasize those aspects of the equation that are important in special circumstances. The means employed are unitary transformations that carry the Dirac Hamiltonian over into new forms. Examples are: the Foldy-Wouthuysen[1] (F.W.) transformation which has been useful in bringing out clearly the properties of the equation in the non-relativistic limit; and the Cini-Touschek[2] (C.T.) transformation which yields essentially a projection to the extreme relativistic limit. The physical content of the theory rests on the specific choice of operators occurring in the theory to represent dynamical variables. The choice of operators in the usual Dirac theory has led to some of the dynamical variables, for instance, the particle velocity, having properties that have no correspondence with those of their classical counterparts. The form of the transformed F. W. Hamiltonian, on the other hand, suggested new operators whose properties are in accordance with one's expectations based on the relativistic classical mechanics and non-relativistic quantum mechanics. In particular, the position operator (called 'mean position' by F. W.) is such as to lead to a velocity operator equal to $\boldsymbol{p}/E$ (except for a negative sign that comes in, as expected, in the case of negative energy states). However, the choice of operators made in the E representation* by Bose, Gamba and Sudarshan[3] (B.G.S.) and Pac[4] does not seem entirely happy, and it is the object of the present paper to reconsider this question and to suggest a different assignment of operators to observables in the E representation.

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* Following Bose, Gamba and Sudarshan, we shall prefix the classical (non-relativistic) and the extreme relativistic representations arising from the F. W. and C. T. transformations by the letters C and E respectively and the usual (Dirac) representation by D.

§ 2. The *C*, *D* and *E* representations

The passage from the D representation to the C and E representations is accomplished by transformation with the unitary matrices e^{iS} and e^{iT} respectively, where

$$S=-\frac{i}{2}\beta\frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}}{p}\tan^{-1}\left(\frac{p}{m}\right), \tag{1a}$$

$$e^{\pm iS}=\frac{E+m\pm\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})}{[2E(E+m)]^{1/2}}, \tag{1b}$$

$$T=\frac{i}{2}\beta\frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}}{p}\tan^{-1}\left(\frac{m}{p}\right), \tag{2a}$$

$$e^{\pm iT}=\frac{E+p\mp m\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})/p}{[2E(E+p)]^{1/2}}. \tag{2b}$$

The Dirac Hamiltonian

$$H^D=\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta m, \tag{3}$$

then gets transformed into

$$H^C=e^{iS}H^De^{-iS}=\beta E, \tag{4}$$

and

$$H^E=e^{iT}H^De^{-iT}=E(\boldsymbol{\alpha}\cdot\boldsymbol{p})/p, \tag{5}$$

respectively, where

$$E=+(p^2+m^2)^{1/2}. \tag{6}$$

Further, the relation between the C and E representations is typified by the transformation of the Hamiltonian:

$$H^E=e^{i(T-S)}H^Ce^{-i(T-S)}, \tag{7}$$

where

$$T-S=i\frac{\pi}{4}\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})/p, \tag{8a}$$

and

$$e^{\pm i(T-S)}=\frac{1}{\sqrt{2}}[1\mp\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})/p]. \tag{8b}$$

§ 3. Observables in the *E* representation

The freedom in the choice of operators exists only in respect of the dynamical variables representing the position of the particle, for the operators representing momentum $\boldsymbol{p}$ and angular momentum $\boldsymbol{J}$ are fixed by their relation to the infinitesimal generators of translation and rotation respectively in the

ordinary three-dimensional space whose points are labelled by the variable $\boldsymbol{x}$. Thus, in the usual representation of the Dirac equation,

$$\boldsymbol{p}=-i\boldsymbol{\nabla}, \tag{9}$$

and

$$\boldsymbol{J}=\boldsymbol{x}\times\boldsymbol{p}+\frac{\boldsymbol{\sigma}}{2}, \tag{10}$$

where

$$\boldsymbol{\sigma}=\frac{1}{2i}(\boldsymbol{\alpha}\times\boldsymbol{\alpha}). \tag{10a}$$

$\boldsymbol{x}$ itself appears merely as a label for an arbitrary point in space, and there is no compelling reason to identify it as the representative of the position observable of the particle described by the Dirac equation. Indeed, as is well known, such an identification leads to difficulties of interpretation. Specifically, it does not have the first two of the following properties considered desirable for a position operator :[5),6)]

(i) Any one-particle operator, including the position operator, must be defined separately on positive and negative energy eigenstates, i. e. any such operator A must commute with the sign of the energy, which is represented by (H/E):

$$[A,\ (H/E)]=0; \tag{11}$$

(ii) The position operator, say $\boldsymbol{X}$, must lead to a velocity operator $\dot{\boldsymbol{X}}$ such that

$$\dot{\boldsymbol{X}}=(\boldsymbol{p}/E)\,(H/E); \tag{12}$$

(iii) The components of $\boldsymbol{X}$ must commute among themselves;

(iv) $\boldsymbol{X}$ must be the space part of a covariant four-vector.

It may not be possible, in general, to find an operator that satisfies all these requirements.[5)] However, by choosing the representative of the position operator in the C representation as $\boldsymbol{X}^C=\boldsymbol{x}$, F. W. were able to satisfy the first three requirements (while the status of this operator as regards covariance remains unclear). They further showed that a choice of $\boldsymbol{X}$ on these lines leads to the correct non-relativistic form of the Hamiltonian of an electron in interaction with the electromagnetic field.

In the choice of the position operator in the E representation, B.G.S.[3)] and Pac[4)] have proceeded apparently by analogy with the F. W. procedure. Thus Pac started by examining the possibility of choosing $\boldsymbol{x}$ as the position operator in the E representation, but discarded it in favour of

$$\boldsymbol{X}'^E=\boldsymbol{x}-(\boldsymbol{\sigma}\times\boldsymbol{p})/(2p^2)$$

which satisfies both (i) and (ii) in the E representation unlike $\boldsymbol{x}$ itself. B.G.S.

also represent the position variable in the E representation by $\boldsymbol{x}$ and the other observables by operators usually associated with them in the D representation; but since $\boldsymbol{x}$ does not commute with (H^E/E), this causes mixing of positive and negative energy states, and they seek to avoid this by attributing a pair of operators to each dynamical variable, one to operate on positive energy states only and the other on negative energy states only. Thus, for instance, they postulate the position operator in the E representation to be given by

$$\boldsymbol{x}_\pm = \Lambda_\pm^E \boldsymbol{x} \Lambda_\pm^E, \tag{13}$$

where

$$\Lambda_\pm^E = 1/2(1 \pm H^E/E) \tag{14}$$

are the projection operators to positive and negative energy states respectively in the E representation. This association of a pair of operators with each dynamical variable is evidently unsatisfactory, particularly in view of the fact that one can define, following Pryce,[5] a single operator

$$\Lambda_+ A \Lambda_+ + \Lambda_- A \Lambda_- \equiv A_+ + A_-,$$

which does not mix positive and negative energy states, to represent the dynamical variable A. If this procedure is followed, the operators defined by B.G.S. coincide with Pac's operators:

$$\boldsymbol{X}'^E = \boldsymbol{x}_+ + \boldsymbol{x}_-, \quad \text{etc.}$$

It is therefore sufficient for us to consider the E representation operators defined by Pac. A list of these is included in Table I for easy reference.

Pac's E representation position operator $\boldsymbol{X}'^E$ does commute with the sign of the energy and also leads to the correct expression for the velocity operator. But its components do not commute. (The question of covariance is an involved one and will not be dealt with here.) A more serious objection to Pac's choice of $\boldsymbol{X}'^E$ as position operator (and hence to the B.G.S. operators) arises in connection with the spin angular momentum operator $\boldsymbol{\Sigma}'^E$. The latter is uniquely determined once the position operator is chosen. Pac's expression $\boldsymbol{\Sigma}'^E = (\boldsymbol{\sigma}\cdot\boldsymbol{p})\boldsymbol{p}/p^2$, according to which the 'spin' is purely longitudinal and hence simply related to helicity, is interesting, to be sure, in view of the importance of helicity in the domain of weak interactions. Nevertheless, the fact that the components of $\boldsymbol{\Sigma}'^E$ commute makes the identification of $\boldsymbol{\Sigma}'^E$ with spin angular momentum questionable, for it is well known that the components of any angular momentum operator, say $\boldsymbol{M}$, must satisfy relations of the type $[M_x, M_y] = iM_z$. Consequently the position operator which led to $\boldsymbol{\Sigma}'^E$ must also be considered unsatisfactory.

We are therefore forced to seek a different assignment of E representation operators to the dynamical variables of a Dirac particle. In this we are aided by the knowledge that operators defining dynamical variables in the C repre-

sentation as given by F. W. have all the properties that are desired of them; and a unitary transformation of all the operators would leave those properties unchanged. We therefore define operators A^E in the E representation by

$$A^E = e^{i(T-S)} A^C e^{-i(T-S)}, \tag{15}$$

where A^C are the corresponding F. W. operators. The results are given in Table I.

Table I

Dynamical variables	Operator representatives in the E representation	
	Pac's case	Present case
Position	$\boldsymbol{X}'^E = \boldsymbol{x} - \frac{1}{2p^2}(\boldsymbol{\sigma}\times\boldsymbol{p})$	$\boldsymbol{X}^E = \boldsymbol{x} + \frac{i}{2p}\left[\beta\boldsymbol{\alpha} - \beta\frac{(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{p}}{p^2}\right] - \frac{1}{2p^2}(\boldsymbol{\sigma}\times\boldsymbol{p})$
Momentum	$\boldsymbol{p}'^E = \boldsymbol{p}$	$\boldsymbol{p}^E = \boldsymbol{p}$
Velocity	$\dot{\boldsymbol{X}}'^E = \frac{\boldsymbol{p}}{E}\,\frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}}{p}$	$\dot{\boldsymbol{X}}^E = \frac{\boldsymbol{p}}{E}\,\frac{\boldsymbol{\alpha}\cdot\boldsymbol{p}}{p}$
Orbital angular momentum	$\boldsymbol{L}'^E = \boldsymbol{x}\times\boldsymbol{p} + \frac{\boldsymbol{\sigma}}{2} - \frac{1}{2p^2}(\boldsymbol{\sigma}\cdot\boldsymbol{p})\boldsymbol{p}$	$\boldsymbol{L}^E = \boldsymbol{x}\times\boldsymbol{p} + \frac{\boldsymbol{\sigma}}{2} + \frac{i\beta}{2p}(\boldsymbol{\alpha}\times\boldsymbol{p}) - \frac{1}{2p^2}(\boldsymbol{\sigma}\cdot\boldsymbol{p})\boldsymbol{p}$
Spin angular momentum	$\boldsymbol{\Sigma}'^E = \frac{1}{p^2}(\boldsymbol{\sigma}\cdot\boldsymbol{p})\boldsymbol{p}$	$\boldsymbol{\Sigma}^E = \frac{1}{p^2}(\boldsymbol{\sigma}\cdot\boldsymbol{p})\boldsymbol{p} - \frac{i\beta}{p}(\boldsymbol{\alpha}\times\boldsymbol{p})$

It will be noticed that the new set of operators differs from Pac's in the appearance of terms involving $i\beta\boldsymbol{\alpha}$. (Such terms appear also in the D representation of the F. W. operators and have no obvious physical interpretation since $i\beta\boldsymbol{\alpha}$ has no simple relation to any of the significant dynamical variables of the particle.) In particular, our $\boldsymbol{X}^E$ differs from Pac's $\boldsymbol{X}'^E$ in terms which represent the transverse component of $i\beta\boldsymbol{\alpha}$. The resultant difference in the spin operators, viz., term $i\beta(\boldsymbol{\alpha}\times\boldsymbol{p})/p$ makes $\boldsymbol{\Sigma}^E$ partly transverse (unlike the purely longitudinal $\boldsymbol{\Sigma}'^E$) and is essential for obtaining the correct commutation relations for the components of $\boldsymbol{\Sigma}^E$.* The helicity operator is the same in both cases, being just $(\boldsymbol{\sigma}\cdot\boldsymbol{p})/p$. As for orbital angular momentum, the new operator $\boldsymbol{L}^E$, like $\boldsymbol{L}'^E$, is purely transverse. The association of $\boldsymbol{x}\times\boldsymbol{p}$ and $\boldsymbol{\sigma}/2$ in $\boldsymbol{L}^E$, observed by B.G.S. and Pac, continues here also.

Finally, it must be presumed that there can be only one 'correct' set of operators representing dynamical variables in any given representation, and when the choice of one such set is made as the correct one (from among an infinity of unitarily equivalent sets of operators), the corresponding operators in any other representation are automatically determined using the appropriate unitary transformation of the chosen set. The choice of a particular set to go with a given

* Any purely longitudinal operator (which does not involve $\boldsymbol{x}$ explicitly) will have commuting components and will therefore be unsuitable for representing angular momentum.

form of the Hamiltonian (i. e. in some chosen representation) must be guided by requirements like the ones we have considered; choice of a position operator, in particular, is equivalent to the assignment of a specific law of interaction with external fields, as has been pointed out by Acharya and Sudarshan.[7] The ultimate justification for the F. W. choice (and hence for ours) must then depend on its success in yielding the correct results for the interaction with an electromagnetic field. Arguments that the position operator should be that which is represented by $\boldsymbol{x}$ in the 'canonical' representation[8] in which the Hamiltonian takes the form $\beta(p^2+m^2)^{1/2}$ reinforce this choice.

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On the Quantization of Physical Space-Time Operators. II

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The characteristic features of the altered theory which was proposed in our previous paper are analysed by using the interaction representation. It is shown that our theory gives the same result for the transition matrix as the conventional field theory does. Therefore, it is regarded that the latter is nothing but the true theory to our altered theory.

§ 1. Introduction and summary

In order to investigate Dirac and Bergmann's field theory, in which the space-time x^ρ are quantized according to the uncertainty relation, an auxiliary condition on the space-time variable was introduced and a new formulation of the condition was proposed in the previous article.[1)]* In the present article, we shall explore the characteristic features of the formalism, and examine whether it introduces the desirable modification of the conventional field theory, as was expected in the article I, by using the interaction representation with respect to both the field variables and space-time field.

It will be proved that we can define the Fourier decomposition in the same way as in the conventional field theory under the appropriate condition, and that the Fourier components commute with the exponential operator $e^{ikx(u)}$. For the free field we can prove that the commutation relation between the Fourier components has the same form as the one in the conventional field theory. Therefore, the Fourier components are the true observables of the system and they have the same character as the conventional theory. As to the interacting fields the commutation relation between the Fourier components cannot be written down explicitly. Therefore we shall construct the S-matrix element by using the interaction representation, and compare its form with that of the conventional field theory. In the representation, we can use the Fourier components of the free field, which describes elementary particles, having nothing to do with space-time field, $x^\rho(u)$, and they give the same contribution to the matrix elements as the conventional field theory. At first sight, the contribution of the part of space-time field seems to give rise to some kind of cutoff at the inner vertex or in the Green functions of the ordinary field in the neighbourhood

* In the following we quote this article as I, and refer to its equations as, for instance, (I. 4. 11).

of light cone, and the cutoff distance becomes of the order of κ^{-1}, where κ^{-1} is the constant with the dimension of length and introduced in the aforementioned auxiliary condition. However, these effects of the space-time field disappear in the final expression of S-matrix. That is, the transition matrix elements in the interaction representation become the same as those of the conventional field theory. Thus both for the free field and for the interacting fields, we may regard that the Fourier components of the field variables are the true observables of the system and they have the same character as that of the conventional field theory, if some appropriate condition, which will perhaps be fulfilled for all the systems with the local interaction, is satisfied. It will mean that the conventional field theory is nothing but the true theory of the Dirac-Bergmann's theory, which was summarized in § 2 of I, in contrast to our altered theory.

The proof of the commutability between the Fourier components and the x^ρ operator will be given in § 2, and the S-matrix treatment for the interacting fields in § 3.

§ 2. Fourier decomposition

In order that the Fourier decomposition of field operator has definite meaning, it must be proved that the operator $e^{ikx(u)}$ is well defined and satisfies the relation

$$\frac{\partial e^{ikx(u)}}{\partial x^\rho} \equiv ik_\rho e^{ikx(u)}, \tag{2·1}$$

and that $e^{ikx(u)}$ or $x^\rho(u)$ commutes with the Fourier component $y_A(k)$; i. e.

$$[x^\rho(u),\ y_A(k)] \equiv 0. \tag{2·2}$$

This equation must hold for each component $y_A(k)$ and for all $x^\rho(u)$ irrespective of the value of u^s.

Before entering into these problems, an explicit definition of the operator $\partial y_A/\partial x^\rho$, which was defined implicitly by the equation

$$\frac{\partial y_A}{\partial u^s} \equiv \frac{1}{2}\left\{\frac{\partial x^\rho}{\partial u^s}, \frac{\partial y_A}{\partial x^\rho}\right\}, \tag{2·3}$$

may be given. For any operator F, we have the relation

$$\frac{\partial F}{\partial u^s} \equiv \frac{1}{i\hbar}[F,\ \mathfrak{H}_s], \tag{2·4}$$

where $\mathfrak{H}_s \equiv \int d^3u\, G_s$ and G_s is the canonical energy-momentum density

$$G_s \equiv \frac{1}{2}\{x^\rho_{|s}, \eta_\rho\} - \delta^0_s \frac{1}{2}\kappa^{-2} g^{\rho\sigma}\eta_\rho\eta_\sigma, \tag{2·5}$$

or

$$G_i \equiv -\frac{1}{2}\kappa^2 g_{\rho\sigma}\left\{\frac{\partial x^\rho}{\partial u^i}, \frac{\partial x^\sigma}{\partial u^0}\right\} + \kappa^4 \frac{\partial x^\rho}{\partial u^i} l_\rho,$$

and (2·5′)

$$G_0 \equiv JH \equiv \frac{1}{2}\kappa^2\left(g_{\mu\sigma}\frac{\partial x^\rho}{\partial u^0}\frac{\partial x^\sigma}{\partial u^0} - \kappa^4 l^2\right).$$

The relation (2·4) is the four-dimensional extension of the equation of motion (I·4·17) in our altered theory. On the other hand, owing to the relation (I·4·14), i. e.

$$\frac{\partial x^\rho}{\partial u^0} \equiv g^{\rho\sigma}\{\kappa^{-2}(\lambda_\sigma + p_\sigma) - \kappa^2 l_\sigma\}, \tag{2·6}$$

the relation

$$\left[\frac{\partial x^\rho}{\partial u^0}, y_A\right] \equiv \kappa^{-2} g^{\rho\sigma}[p_\sigma, y_A] \tag{2·7}$$

holds for any field variable, $y_A(u)$. Therefore, taking into account the commutability between $x^\rho(u)$ and $y_A(u')$, we have

$$\frac{\partial y_A}{\partial u^s} \equiv \frac{1}{i\hbar}\int d^3u' \frac{1}{2}\left\{\frac{\partial x'^\rho}{\partial u'^s}, [y_A, p_\rho']\right\} \tag{2·8}$$

from Eqs. (2·4) and (2·5′). If we can write for any $y_A(u)$ as

$$[y_A(u), p_\rho(u')] \equiv i\hbar\,\mathfrak{Y}_{A\rho}(u)\delta_3(u-u'), \tag{2·9}$$

Eq. (2·8) reduces to

$$\frac{\partial y_A}{\partial u^s} \equiv \frac{1}{2}\left\{\frac{\partial x^\rho}{\partial u^s}, \mathfrak{Y}_{A\rho}\right\}. \tag{2·10}$$

Comparing this equation with the implicit definition of $\partial y_A/\partial x^\rho$, Eq. (2·3), we can put

$$\frac{\partial y_A}{\partial x^\rho} \equiv \mathfrak{Y}_{A\rho}, \tag{2·11}$$

and it is shown that the definition, Eq. (2·3), or more generally for any operator F, the definition

$$\frac{\partial F}{\partial u^s} \equiv \frac{1}{2}\left\{\frac{\partial x^\rho}{\partial u^s}, \frac{\partial F}{\partial x^\rho}\right\} \tag{2·3′}$$

is reasonable.

For example, the free neutral scalar field $\phi(u)$ and free electromagnetic field $A_\mu(u)$ have the character given by Eq. (2·9), as will be seen by using the expression of p_ρ, Eqs. (I·2·10″) and (I·2·10), and we can write down the explicit form of $\partial\phi/\partial x^\rho$ and $\partial A_\mu/\partial x^\rho$ as

$$\frac{\partial\phi}{\partial x^\rho}\equiv l^{-2}l^\beta\phi_{\rho\beta}-l^{-2}l_\rho\pi \tag{2·12}$$

and

$$\frac{\partial A_\mu}{\partial x^\rho}\equiv l^{-2}A_{\alpha\beta\rho}(l^\alpha\delta_\mu{}^\beta-l^\beta\delta_\mu{}^\alpha-g^{\alpha\beta}l_\mu)-l_\rho l^{-4}\pi^\alpha(g_{\alpha\mu}l^2-2l_\alpha l_\mu). \tag{2·12'}$$

In the course of the derivation of (2·12'), we used the auxiliary condition $S\equiv g^{\alpha\beta}\cdot\partial A_\alpha/\partial x^\beta=0$[2] (and accordingly additional Lagrangian density $L_{add}\equiv 1/2\cdot S^2$), which corresponds to the constraint $\phi\equiv\pi^\mu l_\mu=0$.[3] Similarly, the explicit form of the x-derivative of the free Fermion field can be obtained by more careful treatment (Appendix).

The operator $e^{ikx(u)}$ can be defined by the series

$$e^{ikx(u)}\equiv 1+ik_\rho x^\rho(u)+\frac{1}{2}(ik_\rho x^\rho(u))^2+\cdots. \tag{2·13}$$

If we write

$$\left[ik_\rho\frac{\partial x^\rho(u)}{\partial u^0},\ ik_\sigma x^\sigma(u)\right]\equiv i\hbar\kappa^{-2}k^2\delta_3(0)\equiv iZ, \tag{2·14}$$

where Z is a c-number with the infinite value, by the definition (2·13) we have

$$\frac{\partial e^{ikx(u)}}{\partial u^0}\equiv ik_\rho\frac{\partial x^\rho}{\partial u^0}e^{ikx(u)}-\frac{i}{2}Ze^{ikx(u)}$$

$$\equiv e^{ikx(u)}ik_\rho\frac{\partial x^\rho}{\partial u^0}+\frac{i}{2}Ze^{ikx(u)},$$

and then

$$\frac{\partial e^{ikx(u)}}{\partial u^0}\equiv\frac{1}{2}\left\{ik_\rho\frac{\partial x^\rho}{\partial u^0},\ e^{ikx(u)}\right\}. \tag{2·15}$$

Comparing this relation with the implicit definition of the x-derivative, Eq. (2·3'), it is proved that the definition of $e^{ikx(u)}$, Eq. (2·13), satisfies the desirable relation (2·1). Thus, if the commutability between the Fourier component $y_A(k)$ and $x^\rho(u)$ is proved; for instance, the Fourier decomposition of the free scalar field

$$\phi(u)\equiv\int\frac{dk}{(2\pi)^3}\delta(k^2+\mu^2)e^{ikx(u)}\phi(k) \tag{2·16}$$

is consistent with the field equation

$$g^{\rho\sigma}\frac{\partial}{\partial x^\rho}\frac{\partial}{\partial x^\sigma}\phi-\mu^2\phi\equiv 0, \tag{2·17}$$

where the x-derivative of ϕ is given by Eq. (2·3).

In the following, the free neutral scalar field being taken as an example, the explicit definition of $y_A(k)$ is given, and its commutability with $x^\rho(u)$

is proved. The other fields can be treated in a quite similar way. $\phi(k)$ is defined by

$$\frac{1}{(2\pi)^3}\delta(k^2+\mu^2)\,\phi(k) \equiv \phi(k\,;u)e^{-ikx(u)} \tag{2·18}$$

where

$$\phi(k\,;u) \equiv \int \frac{dq}{(2\pi)^4} e^{-ikq}\Phi(q\,;u) \tag{2·19}$$

and

$$\Phi(q\,;u) \equiv \phi(u) + q^\rho \frac{\partial\phi}{\partial x^\rho} + \frac{1}{2}\,q^\rho q^\sigma \frac{\partial^2\phi}{\partial x^\rho \partial x^\sigma} + \cdots \tag{2·20}$$

with the c-number parameter q^ρ.

The quantity $\phi(k\,;u)$ has the following properties,

$$[\phi(k\,;u);\ x^\rho(u')] \equiv 0 \quad \text{for} \quad u^0 = u^{0\prime}, \tag{2·21}$$

and

$$(k^2+\mu^2)\,\phi(k\,;u) \equiv 0, \tag{2·22}$$

and the quantity on the right-hand side of Eq. (2·18) is independent of u^s, as will be proved later. The last fact means that $\phi(k)$ does not depend on u^s and accordingly it commutes with $x^\rho(u)$ for all u^s, Eq. (2·2).

The relation (2·21) is guaranteed by the commutation relations

$$\left.\begin{aligned} &\left[x^\rho(u), \frac{\partial^n y_A(u')}{\partial x^{\sigma_1}\cdots\partial x^{\sigma_n}}\right] \equiv 0, \\ \text{and}\quad &\left[\frac{\partial x^\rho(u)}{\partial u^0}, \frac{\partial^n y_A(u')}{\partial x^{\sigma_1}\cdots\partial x^{\sigma_n}}\right] \equiv -i\hbar\kappa^{-2} g^{\rho\lambda} \frac{\partial^{n+1} y_A(u)}{\partial x^\lambda \partial x^{\sigma_1}\cdots\partial x^{\sigma_n}} \delta_3(u-u'), \end{aligned}\right\} \tag{2·23}$$

$$(n=0,\ 1,\ 2,\ \cdots) \quad \text{for} \quad u^0 = u^{0\prime}.$$

Eq. (2·23) can be proved for $y_A(u)$ with the character given by Eq. (2·9) as follows: By Eq. (2·7), we have

$$\left.\begin{aligned} &[x^\rho(u),\ y_A(u')] \equiv 0, \\ &\left[\frac{\partial x^\rho(u)}{\partial u^0},\ y_A(u')\right] \equiv -i\hbar\kappa^{-2} g^{\rho\lambda} \frac{\partial y_A(u)}{\partial x^\lambda} \delta_3(u-u'), \end{aligned}\right\} \tag{2·24}$$

and these equations are nothing but the special case of Eq. (2·23) for $n=0$. By virtue of the field equation of x^ρ, Eq. (I·4·12) and Eq. (2·24), we have

$$\left[\frac{\partial z^\rho}{\partial u^0},\ \frac{\partial y_A}{\partial u^{0\prime}}\right] \equiv \frac{\partial}{\partial u^0}\left(\left[\frac{\partial x^\rho}{\partial u^0}, y_A\right]\right) - \left[\frac{\partial^2 x^\rho}{\partial u^{02}}, y_A\right] \equiv -i\hbar\kappa^{-2} g^{\rho\lambda} \frac{\partial}{\partial u^0}\left(\frac{\partial y_A}{\partial x^\lambda}\right)\delta_3,$$

$$\left[\frac{\partial x^\rho}{\partial u^0},\ \frac{\partial y_A}{\partial u^{i\prime}}\right] \equiv -i\hbar\kappa^{-2} g^{\rho\lambda}\left\{\frac{\partial}{\partial u^i}\left(\frac{\partial y_A}{\partial x^\lambda}\right)\delta_3 - \frac{\partial y_A}{\partial x^\lambda}\,\delta_i\right\}, \quad \left(\delta_i = \frac{\partial\delta_3}{\partial u^i}\right)$$

and then

$$\frac{1}{2}\left\{\frac{\partial x^\lambda}{\partial u^{s'}}, \left[\frac{\partial x^\rho}{\partial u^0}, \frac{\partial y_A}{\partial x^\lambda}\right]\right\} = \left[\frac{\partial x^\rho}{\partial u^0}, \frac{1}{2}\left\{\frac{\partial x^\lambda}{\partial u^{s'}}, \frac{\partial y_A}{\partial x^\lambda}\right\}\right] + \frac{1}{2}\left\{\left[\frac{\partial x^\lambda}{\partial u^{s'}}, \frac{\partial x^\rho}{\partial u^0}\right], \frac{\partial y_A}{\partial x^\lambda}\right\}$$

$$\equiv \left[\frac{\partial x^\rho}{\partial u^0}, \frac{\partial y_A}{\partial u^{s'}}\right] - i\hbar(1-\delta_s^{\,0})\kappa^{-2} g^{\rho\lambda} \frac{\partial y_A}{\partial x^\lambda} \frac{\partial \delta_3}{\partial u^s}$$

$$\equiv -i\hbar\kappa^{-2} g^{\rho\lambda} \frac{\partial}{\partial u^s}\left(\frac{\partial y_A}{\partial x^\lambda}\right)\delta_3 \equiv -i\hbar\kappa^{-2} g^{\rho\lambda} \frac{1}{2}\left\{\frac{\partial x^\sigma}{\partial u^s}, \frac{\partial^2 y_A}{\partial x^\lambda \partial x^\sigma}\right\}\delta_3.$$

Comparing the both sides of this equation, the second equation of (2·23) is proved for $n=1$. The first part of Eq. (2·23) can also be proved in the similar way for $n=1$. In the same way, the relations for arbitrary n can be proved by the mathematical induction. In order to prove the relation (2·22), it is convenient to introduce the quantity $\varphi_\rho \equiv i/\hbar \cdot \int \gamma_\rho d^3u$, which has the property

$$\left[\varphi_\rho(u^0), \frac{\partial^n y_A(u')}{\partial x^{\sigma_1} \cdots \partial x^{\sigma_n}}\right] \equiv \frac{\partial^{n+1} y_A(u')}{\partial x^\rho \partial x^{\sigma_1} \cdots \partial x^{\sigma_n}} \quad (\text{for } u^0 = u^{0\prime}), \tag{2·25}$$

owing to (2·23). Therefore we can write as

$$\Phi(q;u) \equiv \phi + [q^\rho \varphi_\rho, \phi] + \frac{1}{2}[q\varphi, [q\varphi, \phi]] + \cdots \equiv e^{q\varphi}\phi(u)e^{-q\varphi}, \tag{2·26}$$

and

$$\frac{\partial \Phi}{\partial q^\rho} \equiv \frac{\partial \phi}{\partial x^\rho} + q^\sigma \frac{\partial^2 \phi}{\partial x^\rho \partial x^\sigma} + \cdots \equiv \frac{\partial \Phi}{\partial x^\rho} \equiv e^{q\varphi}[\varphi_\rho, \phi]e^{-q\varphi} \equiv e^{q\varphi}\frac{\partial \phi}{\partial x^\rho}e^{-q\varphi}. \tag{2·27}$$

By virtue of these properties, we have

$$ik_\rho \phi(k;u) \equiv \int \frac{dq}{(2\pi)^4} e^{-ikq} \frac{\partial \Phi}{\partial q^\rho} \equiv \int \frac{dq}{(2\pi)^4} e^{-ikq} e^{q\varphi} \frac{\partial \phi}{\partial x^\rho} e^{-q\varphi}, \tag{2·28}$$

and then the relation (2·22) is obtained as

$$(k^2+\mu^2)\,\phi(k;u) \equiv \int \frac{dq}{(2\pi)^4} e^{-ikq} e^{q\varphi}\left(-g^{\rho\sigma}\frac{\partial}{\partial x^\rho}\frac{\partial}{\partial x^\sigma} + \mu^2\right)\phi e^{-q\varphi} \equiv 0.$$

Finally, we must prove that the quantity on the right-hand side of Eq. (2·18) is independent of u^s. By definition, we have

$$\frac{\partial \phi(k;u)}{\partial u^s} \equiv \frac{1}{2}\left\{\frac{\partial x^\rho}{\partial u^s}, \int \frac{dq}{(2\pi)^4} e^{-ikq} \frac{\partial \Phi}{\partial x^\rho}\right\} \equiv \frac{1}{2}\left\{ik_\rho \frac{\partial x^\rho}{\partial u^s}, \phi(k;u)\right\}, \tag{2·29}$$

$$\left[ik_\rho \frac{\partial x^\rho}{\partial u^0}, \phi(k;u)\right] \equiv -i\hbar\kappa^{-2} \int \frac{dq}{(2\pi)^4} e^{-ikq} ik_\rho g^{\rho\lambda} \frac{\partial \Phi}{\partial x^\lambda} \delta_3(0) \equiv iZ\phi(k;u),$$

and

$$\left[ik_\rho \frac{\partial x^\rho}{\partial u^0}, e^{-ikx(u)}\right] \equiv -iZe^{-ikx(u)}.$$

Then, the desired relation

$$\frac{\partial}{\partial u^s}(\phi(k;u)e^{-ikx(u)}) \equiv \frac{1}{2}\left\{ik_\rho \frac{\partial x^\rho}{\partial u^s},\ \phi(k;u)\right\}e^{-ikx(u)}$$

$$+\phi(k;u)\frac{1}{2}\left\{-ik_\rho \frac{\partial x^\rho}{\partial u^s},\ e^{-ikx(u)}\right\} \equiv \frac{1}{2}\left[ik_\rho \frac{\partial x^\rho}{\partial u^s},\ \phi(k;u)e^{-ikx(u)}\right] \equiv 0$$

is obtained.

The above discussions remain correct also for the interacting fields, so far as the condition (2·9) is satisfied, as easily seen by following the discussions.

For the free fields we can prove that the commutation relation between the Fourier components has the same form as the one in the conventional theory. For example, for the free scalar field, we have

$$[\phi(k),\ \phi(k')]\delta(k^2+\mu^2) \equiv (2\pi)^3\varepsilon(k)\delta(k+k'). \tag{2·30}$$

The proof may be carried out as follows. First, we consider the quantity

$$[\phi(k;u),\ \phi(k';u')] \equiv \int \frac{dq\,dq'}{(2\pi)^8} e^{-ikq-ik'q'}\,[\Phi(q;u),\ \Phi(q';u')]$$

$$\equiv \int \frac{dq\,dq'}{(2\pi)^8} e^{-i\frac{k+k'}{2}(q+q')-i\frac{k-k'}{2}(q-q')}\left[\phi(u)+q\frac{\partial\phi}{\partial x}+\cdots,\ \phi(u')+q'\frac{\partial\phi'}{\partial x}+\cdots\right] \tag{2·31}$$

for $u^0=u^{0\prime}$. Generally, $[\partial^n\phi/\partial x^n,\ \partial^m\phi'/\partial x^m]$ is the function only of x^ρ_{ij}, because $\partial^n\phi/\partial x^n$ is the linear function of $\phi(u)$ and $\pi(u)$ owing to the bilinear character of p_ρ and of Eqs. (2·6) and (2·23). Therefore the right-hand side of (2·31) commutes with $\phi(u)$ and $\pi(u)$ on the same u^0. Further, from the relations

$$\left[\frac{\partial^{n+1}\phi}{\partial x^\rho \partial x^n},\ \frac{\partial^m\phi'}{\partial x^m}\right]+\left[\frac{\partial^n\phi}{\partial x^n},\ \frac{\partial^{m+1}\phi'}{\partial x^\rho \partial x^m}\right] \equiv \left[\varphi_\rho,\ \left[\frac{\partial^n\phi}{\partial x^n},\ \frac{\partial^m\phi'}{\partial x^m}\right]\right]$$

$$\equiv \int d^3u_1\left[\lambda_\rho(u_1),\ \left[\frac{\partial^n\phi}{\partial x^n},\ \frac{\partial^m\phi'}{\partial x^m}\right]\right] \equiv 0, \tag{2·32}$$

$$\frac{1}{2}q^\rho q^\sigma\left[\frac{\partial^2\phi}{\partial x^\rho \partial x^\sigma},\ \phi'\right]+q^\rho q'^\sigma\left[\frac{\partial\phi}{\partial x^\rho},\ \frac{\partial\phi'}{\partial x^\sigma}\right]+\frac{1}{2}q'^\rho q'^\sigma\left[\phi,\ \frac{\partial^2\phi'}{\partial x^\rho \partial x^\sigma}\right]$$

$$\equiv \frac{1}{2}(q-q')^\rho(q-q')^\sigma\left[\frac{\partial^2\phi}{\partial x^\rho \partial x^\sigma},\ \phi'\right],\ \text{etc.},$$

hold. Thus $[\Phi(q;u),\ \Phi(q';u')]$ depends only on $(q-q')^\rho$ and the integration of the right-hand side of (2·31) must have the factor $\delta^4(k+k')$. If we put

$$[\phi(k),\ \phi(k')]\delta(k^2+\mu^2) \equiv G(k,\ k'), \tag{2·30′}$$

$G(k,\ k')$ must have the form as $\varepsilon(k)\delta(k+k')G(k)$, and $G(k)$ commutes not only with $x^\rho(u)$ and $\dot{x}^\rho(u)$ but also with $\phi(u)$ and $\pi(u)$. Therefore $G(k)$ is a c-number, and from the covariance character it must be a constant independent of k as $k^2=-\mu^2$. Thus we have arrived at the commutation relation, (2·30),

in our theory too. It means that, for the free field system, the physical character of the Fourier components is the same as the conventional field theory and the x^ρ-operator have no effect on it.

On the other hand, according to Eqs. (2·12), (I·2·13) and (I·2·14), we have the commutation relations

$$\left.\begin{aligned} &[\phi(u), \phi(u')] \equiv 0, \\ &\left[\frac{\partial\phi(u)}{\partial x^\rho}, \phi(u')\right] \equiv l^{-2} l_\rho i\hbar\delta_3(u-u') \end{aligned}\right\} \tag{2·33}$$

for $u^0=u^{0\prime}$. In order that the Fourier transformation of Eqs. (2·33) have just the form (2·30), it is necessary for our definition of the Fourier transformation, (2·16), to meet the requirement that the operator relations

$$\begin{aligned} &\int \frac{dk}{(2\pi)^3} e^{ik(x(u)-x(u'))} \varepsilon(k)\, \delta(k^2+\mu^2) \equiv 0, \\ &\int \frac{dk}{(2\pi)^3} e^{ik(x(u)-x(u'))} \varepsilon(k) \delta(k^2+\mu^2) ik_\rho \equiv l^{-2} l_\rho \delta_3(u-u') \end{aligned} \tag{2·34}$$

should be satisfied. This may be considered as a reasonable one in our theory also from the following consideration: Our scheme should be reduced to the conventional field theory in the limit $\kappa^{-1}\to 0$. If $x^\rho(u)$ is parameter, the relations (2·34) are correct only when $x(u)$ and $x(u')$ are located in the mutually spacelike configuration, because the expressions on the left-hand side of these relations are nothing but $iJ(x(u)-x(u'))$ and $i\cdot\partial J(x(u)-x(u'))/\partial x^\rho(u)$, respectively. On the other hand, in our theory the indefinite metric was introduced in the part of the Hilbert space with respect to space-time field as a consequence of the indefinite nature of the metric in Minkowski space as was pointed out in § 4 of the article I. Therefore, it may not be surprising, that the operator relations such as (2·34) coincide with the corresponding c-number relations in the ordinary field theory only when $x(u)-x(u')$ is in a space-like configuration. The concrete formulation of our Hilbert space with indefinite metric should be made in such a way that the relations (2·34) hold for any configuration of $x(u)-x(u')$.

§ 3. Transition matrix

In this section for the purpose of seeing the character of the interacting fields we construct the S-matrix element by using the interaction representation. The equation that corresponds to the Schroedinger equation in the conventional field theory is given by

$$i\hbar\frac{d\Psi}{du^0} = \mathfrak{H}\Psi, \tag{3·1}$$

where

$$\mathfrak{H} \equiv \int d^3u JH \tag{3·1′}$$

and

$$JH \equiv -\frac{1}{2}\{\dot{x}^\rho, (\lambda_\rho + p_\rho)\} - \frac{1}{2}\kappa^{-2} g^{\rho\sigma}(\lambda_\rho + p_\rho)(\lambda_\sigma + p_\sigma)$$

$$\equiv \overline{JH} + \frac{1}{2}\{\overline{\dot{x}^\rho}, p_\rho{}^I\} + \frac{1}{2}\kappa^{-2} g^{\rho\sigma} p_\rho{}^I p_\sigma{}^I, \tag{3·2}$$

which was already given in (I·4·16). In Eq. (3·2), $\dot{x}^\rho$ is given by

$$\dot{x}^\rho \equiv \overline{\dot{x}^\rho} + \kappa^{-2} g^{\rho\sigma} p_\sigma{}^I, \tag{3·3}$$

from Eq. (2·6), and JH and $\overline{\dot{x}^\rho}$ are the expressions obtained by replacing p_ρ by $p_\rho{}^0$ in JH and $\dot{x}^\rho$, respectively, where $p_\rho{}^0$ is the free part of energy-momentum density and its remaining part is expressed as p_ρ^I, that is,

$$p_\rho \equiv p_\rho{}^0 + p_\rho{}^I. \tag{3·4}$$

Here we may insert some remarks concerning the indefinite character of the metric in the x^ρ field part of our Hilbert space. If the metric in the Hilbert space were positive definite and the additional relation $\eta_\rho = 0$ might be regarded as the auxiliary condition for the state vector Ψ,

$$\eta_\rho \Psi = (\lambda_\rho + p_\rho)\Psi = 0, \tag{3·5}$$

Ψ would remain constant for all u^0, as the relation

$$i\hbar \frac{d\Psi}{du^0} = \mathfrak{H}\Psi = 0 \tag{3·6}$$

holds owing to the structure of $\mathfrak{H}$, which was given by Eqs. (3·1′) and (3·2). Further, then in the representation, in which x^ρ is diagonal, the constraints (I·3·5) equivalent to (3·5),

$$i\hbar \frac{\delta\varphi(\{x'(u)\})}{\delta x'^\rho(u)} = p_\rho{}'(u)\varphi(\{x'(u)\}), \tag{3·7}$$

would correspond to the Schroedinger equation. However, owing to the indefiniteness of our Hilbert space, we must use the weaker auxiliary condition,

$$(\Psi^*, \eta_\rho \Psi) = 0, \tag{3·8}$$

than Eq. (3·5), or

$$\eta_\rho^{(+)} \Psi = 0, \tag{3·9}$$

[Eq. (I. 4·30)]. Therefore

$$i\hbar \frac{d\Psi}{du^0} = \mathfrak{H}\Psi \neq 0,$$

and we must calculate the transition matrix concretely to see whether our

scheme deviates from the conventional field theory or not.

In order to define S-matrix, we may use the interaction representation, as usually done. In the representation starting from Eqs. (3·1) and (3·2), the Schroedinger equation can be written as

$$i\hbar\frac{d\Psi''(u^0)}{du^0}=\mathfrak{H}_I(u^0)\Psi''(u^0), \tag{3·10}$$

where

$$\mathfrak{H}_I(u^0)\equiv\int d^3u\left[\frac{1}{2}\{\overline{\dot{x}^\rho},\, p_\rho{}^I\}+\frac{1}{2}\kappa^{-2}g^{\rho\sigma}p_\rho{}^I p_\sigma{}^I\right]. \tag{3·11}$$

The transition matrix $U(u^{0\prime}, u^{0\prime\prime})$ between two different values of u^0, $u^{0\prime}$ and $u^{0\prime\prime}$, which is defined by

$$\Psi''(u^{0\prime})=U(u^{0\prime}, u^{0\prime\prime})\Psi''(u^{0\prime\prime}), \tag{3·12}$$

is expressed as

$$U(u^{0\prime}, u^{0\prime\prime})=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i}{\hbar}\right)^n\int_{u^{0\prime\prime}}^{u^{0\prime}}du_1{}^0\cdots du_n{}^0P(\mathfrak{H}_I(u_1{}^0)\cdots\mathfrak{H}_I(u_n^0)), \tag{3·13}$$

and the S-matrix becomes

$$S=U(+\infty, -\infty), \tag{3·14}$$

in the same way as in the conventional field theory.

Usually the interaction part of energy-momentum density, p_ρ^I, comes from the interaction term of the Lagrangian density L_I in such a way as $p_\rho^I\equiv-l_\rho L_I$.* In the following discussion, it is more convenient to rewrite $\mathfrak{H}_I$ as

$$\mathfrak{H}_I\equiv\int d^3u\left[-\frac{1}{2}\{\bar{J}, L_I\}+\frac{1}{2}\kappa^{-2}l^2L_I^2\right]. \tag{3·15}$$

The second term appears by the elimination of $\overline{\dot{x}^\rho}$ involved in $\bar{J}$ in the first term, and the situation is similar to the case where the interaction Hamiltonian is the derivative type in the conventional theory. At first we consider only the contribution from the first term which have the same form as the conventional theory. As to the free field operators it was proved in the last section that we can perform Fourier transformation of the field operators, and that their Fourier component $y_A(k)$ satisfies the commutation relation (2·2). Therefore, in the calculation of the matrix elements we can treat the $y_A(k)$ field and the $x^\rho(u)$ field independently. Namely, first, we can consider the matrix element for $y_A(k)$ field in the same way as in the conventional field theory, and then deal with the x^ρ field by means of the field equation (I·4·12) and the commutation relation (I·4·15). As to the x^ρ field, the exact treatment is impossible

* For example, for the system of the Fermion field ψ and the scalar field ϕ which interact mutually with scalar coupling, $p_\rho{}^I\equiv l_\rho\cdot g\phi\bar{\psi}\psi$.

owing to the nonlinear character of its field equation, and we must be contented with the linear approximation with respect to the A^ρ-field, which was introduced in Eq. (I·4·23),

$$x^\rho(u) \equiv \kappa^{-1}(\delta_s{}^\rho u^s + A^\rho(u)). \tag{3·16}$$

In this approximation

$$J(u) \equiv \kappa^{-4}\left\{1 + \frac{\partial A^0}{\partial u^0} + \sum_i \frac{\partial A^i}{\partial u^i}\right\}, \quad (i=1, 2, 3) \tag{3·17}$$

and

$$e^{ikx(u)} \equiv e^{i\kappa^{-1}ku}\{1 + i\kappa^{-1}k_\rho A^\rho(u)\}. \tag{3·18}$$

Concerning the Fourier components $y_A(k)$, they have the same character as the conventional field theory, as was shown in the last section. Therefore, when we aim to clarify the relation of our theory to the conventional one, it is sufficient to consider the $x^\rho(u)$ dependent part in the first term of $\mathfrak{H}_I(u^0)$,

$$\frac{1}{2}\{J(u), e^{ikx(u)}\} \equiv \kappa^{-4} e^{i\kappa^{-1}ku}\left\{1 + \frac{\partial A^0}{\partial u^0} + \sum_i \frac{\partial A^i}{\partial u^i} + i\kappa^{-1}k_\rho A^\rho(u)\right\}, \tag{3·19}$$

where the factor $e^{ikx(u)}$ comes from the Fourier decomposition of L_I in Eq. (3·15), and to investigate the expectation value of S-matrix between the vacuum with respect to A^ρ-field, (I·4·29).

For this purpose, in the following we analyse the expression

$$G(u_1, u_2; k_1, k_2) = \left\langle \frac{1}{2}\left\{J(u_1), e^{ik_1x(u_1)}\right\} \frac{1}{2}\left\{J(u_2), e^{ik_2x(u_2)}\right\}\right\rangle_0 \tag{3·20}$$

as a simplest example. Making use of the polarization vectors e_λ introduced in the article I, which have the properties

$$e_3{}^i e_3{}^j = \frac{h_i h_j}{h_0{}^2}, \quad e_1{}^i e_1{}^j + e_2{}^i e_2{}^j = \delta_{ij} - \frac{h_i h_j}{h_0{}^2},$$

and the decomposition of A^ρ, Eqs. (I·4·27), (I·4·27′) and (I·4·27″), we obtain

$$\langle A^0(u_1)\, A^0(u_2)\rangle_0 = -\int \frac{dh}{(2\pi)^3}\theta(h)\,\delta(h^2)\, e^{ih(u_1-u_2)}, \tag{3·21}$$

and

$$\langle A^i(u_1)\, A^j(u_2)\rangle_0 = \int \frac{dh}{(2\pi)^3}\theta(h)\,\delta(h^2)\, e^{ih(u_1-u_2)}$$
$$\times\left[\frac{h_i h_j}{h_0{}^2} e^{-ih_0(u_1{}^0-u_2{}^0)} + (1-ih_0u_1{}^0)(1+ih_0u_2{}^0)\left(\delta_{ij} - \frac{h_i h_j}{h_0{}^2}\right)\right]. \tag{3·21′}$$

Here the relation

$$\langle\{(1-ih_0u_1{}^0)a_\lambda + (1+ih_0u_1{}^0)a_\lambda{}^+\}\{(1-ih_0u_2{}^0)\,a_\lambda + (1+ih_0u_2{}^0)a_\lambda{}^+\}\rangle_0$$
$$= (1-ih_0u_1{}^0)(1+ih_0u_2{}^0)\langle a_\lambda a_\lambda{}^+\rangle_0 = (1-ih_0u_1{}^0)(1+ih_0u_2{}^0)$$

is used. The results (3·21) and (3·21′) can be rewritten as

$$\left.\begin{aligned} \langle A^0(u_1)\,A^0(u_2)\rangle_0 &= -iD^{(+)}(u_1-u_2), \\ \Big\langle \sum_i \frac{\partial A^i(u_1)}{\partial u_1{}^i}\; A^j(u_2)\Big\rangle_0 &= i\frac{\partial D^{(-)}(u_1-u_2)}{\partial u_1{}^j}, \end{aligned}\right\} \tag{3·22}$$

where $D^{(+)}$ is the $D^{(+)}$-function which is used in the conventional field theory. Thus, in our approximation, the concerned quantity (3·20) becomes

$$\begin{aligned} G(u_1, u_2\,;\, k_1, k_2) &= \kappa^{-8} e^{i\kappa^{-1}(k_1u_1+k_2u_2)} \Big\{1+\Big\langle\Big(\frac{\partial A^0}{\partial u^0}+\sum_i\frac{\partial A^i}{\partial u^i}\Big)\Big(\frac{\partial A^0}{\partial u^0}+\sum_j\frac{\partial A^j}{\partial u^j}\Big)\Big\rangle_0 \\ &\quad +\Big\langle\Big(\frac{\partial A^0}{\partial u^0}+\sum_i\frac{\partial A^i}{\partial u^i}\Big) i\kappa^{-1}k_{2\rho}A^\rho(u_2)\Big\rangle_0+\Big\langle i\kappa^{-1}k_{1\rho}A^\rho(u_1)\Big(\frac{\partial A^0}{\partial u^0}+\sum_j\frac{\partial A^j}{\partial u^j}\Big)\Big\rangle_0\Big\} \\ &= \kappa^{-8}e^{i\kappa^{-1}(k_1u_1+k_2u_2)}\Big\{1-i\Box_1 D^{(-)}(u_1-u_2)-\kappa^{-1}k_{2\rho}g^{\rho\sigma}\frac{\partial D^{(+)}}{\partial u_1{}^\sigma}-\kappa^{-1}k_{1\rho}g^{\rho\sigma}\frac{\partial D^{(+)}}{\partial u_2{}^\sigma}\Big\} \\ &= \kappa^{-8}e^{i\kappa^{-1}(k_1u_1+k_2u_2)}\Big\{1+\kappa^{-1}(k_1-k_2)_\rho g^{\rho\sigma}\frac{\partial D^{(+)}}{\partial u_1{}^\sigma}\Big\}, \end{aligned} \tag{3·23}$$

where we used the relation $\Box D^{(+)}=0$. Because the second term

$$\begin{aligned} \kappa^{-1}(k_1-k_2)_\rho g^{\rho\sigma}\frac{\partial D^{(+)}}{\partial u_1{}^\sigma} &= i\kappa^{-1}(k_1-k_2)u\Big\{\frac{1}{2\pi^2}\frac{1}{(\boldsymbol{u}^2-u^{0^2})^2}+\frac{\varepsilon(u)}{2\pi}\delta'(\boldsymbol{u}^2-u^{0^2})\Big\} \\ &\quad -\frac{1}{4\pi}(k_1-k_2)_0\delta_3(u),\quad (u=u_1-u_2) \end{aligned} \tag{3·24}$$

is singular near light cone, it seems to play the very important role in the calculation. At first sight this term might be expected to give rise to some kind of cutoff effect.* However, the appearance of the second term of Eq. (3·23) is provisional, and the contribution from this term to the S-matrix is cancelled out by that from the second term of $\mathfrak{H}_I$ [Eq. (3·15)], as will be seen in the following. The contribution to the second order term of the transition matrix (3·13) from the first part of $\mathfrak{H}_I$, Eq. (3·15), is

$$\frac{1}{2}\Big(\frac{-i}{\hbar}\Big)^2\int_{-\infty}^{\infty} du_1{}^0\,du_2{}^0\,\langle P(\tilde{\mathfrak{H}}_I(u_1{}^0)\tilde{\mathfrak{H}}_I(u_2{}^0))\rangle_0, \tag{3·25}$$

where

$$\tilde{\mathfrak{H}}_I(u^0)\equiv\int d^3u\int d^4k\,\frac{1}{2}\{J(u),\,e^{ikx(u)}\}H(k) \tag{3·26}$$

* If we regard the second term of the function G, (3·24), as the second term of the expanded series of the exponential function, we can replace (3·23) by

$$G(u_1, u_2;\; k_1, k_2)=\kappa^{-8}\exp\Big\{i\kappa^{-1}(k_1u_1+k_2u_2)+i\kappa^{-1}(k_1-k_2)u\,\frac{1}{2\pi^2}\frac{1}{(\boldsymbol{u}^2-u^{0^2})^2}\Big\}.$$

The additional factor behaves as the cutoff factor in the neighbourhood of light cone, and the cutoff distance is the order of κ^{-1}, as is easily seen by the correspondence $\kappa^{-1}u\leftrightarrow x$.

and the symbol $\langle\ \rangle_0$ means the vacuum expectation value with respect to A^ρ-field. By Eq. (3·23) and the partial integration this expression becomes

$$\frac{1}{2}\left(\frac{-i}{\hbar}\right)^2\int dk_1\,dk_2\int_{-\infty}^{\infty}du_1\int_{-\infty}^{u_1{}^0}du_2\,\kappa^{-8}e^{i\kappa^{-1}(k_1u_1+k_2u_2)}H(k_1)H(k_2)$$

$$\times\left\{1+\kappa^{-1}(k_1-k_2)_\rho g^{\rho\sigma}\frac{\partial D^{(+)}(u_1-u_2)}{\partial u_1{}^s}\right\}+(1\cdot2)\ \text{exchange}$$

$$=\frac{1}{2}\left(\frac{-i}{\hbar}\right)^2\int dk_1\,dk_2\int_{-\infty}^{\infty}du_1\left(\int_{-\infty}^{u_1{}^0}du_2\{1+2i\square_1 D^{(+)}(u_1-u_2)\}\right.$$

$$\left.-i\int_{u_1{}^0}^{\infty}d^3u_2\frac{\partial D^{(+)}(u_1-u_2)}{\partial u_1{}^0}\right)\kappa^{-8}e^{i\kappa^{-1}(k_1u_1+k_2u_2)}H(k_1)\,H(k_2)+(1\cdot2)\ \text{exchange}$$

$$=\frac{1}{2}\left(\frac{-i}{\hbar}\right)^2\int dk_1\,dk_2\int_{-\infty}^{\infty}du_1\,du_2\,\kappa^{-8}e^{i\kappa^{-1}(k_1u_1+k_2u_2)}P(H(k_1)\,H(k_2))$$

$$-i\frac{1}{2}\left(\frac{-i}{\hbar}\right)^2\int dk_1\,dk_2\int du^0\int_{u^0}d^3u_1\,d^3u_2\,\kappa^{-8}e^{i\kappa^{-1}(k_1u_1+k_2u_2)}$$

$$\times\left(\frac{1}{2}\{H(k_1),\ H(k_2)\}\frac{\partial D}{\partial u_1{}^0}+\frac{i}{2}[H(k_1),\ H(k_2)]\frac{\partial D^{(1)}}{\partial u_1{}^0}\right),\qquad(3\cdot27)$$

where

$$D^{(+)}=\frac{1}{2}(D-iD^{(1)})\,.$$

The first term of the second integral of (3·27) becomes

$$\frac{1}{2}\left(\frac{-i}{\hbar}\right)^2\int dk_1\,dk_2\int du^0\int d^3u\,\kappa^{-8}e^{i\kappa^{-1}(k_1+k_2)u}H(k_1)\,H(k_2)\,,$$

which is cancelled by the first order contribution from the second term of (3·15). As to the second term of the second integral of (3·27), the quantity

$$\int dk_1\,dk_2\,e^{i\kappa^{-1}(k_1u_1+k_2u_2)}[H(k_1),\,H(k_2)]\qquad(3\cdot28)$$

can be seen as the Fourier decomposition of the expression $[H_I(x_1),\,H_I(x_2)]$ in the conventional field theory, which vanishes for $x_1{}^0=x_2{}^0$, if we remember the correspondence between our results and the conventional field theory, in which x^ρ are treated as parameter $\kappa^{-1}\delta_s{}^\rho u^s\longleftrightarrow x^\rho$, and the interaction Hamiltonian density is given as $H_I(x)=-L_I(x)$. The expression (3·28) vanishes as the u-integral in Eq. (3·27) imposes $u_1{}^0=u_2{}^0$. Therefore there remains only the first integral in Eq. (3·27), and thus we have the same result as that obtained by the conventional theory. This situation will not be changed even if the higher order terms in the perturbation calculation with respect to both $\mathfrak{H}_I$ and

A^ρ-field are taken into account. In practice it is confirmed for the term of the order of A^2, but the calculation is tedious and not reproduced here.

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Appendix

The Fermion field

Because the canonical formalism for the free Fermion field ψ and $\bar{\psi}$ has two constraints,

$$\left.\begin{aligned} \bar{\phi} &\equiv \pi + \frac{\hbar c}{2}\bar{\psi}\gamma^\mu l_\mu = 0, \\ \bar{\phi} &\equiv \bar{\pi} - \frac{\hbar c}{2}\gamma^\mu l_\mu \psi = 0, \end{aligned}\right\} \tag{A·1}$$

and their Poisson bracket $(\phi, \bar{\phi})$ does not vanish, we must use the Dirac bracket

$$(F, G)_D \equiv (F, G) - (F, \phi_i) A^{-1}_{ij} (\phi_j, G), \tag{A·2}$$

instead of the Poisson bracket, where A^{-1}_{ij} is the inverse of the matrix

$$A_{ij} \equiv (\phi_i, \phi_j).^{5)}$$

The commutation relations can be obtained by

$$[F, G] \text{ or } \{F, G\} \equiv i\hbar (F, G)_D, \tag{A·3}$$

where $\{F, G\}$ is to be used only between the Fermion fields. Then, by use of the relations

$$(\lambda_\sigma(u), l_\rho(u')) \equiv l^i_{\sigma\rho}(u)\, \delta_i(u-u'), \tag{A·4}$$

we obtain the commutation relations

$$\left.\begin{aligned} \{\psi(u), \bar{\psi}(u')\} &\equiv -il^{-2}(\gamma^\mu l_\mu)\, \delta_3(u-u'), \\ [\lambda_\sigma(u), \psi(u')] &\equiv -\frac{i\hbar}{2} l^{-2}(\gamma^\mu l_\mu)\gamma^\lambda l^i_{\sigma\lambda}\psi(u')\, \delta_i(u-u'), \\ \text{and} \\ [\lambda_\sigma(u), \bar{\psi}(u')] &\equiv -\frac{i\hbar}{2}\bar{\psi}(u')\gamma^\lambda l^i_{\sigma\lambda} l^{-2}(\gamma^\mu l_\mu)\, \delta_i(u-u'), \end{aligned}\right\} \tag{A·5}$$

* He pointed out to the author that the conventional field theory should be obtained from Dirac-Bergmann's formalism as the true theory, if the auxiliary condition $x^\rho = \kappa^{-1}\delta_s{}^\rho u^s$ is imposed and the Dirac bracket method is used in a similar way to the treatment of the gravitational field.[4)]

and the other commutation relations are not to be modified by the existence of the constraints, (A·1). Then we must use the relations

$$\left[\frac{\partial x^\sigma}{\partial u^0},\ y_A\right]\equiv\kappa^{-2}g^{\sigma\rho}[\lambda_\rho+p_\rho,\ y_A] \tag{A·6}$$

and

$$\frac{\partial y_A}{\partial u^s}\equiv\frac{1}{i\hbar}\int d^3u'\frac{1}{2}\left\{\frac{\partial x'^\rho}{\partial u^{s\prime}},\ [y_A,\ \lambda_\rho'+p_\rho']\right\} \tag{A·7}$$

instead of Eqs. (2·7) and (2·8). By Eqs. (A·5) and the expression of p_ρ, (I·2·10′), we obtain the following forms,

$$\left.\begin{aligned}[\psi,\ \lambda_\rho+p_\rho]&\equiv i\hbar\frac{\partial\psi}{\partial x^\rho}\delta_3,\\ [\bar\psi,\ \lambda_\rho+p_\rho]&\equiv i\hbar\frac{\partial\bar\psi}{\partial x^\rho}\delta_3,\end{aligned}\right\} \tag{A·8}$$

and

$$\left.\begin{aligned}\frac{\partial\psi}{\partial x^\rho}&\equiv l^{-2}(\gamma^\mu l_\mu)\,(\gamma^\lambda\psi_{\rho\lambda}-ml_\rho\psi)\,,\\ \frac{\partial\bar\psi}{\partial x^\rho}&\equiv(\bar\psi_{\rho\lambda}\gamma^\lambda+ml_\rho\bar\psi)\,l^{-2}\,(\gamma^\mu l_\mu).\end{aligned}\right\} \tag{A·9}$$

Therefore, we know that Eqs. (2·10) and (2·23) can be used and all of the proof in § 2 remain correct also for the Fermion field component $\psi(k)$.

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Nuclear Interactions in the $p_{1/2}$-$g_{9/2}$ Configurations

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Zr^{90} offers a very good case for the study of $T=1$ levels in the $p_{1/2}$—$g_{9/2}$ subshells and has recently been explored by several authors. Here earlier calculations of Thankappan and Waghmare are extended and analysed in detail. It is found that a simple central two-body interaction can be constructed which will give correctly the energy levels of the $(g_{9/2})^2$ configuration, and hence also the levels of all $(g_{9/2})^n$ configuration. However, the same interaction fails to give correctly the levels of the other configurations $(p_{1/2})(g_{9/2})$ and $(p_{1/2})^2$. This simple two-body nuclear interaction is thus shown to be configuration dependent. It is pointed out that experimental identification of the as yet unobserved 4^- level would be very helpful for further elucidation of this phenomenon. The results are compared with those of the other authors.

§ 1. Introduction

It is of considerable interest for elucidation of the phenomenology of the nuclear spherical shell model to be able to determine the region of validity of the model, the nature of the coupling scheme for the nucleons and the nature of the two-body effective interaction which will adequately explain and predict the low-lying energy levels of nuclei in this region. Some work in this direction has been done recently in the region $A \cong 90$, $p_{1/2}$-$g_{9/2}$ subshell. The early qualitative predictions of Ford[1] regarding the theoretically expected energy level scheme in Zr^{90} were easily confirmed experimentally.[2] Later a quantitative calculation by Lane[3] for this nucleus, based on the assumptions of a simple *jj* coupling scheme for wave functions and a short-range (δ-type) interaction between the nucleons, gave results which were in poor agreement with the experimentally observed splitting of the levels of the $(g_{9/2})^2$ configuration. Since the observed splitting of the levels 0^+-2^+ of this configuration is of the same order of magnitude as the splitting of the 2^+-4^+ or 4^+-6^+ levels, whereas the short-range interaction would give a much larger depression of the ground state 0^+ relative to the other levels of the configuration, the necessity of taking into account the finite range of the nuclear interaction is quite obvious. This was done by Bayman et al.,[4] and independently by Thankappan and Waghmare[5] at the same time. Finally, Talmi and Unna[6] have recently attempted to determine the matrix elements (diagonal as well as off-diagonal) of the nuclear interaction from the observed spacing of the energy levels in a

number of nuclei in this region. In this paper we would like to present a somewhat improved version of the calculations of Thankappan and Waghmare, compare the various results in some detail and offer some additional comments on the nature of the two-body interaction.

§ 2. Analysis of calculations for Zr^{90}

The theoretical calculation for Zr^{90} is straightforward, and was described by Thankappan and Waghmare and also in more detail by Bayman et al. We sketch a brief outline here for completeness. The levels of the configurations $(p_{1/2})^2$, $J=0^+$; $(g_{9/2})^2$, $J=0^+$, 2^+, 4^+, 6^+, 8^+ ; and $(p_{1/2})(g_{9/2})$, $J=4^-$, 5^- for the last two protons in Zr^{90} are considered. We assume the equality of the radial oscillator parameters $r_p=r_g$. We take the separation energy for the single particle states $p_{1/2}$ and $g_{9/2}$ to be $E[g_{9/2}]-E[p_{1/2}]=1.0$ Mev from the data[7] on Y^{89}. A central two-body interaction between the two-protons of the type $(a+b\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)V(r)$ is assumed where a and b are constants and for convenience in calculations $V(r)$ is taken to be of the Gaussian shape $V(r)=\exp[-(r/r_0)^2]$. The range parameter r_0 only enters the calculations in the combination $\lambda=r_0/r_p=r_0/r_g$. The matrix elements of the two-body interaction for the various states of the two nucleons are calculated by the well-known techniques, which we need not describe here.

We consider a, b and λ as variable parameters and attempt to fit them from the available experimental data by using the following procedure. The energy levels are calculated for the pure configuration $(g_{9/2})^2$, and for different values of λ. For each value of λ, the exchange character of the two-body interaction given by a and b is determined by fitting the experimental energy separations of the levels 2^+, 4^+ and 8^+. The rest of the matrix elements are then evaluated for this choice of the parameters. The results are shown in Fig. 1. The levels 2^+, 4^+ and 8^+ are normalized to the observed values and the positions of the other levels of the pure configura-

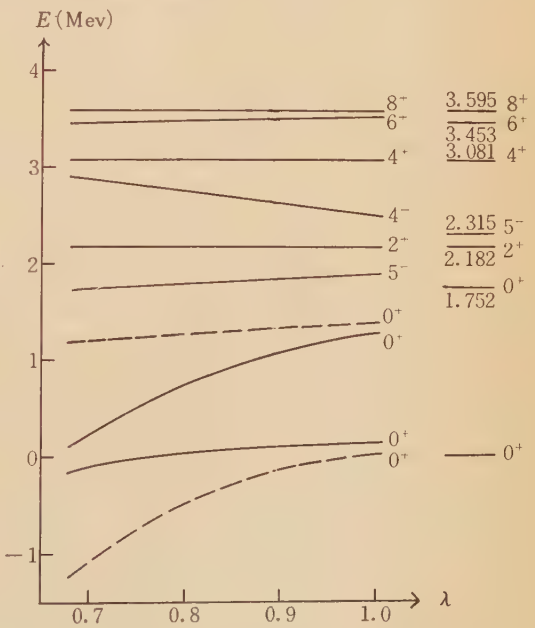

Fig. 1. The calculated and the observed energy levels of Zr^{90}. The observed levels are shown on the right. The full lines show the energies of the pure configuration states and the dotted lines show the energies including the effect of configuration mixing.

tions are shown relative to these. The configuration mixing of the two 0^+ levels, and the consequent repulsion of these is shown by dotted lines. The experimental values of the energy levels as reported by Bjørnholm et al.[8] are also shown.

Firstly, we remark that the position of the 6^+ level is not very sensitive to variations of λ (we remind the reader that for each λ there is a different set of values of parameters a and b). As λ changes from 0.5 to 1.0, the shift in the position of the 6^+ level is only about 0.1 Mev. If the position of the pure 0^+ level of this configuration were known, it would at once enable us to determine a suitable value of λ (and a, b). Since this is not known, we may apply another test to determine a value of λ which will describe the energy levels of the $(g_{9/2})^2$ configuration. It is well known that in the $(g_{9/2})^3$ configuration the $J=7/2^+$ state occurs very close to the ground state $J=9/2^+$ and sometimes even becomes the ground state. For example,[7] in Sr^{85} the $7/2^+$ level occurs at 0.225 Mev above the ground state $9/2^+$. For various values of λ, and the corresponding calculated energy levels of the pure $(g_{9/2})^2$ configuration, we calculate the separation of the $9/2^+ - 7/2^+$ states of $(g_{9/2})^3$, and the results are shown in Table I. It is obvious that for $\lambda=0.9-1.0$, one obtains a low-lying $7/2^+$. We therefore conclude that the levels of the pure $(g_{9/2})^2$ configuration are correctly given at the value $\lambda=1.0$. The corresponding values of the constants a and b are -15.9 Mev and $+4.8$ Mev respectively.

Table I. Levels of $(g_{9/2})^3$ configuration normalized to $E=0$ for the ground state $9/2^+$.

Spin	$\lambda=0.7$	$\lambda=0.8$	$\lambda=0.9$	$\lambda=1.0$
$7/2^+$	0.93	0.51	0.23	0.04
$5/2^+$	1.67	1.27	1.03	0.91
$3/2^+$	2.13	1.71	1.47	1.30
$9/2^+$	2.36	1.96	1.72	1.59

Next, we discuss the relative positions of the two 0^+ levels. It is clear from the Figure that for $\lambda<0.7$, the two 0^+ levels cross over, and hence the ground state would be predominantly $(g_{9/2})_0^2$; thus in what follows we consider only values of $\lambda>0.7$. The analysis of the data on the β-decay of Y^{90} and the γ-decay of the 2^+ state of Zr^{90} by Bayman et al.[4] gives for the wave functions of the two 0^+ states:

$$\left.\begin{aligned}\Psi(0^+) &= 0.8\Psi[(p_{1/2})_0^2] - 0.6\Psi[(g_{9/2})_0^2],\\ \Psi'(0^+) &= 0.6\Psi[(p_{1/2})_0^2] + 0.8\Psi[(g_{9/2})_0^2].\end{aligned}\right\} \quad (1)$$

Now the calculations show that the separation of the two pure states as well as the magnitude of the inter-configuration matrix element varies rapidly with the value of λ. The separation of the pure 0^+ levels decreases from 1.14 Mev at $\lambda=1.0$ to 0.33 Mev at $\lambda=0.7$, while the shift of each due to the inter-configuration matrix element

$$\langle (g_{9/2})_0^2 | V_{12} | (p_{1/2})_0^2 \rangle = V.$$

changes from 0.12 Mev at $\lambda=1.0$ to 0.97 Mev at $\lambda=0.7$. The observed splitting of the two levels (1.76 Mev) and the composition of the wave functions mentioned above can be reasonably well obtained for $\lambda=0.8$, the calculated value for the separation being 1.76 Mev, and the wave function of the ground state

$$0.84\,\mathcal{F}[(p_{1/2})_0^2]-0.55\,\mathcal{F}[(g_{9/2})_0^2].$$

However, it will be noticed that relative to the 2^+, 4^+, 8^+ levels, the 0^+ levels are now predicted too low by about 0.5 Mev. For $\lambda=1.0$, the ground state energy is predicted correctly, but the excited 0^+ state is too low by 0.35 Mev and the configuration mixing is quite small. We later remark on the implications of these results.

Finally, we discuss the odd parity states. Only 5^- is observed, and it is predicted below the 2^+ state for the range of λ of interest in our calculation. However, the excitation energy above the ground state is correctly obtained for $\lambda=0.8$ Mev. The 4^- state is predicted above 5^-, the separation of the two decreases rapidly with increasing λ, and is 0.95 Mev for $\lambda=0.8$ and 0.64 Mev for $\lambda=1.0$. One would thus expect to observe it at $\sim$3.0 Mev excitation, provided the nature of the nuclear interaction in this $(p_{1/2})(g_{9/2})$ configuration is not violently different from that in the other configurations.

To summarize briefly, then, it appears that with a unique choice of the parameters of the central two-body interaction assumed here, one cannot explain all the low-energy levels of the Zr^{90} nucleus with complete accuracy. The choice $\lambda=1.0$ adequately explains the energy levels of the $(g_{9/2})^n$ configurations, but the same choice would not satisfactorily give the lower levels in agreement with the observed values. On the other hand, for $\lambda=0.8$, the lower levels 0^+, 0^+ and 5^- can be satisfactorily explained, but the remaining levels would be pushed up, so that they would be predicted above the observed positions. It is difficult to avoid the conclusion that such a simple two-body interaction appears to be configuration-dependent. For further elucidation of the situation it would be of considerable interest to locate the as yet unobserved 4^- level experimentally.

We would like to emphasize that since the 6^+-8^+ separation is not very sensitive to the choice of λ, we have to rely on the $9/2^+-7/2^+$ separation of the $(g_{9/2})^3$ configuration to obtain a unique value of λ. It turns out that this parameter is a very sensitive test of the choice of the interaction parameters (as pointed out also by Talmi and Unna[6]), and even a small change in the parameters (particularly b) in going from $\lambda=1.0$ to 0.9 produces a considerable shift of the $7/2^+$ level. We can now look at the fitting of the other levels (in particular the ground state 0^+) from a somewhat different point of view. The two 0^+ states, their energies and eigenfunctions, are described by the matrix

$$\begin{pmatrix} 1.76-x & V \\ V & x \end{pmatrix}.$$

Since the value of $(1.76-x)$ (the energy of the $(g_{9/2})_0^2$ state) is effectively known from the analysis of the $(g_{9/2})^2$ configuration, we may ask ourselves the question: where should the unperturbed $(p_{1/2})_0^2$ level be, and what should be the strength of the configuration mixing matrix element V to produce the experimental level spectrum and the eigenfunctions given by Eq. (1) for these states? It is then easily found that for $\lambda=1.0$ and 0.9 the separation of the unperturbed 0^+ states, $\delta=-(2x-1.76)$, should be roughly 0.76 and 0.36 Mev, and for the matrix element $V=0.8$ and 0.86 Mev respectively. These may be compared with the calculated values $\delta=1.15$ and 0.95 Mev and $V=0.35$ and 0.50 Mev. Thus the nuclear interaction which gives correct matrix elements for the $(g_{9/2})^2$ configuration gives for other configurations the results that are in error by $\sim$0.5 Mev. In nuclear spectroscopy calculations where one is satisfied with approximate agreements within $\sim 0.2-0.3$ Mev, such configuration dependence of the interaction would be masked. It is only when one attempts to make a refined analysis and look for precise predictions such as $9/2^+-7/2^+$ separation that the information on detailed nature of the nuclear interaction becomes available.

If the nuclear interaction V_{12} is written in a conventional way as

$$V_{12}=\sum_k f_k(r_1, r_2; \sigma_1, \sigma_2) P_k(\cos\theta_{12}),$$

it is easily seen that the matrix element of the interaction for the $(p_{1/2})_0^2$ state, as well as for the $(p_{1/2})(g_{9/2})$ configuration states, will depend only upon $k=0$, 2 terms, whereas the matrix elements in the $(g_{9/2})^2$ states will depend upon $k=0$, 2, 4, 6, 8 terms. On the other hand, the matrix element V will depend only upon $k=3$, 5. We can push our analysis further to derive from the empirical matrix elements the nature of the interaction in these various substates of relative angular momentum. An analysis along these lines for this nucleus as well as in the *s-d* shell is now in progress. To obtain a consistent interpretation of these matrix elements one may be led to an empirical non-local potential.

§ 3. Comparison with other results

Our results are, in a broad sense, similar to those of Bayman et al.[4] We have not taken into account the Coulomb interaction of the two protons. Our method of choosing the 'best fit' parameters of the two-body interaction is different from theirs, and gives somewhat different results for the parameters. However, the separation of the odd-parity levels predicted by them agrees well with our estimate. The best fit obtained by Bayman et al. shows the 5^- and the excited 0^+ levels lower and the 2^+, 4^+, and 6^+ levels a little higher than

the observed positions. Table II shows the variation of our parameters a, b with λ, and also the 'best fit' values of Bayman et al. It is clear that the effective nuclear interaction obtained for Zr^{90} has an exchange nature similar to that obtained by French and Raz[9] from an analysis of the data on Calcium isotopes, but the well-depth for Zr^{90} is about twice as much. We should like to emphasize, however, that this interaction is quite different in its exchange nature from the other effective interactions used in light nuclei (Rosenfeld[10] or Elliott and Flowers[11]) or in heavy nuclei in the Pb region (Carter et al.[12]). The simple central two-body interaction conventionally used in the nuclear shell model appears to be not only configuration-dependent in the same nucleus, but also varies with the mass of the nucleus. We hope to deal with this extremely interesting phenomenon in a separate paper.

Table II. Nuclear interaction parameters obtained in our calculations. The values obtained by Bayman et al. are shown in the last column.

λ	0.7	0.8	0.9	1.0	0.75
a	−18.0	−15.7	−15.5	−15.9	$-31.2<a<-11.7$
b	17.4	11.1	7.2	4.8	$5.5<b<11.7$

Talmi and Unna[6] have recently carried out an analysis of the level spectra of Zr^{90} and other neighbouring nuclei. The analysis is based on an attempt to rewrite the experimental data in terms of the numerical values of the matrix elements of the effective nuclear interaction in various states, rather than the parameters of the interaction itself. Such an approach is very valuable in correlating a large number of experimental data and in predicting new levels. The advantage of this technique is obviously its independence of the detailed assumptions regarding the explicit nature of the two-body interaction and the independent-particle wave functions. Hence, such an analysis cannot, by its very nature, yield any information on the properties of the phenomenological two-body interaction which it is the aim of our study. It may be seen that the results of Talmi and Unna interpreted in terms of a two-body central interaction do confirm our conclusion that such an interaction should be regarded as non-local or configuration-dependent. The matrix elements relevant to our calculations are

$$\gamma = E[(g_{9/2})^2_{J=2}] - E[(g_{9/2})^2_{J=0}] = 0.866 \text{ Mev},$$

$$\delta_2 = E[(g_{9/2})^2_{J=0}] - E[(p_{1/2})^2_{J=0}] = 0.836 \text{ Mev},$$

$$|V| = |\langle (g_{9/2})^2_{J=0} | H_{12} | (p_{1/2})^2_{J=0} \rangle| = 0.708 \text{ Mev},$$

where the numerical values are as required by Talmi and Unna to explain the experimental level schemes, and we have included in δ_2 the single-particle energy

difference 2β. From Fig. 1 it may be seen that the value of γ is obtained for $\lambda=1.0$. It is thus clear, as we previously remarked, that the energy levels of the pure $(g_{9/2})^2$ configuration can be correctly predicted by the interaction H_{12} with parameters corresponding to $\lambda=1.0$. A three-parameter model is then sufficient to predict all the energy levels of $(g_{9/2})^n$ configurations of identical particles. This interaction would also predict a low-lying $7/2^+$ state of the $(g_{9/2})^3$ configuration, and the failure of the earlier attempts in the search of a simple phenomenological potential that will give such a state is not at all surprising in view of our earlier remarks. The empirical and calculated values of δ and V have already been discussed earlier.

The splitting of the odd-parity states 4^- and 5^- is predicted by Talmi and Unna to be 0.04 Mev, whereas our culculations (as also those of Bayman et al.) based on explicit interactions predict it to be at least ten times larger. We feel that an experimental identification of the 4^- states assumes added importance in view of these conflicting predictions. In this connection, it would also be very helpful to determine low-lying odd-parity states of nuclei such as Nb^{91} or Sr^{85}, for these would belong to the configurations $(p_{1/2})(g_{9/2})^2_J$ and would give rise to doublets $J\pm 1/2$. The splitting of such doublets can immediately be calculated in terms of the splitting of $J=4^-, 5^-$ states of $(p_{1/2})(g_{9/2})$ configuration, and would therefore indirectly give a measure of it.

Acknowledgements

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Self-Trapping of an Electron by the Acoustical Mode of Lattice Vibration. I

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The importance of the acoustical mode of lattice vibration for self-trapping of an electron is emphasized. It is shown that when the coupling constant between the electron and the acoustical mode vibration exceeds a certain critical value, the effective mass of the electron changes discontinuously to such an enormous value that it is practically allowed to take a localized self-trapping state as an eigenstate, in contrast to the case of polaron, in which the effective mass changes continuously with coupling constant. This difference is attributed to the different force range of electron-lattice interaction in the two cases.

§ 1. Introduction

Since Landau[1] pointed out the possibility of self-trapping of an electron in deformable lattice as early as in 1933, much attention has been focused on the interaction between an electron and polar mode of lattice vibration.[2)~6)] One may well suppose that in ionic crystals the electrostatic field caused by the polarization wave interacts with the conduction electron more strongly than the acoustical mode does. In fact, the interaction in the former case is usually so strong[3),6)] that the perturbation theoretical treatment is not a very good approximation whereas it is in the latter case, as long as one starts from the band picture.

If the interaction of an electron with lattice vibration is very strong, the electron is supposed to 'dig its own hole and be trapped there'.[1),2),7)] On the other hand, true eigenstate of the electron-lattice system must be such that it has translational symmetry with wave vector $\boldsymbol{k}$, that is, it is a plane-wave like linear combination of localized electron states accompanied by surrounding lattice distortion,[8)] even if the localized states themselves are good approximations. When the interaction is not very strong, this way of description is not appropriate. One should rather describe the electron to be virtually emitting and absorbing phonons, suffering recoil each time, and as a whole it moves through the crystal with a certain wave vector $\boldsymbol{k}$.[3)] In any case, the word 'self-trapping' or 'localization' should not be taken so literally, and the problem of the possibility of self-trapping should be replaced by the discussion of the magnitude of the effective mass of this quasi-particle, that is, an electron accompanied by virtual phonons or lattice distortion.

In the case of a slow electron interacting with polar mode of lattice vi-

bration, this quasi-particle has been called 'polaron', and several authors investigated its properties, such as self-energy, effective mass, spatial extension and so on.[3)~6)] According to them, especially to Feynman,[5)] the effective mass of the polaron increases rather rapidly but continuously as the interaction becomes stronger. It proved that neither discontinuity nor sudden increase of the effective mass appears, contrary to the expectation by some people. Thus, the word 'self-trapping' has only a relative meaning here, or has to be taken as a limiting case of very large effective mass of the quasi-particle. Moreover, in real crystals, it is limited to rather few cases that one can expect the effective mass or the time of staying of an electron within a unit cell is so large that the word self-trapping is appropriate.

According to the experimental works carried out hitherto, the electrons in alkali[9)] and silver halides[10)] are mobile though the mobility is much smaller than in the cases of Ge and Si, and through the analysis of the temperature dependence of the mobility it has been shown that the scattering of the electron (or exactly polaron) by polar mode vibration plays the main rôle except at rather low temperature where the scattering by acoustical mode vibration and by impurities becomes more important. As for the holes, the situation is the same in the case of AgBr,[11)] but it is reported that the hole in AgCl is several orders of magnitude less mobile than the electron.[10)] The holes in alkali-halides are now believed to be practically in self-trapped state since the experimental works by Känzig and others[12)] appeared.

This essentially qualitative, difference, or quantitatively quite large difference in the behaviors of holes in the cases of alkali-halides and AgCl on the one hand and of AgBr on the other hand, the other properties of both of which do not seem to have essential difference, suggests that the motion of an electron (or a hole) in deformable lattice is divided rather distinctly into mobile type and practically immobile type (self-trapping state) according as the electron-lattice interaction is smaller or larger than a certain critical value.

The interaction of an electron with polar mode, the importance of which has been emphasized by many authors and in fact has been proved experimentally for mobile electrons, fails to explain this discontinuity, as was mentioned above. In this paper we shall emphasize that the interaction with acoustical mode plays an essential rôle in self-trapping of an electron. The most important difference between the two kinds of interaction is that in the case of polar mode the interaction is of long range (electrostatic Coulomb potential) while it is of short range in the case of acoustical mode (deformation potential). The effect of the force range will be discussed in § 2, where we shall show, in rather intuitive way based on the continuum model, that the electron changes its behavior discontinuously from mobile type to self-trapping type when the strength of the interaction with acoustical mode exceeds certain critical value. In view of the strong localization of the electron and the lattice distortion in the self-trapping

state, we shall take into account the discrete structure of the lattice explicitly in § 3, and investigate the eigenstate of the electron-lattice system with wave number $\boldsymbol{k}$, making use of the variation method, with the result that the effective mass of the electron increases discontinuously to an enormous value as the interaction exceeds the critical value. The validity of the approximations used in § 3 are discussed in the final section, more thorough calculations being left to the following paper.

§ 2. Preliminary discussions based on the continuum model

As was mentioned in § 1, the exact eigenstates of the electron-lattice system have the translational symmetry with wave number $\boldsymbol{k}$, and the localized self-trapping state is only an approximate eigenstate. Nevertheless, it seems suggestive in elucidating the effect of the difference in the force ranges of acoustical and polar modes to calculate the adiabatic potential for the localized self-trapping state. We shall take the continuum model for the lattice, for simplicity.

Acoustical mode of lattice vibration is nothing but the elastic wave as long as the wavelength is large compared with the interatomic distance. Let us assume that the crystal has deformed in such a way that the volume dilation $\Delta(\boldsymbol{r})$ is given by

$$\Delta(\boldsymbol{r})=\begin{cases}\Delta & (r<R),\\ 0 & (r>R).\end{cases} \tag{2·1}$$

The elastic potential energy for this deformation is given by

$$U=\frac{1}{2}C\frac{4\pi}{3}R^3\Delta^2, \tag{2·2}$$

where C is an appropriate elastic constant.*

According to the deformation potential theory,[13] an electron in the conduction band suffers the potential energy

$$E_1\Delta(\boldsymbol{r}) \tag{2·3}$$

due to this deformation, where E_1 is the deformation potential constant. If the

* It is to be noted that the local dilation (2·1) induces shearing strain field outside the sphere such that the medium is compressed along the radial direction and stretched in directions normal to it. Since the displacement of the medium at $r=R$ is equal to $\Delta R/3$, the displacement outside the sphere is given by $(\Delta R/3)\ (R/r)$, and the elastic energy stored outside the sphere is proportional to $\Delta^2R^4\int_R^\infty\left(\nabla\frac{1}{r}\right)^2 r^2\mathrm{d}r\propto\Delta^2R^3$. This energy is to be included in (2·2). C, therefore, is a somewhat complicated combination of elastic constants. As for the electron-lattice interaction, the shearing strain has no first order interaction with the electron in the case where the lattice is cubic and the minimum of the electronic energy band is located at $k=0$.[13] Hence we have the form (2·3) for the interaction energy.

sign of Δ is such that $E_1\Delta<0$, we have a three-dimensional square well potential problem for the electron. Let us denote the band effective mass of the electron by m^*. When the depth of the well is sufficiently large, that is, when

$$\delta\equiv-\frac{2m^*}{\hbar^2}R^2E_1\Delta>\left(\frac{\pi}{2}\right)^2, \tag{2·4}$$

there is a bound state, with binding energy

$$-E=\frac{\hbar^2}{2m^*R^2}(\xi\cot\xi)^2, \tag{2·5}$$

where ξ is the root of

$$\left(\frac{\xi}{\sin\xi}\right)^2=\delta \qquad \left(\frac{\pi}{2}<\xi<\pi\right). \tag{2·6}$$

If (2·4) is not valid, there is no bound state and

$$E=0 \tag{2·7}$$

for the ground state of the electron.

The adiabatic potential for the ground state of the electron is now written as

$$W(\Delta, R)=U+E$$
$$=\begin{cases}\dfrac{\hbar^2}{2m^*}\dfrac{\delta^2}{R_0R} & \left(\delta<\left(\dfrac{\pi}{2}\right)^2\right),\\[2ex] \dfrac{\hbar^2}{2m^*}\left[\dfrac{\delta^2}{R_0R}-\dfrac{\delta\cos^2\xi}{R^2}\right] & \left(\delta>\left(\dfrac{\pi}{2}\right)^2\right),\end{cases} \tag{2·8}$$

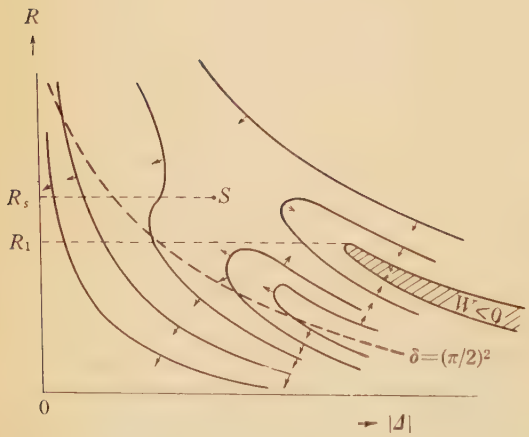

Fig. 1. Schematic representation of the equipotential lines for the adiabatic potential of the electron-acoustical mode system on the elastic continuum model. R is the radius of the sphere in which there is a uniform volume dilation Δ. The arrows give the direction of *decreasing* energy.

where

$$R_0=\frac{3m^*|E_1|^2}{\pi\hbar^2C}. \tag{2·9}$$

Taking R constant, (2·8) increases monotonously with δ for larger values of R, while it takes a maximum and a minimum, and then increases monotonously for smaller values of R. The equipotential line of (2·8) on the (Δ, R)-plane is shown schematically in Fig. 1. When δ is larger than $(\pi/2)^2$ (the right side of the broken line), a bound state for the electron appears, and when

$$R<R_1\equiv R_0\cos^2(2\xi_1)=0.0476R_0 \tag{2·10}$$

where $\xi_1=2.25$ is the root of

$$\frac{\tan(2\xi)}{2\xi}=1 \qquad \left(\frac{\pi}{2}<\xi<\pi\right),$$

W takes negative values for appropriate values of Δ (the shaded region in Fig. 1).

If the crystal were perfectly continuous elastic medium, the stable state of the system corresponds to infinitesimally small R and infinitely large Δ, with infinitely deep adiabatic potential. Due to the atomic structure of the crystal, R must be larger than a' where a' is of the same order of magnitude with interatomic distance a.

Consequently, we can conclude as follows. If R_1 is larger than a', that is, if

$$\frac{m^*|E_1|^2}{\hbar^2 C a'}>22.0, \tag{2·11}$$

the stable state of the system is such that the electron is trapped at the lattice distortion which takes as small a spatial extension as is compatible with atomic structure of the crystal, while the state in which the electron moves through the lattice without lattice distortion is metastable. The latter statement is evident because these two states are completely different from each other being separated by a *barrier* of adiabatic potential.

The saddle point of this barrier is given by

$$\left.\begin{aligned} R_s&=2\left(\frac{\sin 2\xi_s}{2\xi_s}\right)^2 R_0=0.0578R_0 \\ \delta_s&=\left(\frac{\xi_s}{\sin\xi_s}\right)^2 \\ W_s&=\frac{\hbar^2}{8m^*R_0^2}\,\frac{\xi_s^6}{\sin^6\xi_s\cos^2\xi_s} \end{aligned}\right\}, \tag{2·12}$$

where $\xi_s=1.929$ is the root of

$$\frac{1}{\xi}-\mathrm{ctn}\xi+\frac{1}{3}\tan\xi=0 \qquad \left(\frac{\pi}{2}<\xi<\pi\right).$$

It is important to note that at the saddle point a bound state is already split off from the conduction band, and that the adiabatic approximation is valid there.

The existence of the potential barrier between the two types of states is due to the fact that for the square well potential the bound state appears only when the depth or the radius is larger than a finite value. Before the energy gain due to the binding appears, the lattice must suffer finite distortion with finite elastic energy. However, one may well ask why one has to confine the distortion to the shape given by (2·1). Were there no path in the functional space of $\Delta(\boldsymbol{r})$ through which one can get to the negative W region starting from the undistorted state without any potential barrier? In order that the

bound state appears for any small distortion, $\Delta(\boldsymbol{r})$ must decay as r^{-1} or more slowly with increasing r. This makes the elastic energy $U=(C/2)\int \Delta(\boldsymbol{r})^2 d\boldsymbol{r}$ diverge at $r=\infty$. The energetically favorable distortion is such that $\Delta(\boldsymbol{r})$ is more or less localized. That is the reason why we took the form (2·1).

The situation is quite different in the case of polar mode. Denote by $\boldsymbol{P}(\boldsymbol{r})$ the local electric polarization due to the relative displacement of positive and negative ions. The space charge $\rho(\boldsymbol{r})$ due to this polarization is given by $\rho(\boldsymbol{r})=-\mathrm{div}\,\boldsymbol{P}(\boldsymbol{r})$, while the electrostatic potential energy for the electron is given by $-e\varphi(\boldsymbol{r})$ where $\Delta\varphi(\boldsymbol{r})=-4\pi\rho(\boldsymbol{r})$.

Let us take the polarization field such that

$$-e\varphi(\boldsymbol{r})=\begin{cases} -\dfrac{Ze^2}{r} & r>b, \\ -\dfrac{Ze^2}{b} & r<b, \end{cases} \tag{2·13}$$

where Z and b are parameters. $\boldsymbol{P}(\boldsymbol{r})$ is then given by

$$\boldsymbol{P}(\boldsymbol{r})=\begin{cases} -\dfrac{Ze}{4\pi}\dfrac{\boldsymbol{r}}{r^3} & r>b, \\ 0 & r<b, \end{cases} \tag{2·14}$$

and the electrostatic energy stored in the lattice is

$$U=\text{const.}\int \boldsymbol{P}(\boldsymbol{r})^2 d\boldsymbol{r}=\frac{1}{2K}\frac{Z^2}{b}, \tag{2·15}$$

where K is an appropriately defined polarizability of the lattice. It is important to note that in the present case it is possible to choose a pattern of lattice distortion such that the lattice energy integral converges and yet there are bound states for any small magnitude of the distortion. The binding energy of the ground state is given by

$$-E=\frac{Ze^2}{2r_0}, \quad r_0=\frac{\hbar^2}{m^*Ze^2}, \tag{2·16}$$

if $r_0 \gg b$ so that the flattening of the potential (2·13) at $r<b$ has no sizable effect, while if $r_0 \ll b$ there is no effect of uncertainty principle and $-E$ is given by Ze^2/b itself. Combining these limiting cases we can write

$$-E=\frac{Ze^2}{2r_0+b} \tag{2·17}$$

for qualitative purpose.

The equipotential line of the adiabatic potential $W=U+E$ on the (Z, b)-plane is shown in Fig. 2. There is a negative energy region (shaded in the Figure) which begins from the undistorted state and there is only one minimum point,

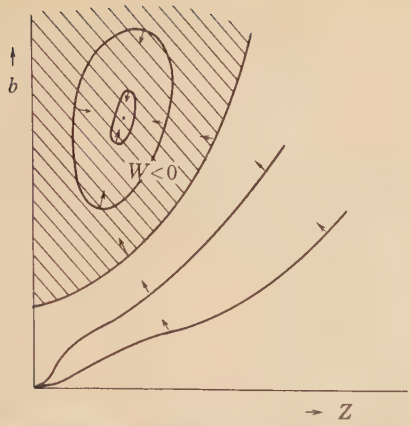

Fig. 2. Schematic representation of the equipotential line for the adiabatic potential of the electron-polar mode system on the dielectric continuum model. b is the radius of the sphere on the surface of which there is a uniform charge, $+Ze$ in total, due to the polarization field surrounding the sphere. Z is a parameter representing the strength of polarization.

$$Z=\frac{8}{9}e^2K, \qquad b=\frac{9}{2}\frac{\hbar^2}{m^*e^4K}. \tag{2·18}$$

There is no potential barrier between this stable state and the undistorted state.

As the coupling constant K increases, the polarization Z of the stable state increases, and the radius b of the space charge and also the radius of electron orbit decreases. That is, the stable configuration changes continuously with coupling constant, in contrast to the case of acoustical mode where the stable state changes abruptly from the undistorted configuration into the strongly distorted configuration.

It is easy to see from the above discussions that the difference between these two modes of lattice vibration results from the different force ranges of electron-lattice interaction in the two cases. As is given by (2·3), a local dilation induces the potential for the electron at the same position only, that is, the interaction is a short-range force. The local space charge due to the electric polarization, on the other hand, gives rise to electrostatic potential at a distant point, too. A long-range potential, such as Coulomb potential, can always accommodate an electron, while a short-range potential, such as square well potential can bind an electron only when the depth of the potential becomes sufficiently large, at the cost of finite energy of the lattice.

Of course, the above discussion is not fully satisfactory in that we have not taken into account the translational symmetry of the eigenstate, and in that we have made use of the adiabatic approximation. Moreover, we found that for the self-trapping state in the case of acoustical mode, the continuum theory breaks down which compelled us to introduce a cut-off length a'. In the following section we shall explicitly take into account the discrete nature of the lattice which is essential for the self-trapping state, and when we apply a variation principle, we shall choose *ab initio* the trial eigenfunctions which is consistent with the translational symmetry of the lattice, in order to see how the effective mass changes with the coupling constant. We confine ourselves to the case of acoustical mode, because the case of polar mode has been fully

explored by a number of authors.

§ 3. Discontinuous change of the effective mass with coupling constant

Let us take tight binding approximation for the electron in the crystal lattice, neglecting the overlap energies other than that for the nearest neighbors: $-V$. Making use of the second quantization formalism for the electron and phonons, we can write the total Hamiltonian as follows:

$$\begin{aligned} H &= K_e + I + H_L \\ &= -V \sum_m \sum_n{}^{\wedge} a_m^* a_n \\ &\quad + \sum_m \sum_{\boldsymbol{w}} E_1 \left(\frac{\hbar}{2NMu}\right)^{1/2} iw^{1/2} \{b_{\boldsymbol{w}} \exp(i\boldsymbol{w}\cdot\boldsymbol{m}) - b_{\boldsymbol{w}}^* \exp(-i\boldsymbol{w}\cdot\boldsymbol{m})\} a_m^* a_m \\ &\quad + \sum_{\boldsymbol{w}} \hbar u w b_{\boldsymbol{w}}^* b_{\boldsymbol{w}}. \end{aligned} \tag{3·1}$$

Here a_m^* and a_m are those operators which create and annihilate, respectively, the electron at the m-th lattice, the position vector of which is denoted by $\boldsymbol{m}$. $b_{\boldsymbol{w}}^*$ and $b_{\boldsymbol{w}}$ are those for phonon with wave number $\boldsymbol{w}$. N, M and u are the total number of unit cells, the mass of a unit cell, and sound velocity, respectively. The symbol $\wedge$ over the double sum of the first term means that the summation is to be carried out only when m and n are nearest neighbors to each other. The second term is nothing but the Fourier expansion of the deformation potential given by (2·3), in a discrete lattice space.

We now investigate the lowest eigenstate of (3·1) with wave number $\boldsymbol{k}$, by a variational procedure. We apply the intermediate coupling method[15] which is usually expected to be useful when one wants to see how the weak coupling and strong coupling solutions are connected, and which has been successfully applied to the polaron problem.[4] One important difference between the polaron problem and the present problem is that the total pseudo-momentum operator,

$$\sum_{\boldsymbol{k}} \hbar \boldsymbol{k} a_{\boldsymbol{k}}^* a_{\boldsymbol{k}} + \sum_{\boldsymbol{w}} \hbar \boldsymbol{w} b_{\boldsymbol{w}}^* b_{\boldsymbol{w}},$$

where $a_{\boldsymbol{k}} \equiv N^{-1/2} \sum_m \exp(-i\boldsymbol{k}\cdot\boldsymbol{m}) a_m$ is the annihilation operator for the electron in the Bloch state, does not commute with (3·1), because (3·1) includes the matrix elements for the Umklapp-process, as the result of explicit consideration of periodic structure of the lattice for the motion of the electron. It is inconvenient to cancel the electron coordinate by specifying the total momentum $\boldsymbol{k}$, as was done in the polaron problem.[4] Instead, we write the trial wave function of the total system as

$$|\Psi_{\boldsymbol{k}}\rangle = N^{-1/2} \sum_m \exp(i\boldsymbol{k}\cdot\boldsymbol{m}) S_{\boldsymbol{k}}(\boldsymbol{m}) T_{\boldsymbol{k}}(\boldsymbol{m}) |0\rangle, \tag{3·2}$$

$$T_k(\boldsymbol{m}) \equiv \sum_{n} \phi_k(\boldsymbol{n})\, a^*_{m+n}, \quad \sum_{n} |\phi_k(\boldsymbol{n})|^2 = 1, \tag{3·3}$$

$$S_k(\boldsymbol{m}) \equiv \exp[\sum_{w} f_k{}^*(\boldsymbol{w}) \exp(i\boldsymbol{w}\cdot\boldsymbol{m}) b_w - (\text{hermite conjugate})] \tag{3·4}$$

with $\phi(\boldsymbol{n})$ and $f(\boldsymbol{w})$ are trial functions for the electron and lattice configurations, respectively. $|0\rangle$ means the vacuum state for both electron and phonon.

Making use of the theorem that

$$\text{if } S \equiv \exp(\alpha^* b - \alpha b^*) \quad (\alpha: \text{ c-number}),$$
$$\text{then } S^{-1} b S = b - \alpha \text{ and } S^{-1} b^* S = b^* - \alpha^*, \tag{3·5}$$

we see that the displacement operator of the l-th atom

$$X(\boldsymbol{l}) = \sum_{w} \{h(\boldsymbol{w}) b_w \exp(i\boldsymbol{w}\cdot\boldsymbol{l}) + (\text{h.c.})\}$$

is transformed by $S_k(\boldsymbol{m})$ into

$$S_k(\boldsymbol{m})^{-1} X(\boldsymbol{l}) S_k(\boldsymbol{m}) = X(\boldsymbol{l}) - \sum_{w} \{h(\boldsymbol{w}) f_k(w) \exp(i\boldsymbol{w}\cdot\boldsymbol{l} - \boldsymbol{m}) + (\text{c.c.})\}.$$

in other words, $S_k(\boldsymbol{m})$ is the operator which makes the lattice deform, and the fact that the displacement represented by the second term is a function of $(\boldsymbol{l}-\boldsymbol{m})$ only, means that the pattern of distortion generated by $S_k(\boldsymbol{m}+\boldsymbol{n})$ is nothing but the pattern obtained as the result of translation of pattern $S_k(\boldsymbol{m})$ by lattice vector $\boldsymbol{n}$. This assures the translational symmetry of (3·2) with wave number $\boldsymbol{k}$. The m-th term of (3·2), $S_k(\boldsymbol{m}) T_k(\boldsymbol{m}) |0\rangle$, represents the state in which the electron is distributed with amplitude $\phi_k(\boldsymbol{n})$ around the m-th lattice point, while the lattice is in the zero point vibrational state around the displaced position corresponding to pattern $S_k(\boldsymbol{m})$.

Making use of elementary operational calculus (see the Appendix), we have

$$\langle 0 | S_k{}^*(\boldsymbol{m}) S_k(\boldsymbol{m}+\boldsymbol{l}) | 0 \rangle = \exp(-\sigma_k(\boldsymbol{l})), \tag{3·6}$$

$$\sigma_k(\boldsymbol{l}) \equiv \sum_{w} f_k{}^*(\boldsymbol{w}) f_k(\boldsymbol{w}) \{1 - \exp(-i\boldsymbol{w}\cdot\boldsymbol{l})\} = \sigma_k{}^*(-\boldsymbol{l}) \tag{3·7}$$

for the overlap integral between the zero point vibrational states for the pattern $S_k(\boldsymbol{m})$ and $S_k(\boldsymbol{m}+\boldsymbol{l})$. In the same way we have

$$\langle \Psi_k | \Psi_k \rangle = \sum_{l} \exp\{i\boldsymbol{k}\cdot\boldsymbol{l} - \sigma_k(\boldsymbol{l})\} \sum_{n} \phi_k^*(\boldsymbol{n}) \phi_k(\boldsymbol{n}-\boldsymbol{l}),$$

$$\langle \Psi_k | K_e | \Psi_k \rangle = -V \sum_{l} \exp\{i\boldsymbol{k}\cdot\boldsymbol{l} - \sigma_k(\boldsymbol{l})\} \sum_{n} \widehat{\sum_{\delta}} \phi_k{}^*(\boldsymbol{n}) \phi_k(\boldsymbol{n}-\boldsymbol{l}+\boldsymbol{\delta}),$$

$$\langle \Psi_k | I | \Psi_k \rangle = -\sum_{l} \exp\{i\boldsymbol{k}\cdot\boldsymbol{l} - \sigma_k(\boldsymbol{l})\} \sum_{n} \phi_k{}^*(\boldsymbol{n}) \phi_k(\boldsymbol{n}-\boldsymbol{l}) \tag{3·8}$$

$$\times \sum_{w} E_1 \left(\frac{\hbar}{2NMu}\right)^{1/2} iw^{1/2} \{\exp(-i\boldsymbol{w}\cdot\boldsymbol{n}) f_k(\boldsymbol{w}) - \exp(i\boldsymbol{w}\cdot\boldsymbol{n}-\boldsymbol{l}) f_k{}^*(\boldsymbol{w})\},$$

$$\langle \Psi_k | H_L | \Psi_k \rangle = \langle \Psi_k | \Psi_k \rangle \sum_w \hbar u w f_k^*(\boldsymbol{w}) f_k(\boldsymbol{w}),$$

where $\widehat{\sum}$ on the summation means that $\boldsymbol{\delta}$ is to run over the nearest neighbours.

As the first approximation, let us neglect the spatial broadening of the electronic wave function and set $\phi(\boldsymbol{n}) = \delta_{n,0}$ where $\delta_{n,m}$ is three-dimensional Kronecker's δ-function. (3·8) is greatly simplified then and we can write the expectation value of the total energy as

$$\begin{aligned} E(\boldsymbol{k}) &= \langle \Psi_k | H | \Psi_k \rangle \\ &= -V \widehat{\sum_\delta} \exp\{i\boldsymbol{k}\cdot\boldsymbol{\delta} - \sigma_k(\boldsymbol{\delta})\} + \sum_w \hbar u w f_k^*(\boldsymbol{w}) f_k(\boldsymbol{w}) \\ &\quad - \sum_w E_1 \left(\frac{\hbar}{2NMu}\right)^{1/2} i w^{1/2} \{f_k(\boldsymbol{w}) - f_k^*(\boldsymbol{w})\}. \end{aligned} \tag{3·9}$$

Minimizing (3·9) with respect to $f_k(\boldsymbol{w})$, we have

$$f_k(\boldsymbol{w}) = \frac{-iw^{1/2} E_1 \left(\frac{\hbar}{2NMu}\right)^{1/2}}{V \widehat{\sum_\delta} \{1 - \exp(-i\boldsymbol{w}\cdot\boldsymbol{\delta})\} \exp\{i\boldsymbol{k}\cdot\boldsymbol{\delta} - \sigma_k(\boldsymbol{\delta})\} + \hbar u w}. \tag{3·10}$$

Inserting this into (3·7) and (3·9), we have

$$\sigma_k(\boldsymbol{\delta}) = \sum_w \frac{E_1^2 \frac{\hbar}{2NMu} w \{1 - \exp(-i\boldsymbol{w}\cdot\boldsymbol{\delta})\}}{\left[V \widehat{\sum_{\delta'}} \{1 - \exp(-i\boldsymbol{w}\cdot\boldsymbol{\delta}')\} \exp\{i\boldsymbol{k}\cdot\boldsymbol{\delta}' - \sigma_k(\boldsymbol{\delta}')\} + \hbar u w\right]^2},$$

$$\begin{aligned} E(\boldsymbol{k}) &= -V \widehat{\sum_\delta} \{1 + \sigma_k(\boldsymbol{\delta})\} \exp\{i\boldsymbol{k}\cdot\boldsymbol{\delta} - \sigma_k(\boldsymbol{\delta})\} \\ &\quad - \sum_w \frac{E_1^2 \frac{\hbar}{2NMu} w}{V \widehat{\sum_\delta} \{1 - \exp(-i\boldsymbol{w}\cdot\boldsymbol{\delta})\} \exp\{i\boldsymbol{k}\cdot\boldsymbol{\delta} - \sigma_k(\boldsymbol{\delta})\} + \hbar u w}. \end{aligned} \tag{3·12}$$

The denominators of (3·11) and (3·12) are real due to (3·7). (3·11) are ν simultaneous equations for ν unknowns $\sigma_k(\boldsymbol{\delta})$, where ν is the number of the nearest neighbors.

To the first order with respect to the coupling constant E_1^2, we have

$$E(\boldsymbol{k}) = -V \widehat{\sum_\delta} \exp(i\boldsymbol{k}\cdot\boldsymbol{\delta}) - \sum_w \frac{E_1^2 \frac{\hbar}{2NMu} w}{V \widehat{\sum_\delta} \exp(i\boldsymbol{k}\cdot\boldsymbol{\delta}) - V \widehat{\sum_\delta} \exp(i(\boldsymbol{k} - \boldsymbol{w})\cdot\boldsymbol{\delta}) + \hbar u w} \tag{3·13}$$

of which the first term is the electronic band energy of the state $\boldsymbol{k}$, and the second term is the self-energy (at absolute zero of temperature) of this state due to the electron-lattice interaction. On the other hand, for very large coupling constant, we can neglect $\exp(-\sigma_k(\boldsymbol{\delta}))$ compared with unity, and we have

$$E(\boldsymbol{k})=-\frac{E_1^2}{2Mu^2}\equiv-\frac{1}{2}D. \tag{3·14}$$

It is the main purpose of this section to see how the solution changes from the weak coupling limit (3·13) to the strong coupling limit (3·14) as we increase the coupling constant continuously.

In solving (3·11) we approximate the band energy by a parabolic form:

$$\varepsilon(\boldsymbol{k})-\varepsilon(0)=-V\sum_{\delta}\{\exp(i\boldsymbol{k}\cdot\boldsymbol{\delta})-1\}\approx\frac{\hbar^2k^2}{2m^*} \tag{3·15}$$

without spoiling the essential feature of the matter. First we solve (3·11) for the state: $\boldsymbol{k}=0$. It is evident that in this case all $\sigma_0(\boldsymbol{\delta})$ are equal:

$$\sigma_0(\boldsymbol{\delta})=\sigma, \tag{3·16}$$

and (3·11) reduces to a single equation

$$\sigma=3g\gamma\exp(2\sigma)\int_0^1\frac{t^3dt}{\{t+\gamma\exp(\sigma)\}^2}, \tag{3·17}$$

while (3·12) can be written as

$$-\frac{E(0)}{\nu V}=(1+\sigma)\exp(-\sigma)+3g\gamma\exp(\sigma)\int_0^1\frac{t^2dt}{t+\gamma\exp(\sigma)} \tag{3·18}$$

where ν is the number of the nearest neighbors, and

$$\gamma\equiv\frac{\hbar uw_0}{\hbar^2w_0^2/2m^*}, \tag{3·19}$$

$$g\equiv\frac{D}{2\nu V}, \tag{3·20}$$

w_0 being the Debye cut-off wave number. Hereafter we shall call g the coupling constant. It is important to note that γ, the ratio of maximum phonon energy to the band width, is usually of the order of 1/100~1/10. We shall call it the non-adiabatic parameter, since the lattice vibration can be treated adiabatically in the limit of $\gamma\to0$.

Instead of solving (3·17) with respect to σ as a function of g and then substituting the σ in (3·18), we shall use the parametric representation of g and E with σ as a parameter. In the limiting case of $\gamma\exp(\sigma)\ll1$, we have

$$g^{-1}\sim\frac{3}{2}\gamma\frac{\exp(2\sigma)}{\sigma},$$

$$\frac{-E(0)}{\nu V}\sim(1+2\sigma)\exp(-\sigma),$$

while when $\gamma\exp(\sigma)\gg1$, we have

$$g^{-1}\sim\frac{3}{4}\gamma^{-1}\frac{1}{\sigma},$$

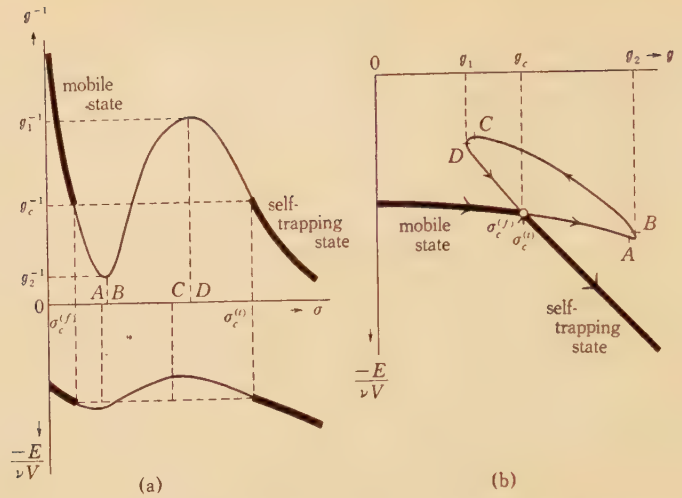

Fig. 3. Schematic representation of the solution of the variation problem. (a) : Parametric representation of the interaction constant g and the calculated energy E with σ as a parameter which is a measure of the magnitude of distortion. (b) : E as a function of g. The arrows on the curve are towards the direction of *increasing* σ.

$$\frac{-E(0)}{\nu V} \sim \frac{4}{3}\gamma\sigma.$$

Since $\gamma \ll 1$, we get $g^{-1}-\sigma$ and $E-\sigma$ curves as shown in Fig. 3a, from which we get the $E-g$ curve shown in Fig. 3b. It is important to note that σ is a three-valued function of g between g_1 and g_2. As the result, $E-g$ curve has a closed loop with a node at $g=g_c$, σ taking the value $\sigma_c^{(f)}$ and $\sigma_c^{(t)}$ on the two branches. Making use of the fact that $\gamma \ll 1$, we have

$$g_c \sim 1, \quad \sigma_c^{(f)} \sim \frac{3}{2}\gamma, \quad \sigma_c^{(t)} \sim \frac{3}{4}\gamma^{-1}, \tag{3·21}$$

$$g_1 \ll g_c \ll g_2. \tag{3·22}$$

As is evident from (3·7), σ represents the magnitude of lattice distortion around the electron. In fact, the expectation value of the number of phonons induced around the electron is given by

$$\bar{n} = \langle \Psi_k | \sum_w b_w^* b_w | \Psi_k \rangle = \sum_w |f_k(\boldsymbol{w})|^2, \tag{3·23}$$

which is of the same order of magnitude with σ's.

When the coupling constant g is smaller than g_c, the stable solution (the one with the lowest energy) corresponds to small $\sigma(<\sigma_c^{(f)})$ while for $g>g_c$ it corresponds to large $\sigma(>\sigma_c^{(t)})$. There appears discontinuity at $g=g_c$, σ jumping from $\sigma_c^{(f)}(\ll 1)$ to $\sigma_c^{(t)}(\gg 1)$. It is evident that the former solution corresponds to the mobile state and the latter to the (plane wave of) self-trapping state. The

second lowest solution for $g_1<g<g_2$ represents a metastable state which is the self-trapping state or the mobile state according as $g<g_c$ or $g>g_c$. (See the discussion in § 2.) Whether the third solution has a definite physical meaning is not clear at present.

Let us calculate the effective mass of the self-trapped electron. Since $\exp(-\sigma)\ll 1$ in this case, the $\boldsymbol{k}$-dependent term of $\sigma_k(\boldsymbol{\delta})$ in (3·11) is of the order of $\exp(-\sigma)$ in comparison with the main term $\sigma_0(\delta)=\sigma$, and we have, from (3·12),

$$E(\boldsymbol{k})-E(0) = -V\exp(-\sigma)\widehat{\sum_{\delta}}\{\exp(i\boldsymbol{k}\cdot\boldsymbol{\delta})-1\}+0(\exp(-2\sigma)), \quad (3\cdot 24)$$

that is, the effective mass of the self-trapped electron is $\exp\sigma$ times the band effective mass. Since $\sigma^{(t)}\geqq\sigma_c^{(t)}=3/4\cdot\gamma^{-1}$, the ratio is an enormous quantity of the order of $e^{10}\sim e^{100}$. It is almost nonsense from the practical point of view to speak of the band formation of the self-trapped electron; it is practically immobile at absolute zero of temperature. The situation is quite different for the case of polaron, where neither the discontinuity of the effective mass (as the coupling increases) nor such a large distortion is expected to take place.

Finally, it is to be noted that the large discontinuity and the coexistence of essentially different states, is partly due to the smallness of non-adiabatic parameter γ. In fact, if γ is as large as $\gtrsim 1$, the $g^{-1}-\sigma$ curve of Fig. 3a becomes monotonous, and the discontinuity disappears. The situation is analogous to the problem of gas-liquid transition if we let temperatare correspond to γ.

§ 4. Discussions

We must now discuss the approximations in the preceding section. The first point is concerned with the Ansatz $\phi(\boldsymbol{n})=\delta_{n,0}$. If we treat the case of polar mode in the same way as in § 3, with the same Ansatz, there appears a discontinuity of the same kind as was obtained there. This discontinuity seems to be illusory in view of the results obtained by many authors on the continuum dielectric model. That the Ansatz $\phi(\boldsymbol{n})=\delta_{n,0}$ is inappropriate in the case of polar mode is evident from the discussions in § 2, where we found that there is one and only one optimum broadening of the electron wavefunction, which is usually much larger than the interatomic distance and can be determined independently of the atomic structure of the lattice.

On the other hand this Ansatz seems appropriate or at least tolerable in the case of the acoustical mode, because we found, in § 2, that in the self-trapping state the electron has a tendency to shrink as small as possible within the compatibility with the atomic structure of the lattice. Moreover, the results of § 3 in the weak-coupling case coincide with the results obtained by the conventional perturbation theory which is sufficiently good approximation in the case of acoustical mode.

Thus it seems rather probable that discontinuity obtained in § 3 is a genuine one in the case of acoustical mode, although the approximations used there lead to dangerous results in the case of polar mode. It might be desirable to relax the Ansatz $\phi(\boldsymbol{n})=\delta_{\boldsymbol{n},0}$, but in that case there appear many $\exp(-\sigma_k(\boldsymbol{l}))$ in the expression for the energy expectation value, and it is practically impossible to solve a number of simultaneous equations for $\sigma(\boldsymbol{l})$'s.

The second point is concerned with the Ansatz (3·4). This operator makes the origin of vibration displace; however, it does not include the effect of change of frequencies and the effects of higher order.

In the following paper we shall investigate the broadening of electron wave function and the corresponding lattice distortion and frequency change in the self-trapping state more thoroughly, at the cost of neglecting the effect of band formation which, in fact, has been shown to be quite small in § 3 of the present paper. Thereby we shall use the perturbation expansion in V (that is, in g^{-1}) and show that the convergence is fairly rapid, that is, the electronic wave function is in fact almost concentrated on the center atom even when g^{-1} is as large as a few times of g_c^{-1}. This partly justifies the Ansatz $\phi(\boldsymbol{n})=\delta_{\boldsymbol{n},0}$, and is also consistent with the nature of discontinuity discussed in § 3; that is, there exist two different types of well-defined states (one stable and the other metastable), and as g passes g_c, the only thing to take place is the reversal of energies of the two states, nothing discontinuous occurring in each state.

By the same method we shall also investigate the activated state through which the self-trapping state at some lattice point jumps to its nearest neighbor, and point out that it is when $g>g_c'(>g_c)$ that the electron moves through the lattice by the 'hopping motion',[8] while for intermediate coupling region: $g_c<g<g_c'$, the electron will rather make a 'Schub-type motion', that is, it is once released from the self-trapping state and moves a few lattice distances before it becomes again self-trapped, because the activated state mentioned above has a higher energy than that of the mobile state.

Finally we must answer the most important question: Is it possible for real crystals that the coupling constant is as large as $g_c(\sim 1)$? Assuming simple cubic lattice with nearest neighbor distance a, we have

$$Va^3=\frac{\hbar^2}{2m^*} \tag{4·1}$$

by (3·15), and (3·20) becomes

$$g=\frac{1}{6}\frac{E_1^2m^*}{\hbar^2Ca} \quad (C=Mu^2/a^3 \text{ is elastic constant})$$

$$=0.035\frac{E_1(\text{in eV})^2(m^*/m)}{C(\text{in } 10^{12}\,\text{dyne/cm}^2)a(\text{in Å})}. \tag{4·2}$$

Since E_1 is usually of the order of several electron volts, it is fairly possible

in relatively soft crystal with narrow band that the electron or hole becomes self-trapped. Quantitatively, of course, the criterion may vary to some extent if we change the model (for instance, other lattice structures, the consideration of the overlap energy V' for the second neighbor, or of the non-diagonal element $\langle m|I|n\rangle$ of the interaction Hamiltonian which seems important for the Känzig center), or if we improve the approximations of § 3, as we shall do in the following paper.

Appendix

Consider the following two operators;

$$A=\alpha^* b-\alpha b^*,\quad A'=\alpha'^* b-\alpha' b^*$$

where b, b^* are phonon operators and α, α' are complex numbers. The expectation value

$$F(\lambda)=\langle 0|\exp(-\lambda A)\exp(\lambda A')|0\rangle_L$$

can be evaluated by differentiation, for it gives

$$F'(\lambda)=\{\alpha^*(\alpha'-\alpha)-\alpha'(\alpha'^*-\alpha^*)\}\lambda F(\lambda)$$

if one makes use of (3·5). After integration we have

$$F(\lambda)=\exp[\{\alpha^*(\alpha'-\alpha)-\alpha'(\alpha'^*-\alpha^*)\}\lambda^2/2]. \qquad (A\cdot 1)$$

In the same way we have

$$\begin{aligned}
&\langle 0|\exp(-\lambda A)b\exp(\lambda A')|0\rangle=-\lambda\alpha' F(\lambda)\\
&\langle 0|\exp(-\lambda A)b^*\exp(\lambda A')|0\rangle=-\lambda\alpha^* F(\lambda) \qquad (A\cdot 2)\\
&\langle 0|\exp(-\lambda A)b^* b\exp(\lambda A')|0\rangle=\lambda^2\alpha^*\alpha' F(\lambda).
\end{aligned}$$

(A·1) is made use of in deriving (3·6) and (3·7) while (A·2) is used in deriving (3·8).

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The Pair Correlation Function of an Imperfect Electron Gas in High Densities

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The pair correlation function, or the relative pair distribution function, of a high density imperfect electron gas in the ground state is computed numerically versus the distance between two electrons with antiparallel spins on the one hand and with parallel spins on the other, for three values of density for each. In the latter case, it is computed, using an approximate expression, only over the distance small compared with the reciprocal of the Fermi momentum divided by $\hbar$. In both cases it increases monotonously with distance for a value of density and decreases with density for a fixed distance due to the increasing effect of Coulomb repulsion with decreasing density.

§ 1. Introduction

It has been well known that, in an ideal electron gas, the pair correlation function $g(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2)$, that is, the probability of finding the second electron of spin σ_2 at the position $\boldsymbol{r}_2$ when the first of spin σ_1 is known to be at $\boldsymbol{r}_1$, is, in the ground state, unity over the whole value of $|\boldsymbol{r}_1-\boldsymbol{r}_2|$ if $\sigma_1\neq\sigma_2$, and, if $\sigma_1=\sigma_2$, increases gradually from zero with $|\boldsymbol{r}_1-\boldsymbol{r}_2|$, as given in Eq. (4), showing the hole, the so-called Fermi hole,[1] near $|\boldsymbol{r}_1-\boldsymbol{r}_2|=0$.

If the Coulomb repulsion among electrons is taken into account, the pair correlation function may, of course, be expected to show a behavior considerably different from the ideal case, and it is difficult to obtain its expression in the moderate densities. Only in the case of extremely high densities, the gas behaves nearly ideally, and we can calculate the pair correlation function using the Green function formalism, as was the case in calculating the correlation energy up to the terms proportional to $\ln r_s$ and also independent of r_s, following the basic idea of Gell-Mann and Brueckner.[2]

In this report, confining ourselves to such a case of high densities, we first give the general expression of the pair correlation function in § 2 and compute it numerically in the cases of antiparallel spin (§ 3) and of parallel spin only with an approximate expression (§ 4). The results are given in Fig. 1, Fig. 2, Fig. 3 and Fig. 4.

§ 2. Pair correlation function

In the ground state of an electron gas, the pair correlation function (hereafter abbreviated as p.c.f.) $g(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2)$ of the two electrons at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ and

with spin σ_1 and σ_2, respectively, is defined as

$$g(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2) = (n/2)^{-2}\langle\psi_{\sigma_1}^*(\boldsymbol{r}_1)\psi_{\sigma_2}^*(\boldsymbol{r}_2)\psi_{\sigma_2}(\boldsymbol{r}_2)\psi_{\sigma_1}(\boldsymbol{r}_1)\rangle, \tag{1}$$

where ψ and ψ^* are field operators, n the mean number density and $\langle\ \ \rangle$ means the expectation value of the operator inside the bracket in the ground state of the gas.

By the two-electron Green function $G(\boldsymbol{r}_1\sigma_1 t_1, \boldsymbol{r}_2\sigma_2 t_2; \boldsymbol{r}_1'\sigma_1' t_1', \boldsymbol{r}_2'\sigma_2' t_2')$, defined as

$$G(\boldsymbol{r}_1\sigma_1 t_1, \boldsymbol{r}_2\sigma_2 t_2\,;\, \boldsymbol{r}_1'\sigma_1' t_1', \boldsymbol{r}_2'\sigma_2' t_2')$$

$$= i^2\langle \boldsymbol{T}(\psi_{\sigma_1}(\boldsymbol{r}_1 t_1)\psi_{\sigma_2}(\boldsymbol{r}_2 t_2)\psi_{\sigma_2'}^*(\boldsymbol{r}_2' t_2')\psi_{\sigma_1'}^*(\boldsymbol{r}_1' t_1'))\rangle,$$

$\psi_\sigma(\boldsymbol{r}, t)$ and $\psi_\sigma^*(\boldsymbol{r}, t)$ being the Heisenberg representation of $\psi_\sigma(\boldsymbol{r})$ and $\psi_\sigma^*(\boldsymbol{r})$, the p.c.f. $g(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2)$ is expressed as

$$g(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2) = (n/2)^{-2} i^{-2} \lim_{t'\to t+} G(\boldsymbol{r}_1\sigma_1 t, \boldsymbol{r}_2\sigma_2 t\,;\, \boldsymbol{r}_1\sigma_1 t', \boldsymbol{r}_2\sigma_2 t'), \tag{2}$$

the right-hand side becoming independent of t. In a free electron gas, replacing a set of $\boldsymbol{r}t$ by x,

$$G_0(x_1\sigma_1, x_2\sigma_2\,;\, x_1'\sigma_1, x_2'\sigma_2)$$

$$= G_0(x_1\sigma_1\,;\, x_1'\sigma_1)G_0(x_2\sigma_2\,;\, x_2'\sigma_2) - G_0(x_1\sigma_1\,;\, x_2'\sigma_2)G_0(x_2\sigma_2\,;\, x_1'\sigma_1), \tag{3}$$

where $G_0(x\sigma\,;\, x'\sigma')$ stands for the one-particle Green function without interaction defined by

$$G_0(x\sigma\,;\, x'\sigma') = i\langle \boldsymbol{T}(\psi_\sigma(\boldsymbol{r}t)\psi_\sigma^*(\boldsymbol{r}'t'))\rangle.$$

In the limit of $t'\to t+$, after easy calculations we obtain

$$g_0(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2) = 1 - \delta(\sigma_1, \sigma_2)9(rp_F)^{-2} j_1^2(rp_F), \tag{4}$$

showing the Fermi hole in the case of parallel spin.

Now in an interacting electron gas, the sum of E_c (the correlation energy) and E_x (the exchange energy) is given as

$$E_x + E_c = (1/2)\sum_{\sigma_1\sigma_2}\int_0^1 d\lambda/\lambda \iint d\boldsymbol{r}_1\, d\boldsymbol{r}_2\, \lambda v(\boldsymbol{r}_1 - \boldsymbol{r}_2)(n/2)^2[g(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2\,;\, \lambda) - 1],$$

assuming the presence of a uniform background of positive charges that makes the system neutral. Here $v(\boldsymbol{r}_1 - \boldsymbol{r}_2)$ is the Coulomb potential and $g(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2\,;\, \lambda)$ denotes the p.c.f. in the interacting gas with mutual potential λv. We seek for the expression of g which gives the value of E_c correctly up to the terms proportional to $\ln r_s$ and also independent of r_s (r_s is the radius, in the unit of Bohr radius, of the sphere of the volume per electron).

As is known,[3] the two-electron Green function is given, to the above-mentioned approximation, as

$$G(x_1\sigma_1, x_2\sigma_2\,;\, x_1'\sigma_1, x_2'\sigma_2)$$

$$=G_0(x_1\sigma_1,\ x_2\sigma_2\,;\ x_1'\sigma_1,\ x_2'\sigma_2)$$

$$+\sum_{\sigma'\sigma''}\iint G_0(x_1\sigma_1,\ y\sigma')G_0(x_2\sigma_2,\ z\sigma'')\,V(y-z)G_0(y\sigma'\,;\ x_1'\sigma_1)G_0(z\sigma'',x_2'\sigma_2)\,dydz$$

$$-\sum_{\sigma'\sigma''}\iint G_0(x_1\sigma_1,\ y\sigma')G_0(x_2\sigma_2,\ z\sigma'')\,V(y-z)G_0(y\sigma',x_2'\sigma_2)G_0(z\sigma'',\ x_1'\sigma_1)\,dydz, \tag{5}$$

where $\int\cdots dy$ means the integration by space and time coordinates and $V(y-z)$ is the effective interaction, of which the four-dimensional Fourier transform, $V(\boldsymbol{q},\ \omega)$, is given in Eq. (7).

From Eq. (2), therefore, the expression of $g(\boldsymbol{r}_1\sigma_1\,;\ \boldsymbol{r}_2\sigma_2)$ is obtained by the straightforward calculation taking the limit of $t'\to t+$ in Eq. (5), as

$$g(\boldsymbol{r}_1\sigma_1\,;\ \boldsymbol{r}_2\sigma_2)=g_0(\boldsymbol{r}_1\sigma_1\,;\ \boldsymbol{r}_2\sigma_2)$$

$$+(i/\hbar)\,(n/2)^{-2}(2\pi)^{-4}\int d\boldsymbol{q}\int_{-\infty}^{\infty}d\omega V(\boldsymbol{q},\ \omega)\,W^2(\boldsymbol{q},\ \omega)e^{-i\boldsymbol{q}\boldsymbol{r}}$$

$$+\delta(\sigma_1,\ \sigma_2)\,(i/\hbar)\,(n/2)^{-2}(2\pi)^{-4}\int d\boldsymbol{q}\int_{-\infty}^{\infty}d\omega V(\boldsymbol{q},\ \omega)\,A(\boldsymbol{q},\ \omega\,;\ \boldsymbol{r})B(\boldsymbol{q},\ \omega;\ \boldsymbol{r}), \tag{6}$$

where

$$V(\boldsymbol{q},\ \omega)=v(\boldsymbol{q})[1-2v(\boldsymbol{q})\hbar^{-1}W(\boldsymbol{q},\ \omega)]^{-1},$$

$$W(\boldsymbol{q},\ \omega)=(-i)\int G_0(p)G_0(p-q)\,d^4p$$

$$=\Omega^{-1}\sum_{\substack{p\leqq p_F\\ |p+q|>p_F}}[(\omega-(\omega_{p+q}-\omega_p)+i\varepsilon)^{-1}-(\omega+(\omega_{p+q}-\omega_p)-i\varepsilon')^{-1}],$$

$$A(\boldsymbol{q},\ \omega\,;\ \boldsymbol{r})=(-i)\int G_0(p)G_0(p-q)e^{-ipr}\,d^4p$$

$$=\Omega^{-1}\sum_{\substack{p\leqq p_F\\ |p+q|>p_F}}[(\omega-(\omega_{p+q}-\omega_p)+i\varepsilon)^{-1}\exp[-i(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{r}]$$

$$-(\omega+(\omega_{p+q}-\omega_p)-i\varepsilon')^{-1}\exp(i\boldsymbol{p}\boldsymbol{r})],$$

$$B(\boldsymbol{q},\ \omega\,;\ \boldsymbol{r})=(-i)\int G_0(p)G_0(p-q)e^{ipr}\,d^4p$$

$$=\Omega^{-1}\sum_{\substack{p\leqq p_F\\ |p+q|>p_F}}[(\omega-(\omega_{p+q}-\omega_p)+i\varepsilon)^{-1}\exp[i(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{r}]$$

$$-(\omega+(\omega_{p+q}-\omega_p)-i\varepsilon')^{-1}\exp(-i\boldsymbol{p}\boldsymbol{r})],$$

$v(\boldsymbol{q})=4\pi e^2/|\boldsymbol{q}|^2$,

$G_0(p)=$the four-dimensional Fourier transform of one-particle Green

function without spin,

Ω=the volume of the system, $\omega_p=\hbar\boldsymbol{p}^2/2m$, and $\varepsilon, \varepsilon'>0$. (7)

§ 3. Calculation I. (The case of antiparallel spin)

In the case of antiparallel spin, i.e. $\sigma_1\neq\sigma_2$, the first and second terms of Eq. (6) contribute to g. The integral over ω is altered to the integral over $\lambda=\omega/i$, deforming the contour to the imaginary axis. The result is

$$g(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2)=1-(3/4\pi)^2\alpha\, r_s\int_0^\infty d\xi\int_0^\infty d\eta$$

$$\times F^2(\xi,\eta)\{1-(\alpha r_s/\pi\xi^2)F(\xi,\eta)\}^{-1}\sin(\xi s/\alpha)/(\xi s/\alpha),$$

where

$$\xi=|\boldsymbol{q}|/p_F,\ \eta=\lambda/(\hbar p_F{}^2/2m),\ s=|\boldsymbol{r}_1-\boldsymbol{r}_2|/r_0=r/r_0,$$

p_F=Fermi momentum divided by $\hbar$, m=electron mass,

r_0=the radius of the sphere of the volume per electron,

$\alpha=(4/9\pi)^{1/3}=0.521$

and

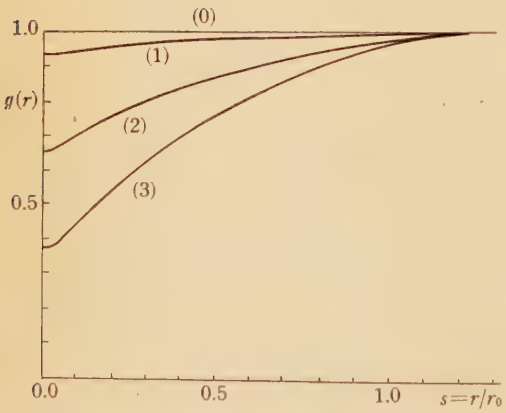

Fig. 1. The p.c.f. $g(r)$, the one electron having the spin antiparallel to the other, versus the distance measured in the unit of r_0, i.e. $s=r/r_0$, between the two electrons. Curve (1) is for $r_s=0.1$, (2) for $r_s=0.5$ and (3) for $r_s=1.0$. Values of $g(r)$ are computed numerically for $s=0.0$, 0.1, 0.2, 0.3, 0.4, 0.8 and 1.2, for each. Curve (0), i.e. $g(r)=1$, is the p.c.f. for ideal gas. $g(r=0)=0.93$ for (1), 0.66 for (2) and 0.37 for (3).

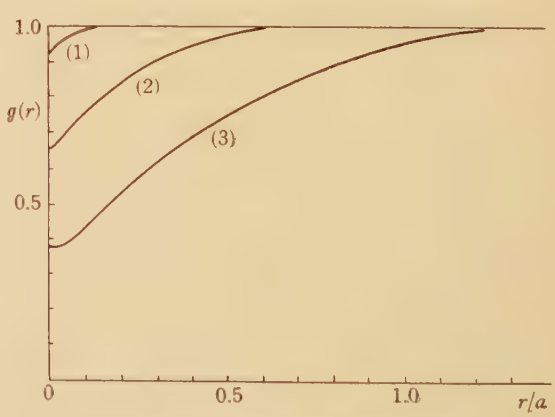

Fig. 2. The p.c.f. $g(r)$, the one electron having the spin antiparallel to the other, versus the distance in the unit of Bohr radius. Numbers given to the curves have the same meaning as in Fig. 1.

$$F(\xi,\eta)=2(2\pi)^2(\hbar/mp_F)\,W(\boldsymbol{q},i\lambda)$$
$$=-2+(4\xi^3)^{-1}(\xi^4-\eta^2-4\xi^2)\ln[((\xi^2+2\xi)^2+\eta^2)/((\xi^2-2\xi)^2+\eta^2)]$$
$$+(\eta/\xi)[\tan^{-1}\{\xi(\xi+2)/\eta\}-\tan^{-1}\{\xi(\xi-2)/\eta\}].$$

Here the p.c.f. $g(\boldsymbol{r}_1\sigma_1,\boldsymbol{r}_2\sigma_2)=g(|\boldsymbol{r}_1-\boldsymbol{r}_2|)$ is a function of r_s as well as s and the computation is carried out of three values of r_s, i.e. $r_s=0.1$, 0.5 and 1.0, in each case for values of $s=0.0$, 0.1, 0.2, 0.3, 0.4, 0.8 and 1.2. The result is given in Fig. 1 versus s and is also given in Fig. 2 versus $r/a\,(=r_s s)$ (a=Bohr radius).

Since $r_0(=r_s a)$ is equal to $(3/4\pi n)^{1/3}$, we see from Fig. 2 the increasing effect of Coulomb repulsion with decreasing density.

§ 4. Calculation II. (The case of parallel spin)

In the case of parallel spin, we obtain the result by setting $\sigma_1=\sigma_2$ in Eq. (6). As it is very difficult to compute the third term of the equation numerically, we handle it here with the approximate expression of $A(\boldsymbol{q},\omega;\boldsymbol{r})$ and $B(\boldsymbol{q},\omega;\boldsymbol{r})$ confining ourselves to the range of $rp_F\ll 1$, or $s/\alpha\ll 1$. In this range, to a good approximation, we can use the expressions

$$A(\boldsymbol{q},\omega;\boldsymbol{r})=\exp((-iqr\cos\vartheta)/2)[\cos((qr\cos\vartheta)/2)\,W(\boldsymbol{q},\omega)$$
$$-\sin((qr\sin\vartheta)/2)\,U(\boldsymbol{q},\omega)]$$

and

$$B(\boldsymbol{q},\omega;\boldsymbol{r})=\exp((irq\cos\vartheta)/2)[\cos((qr\cos\vartheta)/2)\,W(\boldsymbol{q},\omega)]$$
$$+\sin((qr\sin\vartheta)/2)\,U(\boldsymbol{q},\omega)].$$

Here ϑ is the angle between $\boldsymbol{q}$ and $\boldsymbol{r}$, and

$$U(\boldsymbol{q},\omega)=\Omega^{-1}\sum_{\substack{p\le p_F\\ |p+q|>p_F}}[(\omega-(\omega_{p+q}-\omega_p)+i\varepsilon)^{-1}+(\omega+(\omega_{p+q}-\omega_p)-i\varepsilon')^{-1}].$$

The third term of Eq. (6) becomes, deforming again the contour of the integration by ω to the imaginary axis,

$$\delta(\sigma_1\sigma_2)\,(1/2)\,(3/4\pi)^2\alpha r_s\int_0^\infty d\xi\int_0^\infty d\eta[(1+\sin(\xi s/\alpha)/(\xi s/\alpha))F^2(\xi,\eta)$$
$$+(1-\sin(\xi s/\alpha)/(\xi s/\alpha))G^2(\xi,\eta)][1-(\alpha r_s/\pi\xi^2)F(\xi,\eta)]^{-1},$$

where

$$G(\xi,\eta)=2(2\pi)^2(\hbar/mp_F)\,U(\boldsymbol{q},i\lambda)$$
$$=(2\xi^3)^{-1}(\xi^4-4\xi^2-\eta^2)[\tan^{-1}\{\xi(\xi+2)/\eta\}-\tan^{-1}\{\xi(\xi-2)/\eta\}]$$
$$-(\eta/2\xi)\ln[((\xi^2+2\xi)^2+\eta^2)/((\xi^2-2\xi)^2+\eta^2)]+2\eta/\xi^2.$$

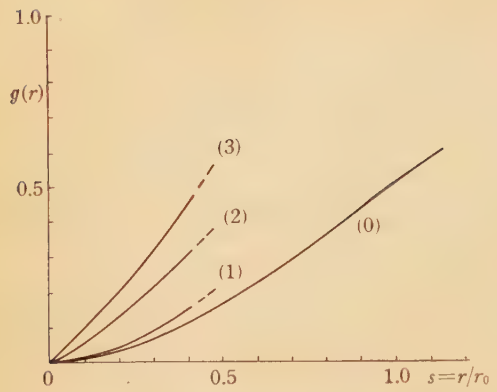

Fig. 3. The p.c.f. $g(r)$,the one electron having the spin parallel to the other. s is the distance between the electrons in the unit of r_0 as in Fig. 1. Curve (1) is for $r_s=0.1$, (2) for $r_s=0.5$ and (3) for $r_s=1.0$. For each, values of $g(r)$ are computed numerically for $s=0.0$, 0.1, 0.2, 0.3 and 0.4, using an approximate expression valid for $s\ll\alpha$. Curve (0) is the p.c.f. for ideal gas, i.e. Eq. (4) for $\sigma_1=\sigma_2$.

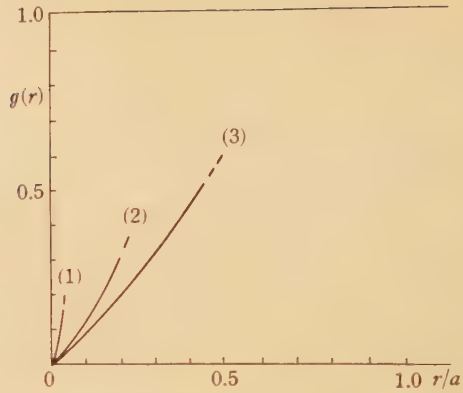

Fig. 4. The p.c.f. $g(r)$, the one electron having the spin parallel to the other, versus the distance in the unit of Bohr radius. Numbers given to the curves have the same meaning as in Fig. 3. For density infinitely large ($r_s\to 0$ or $p_F\to\infty$), $g(r)$ is unity for $r>0$, as is seen from Eq. (4).

Here again, three values of r_s, 0.1, 0.5 and 1.0, were chosen and in each case the values of p.c.f. were computed for $s=0.0$, 0.1, 0.2, 0.3 and 0.4. The result is given in Fig. 3. The condition $s/\alpha\ll 1$ is not so well satisfied as s comes nearer to 0.4 ($s/\alpha=0.77$ for $s=0.4$), so that the curves of p.c.f. in this case are to be regarded as only the extrapolation from the smaller values of s.

The p.c.f. is also given in Fig. 4 versus r/a. Here again, $g(r)$ decreases with density for a fixed r, due to the increasing effect of Coulomb repulsion.

As is seen from the results obtained above (§ 3 and § 4), the mean of p.c.f. of the two cases for a value of r, i.e. the probability of finding the second electron, regardless of its spin, at a distance r from the first one, is positive near $r=0$. This is to be compared with that obtained by Glick and Ferrell[4] using the random phase approximation with the negative values near $r=0$.

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Space-Time Correlation Function in the Theory of Electrical Conductivity*

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A formula is given for the electrical resistivity of metals with the use of the result of excitation-response theory. It is shown that the electrical resistivity is expressed in terms of the four-dimensional Fourier transform of the appropriate pair correlation function in space and time of the scattering systems, by which conduction electrons are scattered to lose their initial velocity. The dependence of the electrical conductivity on the effects of the correlation in the scattering system is discussed systematically by using the correlation functions in some of the typical examples. Especially the electrical resistivity of the ferromagnetic metals is discussed in some detail by using the so-called *s-d* interaction model. In this case it is shown that the inclusion of the effect of the correlation between *d* spins gives rise to deviations in electrical resistivity from the one given by the simple molecular field approximation.

§ 1. Introduction

As was shown in an earlier paper[1] adopting the so-called *s-d* interaction model, the temperature dependence of electrical resistivity of a ferromagnet is considerably affected by the inclusion of the correlation effect between *d* spins at low temperatures. In the present paper the effects of the correlation in the scattering systems are explicitly taken into account in calculating the electrical resistivity of some of the typical examples such as *s-d* interaction in ferromagnet, electron-phonon interaction in normal metal, etc.

The purpose of the present paper is two-fold: The one is to show that with the use of the result of the excitation-response theory[2),3)] the electrical resistivity is expressible in terms of a suitably defined pair correlation function[4)] depending on a space coordinate and a time variable, or explicitly in terms of a four-dimensional Fourier transform of this function, and the other is to examine the effects of the correlation in the scattering systems to the electrical conductivity in some detail for a few systems, especially for a system of ferromagnet. The formula thus obtained for the electrical resistivity provides a physically intuitive method in discussing the electrical conductivity. Since

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the electrical resistivity is determined by the scattering cross section of conduction electron by the scattering system[5] and since the four-dimensional Fourier transform of pair correlation function in space and time is directly connected to the scattering cross section,[4] it is rather natural to expect that we have a formula for the electrical resistivity expressed in terms of pair correlation function.

In § 2 a formula for the electrical resistivity is given in a number of systems and in § 3 the case of *s-d* interaction in a ferromagnet is discussed with the use of this formula in some detail, and § 4 is devoted to the discussion of the electrical resistivity of a dilute alloy system.

§ 2. Formula for electrical conductivity

According to the standard treatment of the linear excitation-response theory such as proposed by Kubo[2] and Nakano,[3] we have the following formula for a static electrical conductivity,

$$\sigma=\beta\lim_{\epsilon\to+0}\int_0^\infty dt\, e^{-\epsilon t}\,\mathrm{Re}\langle j_x j_x(t)\rangle \tag{2·1}$$

where $\beta=(k_BT)^{-1}$, ReA stands for real part of A, $\langle B\rangle=\mathrm{Tr}(e^{-\beta H}B)/\mathrm{Tr}(e^{-\beta H})$, and j_x is the x-component of the electric current operator and $j_x(t)$ is the electric current operator in the Heisenberg picture

$$j_x(t)=e^{iHt/\hbar}j_x\,e^{-iHt/\hbar}. \tag{2·2}$$

H is the Hamiltonian of the total system in the absence of external electric field and is written as

$$H=H_e+H_s+V. \tag{2·3}$$

H_e and H_s respectively represent the Hamiltonian of the system of conduction electron and that of the scattering system by which conduction electron is scattered, and V stands for the interaction between them.

In what follows we shall introduce such an approximation as

$$\langle j_x j_x(t)\rangle\to\langle j_x j_x(t)\rangle_0\equiv\mathrm{Tr}\{e^{-\beta H_0}j_x j_x(t)\}/\mathrm{Tr}(e^{-\beta H_0}) \tag{2·4}$$

with $H_0=H_e+H_s$. The validity of this approximation is intimately connected with the commutation property $[H_0, j_x]=0$,[6] so we may use the above approximation in our case. With the aid of this commutation property we need only the diagonal element of $j_x(t)$ in Eq. (2·1) in the representation in which H_0 is diagonal. Let us introduce the Laplace transform

$$J(s)=\int_0^\infty dt\, e^{-st}j_x(t)\quad(\mathrm{Re}\,s>0) \tag{2·5a}$$

and define

$$\varphi(s)=\langle j_x J(s)\rangle_0. \tag{2·5b}$$

Then σ is written as

$$\sigma=\beta\lim_{\epsilon\to+0}\mathrm{Re}\,\varphi(\epsilon). \tag{2·6}$$

If we denote the diagonal part of J by J_d, we have for J_d[7]

$$J_d(s)=\frac{1}{s-\mathcal{D}\mathcal{L}_1\dfrac{1}{s+\mathcal{L}_0+\mathcal{N}\mathcal{L}_1}\mathcal{L}_1}j_x \tag{2·7}$$

where for any operator A $\mathcal{L}$'s are defined by

$$\mathcal{L}_0A=\frac{i}{\hbar}[A, H_0],\quad \mathcal{L}_1A=\frac{i}{\hbar}[A, V],$$

and $\mathcal{D}$ and $\mathcal{N}$ are operators which operate from the left on any operator B as

$$B=\mathcal{D}B+\mathcal{N}B,$$

and $\mathcal{D}B$ and $\mathcal{N}B$ stand for diagonal and off-diagonal parts of operator B. Then one can derive a power series expansion from Eq. (2·7), and confining oneself up to the second order perturbation one may put $\mathcal{N}\mathcal{L}_1=0$ in the denominator of Eq. (2·7); then one writes

$$\varphi(s)=\frac{1}{s}\langle j_x\{1+L(s)+L^2(s)+\cdots\}j_x\rangle_0 \tag{2·8}$$

where

$$L(s)=\frac{1}{s}\mathcal{D}\mathcal{L}_1\frac{1}{s+\mathcal{L}_0}\mathcal{L}_1. \tag{2·9}$$

The applicability of this approximation is discussed by Tomita et al.[8] By the definition (2·9) we have[7]

$$L(s)j_x=-\frac{1}{s\hbar^2}\mathcal{D}[V[V(s), j_x]], \tag{2·10}$$

where $V(s)$ is the Laplace transform of the operator V in the interaction representation:

$$V(s)=\int_0^\infty dt\, e^{-st}\, e^{iH_0t/\hbar}\,V\,e^{-iH_0t/\hbar}. \tag{2·11}$$

With the use of Eqs. (2·8) and (2·10) we have in the lowest order approximation

$$\mathrm{Re}\,\varphi(s)=\frac{1}{s}\langle j_x j_x\rangle_0\left\{1+\mathrm{Re}\,\frac{\mathrm{Tr}(e^{-\beta H_0}j_x\mathfrak{D}[V[V(s),j_x]])}{s\hbar^2\,\mathrm{Tr}(e^{-\beta H_0}j_x j_x)}\right\}^{-1} \quad (2\cdot12)^*$$

and we have from Eq. (2·6)

$$\sigma=\beta\hbar^2\langle j_x j_x\rangle_0^2/\mathrm{Re}\langle j_x\mathfrak{D}[V[V(s),j_x]]\rangle_0|_{s=0}. \quad (2\cdot13)$$

Upon introducing the Fourier component $V(\omega)$ by the relation

$$V(t)=\int_{-\infty}^{\infty}V(\omega)\,e^{i\omega t}\,d\omega, \quad (2\cdot14)$$

Eq. (2·13) can be written in a little more familiar form.[3),6)] With the use of the relations

$$V(s)=\int_{-\infty}^{\infty}\frac{1}{s-i\omega}V(\omega)\,d\omega$$

and

$$\lim_{s\to+0}\mathrm{Re}\langle j_x\mathfrak{D}[V[V(s),j_x]]\rangle_0$$
$$=\mathrm{Re}\int d\omega'\int d\omega\left(\pi\delta(\omega)+P\frac{i}{\omega}\right)\langle j_x\mathfrak{D}[V(\omega')[V(\omega),j_x]]\rangle_0,$$

and of the property that $\langle j_x\mathfrak{D}[V(\omega')[V(\omega),j_x]]\rangle_0\neq0$ for $\omega+\omega'=0$, one can define $\phi(\omega)$ by

$$\langle j_x\mathfrak{D}[V(\omega')[V(\omega),j_x]]\rangle_0=\phi(\omega)\delta(\omega+\omega') \quad (2\cdot15)$$

and finally one has that[3),6)]

$$\sigma=\frac{\beta\hbar^2}{\pi}\frac{\langle j_x j_x\rangle_0^2}{\phi(0)} \quad (2\cdot16)^{**}$$

where use has been made of the property that $\langle j_x[V(-\omega)[V(\omega),j_x]]\rangle_0$ is an even function of ω. Now we will turn to the problem how to express σ in terms of pair correlation functions.

---

* It has been assumed that the relation

$$\langle j_x L^n(s)j_x\rangle_0\langle j_x j_x\rangle_0^{n-1}=\langle j_x L(s)j_x\rangle_0^n$$

holds. For example, in the case of impurity scattering in which impurities are distributed at random, each being located rigidly (cf. § 2−(d)), it is an easy matter to show that for $n=2$

$$\frac{\langle j_x L^2(s)j_x\rangle_0\langle j_x j_x\rangle_0}{\langle j_x L(s)j_x\rangle_0^2}=2\frac{\int_{-1}^{1}d(\cos\theta)\,(1-\cos\theta)\,|U\left(2k_f\sin\frac{\theta}{2}\right)|^4}{\left\{\int_{-1}^{1}d(\cos\theta)\,(1-\cos\theta)\,|U\left(2k_f\sin\frac{\theta}{2}\right)|^2\right\}^2}$$

which is equal to unity when $U(2k_f\sin(\theta/2))$ is independent of θ.

** Recently, Zigenlaub[9)] derived a somewhat different formula for σ starting from the formula (2·1) in the above. However, his result does not seem to be accurate enough to give the factor $(1-\cos\theta)$ in the explicit formula for the electrical conductivity.

(a) *The case of s-d interaction*

In this subsection we deal with such an anomalous[10] part of the resistivity of a ferromagnet that is the difference between the total resistivity and the one which comes from the scattering by phonon system. Adopting the *s-d* interaction model for ferromagnetic metals we calculate the electrical resistivity. The model states that the electrical current is carried by 4*s* (or 6*s*) electrons which are assumed to be described in the band scheme, and unpaired electrons (3*d* or 4*f*) responsible for ferromagnetism are assumed to be localized on the lattice points, and there is an exchange interaction V between these electrons.[10] V is written as

$$V=-\sum_{\boldsymbol{k}\boldsymbol{k}'}\sum_{n} J(\boldsymbol{k}'-\boldsymbol{k})\exp[i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{R}_n]\{(a^*_{\boldsymbol{k}'\uparrow}a_{\boldsymbol{k}\uparrow}-a^*_{\boldsymbol{k}'\downarrow}a_{\boldsymbol{k}\downarrow})S_n^{\,z} +a^*_{\boldsymbol{k}'\downarrow}a_{\boldsymbol{k}\uparrow}S_n^{\,+}+a^*_{\boldsymbol{k}'\uparrow}a_{\boldsymbol{k}\downarrow}S_n^{\,-}\}, \tag{2·17}$$

where $a^*_{\boldsymbol{k}\uparrow}$ and $a_{\boldsymbol{k}\uparrow}$ are creation and annihilation operators for the conduction electron with spin upwards and with wave vector $\boldsymbol{k}$, and $a^*_{\boldsymbol{k}\downarrow}$, $a_{\boldsymbol{k}\downarrow}$ refer to the electron with spin downwards, $\boldsymbol{S}_n$ is the spin operator of the unpaired electrons which are assumed to be localized on the n-th lattice point, and $J(\boldsymbol{k}'-\boldsymbol{k})$ means the Fourier component of the exchange integral between s and d electrons. H_0 now takes the form

$$\begin{aligned} H_0&=H_e+H_d+H_p,\\ H_e&=\sum_{\boldsymbol{k}\sigma}E_{\boldsymbol{k}}{}^{\sigma}a^*_{\boldsymbol{k}\sigma}a_{\boldsymbol{k}\sigma},\\ H_d&=-I\sum_{\langle nl\rangle}\boldsymbol{S}_n\cdot\boldsymbol{S}_l,\\ H_p&=\sum_{\boldsymbol{q}}\sum_{j=1}^{3}\hbar\omega_{\boldsymbol{q}j}b^*_{\boldsymbol{q}j}b_{\boldsymbol{q}j}, \end{aligned} \tag{2·18}$$

where $b^*_{\boldsymbol{q}j}$ and $b_{\boldsymbol{q}j}$ are usual creation and annihilation operators for the j-th mode of the phonon with wave vector $\boldsymbol{q}$. Then one has with Eqs. (2·17) and (2·18) that

$$\begin{aligned} V(t)=-\sum_{\boldsymbol{k}\boldsymbol{k}'}\sum_{n} J(\boldsymbol{k}'-\boldsymbol{k})\exp[i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{R}_n(t)]\{&a^*_{\boldsymbol{k}'\uparrow}a_{\boldsymbol{k}\uparrow}S_n^{\,z}(t)\exp[i(E^+_{\boldsymbol{k}'}-E_{\boldsymbol{k}}{}^+)t/\hbar]\\ -a^*_{\boldsymbol{k}'\downarrow}a_{\boldsymbol{k}\downarrow}S_n^{\,z}(t)&\exp[i(E^-_{\boldsymbol{k}'}-E^-_{\boldsymbol{k}})t/\hbar]+a^*_{\boldsymbol{k}'\downarrow}a_{\boldsymbol{k}\uparrow}S_n^{\,+}(t)\exp[i(E^-_{\boldsymbol{k}'}-E^+_{\boldsymbol{k}})t/\hbar]\\ +a^*_{\boldsymbol{k}'\uparrow}a_{\boldsymbol{k}\downarrow}S_n^{\,-}(t)&\exp[i(E^+_{\boldsymbol{k}'}-E^-_{\boldsymbol{k}})t/\hbar]\}, \end{aligned} \tag{2·17a}$$

where

$$\boldsymbol{R}_n(t)=\exp[iH_pt/\hbar]\boldsymbol{R}_n\exp[-iH_pt/\hbar],\quad \boldsymbol{S}_n(t)=\exp[iH_dt/\hbar]\boldsymbol{S}_n\exp[-iH_dt/\hbar],$$

and one can easily obtain the formula for $\phi(0)$ (see Appendix I)

$$\begin{aligned} \phi(0)=N\left(\frac{e\hbar}{m}\right)^2\sum_{\boldsymbol{k}\boldsymbol{k}'}|J(\boldsymbol{k}-\boldsymbol{k}')|^2(k_x'-k_x)^2\\ \times\left\{f^+_{\boldsymbol{k}'}(1-f^+_{\boldsymbol{k}})G^{zz}_{\boldsymbol{k}'-\boldsymbol{k}}\left(\frac{E^+_{\boldsymbol{k}'}-E^+_{\boldsymbol{k}}}{\hbar}\right)+f^-_{\boldsymbol{k}'}(1-f^-_{\boldsymbol{k}})G^{zz}_{\boldsymbol{k}'-\boldsymbol{k}}\left(\frac{E^-_{\boldsymbol{k}'}-E^-_{\boldsymbol{k}}}{\hbar}\right)\right. \end{aligned}$$

$$+f_{k'}^{-}(1-f_{k}^{+})G_{k'-k}^{+-}\left(\frac{E_{k'}^{-}-E_{k}^{+}}{\hbar}\right)+f_{k'}^{+}(1-f_{k}^{-})G_{k'-k}^{-+}\left(\frac{E_{k'}^{+}-E_{k}^{-}}{\hbar}\right)\Big\}, \qquad (2\cdot19)$$

where

$$G_{k}^{\mu\nu}(\omega)=\sum_{n}\int\frac{dt}{2\pi}\exp[i\boldsymbol{\kappa}\cdot\boldsymbol{R}_{n}-i\omega t]\langle S_{0}^{\mu}(0)\,S_{n}^{\nu}(t)\rangle$$
$$\times\exp[-\sum_{\alpha\beta}\{M^{\alpha\beta}(0,0)-M^{\alpha\beta}(\boldsymbol{R}_{n},t)\}\kappa_{\alpha}\kappa_{\beta}] \qquad (2\cdot20)$$

and

$$M^{\alpha\beta}(\boldsymbol{R}_{n},\,t)=\langle u_{0}^{\alpha}(0)u_{n}^{\beta}(t)\rangle \quad (\alpha,\,\beta=x,\,y,\,z), \qquad (2\cdot21)$$

u_{n}^{β} being the β-component ($\beta=x,\,y,\,z$) of the displacement vector of the n-th ion from its equilibrium position.

These equations state that the electrical resistivity is expressed in terms of the Fourier transform of the suitably defined pair correlation function of the scattering system. If one considers the rigid lattice case one may put $M^{\alpha\beta}(\boldsymbol{R}_{n},\,t)=M^{\alpha\beta}(0,\,0)$ and the Fourier transform of the relevant correlation function for the electrical resistivity reduces simply to

$$G_{\kappa}^{\mu\nu}(\omega)\rightarrow g_{\kappa}^{\mu\nu}(\omega)=\sum_{n}\int\frac{dt}{2\pi}\exp[i\boldsymbol{\kappa}\cdot\boldsymbol{R}_{n}-i\omega t]\langle S_{0}^{\mu}(0)\,S_{n}^{\nu}(t)\rangle. \qquad (2\cdot22)$$

On the other hand, if one considers the case in which the range of the correlation function $M^{\alpha\beta}(\boldsymbol{R}_{n},\,t)$ is considerably smaller than the one of $\langle S_{0}^{\mu}(0)S_{n}^{\nu}(t)\rangle$, one may effectively put $M^{\alpha\beta}(\boldsymbol{R}_{n},\,t)=0$ in Eq. (2·20) and G becomes

$$G_{\kappa}^{\mu\nu}(\omega)\rightarrow g_{\kappa}^{\mu\nu}(\omega)\exp[-\sum_{\alpha\beta}M^{\alpha\beta}(0,0)\kappa_{\alpha}\kappa_{\beta}], \qquad (2\cdot23)$$

that is, in this case G is equal to g multiplied by the familiar Debye-Waller factor. This factor nearly equals unity for the usual condition (cf. Eq. (19) of reference 1)). In general, the difference between G and g gives rise to the so-called magneto-vibrational effect, but this type of effect is very small.[1] In what follows, therefore, $g_{\kappa}(\omega)$ is used in place of G. Further discussions for this case are given in § 3.

(b) *The case of dilute alloy*

In the scheme of the present formulation we briefly discuss the electrical resistivity of the dilute alloy[11] such as CuMn whose resistivity is known to show an anomalous temperature dependence at low temperatures.[12] We adopt such a model[13] for these dilute alloys that the electrical current is carried by 4*s* electrons described in the band scheme and the unpaired 3*d* electrons of Mn ions are assumed to be localized on their own lattice points, and that 4*s* electrons are scattered by the disorder in spin orientation of Mn ions through the *s-d* interaction. Then neglecting the spin polarization of conduction electron and also the effect of lattice vibration, we have

$$\phi(0)=2\left(\frac{e\hbar}{m}\right)^2\sum_{kk'}\sum_{mn}|J(\boldsymbol{k}'-\boldsymbol{k})|^2(k_x'-k_x)^2\exp[-i(\boldsymbol{k}'-\boldsymbol{k})\cdot(\boldsymbol{R}_n-\boldsymbol{R}_m)]$$

$$\times f_{k'}(1-f_k)\int\frac{dt}{2\pi}\langle\boldsymbol{S}_n(0)\cdot\boldsymbol{S}_m(t)\rangle\exp[-i(E_{k'}-E_k)t/\hbar], \quad (2\cdot24)$$

where the summation about (m, n) extends over all the lattice points occupied by Mn ions including the case $m=n$.

(c) *The case of electron-phonon interaction*

In this case the interaction Hamiltonian V is given by[14]

$$V=-\int\psi^*(\boldsymbol{r})\boldsymbol{u}(\boldsymbol{r})\cdot\nabla U_0(\boldsymbol{r})\psi(\boldsymbol{r})\,d\boldsymbol{r}, \quad (2\cdot25)$$

where $\psi(\boldsymbol{r})$ is the quantized wave function for conduction electron and is expressed as

$$\psi(\boldsymbol{r})=\sum_k a_k\phi_k(\boldsymbol{r})\equiv\frac{1}{\sqrt{N}}\sum_k a_k u_k(\boldsymbol{r})\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}},$$

where $\phi_k(\boldsymbol{r})$ is the Bloch function for conduction electron, $\boldsymbol{u}$ is the displacement vector of lattice, and $U_0(\boldsymbol{r})$ is the potential energy due to the regular lattice. The Fourier transform of Eq. (2·25) becomes

$$V(\omega)=-\frac{1}{2\pi N}\sum_{kk'}a_{k'}^*a_k\int d\boldsymbol{r}\int dt\exp\left[i\left(\frac{E_{k'}-E_k}{\hbar}-\omega\right)t\right]$$

$$\times u_{k'}^*(\boldsymbol{r})\,e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}}\boldsymbol{u}(\boldsymbol{r},t)\cdot\nabla U_0(\boldsymbol{r})u_k(\boldsymbol{r})\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}. \quad (2\cdot26)$$

j_x is expressed in the one band picture as

$$j_x=\sum_k(-e\hbar/m)k_x a_k^*a_k \quad (2\cdot27)$$

with the effective mass m of the conduction electron. From Eq. (2·16) we only need to know $\phi(0)$, then making use of the relations

$$\langle j_x[V(-\omega)[V(\omega),j_x]]\rangle_0=\langle[j_x,V(\omega)][V(-\omega),j_x]\rangle_0 \quad (\omega\to0)$$

and

$$[a_{k'''}^*a_{k''},\ a_{k'}^*a_k]=\delta_{k''k'}a_{k'''}^*a_k-\delta_{k'''k}a_{k'}^*a_{k''},$$

we easily obtain

$$\phi(0)=-\frac{4e^2\hbar^2C^2}{9m^2}\sum_{kk'}(k_x'-k_x)^2f_{k'}(1-f_k)\sum_{\alpha\beta}^{x,y,z}M^{\alpha\beta}\left(\boldsymbol{k}'-\boldsymbol{k},\frac{E_{k'}-E_k}{\hbar}\right)$$

$$\times(k_\alpha'-k_\alpha)(k_\beta'-k_\beta), \quad (2\cdot28)$$

where C is the Sommerfeld-Bethe coupling constant,[14] f_k is the distribution function for the conduction electron with wave number vector $\boldsymbol{k}$, and M is the four-dimensional Fourier transform of the correlation function $\langle u_0^\alpha(0)u_n^\beta(t)\rangle$:

$$M^{\alpha\beta}(\boldsymbol{\kappa},\omega)=\frac{1}{N}\sum_{n}\int\frac{dt}{2\pi}\exp[i\boldsymbol{\kappa}\cdot\boldsymbol{R}_n-i\omega t]\langle u_0^{\alpha}(0)u_n^{\beta}(t)\rangle. \qquad (2\cdot29)$$

Introducing Eq. (2·28) into Eq. (2·16), we have the formula for the electrical conductivity expressed in terms of the four-dimensional Fourier transform of the pair correlation function of the displacement of ion.

If one adopts the Debye approximation for the lattice motion, one has

$$\sum_{\alpha\beta}M^{\alpha\beta}(\boldsymbol{\kappa},\omega)\kappa_\alpha\kappa_\beta=\frac{\hbar}{2NM}\sum_{j}\frac{(\boldsymbol{\kappa}\cdot\boldsymbol{e}_{\kappa j})^2}{\omega_{\kappa j}}\{n_{-\kappa j}\delta(\omega+\omega_{-\kappa j})+(n_{\kappa j}+1)\delta(\omega-\omega_{\kappa j})\} \qquad (2\cdot30)$$

where $\boldsymbol{e}_{\kappa j}$ is the unit vector determining the direction of polarization of the j-th species of phonon with wave number $\boldsymbol{\kappa}$ and with energy $\hbar\omega_{\kappa j}$, and $n_{\kappa j}$ is the average number of that phonon at thermal equilibrium. The presence of factor $(\boldsymbol{\kappa}\cdot\boldsymbol{e}_{\kappa j})^2$ tells that only the longitudinal phonon is effective in determining the resistivity. With Eqs. (2·16), (2·28) and (2·30) one easily has the Bloch-Grüneisen formula for resistivity.[3)]

(d) *The case of impurity scattering*

We now consider the case in which the conduction electron is scattered by the presence of impurity ions, and we have the following form for V,

$$V=\sum_{\boldsymbol{k}\boldsymbol{k}'}\sum_{n}\exp[i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{R}_n]U(\boldsymbol{k}'-\boldsymbol{k})a_{\boldsymbol{k}'}^{*}a_{\boldsymbol{k}}, \qquad (2\cdot31)$$

$U(\boldsymbol{k}'-\boldsymbol{k})$ being the Fourier component of the potential for electron due to the impurity ion at the origin. The summation $\sum_n$ extends over all the impurity ions. And we have

$$\phi(0)=n_i\left(\frac{e\hbar}{m}\right)^2\sum_{\boldsymbol{k}\boldsymbol{k}'}|U(\boldsymbol{k}-\boldsymbol{k}')|^2(k_x'-k_x)^2f_{\boldsymbol{k}'}(1-f_{\boldsymbol{k}})\mathcal{M}\left(\boldsymbol{k}'-\boldsymbol{k},\ \frac{E_{\boldsymbol{k}'}-E_{\boldsymbol{k}}}{\hbar}\right) \qquad (2\cdot32)$$

with

$$\mathcal{M}(\boldsymbol{\kappa},\omega)=\sum_{n}\int\frac{dt}{2\pi}\exp[i\boldsymbol{\kappa}\cdot\boldsymbol{R}_n-i\omega t]\exp[-\sum_{\alpha\beta}(M^{\alpha\beta}(0,0)-M^{\alpha\beta}(\boldsymbol{R}_n,t))\kappa_\alpha\kappa_\beta], \qquad (2\cdot33)$$

where n_i is the number density of impurity ions, and the summation in Eq. (2·33) extends over $\boldsymbol{R}_n$ which is the lattice vector drawn from a certain impurity ion to the n-th impurity ion (including $\boldsymbol{R}_n=0$). So in this case too we have arrived at a formula for the electrical resistivity written in terms of the pair correlation functions.

If one considers the case in which the lattice motion is neglected and the impurity ions are distributed at random, one may write as $\mathcal{M}(\boldsymbol{\kappa},\omega)=\delta(\omega)$ and using the formula $\langle j_x j_x\rangle_0=2(e\hbar/m)^2\sum_{\boldsymbol{k}}k_x^2f_{\boldsymbol{k}}(1-f_{\boldsymbol{k}})$ one obtains the resistivity in the usual manner,[3)]

$$\rho=\frac{3\pi\hbar n_i}{2e^2}\left(\frac{dE}{dk_f}\right)^{-2}\int_{-1}^{1}d(\cos\theta)\,(1-\cos\theta)\,|\,U\left(2k_f\sin\frac{\theta}{2}\right)|^2,$$

where the suffix f is used to represent the value at the Fermi surface.

(e) *The case of liquid metal*

In this case, as a model Hamiltonian we choose for the system of the conduction electrons

$$H_e=\sum_{k}E_k a_k^* a_k \tag{2·34}$$

and for the system of ions

$$H_i=\sum_{n}\frac{\boldsymbol{P}_n^2}{2M}+(\text{P. E.}) \tag{2·35}$$

and for the interaction between them we write

$$V=\sum_{kk'}\sum_{n}\exp[i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{R}_n]\,U(\boldsymbol{k}'-\boldsymbol{k})\,a_{k'}^* a_k \tag{2·36}$$

where $\boldsymbol{R}_n$ and $\boldsymbol{P}_n$ stand for the position vector and the momentum vector of the n-th ion. Then it is easy to show that

$$\phi(0)=\left(\frac{e\hbar}{m}\right)^2\sum_{kk'}|\,U(\boldsymbol{k}'-\boldsymbol{k})\,|^2(k_x'-k_x)^2 f_{k'}(1-f_k)\,\delta\left(\boldsymbol{k}'-\boldsymbol{k},\frac{E_{k'}-E_k}{\hbar}\right) \tag{2·37}$$

and

$$\delta(\boldsymbol{\kappa},\omega)=\sum_{mn}\int\frac{dt}{2\pi}\langle\exp[-i\boldsymbol{\kappa}\cdot\boldsymbol{R}_n(0)]\exp[i\boldsymbol{\kappa}\cdot\boldsymbol{R}_m(t)]\rangle e^{-i\omega t}. \tag{2·38}$$

Now we may say that the resistivity of this system is also written in terms of the Fourier transform of the pair correlation function. Replacing the summation in Eq. (2·37) by the integration, we write $\phi(0)$ in the form

$$\phi(0)=\frac{e^2k_f^4}{12\pi^4\hbar}\int d\omega\int d(\cos\theta)\,(1-\cos\theta)\,|\,U(\kappa)\,|^2\frac{\hbar\omega}{\exp[\beta\hbar\omega]-1}\,\delta(\kappa,\omega). \tag{2·39}$$

$$\left(\kappa=2k_f\sin\frac{\theta}{2}\right)$$

Now we briefly mention the formulas for ρ obtained in the above cases (a)–(e). It is shown that ρ has the following form,

$$\rho\propto\sum_{k\kappa}|\,I(\boldsymbol{\kappa})\,|^2\kappa_x^2 f_{k+\kappa}(E_k+\hbar\omega)\,(1-f_k(E_k))\,G_\kappa(\omega)$$

where $I(\boldsymbol{\kappa})$ is the Fourier component of the interaction constant between the conduction electron and the scattering system, and $G_\kappa(\omega)$ is the four-dimensional Fourier transform of the appropriate pair correlation function of the scattering system. $|I(\boldsymbol{\kappa})|^2G_\kappa(\omega)$ is simply related to the differential cross section in the Born approximation.[4] And the factor $f_{k+\kappa}(E_k+\hbar\omega)\,(1-f_k(E_k))$ is a statistical

weight factor for the scattering process $(\boldsymbol{k}+\boldsymbol{\kappa}, E_k+\hbar\omega)\to(\boldsymbol{k}, E_k)$. The presence of the factor κ_x^2 reflects the fact that the static conductivity is determined by the decaying behaviour of the current operator and not by that of a state of the conduction electron, and this factor gives rise to the familiar factor $1-\cos\theta$ in the result (see the cases (d) and (e) and also the next section).

§ 3. The case of *s-d* interaction in ferromagnet

In this section we will discuss the case of *s-d* interaction in some detail. As was seen in § 2-(a), the electrical conductivity is expressed in terms of the four-dimensional Fourier transform $g_\kappa^{\mu\nu}(\omega)$ of the pair correlation function $\langle S_0^\mu(0)S_n^\nu(t)\rangle$ and the main problem to be solved is reduced to the calculation of $g_\kappa^{\mu\nu}(\omega)$. Unfortunately, however, the exact calculation of $g_\kappa^{\mu\nu}(\omega)$ is very difficult and one has to content oneself with some approximation. Now let us begin with the familiar approximation of molecular field.

(i) *Molecular field approximation**

This approximation states that each spin fluctuates independently in a molecular field qM, q being the coefficient of the molecular field and M the average magnetization of the d spin system. Then the Hamiltonian for the spin system of d-electrons takes the form

$$H_0=-\hbar\tilde{\omega}\sum_n S_n^z,$$
$$\hbar\tilde{\omega}=2\mu_B qM, \tag{3·1}$$

where μ_B is the Bohr magneton. $S(t)$'s are written as

$$S_n^+(t)=\exp[iH_0t/\hbar]S_n^+\exp[-iH_0t/\hbar]=S_n^+e^{-i\tilde{\omega}t},$$
$$S_n^-(t)=S_n^-e^{i\tilde{\omega}t},\quad S_n^z(t)=\text{independent of } t,$$

so the function $g_\kappa^{\mu\nu}(\omega)$ is easily calculated to be

$$g_\kappa^{+-}(\omega)=\langle(S+S_0^z)(S-S_0^z+1)\rangle\delta(\omega-\tilde{\omega}),$$
$$g_\kappa^{-+}(\omega)=\langle(S-S_0^z)(S+S_0^z+1)\rangle\delta(\omega+\tilde{\omega}), \tag{3·2}$$
$$g_\kappa^{zz}(\omega)=N\delta_{\kappa,0}\delta(\omega)\sigma^2+\langle(S_0^z-\sigma)^2\rangle\delta(\omega),$$

where $\sigma=\langle S_0^z\rangle$. For example, we calculate $\phi^{+-}(0)$ that is the contribution of the g^{+-} term to $\phi(0)$, which corresponds to the process in which the conduction electron in the state $(\boldsymbol{k}'\downarrow)$ is scattered inelastically to $(\boldsymbol{k}\uparrow)$ accompanying the energy transfer of the amount $\hbar\tilde{\omega}=E_{k'}^--E_k^+$ to any one of d-spins. Since the effect of the spin polarization of conduction electron is very small for the resistivity (cf. the next subsection), we neglect this effect and put $f^+=f^-$. With the

* Although with this approximation the electrical resistivity has been calculated by Kasuya,[10] we will re-calculate it in the scheme of the present formulation.

use of Eq. (2·19) with G's replaced by g's expressed as in Eqs. (3·2), we have that

$$\phi^{+-}(0)=\frac{e^2N}{m^2\hbar}\left(\frac{1}{2\pi}\right)^6\int\frac{dE\,dE'\,k^2k'^2\,d\Omega\,d\Omega'}{|\nabla_k E|\,|\nabla_{k'}E'|}\,|J(\boldsymbol{k}'-\boldsymbol{k})|^2(k_x'-k_x)^2$$
$$\times f_{k'}(1-f_k)\langle(S+S_0^z)(S-S_0^z+1)\rangle\delta(E_k-E_{k'}+\hbar\tilde{\omega})$$
$$=\frac{e^2Nk_f^4}{12\pi^4\hbar^5}J_{eff}^2\frac{\hbar\tilde{\omega}}{\exp[\beta\hbar\tilde{\omega}]-1}\langle(S+S_0^z)(S-S_0^z+1)\rangle, \qquad (3\cdot3)$$

and

$$J_{eff}^2=\int_{-1}^{1}d(\cos\theta)\,(1-\cos\theta)\,|J\left(2k_f\sin\frac{\theta}{2}\right)|^2.$$

The calculation of the other terms can also be carried out in the same way and the resistivity ρ becomes

$$\rho=\frac{3\pi NmJ_{eff}^2}{4\hbar e^2E_f}\left\{\langle(S_0^z-\sigma)^2\rangle+\frac{1}{2}\langle(S+S_0^z)(S-S_0^z+1)\rangle\frac{\beta\hbar\tilde{\omega}}{\exp[\beta\hbar\tilde{\omega}]-1}\right.$$
$$\left.+\frac{1}{2}\langle(S-S_0^z)(S+S_0^z+1)\rangle\frac{\beta\hbar\tilde{\omega}}{1-\exp[-\beta\hbar\tilde{\omega}]}\right\} \qquad (3\cdot4)$$

$$=\rho_0\begin{cases}\times\dfrac{2S+(1/2)}{S(S+1)}\dfrac{\hbar\tilde{\omega}}{k_BT}e^{-\hbar\tilde{\omega}/k_BT} & (k_BT\ll\hbar\tilde{\omega})\\ \times\left(1-\dfrac{\sigma^2}{S(S+1)}\right) & (k_BT\gtrsim\hbar\tilde{\omega}),\end{cases} \qquad (3\cdot4a)$$

where $\rho_0=3\pi J_{eff}^2NmS(S+1)/(4\hbar e^2E_f)$. The first term in the curly brackets of Eq. (3·4) comes from the elastic process, and the second and third terms come from the inelastic processes which are shown respectively in Fig. 1 (a) and (b).

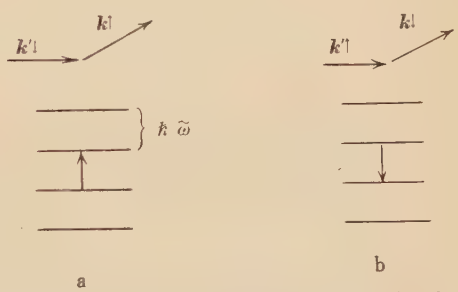

Fig. 1. The four equi-distant lines show the level scheme of each d spin ($S=3/2$) in the molecular field qM.

(ii) *Spin wave approximation*

For temperatures far below the Curie temperature T_c, the excitation in a ferromagnet is naturally described in terms of the free spin waves,[15] and the g's take the form

$$g_{\kappa}^{+-}(\omega)=2S(n_{\kappa}+1)\delta(\omega-\omega_{\kappa}),$$
$$g_{\kappa}^{-+}(\omega)=2Sn_{-\kappa}\delta(\omega+\omega_{-\kappa}), \qquad (3\cdot5)$$

$$g_k^{zz}(\omega)=N\left(S^2-\frac{2S}{N}\sum_\lambda n_\lambda\right)\delta_{\kappa,0}\delta(\omega)\ ,$$

where $n_\kappa=\{\exp(\beta\hbar\omega_\kappa)-1\}^{-1}$, and $\hbar\omega_\kappa$ is the energy quantum of the spin wave with wave vector $\boldsymbol{\kappa}$. In the low temperature region the spin waves thermally excited are mainly of long wavelength and its energy is approximated simply by $\hbar\omega_\kappa=(\hbar^2\boldsymbol{\kappa}^2/2\mu)$, μ being the effective mass of spin wave. Then if we neglect the effect of the spin polarization of conduction electron, with the use of Eqs. (2·16), (2·19) and (3·5) we have the electrical resistivity in the present approximation (see Appendix II)

$$\rho=\frac{3\pi^5}{16}\left(\frac{\mu}{m}\right)^2\frac{S}{e^2}\frac{[NJ(0)]^2[k_BT]^2}{E_f^4}\frac{\hbar}{k_f}\ , \tag{3·6}$$

where $J(\boldsymbol{\kappa})$, the Fourier component of the s-d exchange integral, has been assumed to be independent of $\boldsymbol{\kappa}$ and has been set equal to $J(0)$. This is just the same as the result obtained in reference 1) by solving the Boltzmann equation.

Now we will briefly discuss the effect of the spin polarization of conduction electron. For simplicity if we adopt the energy spectrum of conduction electron to be of the type $E_k^{\pm}=(\hbar^2\boldsymbol{k}^2/2m)\mp J$, by the conservation relations for energy and for wave vector we have that for the backward scattering $\theta>\pi/2$, θ being the scattering angle, the minimum value of the momentum transfer is nearly equal to $(J/E_f)k_f$, and for the forward scattering $\theta<\pi/2$ that value is null. Then using the approximation $\hbar\omega_\kappa=(\hbar^2\kappa^2/2\mu)$, if we denote the temperature by T_a below which the electrical resistivity decreases exponentially with decreasing temperature, we have $T_a\simeq(J/E_f)^2T_C$. This temperature is very low in the usual condition and in practice one can neglect the effect of the spin polarization of conduction electron.

(iii) *Use of the method of the Green's function* $(T<T_C)$

Recently Bogoliubov and Tjablikov[16),17)] have given a systematic method in solving the correlation function of the type $\langle S_m^{\mu}(t)S_n^{\nu}(t')\rangle$ by introducing the two time Green's functions. Their result necessary for the present purpose is that

$$\left.\begin{aligned} g_\kappa^{+-}(\omega)&=2\sigma(n_\kappa+1)\delta(\omega-\tilde{\omega}_\kappa),\\ g_\kappa^{-+}(\omega)&=2\sigma n_\kappa\delta(\omega+\tilde{\omega}_\kappa),\\ g_\kappa^{zz}(\omega)&=N\sigma^2\delta_{\kappa,0}\delta(\omega),\end{aligned}\right\} \tag{3·7}$$

$$\hbar\tilde{\omega}_\kappa=2\sigma\sum_m I(m,0)\,(1-\exp[i\boldsymbol{\kappa}\cdot\boldsymbol{R}_m]), \tag{3·8}$$

$$n_\kappa=(\exp[\beta\hbar\tilde{\omega}_\kappa]-1)^{-1}, \tag{3·9}$$

where $I(m,0)$ is the exchange integral between d electrons located at the lattice points 0 and m, and σ is the average value of S_n^z : $\sigma=\langle S_n^z\rangle$. At first sight

Eqs. (3·7) seem to have a strong resemblance to Eqs. (3·5), but the former differs from the latter in that $g_\kappa^{+-}(\omega)$ and $g_\kappa^{-+}(\omega)$ are proportional to σ in Eqs. (3·7) and to S in Eqs. (3·5), and in Eqs. (3·8) the energy quantum of the elementary excitation $\hbar\tilde{\omega}_\kappa$ is proportional to σ and decreases with increasing temperature and is equal to null at T_C for any wave vector.

If we adopt the so-called effective mass approximation $\hbar\tilde{\omega}_\kappa=\hbar^2\kappa^2/2\tilde{\mu}$, $\tilde{\mu}$ being dependent on T, we get the resistivity formula in the same way as in (ii),

$$\rho=\frac{9\pi^3}{16}\left(\frac{\tilde{\mu}}{m}\right)^2\frac{\sigma}{e^2}\frac{[NJ(0)]^2[k_BT]^2}{E_f^4}\frac{\hbar}{k_f}\int_0^{x_0}\frac{x^2dx}{(e^x-1)(1-e^{-x})} \tag{3·10}$$

with $x_0=\hbar^2q_0^2/2\tilde{\mu}k_BT$, q_0 being the wave vector at the zone boundary. In the low temperature limit, $\sigma\to S$, $\tilde{\mu}\to\mu$, $x_0\to\infty$ and the integral in Eq. (3·10) becomes $\pi^2/3$; so Eq. (3·10) reduces to exactly the same result as in the spin wave approximation. On the other hand if the limit $T\to T_C$ is taken, one has that $\sigma\to 0$, $x_0\to 0$ and one is able to estimate the integral appearing in Eq. (3·10) in the form of the power series in x_0. And one has that

$$\frac{\rho}{\rho_0}=A\frac{T}{T_C}\left(1-\frac{x_0^2}{36}+\cdots\right), \tag{3·11}$$

where ρ_0 is the resistivity calculated by the molecular field approximation in the temperature region $T>T_C$ and is independent of temperature, and A stands for a dimensionless numerical factor of order unity. Eq. (3·11) states that ρ is nearly proportional to T for the temperature region $T\lesssim T_C$.

(iv) *The case $T>T_C$ (the moment method)*

For the present purpose we rewrite $\phi(0)$ in the form

$$\phi(0)=2N\left(\frac{e\hbar}{m}\right)^2\sum_{\boldsymbol{k}\boldsymbol{k}'}|J(\boldsymbol{k}'-\boldsymbol{k})|^2(k_x'-k_x)^2f_{k'}(1-f_k)g_{k'-k}\left(\frac{E_{k'}-E_k}{\hbar}\right) \tag{3·12}$$

in which the effect of the spin polarization of conduction electron has been neglected as before, and $g_\kappa(\omega)$ stands for the diagonal sum of $g_\kappa^{\mu\nu}(\omega)$:

$$g_\kappa(\omega)=\sum_\mu^{x,y,z}g_\kappa^{\mu\mu}(\omega)=\sum_n\int\frac{dt}{2\pi}\exp[i\boldsymbol{\kappa}\cdot\boldsymbol{R}_n-i\omega t]\langle\boldsymbol{S}_0(0)\cdot\boldsymbol{S}_n(t)\rangle. \tag{3·13}$$

Replacing the summation appearing in Eq. (3·12) by the integration we can easily write

$$\phi(0)=\frac{Ne^2k_f^4J_{eff}^2}{6\pi^4\hbar}\int d\omega\,\tilde{g}(\omega)\frac{\hbar\omega}{\exp[\beta\hbar\omega]-1} \tag{3·14}$$

and

$$\tilde{g}(\omega)=\int d(\cos\theta)\,(1-\cos\theta)\,|J(\kappa)|^2g_\kappa(\omega)\Big/\int d(\cos\theta)\,(1-\cos\theta)\,|J(\kappa)|^2. \tag{3·15}$$

$$\left(\kappa=2k_f\sin\frac{\theta}{2}\right)$$

If we use ρ_0 given in Eq. (3·4a) we can write for ρ

$$\frac{\rho}{\rho_0}=\frac{1}{S(S+1)}\int d\omega\,\tilde{g}(\omega)\,\frac{\beta\hbar\omega}{\exp[\beta\hbar\omega]-1}. \tag{3·16}$$

It is worth-while to note the property of Eq. (3·16) in the high temperature limit. In this limit we have that

$$\frac{\rho}{\rho_0}=\frac{1}{S(S+1)}\int d\omega\,\tilde{g}(\omega)$$

and that from Eq. (3·13)

$$\int d\omega\, g_\kappa(\omega)=\sum_n \exp[i\boldsymbol{\kappa}\cdot\boldsymbol{R}_n]\langle \boldsymbol{S}_0(0)\cdot\boldsymbol{S}_n(0)\rangle=S(S+1),$$

where in the last equality the property $\langle \boldsymbol{S}_0\cdot\boldsymbol{S}_n\rangle=\delta_{n,0}S(S+1)$ has been taken into account because of the mutual independence of the system of d spins at high temperatures. So we have in this limit $\rho=\rho_0$.

We now turn to the problem to calculate ρ for $T>T_C$. In this temperature region the spin system is of course paramagnetic, but the spin ordering in short distances still remains. Let us examine the effects of this local spin ordering to the electrical resistivity, and look for the functional form of $g_\kappa(\omega)$ by calculating some of the moments $\langle\omega^m\rangle_\kappa$ defined by

$$\langle\omega^m\rangle_\kappa=\int_{-\infty}^{\infty} g_\kappa(\omega)\,\omega^m\,d\omega\Big/\int_{-\infty}^{\infty} g_\kappa(\omega)\,d\omega, \tag{3·17}$$

which can be rewritten as

$$\langle\omega^m\rangle_\kappa=(-i)^m\frac{\sum_n \exp[i\boldsymbol{\kappa}\cdot\boldsymbol{R}_n]\{(d/dt)^m\langle \boldsymbol{S}_0(0)\cdot\boldsymbol{S}_n(t)\rangle\}_{t=0}}{\sum_n \exp[i\boldsymbol{\kappa}\cdot\boldsymbol{R}_n]\langle \boldsymbol{S}_0(0)\cdot\boldsymbol{S}_n(0)\rangle}. \tag{3·18}$$

Because of the presence of the factor $1-\cos\theta$ in $\tilde{g}(\omega)$ the forward scattering is less effective in determining the resistivity. In the usual conditions that the conduction electrons are strongly degenerate, only the surface to surface transitions on the Fermi sphere can take place. So the average momentum transfer associated with the most effective transitions in determining the resistivity is large, say, of the order of k_f. Then one can approximate Eq. (3·18) only by calculating the correlation functions between a few neighbours and their time derivatives.

The moments are calculated only in the form of the power series of β and we confine ourselves to the estimations of ρ by the high temperature expansion. As is shown in Appendix III, in the limit $\beta\to 0$, $\langle\omega\rangle_\kappa$ goes to null but $\langle\omega^2\rangle_\kappa$ remains finite. It has been shown that[18] for small momentum transfers ($\kappa a\lesssim 1$, a being the lattice constant) the comparison of $\langle\omega^2\rangle_\kappa$ with $\langle\omega^4\rangle_\kappa$ indicates that $g_\kappa(\omega)$ may be approximated by a Lorentzian function, and, on the other hand,

for large momentum transfers ($\kappa a \gtrsim \pi$), $g_\kappa(\omega)$ may be approximated by a Gaussian function. These arguments are less satisfactory in the former case ($\kappa a \lesssim 1$), because the smallness of momentum transfer requires the inclusion of the correlation function for a pair of spins separated by larger distance and its time derivatives.

The calculation is somewhat lengthy but is done straight-forwardly (see Appendix III), and only the results are quoted here:

$$g_\kappa(\omega) = \frac{S(S+1)}{\sqrt{2\pi(\langle\omega^2\rangle_\kappa - \langle\omega\rangle_\kappa^2)}} \exp\left(-\frac{(\omega-\langle\omega\rangle_\kappa)^2}{2(\langle\omega^2\rangle_\kappa - \langle\omega\rangle_\kappa^2)}\right), \qquad (3\cdot19)$$

for $\kappa a \gtrsim \pi$

$$g_\kappa(\omega) = \begin{cases} \mathcal{H}\,\dfrac{S(S+1)}{2\pi}\,\dfrac{2b}{(\omega-\langle\omega\rangle_\kappa)^2+b^2}, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases} \qquad (3\cdot20)$$

for $\kappa a \lesssim 1$

with $b=(\pi/2\omega_c)(\langle\omega^2\rangle_\kappa - \langle\omega\rangle_\kappa^2)$. The factor $\mathcal{H}$, which is order of unity, is introduced to assure the normalization of $g_\kappa(\omega)$ the Lorentzian function truncated at $\omega=\pm\omega_c$; $\int g_\kappa(\omega)\,d\omega = S(S+1)$. For the limit $\kappa a \to 0$, if one adopts[18] $\hbar\omega_c = 9I\sqrt{S(S+1)}$ one has that $\beta\hbar b = \dfrac{\pi}{27}\dfrac{(\kappa a)^2}{\sqrt{S(S+1)}}\dfrac{T_c}{T}$, $\beta\hbar\omega_c = \dfrac{9}{4\sqrt{S(S+1)}}\dfrac{T_c}{T}$, hence $b \cong \dfrac{(\kappa a)^2}{20}\omega_c$. And for $\kappa a \lesssim 1$, we may safely use the truncated Lorentzian function such as Eq. (3·20).

Since the moments are obtained in the form of the power series in βI, we now calculate ρ by the series expansion with respect to βI. However, the function $x/(e^x-1)$ ($x=\beta\hbar\omega$) in Eq. (3·16) is expressible in the power series in x only for $|x|<2\pi$. In the Lorentzian case truncated at $\omega=\pm\omega_c$, $x_c=\beta\hbar\omega_c$ is less than 2.6 for $T=T_c\sim\infty$ and $S\geq 1/2$, and hence $x/(e^x-1)$ can be expanded in the power series in x to perform the integration in Eq. (3·16). On the other hand, in the Gaussian case if the half width of $g_\kappa(\omega)$ is measured by $\sqrt{\langle\omega^2\rangle_\kappa}$, the quantity $\beta\hbar\sqrt{\langle\omega^2\rangle_\kappa} \cong (1/\sqrt{S(S+1)})(T_c/T)$ is less than 1.2 for $T=T_c\sim\infty$ and $S\geq 1/2$. For $x=\beta\hbar\omega=2$ one has $g_\kappa(\omega)\sim e^{-2.0}\cdot g_\kappa(0)$, and the Gaussian function truncated at $\beta\hbar\omega=\pm 2\pi$ is used to calculate ρ.

It must be noted that even in the case $k_f a \gtrsim \pi$ there are some of the scattering processes associated with small momentum transfers; however, because of the presence of the factor $1-\cos\theta$, these scattering processes have nothing to do with the electrical conductivity as far as $J(\kappa)$ remains finite for small κ. And we use the Gaussian form for $g_\kappa(\omega)$ to calculate ρ in the case of a "large" Fermi sphere ($k_f a \gtrsim \pi$). On the other hand for small $k_f a$ we use the Lorentzian form for $g_\kappa(\omega)$ throughout every scattering process. Then the substitutions Eqs. (3·19) and (3·20) in Eq. (3·16) give the following formula for ρ:

$$\frac{\rho}{\rho_0}=1-A, \tag{3·21}$$

$$A=\frac{1-\gamma_1}{zS(S+1)}\left(\frac{T_C}{T}\right)^2\left\{1-\frac{T_C}{T}\left[\frac{1}{8S(S+1)}+\frac{\gamma_1-\gamma_2}{1-\gamma_1}\right]+\left(\frac{T_C}{T}\right)^2 B-\cdots\right\},$$

$$(k_f a \gtrsim \pi) \tag{3·21a}$$

$$A=\frac{4(k_f a)^2}{9zS(S+1)}\left(\frac{T_C}{T}\right)^2\left\{1-\frac{T_C}{T}\left[\frac{1}{8S(S+1)}+1-\frac{(k_f a)^2}{2}\right]+\left(\frac{T_C}{T}\right)^2 C-\cdots\right\},$$

$$(k_f a \lesssim 1) \tag{3·21b}$$

$$B=\frac{1}{20}\left(\frac{17}{9}-\frac{53}{12}\frac{1}{S(S+1)}+\frac{7}{16}\frac{1}{S^2(S+1)^2}\right)+\frac{1}{2}\left(\frac{19}{20}\frac{1}{S(S+1)}\frac{\gamma_1-\gamma_2}{1-\gamma_1}-\frac{4}{3}\frac{\delta_1-\gamma_3}{1-\gamma_1}-\frac{1}{3}\frac{\delta_2-\gamma_4}{1-\gamma_1}\right)+\frac{\gamma_2-\gamma_5}{1-\gamma_1},$$

$$C=\frac{1}{20}\left(\frac{47}{9}+\frac{515}{96}\frac{1}{S(S+1)}+\frac{7}{16}\frac{1}{S^2(S+1)^2}\right)-\frac{1}{4S(S+1)}(k_f a)^2,$$

where z is the number of first neighbours and $T_C=2Iz S(S+1)/3k_B$ is the Curie temperature given by the molecular field approximation. γ's and δ's have been defined as

$$\gamma_1=\gamma_0^{-1}\int_0^{2k_f} d\kappa\,\kappa^3|J(\kappa)|^2\frac{\sin\kappa a}{\kappa a},$$

$$\gamma_2=\gamma_0^{-1}\int_0^{2k_f} d\kappa\,\kappa^3|J(\kappa)|^2\left(\frac{\sin\kappa a}{\kappa a}\right)^2,$$

$$\gamma_3=\gamma_0^{-1}\int_0^{2k_f} d\kappa\,\kappa^3|J(\kappa)|^2\frac{\sin\kappa a}{\kappa a}\frac{\sin\sqrt{2}\,\kappa a}{\sqrt{2}\,\kappa a},$$

$$\gamma_4=\gamma_0^{-1}\int_0^{2k_f} d\kappa\,\kappa^3|J(\kappa)|^2\frac{\sin\kappa a}{\kappa a}\frac{\sin 2\kappa a}{2\kappa a},$$

$$\gamma_5=\gamma_0^{-1}\int_0^{2k_f} d\kappa\,\kappa^3|J(\kappa)|^2\left(\frac{\sin\kappa a}{\kappa a}\right)^3, \tag{3·21c}$$

$$\delta_1=\gamma_0^{-1}\int_0^{2k_f} d\kappa\,\kappa^3|J(\kappa)|^2\frac{\sin\sqrt{2}\,\kappa a}{\sqrt{2}\,\kappa a},$$

$$\delta_2=\gamma_0^{-1}\int_0^{2k_f} d\kappa\,\kappa^3|J(\kappa)|^2\frac{\sin 2\kappa a}{2\kappa a},$$

where

$$\gamma_0 = \int_0^{2k_f} d\kappa\, \kappa^3 |J(\kappa)|^2.$$

In the above expression for ρ, the terms proportional to $(T_C/T)^3$, $(T_C/T)^4$ are correct only in a simple cubic lattice. It may be noted that in the limit $k_f a \to 0$ Eq. (3·21a) reduces to the Lorentzian case (3·21b), and Eq. (3·21a) is plotted in Fig. 2 on the assumption that $J(\kappa)$ is independent of κ. It is easily seen from Eqs. (3·21), (3·21a) and (3·21b) that

(a) in both Gaussian and Lorentzian cases ρ is smaller than ρ_0 and ρ versus T curve is concave to the abscissa T,

(b) ρ in the Gaussian approximation is somewhat smaller than the one in the Lorentzian approximation and in this latter case, if the limit $k_f a \to 0$ is taken, ρ becomes ρ_0, and

(c) if one assumes simply that $J(\kappa)$ is independent of κ, γ_1 versus $2k_f a$ curve is given in Fig. 3 and one obtains in the Gaussian case with typical numerical values ($S=1$, $z=6$, $T \simeq T_C$) the relative correction $A=(\rho_0-\rho)/\rho_0$ which amounts to 10% or so.

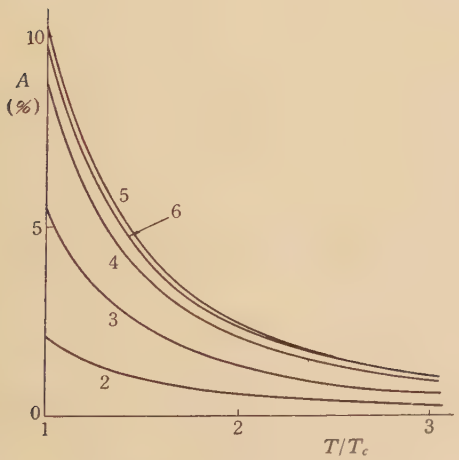

Fig. 2. Relative decrease A (in %) in ρ is plotted against T/T_c in the case $z=6$ and $S=1$. Figures 2~6 stand for the values of $2k_f a$.

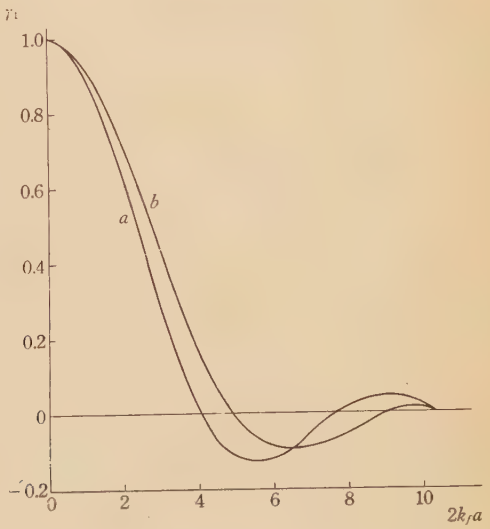

Fig. 3. Typical examples for the parameter γ_1 as a function of $2k_f a$. Curve a is obtained from the assumption $J(\kappa)=$independent of κ, and b from $|J(\kappa)|^2=|J(0)|^2 \times (1-(\kappa/2k_f)^2)$.

These may be explained as follows: In the region $T > T_C$, the long distance order in d spin system disappears in an equilibrium state but the local order in short distances still remains finite. Because of the exchange character of

the s-d interaction by which the conduction electrons are scattered, the presence of the local order will diminish the scattering of conduction electron to give $\rho<\rho_0$ and will diminish especially the one associated with relatively large momentum transfer ($\kappa a\sim\pi$). This seems to lead to the smallness of ρ obtained in the Gaussian case. On the other hand, in the Lorentzian case there are only scattering processes with relatively small momentum transfers ($\kappa a<1$) and these processes are not so much influenced by the presence of the local ordering. This seems to explain the smallness of A in the Lorentzian case.

(v) *The case $T\gtrsim T_C$ (k_f small)*

In the case of small k_f, we need the correlation function for a pair of d spins separated by a rather long distance. This correlation function for the long distant spin pair seems to be a very slowly varying one as a function of t in the temperature region immediately above T_c.[4),19)] So we may approximate $\langle S_0^{\mu}(0)S_n^{\nu}(t)\rangle$ by $\langle S_0^{\mu}(0)S_n^{\nu}(0)\rangle$ (static approximation). With this approximation Elliott and Marshall[20)] calculated the correlation function. Now with the neglect of the spin polarization of conduction electron, ρ is expressible as in Eq. (3·16) in terms of $g_{\kappa}(\omega)$ written as in the static approximation:

$$g_{\kappa}(\omega)=\delta(\omega)\{\phi^{xx}(\kappa)+\phi^{yy}(\kappa)+\phi^{zz}(\kappa)\} \tag{3·22}$$

where ϕ's are defined by[20)] $\phi^{\mu\mu}(\kappa)=\sum_n \exp[i\boldsymbol{\kappa}\cdot\boldsymbol{R}_n]\langle S_0^{\mu}S_n^{\mu}\rangle$ ($\mu=x, y, z$). Eqs. (3·16) and (3·22) give

$$\frac{\rho}{\rho_0}=\frac{\int d(\cos\theta)(1-\cos\theta)|J(\kappa)|^2\sum_{\mu}\phi^{\mu\mu}(\kappa)}{\int d(\cos\theta)(1-\cos\theta)|J(\kappa)|^2 S(S+1)}. \tag{3·23}$$

$$(\kappa=2k_f\sin(\theta/2))$$

Using the result obtained by Elliott and Marshall, this equation is reduced to

$$\frac{\rho}{\rho_0}=\frac{8p_0}{3y^4}\int_0^y\frac{x^3dx}{1-(\sin x/x)+(a^2\kappa_1^2/z)}\qquad\left(S=\frac{1}{2}\right) \tag{3·24}$$

in the temperature region immediately above T_C. In the above expression $y=2k_fa$ is assumed to be small and $J(\kappa)$ has been set equal to a constant, and p_0 and κ_1 have the same meanings as in reference 20). These results are plotted in Fig. 4.

(vi) *The case $T\gtrsim T_C$ (k_f large)*

In this case it is necessary to deal with the correlation function for a pair of spins separated by rather short distances. So the approach of the static approximation breaks down completely even in the temperature region immediately above T_C. In this temperature region, however, the time variation of the long distance order parameter is very slow[19)] and this order parameter

is assumed to be a constant in the time scale in which the scattering process of conduction electron takes place. In order to obtain ρ in the case of large k_f we confine ourselves to the calculation of the pair correlation functions only of the nearest neighbours, i. e. we deal only with $\langle \boldsymbol{S}_0(0)\cdot\boldsymbol{S}_0(t)\rangle$ and $\langle \boldsymbol{S}_0(0)\cdot\boldsymbol{S}_\delta(t)\rangle$, δ being any one of the lattice vectors which join an atom to one of its nearest neighbours. Then $g_z(\omega)$ necessay for the present calculation is approximated as

$$g_z(\omega)=\int\frac{dt}{2\pi}e^{-i\omega t}\left\{\langle \boldsymbol{S}_0(0)\cdot\boldsymbol{S}_0(t)\rangle + \frac{\sin\kappa a}{\kappa a}\langle \boldsymbol{S}_0(0)\cdot\boldsymbol{S}_1(t)\rangle\right\} \qquad (3\cdot25)$$

where in the second term of the right-hand member a directional average has been taken, and $\boldsymbol{S}_1=\sum_\delta \boldsymbol{S}_\delta$. In the present approximation the time evolution of $\boldsymbol{S}_0$ and $\boldsymbol{S}_1$ is assumed to be governed by the Hamiltonian[21]

$$H=-2I\boldsymbol{S}_0\cdot\boldsymbol{S}_1-g\mu_B H_1 S_1^z. \qquad (3\cdot26)$$

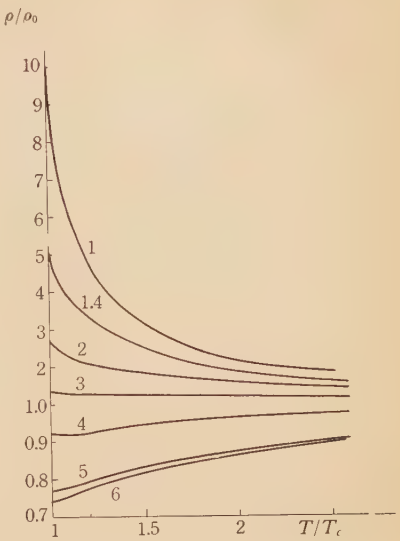

Fig. 4. ρ/ρ_0 versus T/T_c curves in the case $z=6$ and $S=1/2$. Figures 1~6 correspond to the values of $y=2k_f a$.

H_1, the so-called molecular field, is of character of a long distance order parameter and is assumed here to be a constant parameter. Moreover, for the present case ($T\gtrsim T_c$) it is set equal to null.

The way of the calculation is outlined in Appendix IV. Only the results are quoted here:

$$\frac{\rho}{\rho_0}=\frac{4}{3Z}\sum_{S_1}\frac{w(S_1 z)}{2S_1+1}\left\{(1-\gamma_1)2S_1(S_1+1)\left[\exp(-\beta E_{S_1+1/2})-\frac{\beta I(2S_1+1)}{\exp[\beta I(2S_1+1)]-1}\right.\right.$$
$$+\exp(-\beta E_{S_1-1/2})\frac{-\beta I(2S_1+1)}{\exp[-\beta I(2S_1+1)]-1}\Big]+\Big[(S_1+1)\Big(S_1+\frac{3}{2}\Big)(1+2S_1\gamma_1)$$
$$\left.\left.\times\exp(-\beta E_{S_1+1/2})+S_1\Big(S_1-\frac{1}{2}\Big)(1-\gamma_1(2S_1+2))\exp(-\beta E_{S_1-1/2})\Big]\right\} \qquad (3\cdot27)$$

with Z, the partition function,

$$Z=2\sum_{S_1}w(S_1 z)\{(S_1+1)\exp(-\beta E_{S_1+1/2})+S_1\exp(-\beta E_{S_1-1/2})\}, \qquad (3\cdot28)$$

in which γ_1 is the same as that given in § 3-(iv). $w(S_1, z)$ is the multiplicity for a given S_1 constructed by z spins of one half. Eq. (3·27) is plotted in Fig. 5.

(vii) *The summary of the results and the comparison with experiments*

The results of § 3–(i) and (ii) were already discussed in references 10) and 1). We now discuss the results obtained in § 3–(iii) ~ (vi). The electrical resistivity ρ calculated by the method of the Green's function in (iii) is just the same as given by the spin wave approximation at low temperatures, and is nearly equal to the one given by the molecular field approximation for $T \lesssim T_C$. The method of the Green's function with its lowest order solution (3·7) is not sufficient to discuss the paramagnetic region.

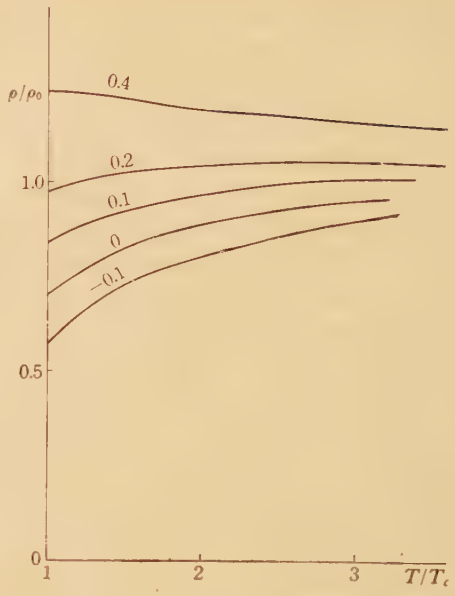

Fig. 5. ρ/ρ_0 versus T/T_c curves in the case $z=6$ and $S=1/2$. Figures $-0.1 \sim 0.4$ stand for the values of γ_1.

The result of § 3–(iv) is that (a) ρ is smaller than ρ_0 and ρ versus T curve is concave to the temperature axis for $T > T_C$, (b) ρ calculated in the case of large k_f is smaller than the one in the case of small k_f, and (c) if the result (3·21) is extrapolated to T_C, $A_C \equiv \frac{\rho_0 - \rho(T_C)}{\rho_0} \cong (1-\gamma_1)/zS(S+1)$ is about 10%.

From § 3–(vi) ρ becomes small for small values of γ_1 which correspond to large values of k_f by virtue of Fig. 3. This result is qualitatively in accordance with the result (b) by the method of moment expansion, and with the result given by the method of § 3–(v) when k_f is large. On the other hand, for small k_f we have $\rho \cong \rho_0$ by the method of moment expansion. However, since small values of k_f must require the spin correlation function for a pair of spins separated by large distance as well as by short distance, then the approximation in § 3–(iv) which takes into account the spin correlation only up to the third neighbours is rather inadequate. From § 3–(v) for small k_f, ρ is larger than ρ_0. This is in accordance with the tendency in the result given by § 3–(vi) (cf. Figs. 3 and 5). We may conclude $\rho > \rho_0$ in the case of small k_f and at temperatures immediately above T_C.

In § 3–(iv) the cases of small k_f and of large k_f are investigated, and in practice ferromagnetic metals correspond to the latter case.[10] The tendency of the result (a) is seen in the ferromagnetic metals[22),23)] of the iron group and of the rare earth group without any exception. From Eq. (3·21) if we extrapolate A to the Curie temperature T_C we have $A_C \cong (1-\gamma_1)/zS(S+1)$ at T_C. The experimental values for A_C are given in Table I, which enable us to estimate $1-\gamma_1$ as listed in the same Table. Values for $1-\gamma_1$ thus obtained are in

reasonable magnitude because the parameter $1-\gamma_1$ takes a value ranging from 0 to 2 (cf. Eqs. (3·21c)).

Table I. A_c's in the column (i) are obtained from references 23), 24) and 25), and those in (ii) are obtained from reference 22). For Fe, $S=1$ is used,[24] and for Ni, the simple Van Vleck model[25] is used to give $S(S+1)=(1/2)(1/2+1)\times 0.6+0(0+1)\times 0.4=0.45$. From these values $1-\gamma_1$ is obtained by assuming the relation $A_c=(1-\gamma_1)/zS(S+1)$.

	A_c		$S(S+1)$	$1-\gamma_1$	
	(i)	(ii)		(i)	(ii)
Fe	0.15	0.0253	2	2.4	0.40
Ni	0.26	0.0963	0.45	1.4	0.52
Gd	0.00996	—	7/2·(7/2+1)	1.9	—

§ 4. Dilute alloy

For an isolated Mn ion Eq. (2·24) amounts to

$$2\left(\frac{e\hbar}{m}\right)^2 S(S+1)\hbar\sum_{kk'}|J(\boldsymbol{k}'-\boldsymbol{k})|^2(k_x'-k_x)^2 f_{k'}(1-f_k)\delta(E_{k'}-E_k) \quad (4\cdot1)$$

which is proportional to T, the absolute temperature. Since $\langle j_x j_x\rangle_0$ is proportional to T (cf. § 2–(d)), the resistivity is independent of T, as we can see basing upon Eq. (2·16). Thus the scattering process by isolated Mn ions gives rise to the resistivity independent of T. Let us consider next the effect of the scattering by a pair of Mn ions. Because of the presence of the factor $1-\cos\theta$ in $\phi(0)$ (cf. Eqs. (3·14) and (3·15)), the scattering process accompanied by small momentum transfer is not effective in determining the resistivity and so the contribution to the resistivity of a pair of Mn ions separated by a large distance is very small. We confine ourselves to a pair of Mn ions separated by the nearest neighbour lattice vectors, and after averaging over angles we have

$$\phi(0)=4\left(\frac{e\hbar}{m}\right)^2\sum_{kk'}|J(\boldsymbol{k}'-\boldsymbol{k})|^2(k_x'-k_x)^2 f_{k'}(1-f_k)\int\frac{dt}{2\pi}\exp[-i(E_{k'}-E_k)t/\hbar]$$
$$\times\left\{\langle \boldsymbol{S}_0(0)\cdot\boldsymbol{S}_0(t)\rangle+\frac{\sin|\boldsymbol{k}'-\boldsymbol{k}|a}{|\boldsymbol{k}'-\boldsymbol{k}|a}\langle \boldsymbol{S}_0(0)\cdot\boldsymbol{S}_\delta(t)\rangle\right\}. \quad (4\cdot2)$$

We assume the coupling between $\boldsymbol{S}_0$ and $\boldsymbol{S}_\delta$ to be

$$H=-A\boldsymbol{S}_0\cdot\boldsymbol{S}_\delta, \quad (4\cdot3)$$

δ standing for any one of the nearest neighbours of the lattice point 0, and we have[26]

$$\langle S_0^z(0)S_0^z(t)\rangle=\{\sum_{JM}\exp(-\beta E_J)|\langle JM|S_0^z|JM\rangle|^2+[\exp(-\beta E_J+i(E_{J-1}-E_J)t/\hbar)$$
$$+\exp(-\beta E_{J-1}+i(E_J-E_{J-1})t/\hbar)]|\langle J-1M|S_0^z|JM\rangle|^2\}/Z,$$

$$\langle S_0^z(0)\,S_\delta^z(t)\rangle=\{\sum_{JM}\exp(-\beta E_J)\,|\langle JM|S_0^z|JM\rangle|^2-[\exp(-\beta E_J+i(E_{J-1}-E_J)t/\hbar)$$
$$+\exp(-\beta E_{J-1}+i(E_J-E_{J-1})t/\hbar)]\,|\langle J-1M|S_0^z|JM\rangle|^2\}/Z \qquad (4\cdot4)$$

with

$$Z=\sum_J (2J+1)\,e^{-\beta E_J}$$

where J and M are the magnitude and the z component of $\boldsymbol{S}_0+\boldsymbol{S}_\delta$ and $E_J=-(A/2)\,(J(J+1)-2S(S+1))$. In Eqs. (4·4) we have used the relation $\langle J'M|S_0^z+S_\delta^z|JM\rangle=\delta_{JJ'}M$. Using the explicit form of the matrix elements[26]

$$\langle JM|S_0^z|JM\rangle=M/2,$$
$$\langle J-1M|S_0^z|JM\rangle=\frac{1}{2}\sqrt{((2S+1)^2-J^2)/(2J-1)\,(2J+1)}$$
$$\times\sqrt{(J-M)\,(J+M)}$$

and Eqs. (2·16), (4·2) and (4·4), we easily obtain

$$\frac{\rho}{\rho_0}\frac{N}{2}=(4S(S+1)Z)^{-1}\sum_{J=1}^{2S}\exp(-\beta E_J)\,\{J(J+1)\,(2J+1)\,(1+\gamma_1)$$
$$+[(2S+1)^2-J^2]J(1-\gamma_1)[2\beta AJ/(\exp(\beta AJ)-1)]\} \qquad (4\cdot5)$$

where ρ_0 and γ_1 have the same meaning as in § 3. Eq. (4·5) is the resistivity per single Mn ion and is normalized to unity at high temperatures. Indeed if the limit $\beta\to0$ is taken the right-hand side of Eq. (4·5) becomes independent of γ_1 and goes to unity. In Eq. (4·5) the terms proportional to $(1+\gamma_1)$ and to $(1-\gamma_1)$ come from the elastic and inelastic scattering processes, respectively. Eq. (4·5) is plotted in Fig. 6.

From Fig. 6 it is seen that for $A>0$ ('ferromagnetic' case) in the temperature region $T\sim A/k_B$ ρ increases or decreases with decreasing temperature according to $\gamma_1>0$ or to $\gamma_1<0$, and for $A<0$ ('antiferromagnetic' case) in the region $T\sim A/k_B$ ρ increases or decreases according to $\gamma_1<0$ or to $\gamma_1>0$. But in that temperature region the increment in ρ with decreasing temperature is less pronounced in the case $A<0$ than in the case $A>0$, because the parameter γ_1 is larger than -0.1 as is seen from Fig. 3.

Consequently, if $A\gamma_1>0$, the electrical resistivity will have a minimum in the temperature region $T\sim A/k_B$ when the phonon resistivity is added to ρ given by Eq. (4·5). Brailsford and Overhauser[11] showed that the present model agrees with the experiment of the electrical resistivity of dilute alloys in a semi-quantitative sense. As mentioned above, the resistivity minimum is much more pronounced in the case $A>0$ than in the case $A<0$, because the values of $|\gamma_1|$ are not so large when $\gamma_1<0$ (cf. Fig. 3). For CuMn alloys, which show an antiferromagnetic behaviour at low temperatures, this mechanism alone does not seem to be sufficient to explain the resistivity minimum.

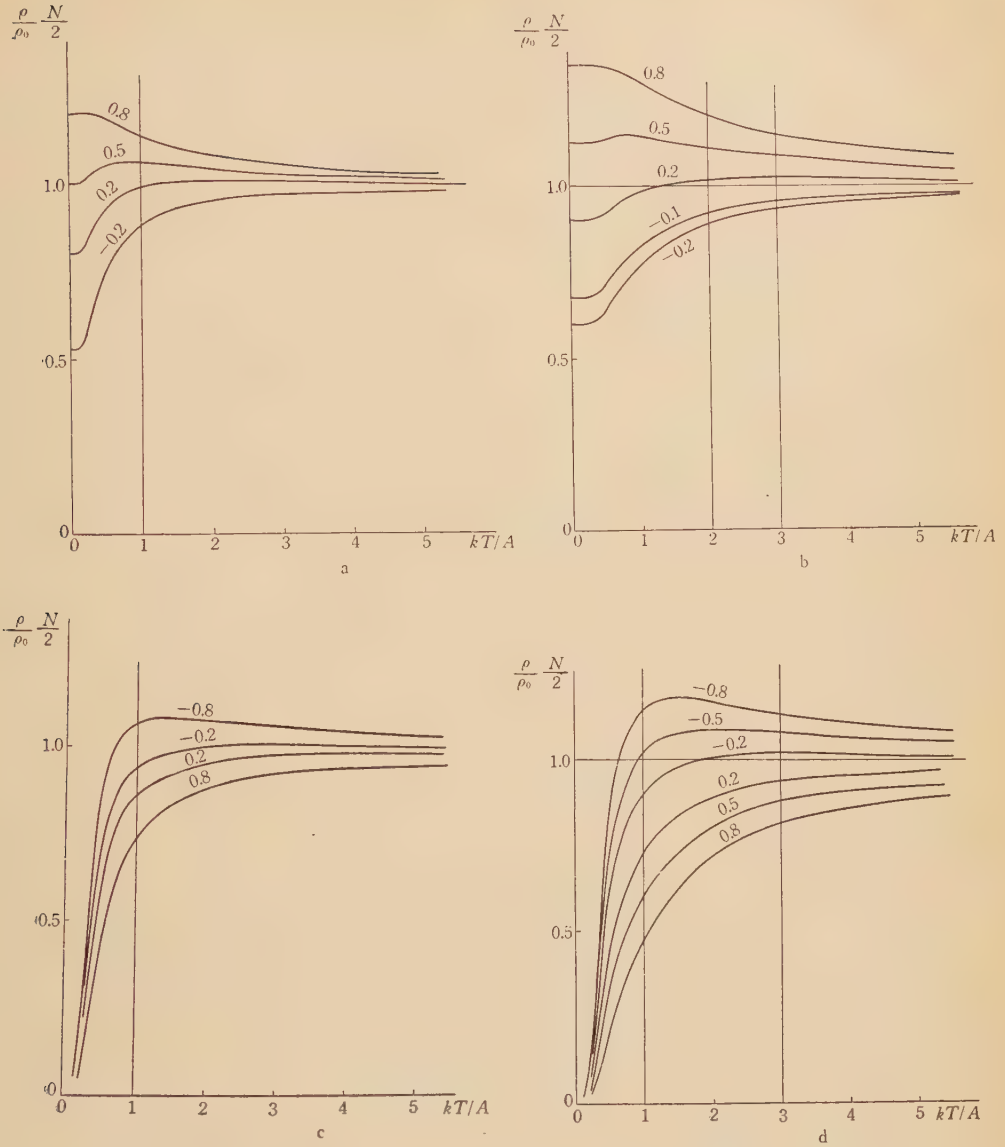

Fig. 6. Resistivity versus temperature curves in the pair approximation. Resistivity ρ is in the unit $2\rho_0/N$, and T in A/k_B. The four figures a, b, c and d correspond to the cases; $(A>0, S=1/2)$, $(A>0, S=1)$, $(A<0, S=1/2)$ and $(A<0, S=1)$, respectively. In each case figures stand for the values of r_1.

§ 5. Concluding remarks

Our aims have been to have a formula for the electrical resistivity written in terms of the space-time correlation function of the scattering system, and to discuss the effects of the correlations in the scattering system to the electrical

resistivity in some typical cases. In the cases of electron-phonon interaction and of impurity scattering the formula for the electrical resistivity gives only the well-known results. And in the case of the *s-d* interaction model for a ferromagnet the formula provides a physically intuitive method in discussing the electrical resistivity and gives some new results as discussed in § 3.

In studying pair correlation function in space and time or its Fourier transform it is not sufficient to have only the experimental data about electrical resistivity, because the electrical resistivity is written in terms of the Fourier transform of the pair correlation function. For this purpose it is useful to use the scattering experiments (*X*-ray scattering, slow neutron scattering) because the Fourier transform of the pair correlation function is directly connected with the differerntial cross section for the scattering process.[4]

It has been observed[27] that in some of the elements in the Uranium group the resistivity versus temperature curve shows an anomalous behaviour as in Fe group, and in the case of Pu metal it has a marked maximum which may be attributed to the critical scattering of conduction electron at the transition temperature of the scattering system.

In rare earth metals except for Gd, the resistivity as a function of temperature exhibits a complicated behaviour in the vicinity of the transition point[23),24] at which these metals make a transition from paramagnetic state to antiferromagnetic state. In these metals there remains an orbital moment and the interaction Hamiltonian (2·17) is not adequate in discussing the resistivity. The effects of orbital moment must be included in explaining the complicated behaviour of the resistivity in the vicinity of the transition point.

Acknowledgement

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Appendix I

In the present case j_x is written as

$$j_x=\sum_{k\sigma}(-e\hbar/m)k_x a^*_{k\sigma}a_{k\sigma} \tag{I·1}$$

where σ stands for spin quantum number. From Eqs. (2·14) and (2·17a) we easily obtain that

$$\lim_{\omega\to 0}\langle[j_x V(\omega)][V(-\omega)j_x]\rangle_0$$

$$=-\left(\frac{e\hbar}{m}\right)^2\sum_{kk'k''k'''}\sum_{mn}J(\boldsymbol{k}'-\boldsymbol{k})J(\boldsymbol{k}'''-\boldsymbol{k}'')(k_x'-k_x)(k_x'''-k_x'')$$

$$\times\int\frac{dt}{2\pi}\int\frac{dt'}{2\pi}\Big\langle\exp[i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{R}_n(t)]\exp[i(\boldsymbol{k}''-\boldsymbol{k}''')\cdot\boldsymbol{R}_m(t')]$$

$$\times\Big\{a^*_{k'\uparrow}a_{k\uparrow}S_n^z(t)\exp[i(E^+_{k'}-E^+_k)t/\hbar]-a^*_{k'\downarrow}a_{k\downarrow}S_n^z(t)\exp[i(E^-_{k'}-E^-_k)t/\hbar]$$

$$+a^*_{k'\downarrow}a_{k\uparrow}S_n^+(t)\exp[i(E^-_{k'}-E^+_k)t/\hbar]+a^*_{k'\uparrow}a_{k\downarrow}S_n^-(t)\exp[i(E^+_{k'}-E^-_k)t/\hbar]\Big\}$$

$$\times\Big\{a^*_{k'''\uparrow}a_{k''\uparrow}S_m^z(t')\exp[i(E^+_{k'''}-E^+_{k''})t'/\hbar]-a^*_{k'''\downarrow}a_{k''\downarrow}S_m^z(t')$$

$$\times\exp[i(E^-_{k'''}-E^-_{k''})t'/\hbar]+a^*_{k'''\downarrow}a_{k''\uparrow}S_m^+(t')\exp[i(E^-_{k'''}-E^+_{k''})t'/\hbar]$$

$$+a^*_{k'''\uparrow}a_{k''\downarrow}S_m^-(t')\exp[i(E^+_{k'''}-E^-_{k''})t'/\hbar]\Big\}\Big\rangle_0\,. \qquad (\mathrm{I}\cdot 2)$$

Because of the presence of the factors $(k_x'-k_x)$ and $(k_x'''-k_x'')$, in the product of the two curly brackets only the following terms have non-vanishing contribution: 1st term×1st term, 2nd term×2nd term, 3rd term×4th term and 4th term×3rd term. For example, we evaluate here the term of the last type:

$$-\left(\frac{e\hbar}{m}\right)^2\sum_{kk'k''k'''}\sum_{mn}J(\boldsymbol{k}'-\boldsymbol{k})J(\boldsymbol{k}'''-\boldsymbol{k}'')(k_x'-k_x)(k_x'''-k_x'')$$

$$\times\int\frac{dt}{2\pi}\int\frac{dt'}{2\pi}\Big\langle\exp[i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{R}_n(t)]\exp[i(\boldsymbol{k}''-\boldsymbol{k}''')\cdot\boldsymbol{R}_m(t')]$$

$$\times\exp[i(E^+_{k'}-E^-_k)t/\hbar+i(E^-_{k'''}-E^+_{k''})t'/\hbar]a^*_{k'\uparrow}a_{k\downarrow}a^*_{k'''\downarrow}a_{k''\uparrow}S_n^-(t)\,S_m^+(t')\Big\rangle_0.$$

This is equal to

$$\left(\frac{e\hbar}{m}\right)^2\sum_{kk'}\sum_{mn}|J(\boldsymbol{k}'-\boldsymbol{k})|^2(k_x'-k_x)^2\int\frac{dt}{2\pi}\int\frac{dt'}{2\pi}f^+_{k'}(1-f^-_k)\exp[i(E^+_{k'}-E^-_k)(t-t')/\hbar]$$

$$\times\langle\exp[i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{R}_n(0)]\exp[-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{R}_m(t'-t)]S_n^-(0)\,S_m^+(t'-t)\rangle_0$$

where $f_k^{\pm}$ are the distribution functions for the conduction electrons $(\boldsymbol{k}\uparrow)$ and $(\boldsymbol{k}\downarrow)$. Denoting the displacement of the n-th ion by $\boldsymbol{u}_n$, we write $\boldsymbol{R}_n(t)=\boldsymbol{R}_n+\boldsymbol{u}_n(t)$ ($\boldsymbol{R}_n$ being the equilibrium position vector of the n-th ion) and we have[4]

$$\langle\exp[i\boldsymbol{\kappa}\cdot\boldsymbol{R}_0(0)]\exp[-i\boldsymbol{\kappa}\cdot\boldsymbol{R}_m(t)]\rangle=\exp[i\boldsymbol{\kappa}\cdot(\boldsymbol{R}_0-\boldsymbol{R}_m)]$$

$$\times\langle\exp[i\boldsymbol{\kappa}\cdot\boldsymbol{u}_0(0)]\exp[-i\boldsymbol{\kappa}\cdot\boldsymbol{u}_m(t)]\rangle$$

$$=\exp[i\boldsymbol{\kappa}\cdot(\boldsymbol{R}_0-\boldsymbol{R}_m)]\exp[-\sum_{\alpha\beta}^{x,y,z}\kappa_\alpha\kappa_\beta\{M^{\alpha\beta}(0,0)-M^{\alpha\beta}(\boldsymbol{R}_m,t)\}]$$

where $M^{\alpha\beta}(\boldsymbol{R}_m,t)=\langle u_0^\alpha(0)u_m^\beta(t)\rangle$. With these relations and taking into account of the translation symmetry of the crystal we have for the $(-+)$ part of $\phi(0)$

$$\phi^{-+}(0)=\left(\frac{e\hbar}{m}\right)^2 N\sum_{kk'}|J(\boldsymbol{k}'-\boldsymbol{k})|^2(k_x'-k_x)^2 f_{k'}^{+}(1-f_k^{-})\,G_{k'-k}^{-+}((E_{k'}^{+}-E_k^{-})/\hbar), \tag{I·3}$$

where $G_\kappa^{-+}(\omega)$ is defined by Eq. (2·20). In the similar way we can evaluate the other parts of $\phi(0)$, and we obtain Eq. (2·19).

Appendix II

In the spin wave approximation, from Eq. (3·5) $g_\kappa^{zz}(\omega)$ is proportional to $\delta_{\kappa,0}$ and $\phi^{zz}(0)$ has nothing to do with the electrical resistivity. Now let us calculate $\phi^{+-}(0)+\phi^{-+}(0)$. With the use of Eqs. (2·19) and (3·5) we have

$$\phi^{+-}(0)+\phi^{-+}(0)=\left(\frac{e\hbar}{m}\right)^2 N\hbar\sum_{kq}|J(\boldsymbol{q})|^2 q_x^2 2S\{f_{k+q}^{-}(1-f_k^{+})\,(n_q+1)$$

$$+f_k^{+}(1-f_{k-q}^{-})\,n_q\}\,\delta(\hbar\omega_q-E_{k-q}^{-}+E_k^{+}). \tag{II·1}$$

Neglecting the effect of spin polarization of conduction electrons we have

$$\phi^{+-}(0)+\phi^{-+}(0)=\left(\frac{e\hbar}{m}\right)^2 4SN\hbar\sum_{kq}|J(\boldsymbol{q})|^2 q_x^2 n_q(n_q+1)\,\hbar\omega_q$$

$$\times\left(-\frac{\partial f_k}{\partial E_k}\right)\delta(\hbar\omega_q-E_{k+q}+E_k). \tag{II·2}$$

Since in the low temperature region the spin waves thermally excited are mainly of long wavelength, the energy quantum $\hbar\omega_q$ is approximated by $\hbar^2\boldsymbol{q}^2/2\mu$, μ being the so-called effective mass of spin wave, and $J(\boldsymbol{q})$ can be assumed to be independent of $\boldsymbol{q}$ and set equal to $J(0)$. If the summation in Eq. (II·2) is replaced by appropriate integrations over $\boldsymbol{k}$ and $\boldsymbol{q}$ this is easily evaluated to give

$$\phi^{+-}(0)+\phi^{-+}(0)=\frac{e^2NS|J(0)|^2\mu^2}{3\pi^4\beta^3\hbar^5}I, \tag{II·3}$$

where

$$I=\int_0^{x_0}\frac{x^2e^x\,dx}{(e^x-1)^2},\quad x_0=\beta\hbar\omega_{q_0}. \tag{II·4}$$

In the low temperature region one may put $x_0=\infty$ and I becomes $\pi^2/3$. And one easily obtains Eq. (3·6).

Appendix III

The Hamiltonian of the system of d spins is assumed to be

$$H=-2I\sum_{\langle jk\rangle}\boldsymbol{S}_j\cdot\boldsymbol{S}_k. \tag{III·1}$$

Let us begin with the calculation of $\langle \boldsymbol{S}_0(0)\cdot\boldsymbol{S}_0(0)\rangle$. This is easily obtained as

$$\langle \boldsymbol{S}_0(0)\cdot\boldsymbol{S}_0(0)\rangle = S(S+1). \tag{III·2}$$

Now we calculate $\langle \boldsymbol{S}_0(0)\cdot\boldsymbol{S}_\Delta(0)\rangle$ in which Δ represents one of the nearest neighbour lattice points. This is calculated in the form of the high temperature expansion

$$\langle \boldsymbol{S}_0(0)\cdot\boldsymbol{S}_\Delta(0)\rangle = \frac{\mathrm{Tr}[1]}{Z}\frac{1}{\mathrm{Tr}[1]}\mathrm{Tr}\left[\left(1-\beta H+\frac{1}{2}\beta^2H^2-\cdots\right)\boldsymbol{S}_0\cdot\boldsymbol{S}_\Delta\right] \tag{III·3}$$

Z being the partition function

$$Z=\mathrm{Tr}(e^{-\beta H}). \tag{III·4}$$

It is easily obtained that[28]

$$\begin{aligned}
&\frac{\mathrm{Tr}[\boldsymbol{S}_0\cdot\boldsymbol{S}_\Delta]}{\mathrm{Tr}[1]}=0,\\
&\frac{\mathrm{Tr}[-H\boldsymbol{S}_0\cdot\boldsymbol{S}_\Delta]}{\mathrm{Tr}[1]}=2I\frac{\mathrm{Tr}[(\boldsymbol{S}_0\cdot\boldsymbol{S}_\Delta)^2]}{\mathrm{Tr}[1]}=\frac{2}{3}IS^2(S+1)^2,\\
&\frac{\mathrm{Tr}[H^2\boldsymbol{S}_0\cdot\boldsymbol{S}_\Delta]}{\mathrm{Tr}[1]}=4I^2\frac{\mathrm{Tr}[(\boldsymbol{S}_0\cdot\boldsymbol{S}_\Delta)^3]}{\mathrm{Tr}[1]}=-\frac{2}{3}I^2S^2(S+1)^2,
\end{aligned} \tag{III·5}$$

where the last equation holds only for S.C. and B.C.C. lattices. On the other hand Z is given by[28]

$$\frac{Z}{\mathrm{Tr}[1]}=1+\frac{Nz}{3}(I\beta)^2S^2(S+1)^2+\cdots. \tag{III·6}$$

From Eqs. (III·3), (III·5) and (III·6) we have

$$\langle \boldsymbol{S}_0\cdot\boldsymbol{S}_\Delta\rangle=-\frac{2}{3}\beta IS^2(S+1)^2\left(1-\frac{1}{2}\beta I+\cdots\right). \tag{III·7}$$

The correlation functions between $\boldsymbol{S}_0$ and the spins located on the further neighbours are calculated only for S.C. lattice to be

$$\begin{aligned}
&\langle \boldsymbol{S}_0\cdot\boldsymbol{S}_{\Delta+\Delta'}\rangle=\frac{8}{9}(\beta I)^2S^3(S+1)^3+\cdots \quad (\Delta\neq\Delta' \text{ and } \Delta+\Delta'\neq 0)\\
&\langle \boldsymbol{S}_0\cdot\boldsymbol{S}_{2\Delta}\rangle=\frac{4}{9}(\beta I)^2S^3(S+1)^3+\cdots\\
&\langle \boldsymbol{S}_0\cdot\boldsymbol{S}_\Gamma\rangle=O(\beta^3)
\end{aligned} \tag{III·8}$$

where Γ stands for any one of the body diagonal lattice vectors in a unit cell.

Now we calculate the correlation function of the type $\langle \boldsymbol{S}_0(0)\cdot\dot{\boldsymbol{S}}_R(0)\rangle$, where $\dot{\boldsymbol{S}}_R$ is written as from Eq. (III·1)

$$\dot{\boldsymbol{S}}_R=(2I/\hbar)\sum_\Delta \boldsymbol{S}_R\times\boldsymbol{S}_{R+\Delta}. \tag{III·9}$$

Then, using commutation relation for spin operators, we have

$$\langle S_0\cdot\dot{S}_0\rangle=(2I/\hbar)\sum_{\Delta}\langle S_0\cdot S_0\times S_{\Delta}\rangle=\frac{2Iz}{\hbar}\,i\langle S_0\cdot S_{\Delta}\rangle. \tag{III·10}$$

With the use of the relation

$$\langle S_0\cdot S_{\Delta}\times S_{\Delta+\Delta'}\rangle=0 \text{ for } \Delta+\Delta'\neq 0$$

which is easily verified to hold up to the term β^3, we have

$$\langle S_0\cdot\dot{S}_{\Delta}\rangle=(2I/\hbar)\sum_{\Delta'}\langle S_0\cdot S_{\Delta}\times S_{\Delta+\Delta'}\rangle=(2I/\hbar)\langle S_0\cdot S_{\Delta}\times S_0\rangle=-(2Ii/\hbar)\langle S_0\cdot S_{\Delta}\rangle. \tag{III·11}$$

$\langle S_0\cdot S_{\Delta}\rangle$ is easily calculated as before and from Eqs. (III·10) and (III·11) we have

$$\begin{aligned}\langle S_0\cdot\dot{S}_{\Delta}\rangle&=(-1/z)\langle S_0\cdot\dot{S}_0\rangle\\&=-i\frac{2I}{\hbar}\frac{2}{3}\beta IS^2(S+1)^2\Big\{1-\frac{1}{2}\beta I+\frac{1}{4}(\beta I)^2S^2(S+1)^2\\&\times\Big[-\frac{8}{3}(2z-1)+\frac{8}{5}\Big(1-\frac{2}{3}\frac{1}{S(S+1)}+\frac{2}{3}\frac{1}{S^2(S+1)^2}\Big)\\&+\frac{16(z-1)}{3}\Big(1-\frac{1}{3S(S+1)}\Big)+\frac{16}{9}(z-2)\Big]+\cdots\Big\}.\end{aligned} \tag{III·12}$$

And we have

$$\langle S_0\cdot\dot{S}_R\rangle=O(\beta^4),\ \text{for}\ |R|\ \text{larger than}\ |\Delta|. \tag{III·13}$$

We then calculate $\langle S_0\cdot\ddot{S}_R\rangle=-\langle\dot{S}_0\cdot\dot{S}_R\rangle$. From Eq. (III·9) we have

$$\langle\dot{S}_0\cdot\dot{S}_0\rangle=\Big(\frac{2I}{\hbar}\Big)^2\frac{\mathrm{Tr}[1]}{Z}\frac{1}{\mathrm{Tr}[1]}\mathrm{Tr}\Big[\Big(1-\beta H+\frac{1}{2}\beta^2H^2-\cdots\Big)\times\sum_{\delta\gamma}(S_0\times S_{\delta})\cdot(S_0\times S_{\gamma})\Big],$$

which is calculated as follows:

$$\begin{aligned}\sum_{\delta\gamma}\frac{\mathrm{Tr}[(S_0\times S_{\delta})\cdot(S_0\times S_{\gamma})]}{\mathrm{Tr}[1]}&=\sum_{\delta}\frac{\mathrm{Tr}[(S_0\times S_{\delta})^2]}{\mathrm{Tr}[1]}\\&=6z\frac{\mathrm{Tr}[(S_0^y)^2(S_{\delta}^z)^2]}{\mathrm{Tr}[1]}=\frac{2}{3}zS^2(S+1)^2,\\\sum_{\delta\gamma}\frac{\mathrm{Tr}[-H(S_0\times S_{\delta})\cdot(S_0\times S_{\gamma})]}{\mathrm{Tr}[1]}&=2Iz\frac{\mathrm{Tr}[(S_0\cdot S_{\delta})(S_0\times S_{\delta})^2]}{\mathrm{Tr}[1]}\\&=12Iz\frac{\mathrm{Tr}[S_0^xS_0^yS_0^zS_{\delta}^xS_{\delta}^yS_{\delta}^z]}{\mathrm{Tr}[1]}=-\frac{Iz}{3}S^2(S+1)^2,\\&\cdots\cdots\end{aligned}$$

and we obtain

$$\langle \boldsymbol{S}_0 \cdot \ddot{\boldsymbol{S}}_0 \rangle = -\left(\frac{2I}{\hbar}\right)^2 \frac{2}{3} zS^2(S+1)^2 \left\{1 - \frac{1}{2}\beta I + \frac{1}{2}(\beta I)^2 \left[\frac{8}{9}\left(z - \frac{13}{5}\right) S^2(S+1)^2 + \frac{4}{5} S(S+1) + \frac{1}{5}\right] - \cdots\right\}. \quad \text{(III·14)}$$

$\langle \boldsymbol{S}_0 \cdot \ddot{\boldsymbol{S}}_\Delta \rangle$ is obtained from Eq. (III·14) and the relation

$$\langle \boldsymbol{S}_0 \cdot \ddot{\boldsymbol{S}}_0 \rangle = -z \langle \boldsymbol{S}_0 \cdot \ddot{\boldsymbol{S}}_\Delta \rangle \quad \text{(III·15)}$$

which is easily verified to hold up to the term β^3. Further we have

$$\langle \boldsymbol{S}_0 \cdot \ddot{\boldsymbol{S}}_{\boldsymbol{R}} \rangle = \mathrm{O}(\beta^3) \text{ for } |\boldsymbol{R}| \text{ larger than } |\Delta|. \quad \text{(III·16)}$$

The first and second moments defined by Eq. (3·18) are obtained from Eqs. (III·2), (III·7), (III·8), (III·12–16) after averaging over angles:

$$\langle \omega \rangle_\kappa = \frac{2}{3} zS(S+1) \frac{2I}{\hbar} \beta I \left(1 - \frac{\sin x}{x}\right) \times \left\{1 - \frac{1}{2}\beta I S(S+1) \left[\frac{1}{S(S+1)} + \frac{4z}{3} \frac{\sin x}{x}\right] + \beta^2 I^2 S^2 (S+1)^2 U + \cdots\right\} \quad \text{(III·17)}$$

$$\langle \omega^2 \rangle_\kappa = \frac{2}{3} zS(S+1) \left(\frac{2I}{\hbar}\right)^2 \left(1 - \frac{\sin x}{x}\right) \times \left\{1 - \frac{1}{2}\beta I S(S+1) \left[\frac{1}{S(S+1)} + \frac{4z}{3} \frac{\sin x}{x}\right] + \beta^2 I^2 S^2 (S+1)^2 V + \cdots\right\} \quad \text{(III·18)}$$

with

$$U = \frac{4}{15}\left(\frac{17}{3} - \frac{28}{3}\frac{1}{S(S+1)} + \frac{1}{S^2(S+1)^2}\right) + 4\left(\frac{1}{S(S+1)}\frac{\sin x}{x} - \frac{8}{3}\frac{\sin\sqrt{2}x}{\sqrt{2}x} - \frac{2}{3}\frac{\sin 2x}{2x}\right) + 16\left(\frac{\sin x}{x}\right)^2,$$

$$V = \frac{1}{10}\left(\frac{136}{9} + \frac{4}{S(S+1)} + \frac{1}{S^2(S+1)^2}\right) - 4\left(\frac{1}{S(S+1)}\frac{\sin x}{x} + \frac{8}{3}\frac{\sin\sqrt{2}x}{\sqrt{2}x} + \frac{2}{3}\frac{\sin 2x}{2x}\right) + 16\left(\frac{\sin x}{x}\right)^2, \quad \text{(III·19)}$$

where $x = \kappa a$, a being the magnitude of any one of the lattice vectors which join an atom to one of its nearest neighbours. Eq. (III·19) is correct only for S.C. lattice.

In the high temperature limit ($\beta \to 0$) the comparison of $\langle \omega^2 \rangle_\kappa$ with $\langle \omega^4 \rangle_\kappa$ shows that[18]

(a) for $x \gtrsim \pi$, $g_\kappa(\omega)$ is approximated by a Gaussian function,

(b) for $x \lesssim 1$, $g_\kappa(\omega)$ is approximated by a truncated Lorentzian function.

Since in Eqs. (III·17) and (III·18) it is observed that the ratio of the second term to the first in the curly brackets is about 10% or so for $S \geq 1/2$ and $T=T_c \sim \infty$, the above arguments (a) and (b) can be assumed to hold for $T \gtrsim T_c$. Then we have Eqs. (3·19) and (3·20).

Appendix IV

From Eqs. (3·15) and (3·16) ρ is given by

$$\frac{\rho}{\rho_0}=\frac{1}{J_{eff}^2 S(S+1)}\int d(\cos\theta)\int d\omega(1-\cos\theta)\,|J(\kappa)|_f^2 g_\kappa(\omega)\,\frac{\beta\hbar\omega}{e^{\beta\hbar\omega}-1}, \qquad \text{(IV·1)}$$

where $g_\kappa(\omega)$ is given by Eq. (3·25) in the present approximation. Now we calculate the correlation functions $\langle \boldsymbol{S}_0(0)\cdot\boldsymbol{S}_0(t)\rangle$ and $\langle \boldsymbol{S}_0(0)\cdot\boldsymbol{S}_1(t)\rangle$ in the representation in which S and M are diagonal. Here we mean S and M are respectively the magnitude and the z-component of the resultant spin angular momentum operator $\boldsymbol{S}=\boldsymbol{S}_0+\boldsymbol{S}_1$, where the magnitude of $\boldsymbol{S}_0$ is of course 1/2 and that of $\boldsymbol{S}_1$ takes an integral value ranging from zero to $z/2$. Then the matrix elements necessary for the present purpose are[26]

$$\langle S-1\,M|S_0^z|S\,M\rangle=\begin{cases}\dfrac{1}{2S_1+1}\sqrt{\left(S_1+\dfrac{1}{2}-M\right)\left(S_1+\dfrac{1}{2}+M\right)} & \left(S=S_1+\dfrac{1}{2}\right)\\ 0 & \left(S=S_1-\dfrac{1}{2}\right)\end{cases}$$

$$\langle S\,M|S_0^z|S\,M\rangle^* =\begin{cases}\dfrac{1}{2S_1+1}M\\ -\dfrac{1}{2S_1+1}M\end{cases}$$

$$\langle S+1\,M|S_0^z|S\,M\rangle=\begin{cases}0\\ \dfrac{1}{2S_1+1}\sqrt{\left(S_1+\dfrac{1}{2}-M\right)\left(S_1+\dfrac{1}{2}+M\right)}\end{cases}$$

$$\langle S-1\,M|S_1^z|S\,M\rangle=\begin{cases}-\dfrac{1}{2S_1+1}\sqrt{\left(S_1+\dfrac{1}{2}-M\right)\left(S_1+\dfrac{1}{2}+M\right)}\\ 0\end{cases}$$

$$\langle S\,M|S_1^z|S\,M\rangle =\begin{cases}\dfrac{2S_1}{2S_1+1}M\\ \dfrac{2S_1+2}{2S_1+1}M\end{cases}$$

* In the right-hand side of this equation and the following one the upper member corresponds to the case $S=S_1+\frac{1}{2}$ and the lower to $S=S_1-\frac{1}{2}$.

$$\langle S+1\ M|S_1^z|S\ M\rangle = \begin{cases} 0 \\ -\dfrac{1}{2S_1+1}\sqrt{\left(S_1+\dfrac{1}{2}-M\right)\left(S_1+\dfrac{1}{2}+M\right)} \end{cases}$$

$$\langle S-1\ M\pm1|S_0^{\pm}|S\ M\rangle = \begin{cases} \pm\dfrac{1}{2S_1+1}\sqrt{\left(S_1\mp M-\dfrac{1}{2}\right)\left(S_1\mp M+\dfrac{1}{2}\right)} \\ 0 \end{cases}$$

$$\langle S\ M\pm1|S_0^{\pm}|S\ M\rangle = \begin{cases} \dfrac{1}{2S_1+1}\sqrt{\left(S_1\mp M+\dfrac{1}{2}\right)\left(S_1\pm M+\dfrac{3}{2}\right)} \\ -\dfrac{1}{2S_1+1}\sqrt{\left(S_1\mp M-\dfrac{1}{2}\right)\left(S_1\pm M+\dfrac{1}{2}\right)} \end{cases}$$

$$\langle S+1\ M\pm1|S_0^{\pm}|S\ M\rangle = \begin{cases} 0 \\ \mp\dfrac{1}{2S_1+1}\sqrt{\left(S_1\pm M+\dfrac{1}{2}\right)\left(S_1\pm M+\dfrac{3}{2}\right)} \end{cases}$$

$$\langle S-1\ M\pm1|S_1^{\pm}|S\ M\rangle = \begin{cases} \mp\dfrac{1}{2S_1+1}\sqrt{\left(S_1\mp M-\dfrac{1}{2}\right)\left(S_1\mp M+\dfrac{1}{2}\right)} \\ 0 \end{cases}$$

$$\langle S\ M\pm1|S_1^{\pm}|S\ M\rangle = \begin{cases} \dfrac{2S_1}{2S_1+1}\sqrt{\left(S_1\mp M+\dfrac{1}{2}\right)\left(S_1\pm M+\dfrac{3}{2}\right)} \\ \dfrac{2S_1+2}{2S_1+1}\sqrt{\left(S_1\mp M-\dfrac{1}{2}\right)\left(S_1\pm M+\dfrac{1}{2}\right)} \end{cases}$$

$$\langle S+1\ M\pm1|S_1^{\pm}|S\ M\rangle = \begin{cases} 0 \\ \pm\dfrac{1}{2S_1+1}\sqrt{\left(S_1\pm M+\dfrac{1}{2}\right)\left(S_1\pm M+\dfrac{3}{2}\right)}. \end{cases}$$

Using these matrix elements, for example, $\langle S_0^z(0)\,S_1^z(t)\rangle$ is easily calculated to be

$$\begin{aligned}\langle S_0^z(0)\,S_1^z(t)\rangle &= \frac{1}{Z}\sum_{SM}\exp(-\beta E_{SM})\Big\{\exp[i(E_{S-1M}-E_{SM})t/\hbar] \\ &\times\langle SM|S_0^z|S-1M\rangle\langle S-1M|S_1^z|SM\rangle+\langle SM|S_0^z|SM\rangle\langle SM|S_1^z|SM\rangle \\ &+\exp[i(E_{S+1M}-E_{SM})t/\hbar]\langle SM|S_0^z|S+1M\rangle\langle S+1M|S_1^z|SM\rangle\Big\} \\ &=\frac{1}{Z}\sum_{S_1}\frac{w(S_1z)}{(2S_1+1)^2}\Big\{\sum_{M=-(S_1-1/2)}^{S_1-1/2}\exp(-\beta E_{S_1-1/2,M}) \\ &\times\Big[\exp[i(E_{S_1+1/2,M}-E_{S_1-1/2,M})t/\hbar](-1)(S_1-M+1/2)(S_1+M+1/2) \\ &-(2S_1+2)M^2\Big]\end{aligned}$$

$$+\sum_{M=-(S_1+1/2)}^{S_1+1/2}\exp(-\beta E_{S_1+1/2,M})\Big[\exp[i(E_{S1-1/2,M}-E_{S_1+1/2,M})t/\hbar]$$
$$\times(-1)(S_1-M+1/2)(S_1+M+1/2)+2S_1M^2\Big]\Big\}, \qquad \text{(IV·2)}$$

where Z is given by Eq. (3·28), and $w(S_1, z)$ has the same meanings as in § 3. And in the same way we have

$$\langle S_0^z(0)S_0^z(t)\rangle=\frac{1}{Z}\sum_{S_1}\frac{w(S_1z)}{(2S_1+1)^2}\Big\{\sum_{M=-(S_1-1/2)}^{S_1-1/2}\exp(-\beta E_{S_1-1/2,M})$$
$$\times\Big[\exp[i(E_{S_1+1/2,M}-E_{S_1-1/2,M})t/\hbar](S_1+1/2-M)(S_1+1/2+M)+M^2\Big]$$
$$+\sum_{M=-(S_1+1/2)}^{S_1+1/2}\exp(-\beta E_{S_1+1/2,M})$$
$$\times\Big[\exp[i(E_{S_1-1/2,M}-E_{S_1+1/2,M})t/\hbar](S_1+1/2-M)(S_1+1/2+M)+M^2\Big]\Big\}. \qquad \text{(IV·3)}$$

In Eqs. (IV·2) and (IV·3) E's are given by

$$E_{S_1\pm1/2,M}=\begin{cases}-IS_1 & (S=S_1+1/2)\\ I(S_1+1) & (S=S_1-1/2)\end{cases}$$

in the temperature region $T>T_c$, where H_1 can be regarded as vanishing. Then we have

$$\langle \boldsymbol{S}_0(0)\cdot\boldsymbol{S}_0(t)\rangle=3\langle S_0^z(0)S_0^z(t)\rangle$$
$$=\frac{1}{Z}\sum_{S_1}\frac{w(S_1z)}{2S_1+1}\Big\{(S_1+1)\exp(-\beta E_{S_1+1/2})[2S_1\exp(iI(2S_1+1)t/\hbar)$$
$$+S_1+3/2]+S_1\exp(-\beta E_{S_1-1/2})[(2S_1+2)\exp(-iI(2S_1+1)t/\hbar)+S_1-1/2]\Big\}$$
$$\langle \boldsymbol{S}_0(0)\cdot\boldsymbol{S}_1(t)\rangle=\frac{2}{Z}\sum_{S_1}w(S_1z)\frac{S_1(S_1+1)}{2(S_1+1)}\Big\{\exp(-\beta E_{S_1+1/2})$$
$$\times[-\exp(iI(2S_1+1)t/\hbar)+S_1+3/2]$$
$$+\exp(-\beta E_{S_1-1/2})[-\exp(-iI(2S_1+1)t/\hbar)-S_1+1/2]\Big\}. \qquad \text{(IV·4)}$$

With the use of Eqs. (IV·1), (IV·4) and (3·25) we can easily obtain Eq. (3·27).

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Particle-Hole Bound States in the Many Fermion System

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Bound states of a particle-hole pair are investigated by means of the Bethe-Salpeter equation. Besides the well-known density oscillation, it is shown that transverse collective oscillations are possible. The conditions for the existence of a transverse wave are studied.

§ 1. Introduction

It has been shown by many authors that in the many fermion system the collective excitation of a density type wave (plasma oscillation and sound wave) can be considered as a bound state of a particle and a hole and one is able to treat such a bound state by means of the Bethe-Salpeter equation.[1)]

In the case of a density wave, the interaction kernel of the Bethe-Salpeter equation is usually given by the lowest order diagram which is shown in Fig. 1.

Fig. 1.

In the case of a plasma oscillation, neglecting the exchange interaction (Fig. 1b), we obtain the well-known Bohm-Pines' dispersion relation. In the short range limit, we cannot distinguish the interactions corresponding to Figs. 1a and 1b. In this limit, e. g. when the potential is considered to be δ-function, it is easy to obtain the solution of the Bethe-Salpeter equation and we can obtain the dispersion relation for the sound wave.

In this note, we shall investigate the general properties of the Bethe-Salpeter equation for a particle-hole pair. We assume that the range of force is finite (non zero and not infinite such as the Coulomb interaction). In the case of a finite range of force, as will be shown later, besides the ordinary density wave which is longitudinal, transverse waves are possible. The dispersion relations for these transverse waves are very similar to those given by Abrikosov and Khalatnikov.[2)] They derived the dispersion relations from Landau's theory[2)] of fermi liquids.

However, in contrast to longitudinal waves, the condition for the existence of transverse waves is strongly dependent on the depth and range of the potential.

The interaction shown in Fig. 1a contributes only to the longitudinal wave

and therefore we cannot obtain a transverse wave in the zero-range approximation.

Although, for the sake of simplicity and concreteness, we assume a local potential, a similar discussion is applicable to the more general case where the effective potential is non-local without retardation.

Since it is very hard to treat the Bethe-Salpeter equation with non-zero interaction range, in general, we confine ourselves to studying it in the limit of long wavelengths. In § 2, we shall derive the Bethe-Salpeter equation for a particle and hole and solve it in the zero range limit. In the long wavelength limit the equation is transformed to a set of simultaneous linear algebraic equations as will be shown in § 3 and we can obtain dispersion relations in some approximation. In § 4, we shall discuss the condition for the existence of transverse excitations by the integral equation method.

§ 2. Bethe-Salpeter equation for a particle-hole pair

As is well-known, the Bethe-Salpeter equation in the lowest order approximation[1] is given by

$$\chi(p)=iG\left(p+\frac{k}{2}\right)G\left(p-\frac{k}{2}\right)\times\left\{-2v(\boldsymbol{k})\int\chi(q)\frac{d^4q}{(2\pi)^4}+\int v(|\boldsymbol{p}-\boldsymbol{q}|)\,\chi(q)\frac{d^4q}{(2\pi)^4}\right\},\qquad(2\cdot1)$$

where $p=(p_0, \boldsymbol{p})$ is the relative energy and momentum,

$k=(k_0, \boldsymbol{k})$ is the total energy and momentum,

$v(\boldsymbol{k})$ is the Fourier amplitude of the interaction, potential,

$\chi(p)$ is the B-S amplitude,

and $G(p)$ is the particle propagator and is given by

$$G(p)=1/\left[p_0-\frac{\boldsymbol{p}^2}{2M}+i\delta\varepsilon(|\boldsymbol{p}|-p_F)\right],\qquad(2\cdot2a)$$

where $\delta>0$, $\varepsilon(x)=\begin{cases}+1 & \text{for } x>0\\ -1 & \text{for } x<0,\end{cases}$

and p_F is the Fermi momentum.

The first and the second terms in Eq. (2·1) correspond to Fig. 1a and Fig. 1b respectively.

Since the potential $v(\boldsymbol{k})$ is independent of the energy k_0, from Eq. (2·1), we can obtain the equation for $F(\boldsymbol{p})$, defined as

$$F(\boldsymbol{p})=\int dp_0\chi(p_0,\boldsymbol{p}). \tag{2·2b}$$

The equation for $F(\boldsymbol{p})$ is as follows:

$$F(\boldsymbol{p})=L(\boldsymbol{p},\boldsymbol{k})\left\{-2v(\boldsymbol{k})\int d^3qF(\boldsymbol{q})+\int d^3qv(|\boldsymbol{p}-\boldsymbol{q}|)F(\boldsymbol{q})\right\} \tag{2·3}$$

where

$$L(\boldsymbol{p},\boldsymbol{k})=\frac{i}{(2\pi)^4}\int dp_0G\left(p+\frac{k}{2}\right)G\left(p-\frac{k}{2}\right)=\frac{1}{2(2\pi)^3}\frac{1}{\omega-(\boldsymbol{pk})/M}$$
$$\times\left[\theta\left(\left|\boldsymbol{p}-\frac{\boldsymbol{k}}{2}\right|-p_F\right)\theta\left(p_F-\boldsymbol{p}+\frac{\boldsymbol{k}}{2}\right)-\theta\left(p_F-\boldsymbol{p}-\frac{\boldsymbol{k}}{2}\right)\theta\left(\boldsymbol{p}+\frac{\boldsymbol{k}}{2}-p_F\right)\right] \tag{2·3a}$$

and

$$\omega=\frac{k_0}{2}.$$

Now let us expand $F(\boldsymbol{p})$ in spherical harmonics:

$$F(\boldsymbol{p})=\sum_{l,m}Y_{l,m}(\Omega_p)f_{lm}(p). \tag{2·4}$$

After some tedious calculations, the equations for $f_{lm}(p)$, derived from (2·4) and (2·3), are as follows:

$$f_{lm}(p)=-2v(k)\,4\pi\delta_{m,0}\int d\Omega_p\,Y_{l,0}^*(\Omega_p)L(p,k,\cos\theta)\,Y_{0,0}(\Omega_p)\int_0^\infty p^2dpf_{00}(p)$$
$$+\sum_{l'=|m|}^{\infty}\frac{4\pi}{2l'+1}\int d\Omega_p\,Y_{lm}^*(\Omega_p)L(p,k,\cos\theta)\,Y_{l'm}(\Omega_p)\int_0^\infty dqq^2v_{ll'}(p,q)f_{l'm}(q) \tag{2·5a}$$

where we have written

$$v(|\boldsymbol{p}-\boldsymbol{q}|)=\sum_{l=0}^{\infty}v_l(p,q)P_l(\cos\theta),$$
$$\cos\theta=\frac{(\boldsymbol{p}\cdot\boldsymbol{q})}{pq}. \tag{2·5}$$

It is remarkable that in Eq. (2·5) only those f_{lm} are coupled together which are designated by the same m value.

For the sake of convenience, we introduce the following dimensionless variables,

$$\frac{k}{p_F}=\delta,\quad\frac{p}{p_F}=x,\quad\frac{q}{p_F}=y,$$
$$p^2f_{lm}(p)=g_{lm}(x). \tag{2·6}$$

In this notation, Eq. (2·5) becomes

$$g_{lm}(x)=-2v(p_F\delta)4\pi\left[-\frac{1}{4}\frac{1}{(2\pi)^3}\right]\frac{M}{p_F}\delta_{m,0}$$
$$\times\int_a^1 dz\left[\frac{1}{\varepsilon'-\delta xz}-\frac{(-1)^l}{\varepsilon'+\delta xz}\right]P_l(z)\int_0^\infty dx\,g_{00}(x)$$
$$+\left[-\frac{1}{4}\frac{1}{(2\pi)^3}\right]\sum_{l'=|m|}^{\infty}\frac{4\pi}{2l'+1}\left[\frac{(2l+1)(2l'+1)(l-|m|)!\,(l'-|m|)!}{(l+|m|)!\,(l'+|m|)!}\right]^{1/2}$$
$$\times\frac{M}{p_F}\int_a^1 dz\left[\frac{1}{\varepsilon'-\delta xz}-\frac{(-1)^{l+l'}}{\varepsilon'+\delta xz}\right]P_l^m(z)P_{l'}^m(z)$$
$$\times\int_0^\infty dy\,v_{l'}(p_Fx,\,p_Fy)g_{l'm}(y) \qquad (2\cdot7)$$

where

$$\varepsilon'=\frac{m\omega}{p_F'}\quad\text{and}\quad a=\frac{1}{pk}\left|p_F^2-p^2-\frac{k^2}{4}\right|=\frac{1}{x\delta}\left|1-x^2-\frac{\delta^2}{4}\right|.$$

Of course, if $a>1$, we define $g(x)$ to be zero. Then the integrals with infinite upper limits in Eq. (2·7) become integrals over finite intervals. Since we are interested in the case where $\delta\ll1$, and since we require $a<1$ we can easily obtain the region of x for which $g_{lm}(x)\neq0$:

$$1+\frac{\delta}{2}>x>1-\frac{\delta}{2}. \qquad (2\cdot8)$$

Now, it is much more convenient to change variables as follows:

$$x=1+\frac{\delta}{2}\xi,\quad y=1+\frac{\delta}{2}\eta$$

where

$$-1<\xi<+1,\quad -1<\eta<+1.$$

Using the variables ξ and η, we obtain

$$g_{lm}(\xi)=\frac{\pi Mp_F}{2(2\pi)^3}\times2v(p_F\delta)\,\delta_{m,0}\int_a^1 dz\left[\frac{1}{\varepsilon-z(1+(\delta/2)\xi)}-\frac{(-1)^l}{\varepsilon+z(1+(\delta/2)\xi)}\right]$$
$$\times\int_{-1}^1 d\xi\,g_{00}(\xi)+\left[-\frac{\pi Mp_F}{2(2\pi)^3}\right]\sum_{l'=|m|}\frac{1}{2l'+1}\left[\frac{(2l+1)(2l'+1)(l-|m|)!\,(l'-|m|)!}{(l+|m|)!\,(l'+|m|)!}\right]^{1/2}$$
$$\times\int_a^1 dz\left[\frac{1}{\varepsilon-z(1+(\delta/2)\xi)}-\frac{(-1)^{l+l'}}{\varepsilon+z(1+(\delta/2)\xi)}\right]P_l^m(z)P_{l'}^m(z)$$
$$\times\int_{-1}^1 d\eta\,v_{l'}\left(p_F\left(1+\frac{\delta}{2}\xi\right),\;p_F\left(1+\frac{\delta}{2}\eta\right)\right)g_{l'm}(\eta) \qquad (2\cdot9)$$

where

$$\varepsilon=\frac{\omega M}{p_F k} \quad \text{and} \quad a=\frac{1}{1+(\delta/2)\xi}\left|\xi+\frac{\delta}{4}(1+\xi^2)\right| .$$

Clearly, the solution of Eq. (2·9) is independent of the sign of m since the equation is only dependent on $|m|$. Thus if solutions with $m\neq 0$ exist, the energy eigenvalues are doubly degenerate. Although it is difficult to solve Eq. (2·9) in general, we can easily solve it in the zero range approximation. In this limit, Fourier amplitude of the potential $v(k)$ is a constant and hence all v_l except v_0 become zero with v_0 being a constant. Then Eq. (2·9) reduces to

$$g(\xi)=\frac{\pi M p_F}{2(2\pi)^3}v_0\int_0^1 dz\left[\frac{1}{\varepsilon-z(1+(\delta/2)\xi)}-\frac{1}{\varepsilon+z(1+(\delta/2)\xi)}\right]\int_{-1}^{1}d\xi\, g(\xi)$$

and we obtain the dispersion relation

$$1=\frac{\pi M p_F}{2(2\pi)^3}v_0\int_{-1}^{1}d\xi\int_{a(\xi)}^{1}dz\ \frac{z(1+(\delta/2)\xi)}{\varepsilon^2-z^2(1+(\delta/2)\xi)^2} . \tag{2·10}$$

In the zero range limit, there exists only a wave with $m=0$, i.e. a density wave and no transverse waves ($m\neq 0$). Even if the primary two-body interaction is a δ-function, the effective two-body interaction in a many-body system need not be a zero range interaction. For example, if we use a chain type approximation for the effective interaction (See Fig. 2.) as was done by Landau,[3] the zero range approximation does not hold. Hereafter we shall take "potential" to mean the effective two-body interaction without retardation in the many-body system.

Fig. 2.

§ 3. The long wavelength approximation

As we are interested in the long wavelength limit, i.e. $\delta\ll 1$, neglecting quantities $O(\delta)$ in Eq. (2·9), we find

$$g_{lm}(\xi)=2v(0)\frac{\pi M p_F}{2(2\pi)^3}\int_{|\xi|}^{1}dz\left(\frac{1}{\varepsilon-z}-\frac{(-1)^l}{\varepsilon+z}\right)P_l(z)\int_{-1}^{1}d\xi\, g_{00}(\xi)\delta_{m,0}$$

$$-\frac{\pi M p_F}{2(2\pi)^3}\sum_{l'=|m|}^{\infty}\frac{v_{l'}(p_F)}{2l'+1}\left[\frac{(2l+1)(2l'+1)(l-|m|)!\,(l'-|m|)!}{(l+|m|)!\,(l'+|m|)!}\right]^{1/2}$$

$$\times\int_{-1}^{1}dz\left(\frac{1}{\varepsilon-z}-\frac{(-1)^{l+l'}}{\varepsilon+z}\right)P_l^m(z)P_{l'}^m(z)\int_{-1}^{1}d\eta\, g_{l'm}(\eta). \tag{3·1}$$

Letting

$$X_{lm}=\int_{-1}^{1} d\xi\, g_{lm}(\xi),$$

Eq. (3·1) reduces to the following algebraic equation:

$$X_{lm}=\frac{\pi M p_F}{(2\pi)^3}\,2v(0)\int_{-1}^{1} dz\,\frac{\xi P_l(z)}{\varepsilon-z}\,X_{00}\,\delta_{m,0}$$

$$-\frac{\pi M p_F}{(2\pi)^3}\sum_{l'=|m|}^{\infty}\frac{v_{ll'}(p_F)}{2l'+1}\left[\frac{(2l+1)(2l'+1)(l-|m|)!\,(l'-|m|)!}{(l+|m|)!\,(l'+|m|)!}\right]^{1/2}$$

$$\times\int_{-1}^{1} dz\,\frac{zP_l^m(z)P_{l'}^m(z)}{\varepsilon-z}\,X_{l'm} \qquad (3\cdot2)$$

where

$$v_l(p_F)=v_l(p_F,\ p_F).$$

Since the eigenvalue equation is a transcendental equation, it is not clear whether a solution exists or not.

We try to solve Eq. (3·2) approximately.

i) *Short range approximation*

In the short range approximation (in which the range of the force μ^{-1} times the Fermi momentum p_F is very large: $\mu/p_F\ll1$),

$$v_l\ll v_{l-1}.$$

ia) *The case with* $m=0$

Because of the condition $v_l\ll v_{l-1}$, the dispersion relation reduces to that in the zero range approximation:

$$1=\frac{\pi M p_F}{(2\pi)^3}\,v(0)\int_{-1}^{1} dz\,\frac{z}{\varepsilon-z} \qquad (3\cdot3)$$

and if the interaction is repulsive ($v(0)>0$), a solution always exists.

ib) *The case with* $m=1$

The dispersion relation becomes

$$1=-\frac{\pi M p_F}{(2\pi)^3}\cdot\frac{1}{2}\,v_1(p_F)\int_{-1}^{1} dz\,\frac{z(1-z^2)}{\varepsilon-z}=-\frac{\pi M p_F}{2(2\pi)^3}\,v_1(p_F)\int_{0}^{1} dz\,\frac{z^2(1-z^2)}{\varepsilon^2-z^2} \qquad (3\cdot4)$$

and the condition for the existence of a solution is

$$1<-\frac{\pi M p_F}{(2\pi)^3}\,\frac{1}{3}\,v_1(p_F). \qquad (3\cdot5)$$

That is, $v_1(p_F)$ must be negative corresponding to an attractive force. Because of the stability of the Fermi sphere, the effective interaction should be repulsive and the interaction in configuration space has the form shown in Fig. 3. From Eq. (3·5), using a square well potential as the surrounding attractive force, the depth of the potential can be estimated and the result is

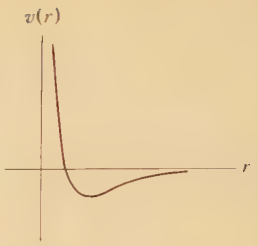

Fig. 3.

$$\frac{1}{45\times 2\pi} MVa^2(p_F a)^3 > 1 \qquad (3\cdot 6)$$

where V is the depth and a is the range. If we take $v_2/v_1 \lesssim 1/10$, then $ap_F \lesssim 1$ and from Eq. (3·6)

$$MVa^2 > 2\pi\times 45. \qquad (3\cdot 6')$$

Two particles in the p-state are able to be bound in a square well potential which satisfies Eq. (3·6′). Consequently, the Fermi sphere should be changed drastically by such a large attractive force even if the scattering length is positive.

ii) *In the case of* $\varepsilon\sim 1$

Now, let us study the case $0<\beta\ll 1$ where $\varepsilon=1+\beta$.

iia) *The case with* $m=0$

Eq. (3·2) with $m=0$ is

$$X_l = 2Av(0)\int_{-1}^{1} dz\,\frac{zP_l(z)}{\varepsilon-z}\,X_0 - A\sum_{l'=0}^{\infty}\sqrt{2l+1}\,\frac{v_{l'}(p_F)}{\sqrt{2l'+1}}\int_{-1}^{1} dz\,\frac{zP_l(z)P_{l'}(z)}{\varepsilon-z}\,X_{l'}.$$

If $\beta\sim 0$, the integrals in the above equation are considered as follows:

$$\int_{-1}^{1} dz\,\frac{zP_l(z)P_{l'}(z)}{\varepsilon-z} \simeq \ln\frac{1}{\beta} - C_0,$$

where the first term is the main one and is independent of l and l'; the second term is in general dependent on l and l'. However, we take C_0 to be a constant (independent of l and l'), because it is less important than the term $\ln(1/\beta)$. Letting $Y=\sum_l (v_l/\sqrt{2l+1})X_l$, we can obtain the following equation:

$$X_0 = A\lambda 2v(0) - A\lambda Y,$$

$$X_l = A\lambda 2v(0) - A\lambda\sqrt{2l+1}\,Y$$

where

$$A=\frac{\pi M p_F}{(2\pi)^3} \quad\text{and}\quad \lambda=\ln\frac{1}{\beta}-C_0.$$

From the above equations and the definition of Y, we derive

$$Y=A\lambda UZ-A\lambda v(0)\,Y,$$

$$Z=2v(0)\,A\lambda Z-A\lambda 2v(0)\,Y$$

where

$$U=\sum_{l=0}^{\infty}\frac{v_l(p_F)}{\sqrt{2l+1}},\quad 2v_0X_0=Z$$

and

$$v(0)=\sum_{l=0}^{\infty}v_l(p_F).$$

The dispersion equation can now be derived and easily solved; the results are

$$\begin{aligned}\beta&=e^{-c_0}\exp\left[-\frac{(2\pi)^3}{\pi Mp_Fv(0)}\cdot\frac{1}{4(1-\alpha)}\{(1+8(1-\alpha))^{1/2}-1\}\right]\\&\qquad\text{for}\quad\alpha<1\\&=e^{-c_0}\exp\left[-\frac{(2\pi)^3}{\pi Mp_Fv(0)}\cdot\frac{1}{4(\alpha-1)}\{1-(1-8(\alpha-1))^{1/2}\}\right]\\&\qquad\text{for}\quad 1<\alpha<1+1/8\end{aligned}\tag{3·7}$$

where

$$\alpha=\frac{1}{v(0)}\sum\frac{v_l(p_F)}{\sqrt{2l+1}}.$$

This result is very similar to that of the zero range approximation. In the zero range approximation, the result is

$$\beta=e^{-2}\exp\left[-\frac{(2\pi)^3}{\pi Mp_Fv(0)}\right],\quad\text{for } 0<\beta\ll 1.\tag{3·7′}$$

iii) *In the limit of* $\varepsilon\gg 1$ *with* $m\neq 0$

In this case the algebraic equations become

$$Y_{lm}=-\frac{A}{\varepsilon}[\sqrt{v_lv_{l+1}}\sqrt{(l+1)^2-m^2}\,Y_{l+1,m}+\sqrt{v_lv_{l-1}}\sqrt{l^2-m^2}\,Y_{l-1,m}]$$

and the dispersion is given by

$$\begin{aligned}1&=\left(\frac{A}{\varepsilon}\right)^2\left[\sum_{l=|m|}^{\infty}v_lv_{l+1}((l+1)^2-m^2)-\sum_{l=m}^{\infty}v_{l+1}\sqrt{v_lv_{l+2}}\sqrt{(l+1)^2-m^2}\sqrt{(l+2)^2-m^2}\right]\\&=\left(\frac{A}{\varepsilon}\right)^2\sum_{l=|m|}^{\infty}v_lv_{l+1}((l+1)^2-m^2)\left[1-\sqrt{\frac{v_{l+2}}{v_l}}\sqrt{\frac{(l+2)^2-m^2}{(l+1)^2-m^2}}\right].\end{aligned}\tag{3·11}$$

If the depth of the potential is sufficiently large, a solution always exists. This conclusion is assured also in the next section and in the Appendix.

§ 4. The integral equation method

Since it is much more convenient to treat a single integral equation than to treat such infinite simultaneous linear algebraic equations as Eq. (3·2), after transforming Eq. (3·2) to an integral equation, we study the conditions for the existence of longitudinal and transverse excitations.

4·1) *The case with* $m=0$

At first, let us consider a longitudinal wave. In this case, the algebraic equation is as follows:

$$X_l=2v(0)\int_{-1}^{1}dz\,\frac{zP_l(z)}{\varepsilon-z}X_0-\sum_{l'=0}^{\infty}v_{l'}\left[\frac{2l+1}{2l'+1}\right]^{1/2}\int_{-1}^{1}dz\,\frac{zP_l(z)P_{l'}(z)}{\varepsilon-z}X_{l'}$$

where we take $A=1$ or $Av_l\to v_l$ for the sake of simplicity. Defining the function $g(z)$ and the constant Y as follows:

$$g(z)=\sum_{l=0}^{\infty}\frac{v_l(p_F)}{\sqrt{2l+1}}X_lP_l(z),$$

$$Y=2v(0)X_0,$$

we can easily obtain an integral equation for $g(z)$ from the above algebraic equation, i.e.

$$Y=2v(0)\int_{-1}^{1}dz\frac{z}{\varepsilon-z}\{Y-g(z)\},$$

$$g(z)=G(z)Y-\int_{-1}^{1}dz'K_0(z|z')\frac{z'}{\varepsilon-z'}g(z'),\qquad(4\cdot1)$$

where

$$G(z)=\sum_{l=0}^{\infty}\frac{v_lP_l(z)}{\sqrt{2l+1}}\int_{-1}^{1}dz'\frac{z'P_l(z')}{\varepsilon-z'},$$

$$K_0(z|z')=\sum_{l=0}^{\infty}v_lP_l(z)P_l(z').\qquad(4\cdot1')$$

Since the integral equation for $g(z)$ is an inhomogeneous Fredholm equation, a solution always exists. By the use of

$$g(z)=\mathfrak{Y}(z)Y$$

and the first equation of (4·1), the dispersion relation is given as below:

$$1=2v(0)\int_{-1}^{1}dz\frac{z}{\varepsilon-z}[1-\mathfrak{Y}(z)].\qquad(4\cdot2)$$

or

$$1=2v(0)\left[\varepsilon\ln\frac{\varepsilon+1}{\varepsilon-1}-2-\int_{-1}^{1}dz\frac{z}{\varepsilon-z}\mathfrak{Y}(z)\right]$$

where $\mathfrak{Y}(z)$ satisfies the equation

$$\mathfrak{Y}(z)=G(z)-\int_{-1}^{1}dz'K_0(z|z')\frac{z'}{\varepsilon-z'}\mathfrak{Y}(z')$$

and is given formally by

$$\mathfrak{Y}(z)=G(z)+\sum_{n=1}^{\infty}(-1)^n\int_{-1}^{1}dz'K^{(n)}(z|z')G(z')$$

where

$$K^{(n)}(z|z')=\int_{-1}^{1}dz''K^{(n-1)}(z|z'')K_0(z''|z') \tag{4·3}$$

and

$$K^{(1)}(z|z')=K_0(z|z')\frac{z'}{\varepsilon-z'}.$$

The kernel $K_0(z|z')$ is expressed more compactly as follows:

$$K_0(z|z')=\frac{1}{2\pi}\int_0^{2\pi}d\varphi v(p_F,\ p_F,\ zz'-\sqrt{1-z^2}\sqrt{1-z'^2}\cos\varphi). \tag{4·4}$$

(The proof of Eq. (4·4) is given in the Appendix.)

4·2) *The case with $m\neq0$ (especially $m=1$)*

In the case of transverse waves, the algebraic equation (3·2) is transformed to a single homogeneous integral equation. When

$$g_m(z)=\sum_{l=|m|}^{\infty}v_l\left[\frac{(l-|m|)!}{(l+|m|)!}\right]^{1/2}P_l^m(z)\frac{z}{\varepsilon-z}X_{lm}, \tag{4·5}$$

the integral equation for $g_m(z)$ is given as

$$g_m(z)=-\int_{-1}^{1}dz'\frac{z}{\varepsilon-z}K_m(z|z')g_m(z') \tag{4·6}$$

where

$$K_m(z|z')=\sum_{l=|m|}^{\infty}v_l\frac{(l-|m|)!}{(l+|m|)!}P_l^m(z)P_l^m(z'). \tag{4·6'}$$

To study the eigenvalue problem of Eq. (4·6), let us consider the following equation:

$$g(z)=-\lambda\frac{z}{\varepsilon-z}\int_{-1}^{1}dz'\,K_m(z|z')g(z'). \tag{4·7}$$

For the existence and properties of the eigenvalues, Eq. (4·7) is inconvenient, because it does not have a symmetric kernel. Thus, we want to transform Eq. (4·7) to a more convenient form. By decomposing $g(z)$ into two parts as follows:

$$g(z)=g_+(z) \quad \text{for } z>0,$$
$$g_-(z) \quad \text{for } z<0,$$

we can obtain the simultaneous equations for $g_+(z)$ and $g_-(z)$ as follows:

$$\begin{aligned} g_+(z)&=-\lambda\frac{z}{\varepsilon-z}\left\{\int_0^1 dz'\,K_m(z|z')g_+(z')+\int_0^1 dz'\,K_m(z|-z')g_-(z')\right\},\\ g_-(z)&=\lambda\frac{z}{\varepsilon+z}\left\{\int_0^1 dz'\,K_m(z|-z')g_+(z')+\int_0^1 dz'\,K_m(z|z')g_-(z')\right\}, \end{aligned} \tag{4·8}$$

where we have used the following properties of $K_m(z|z')$

$$K_m(z|z')=K_m(-z|-z'),$$
$$K_m(z|-z')=K_m(-z|z').$$

The eigenvalues of the simultaneous equations (4·8) are given by the eigenvalues of the following single integral equations.[4]

$$\begin{aligned} g_+(z)&=-\lambda\frac{z}{\varepsilon-z}\int_0^1 dz'\,K_m(z|z')g_+(z'),\\ g_-(z)&=\lambda\frac{z}{\varepsilon+z}\int_0^1 dz'\,K_m(z|z')g_-(z'). \end{aligned} \tag{4·9}$$

From Eq. (4·9), we can obtain integral equations with symmetric kernels:

$$g_+(z)=-\lambda\int_0^1 dz'\sqrt{\frac{zz'}{(\varepsilon-z)(\varepsilon-z')}}\,K_m(z|z')g_+(z'), \tag{4·10a}$$

$$g_-(z)=\lambda\int_0^1 dz'\sqrt{\frac{zz'}{(\varepsilon+z)(\varepsilon+z')}}\,K_m(z|z')g_-(z'), \tag{4·10b}$$

where we assume $\varepsilon>1$. If $\varepsilon<-1$, we should use $|\varepsilon|$ instead of ε and g_+ and g_- should be interchanged. If the potential is always repulsive (i.e. all $v_l>0$), the kernel in Eq. (4·10a) is negative definite and then its eigenvalues are all negative. However, Eq. (4·10b) has a positive definite kernel and its eigenvalues

are all positive. Denoting an eigenvalue of this integral equation by $\lambda_n(\varepsilon)$, the dispersion relation is given by

$$\lambda_n(\varepsilon)=1. \tag{4·11}$$

Now, let us study (4·10b) with $m=1$ more precisely. The properties of the eigenvalues of a homogeneous Fredholm integral equation with a symmetric positive definite kernel are well-known and for the minimum eigenvalue, λ_{min}, the following inequality holds:

$$1\Big/\int_0^1 dz\, K_1(z|z)\,\frac{z}{\varepsilon+z} < \lambda_{min} < 1\Big/\int_0^1 dz\,dz'\sqrt{\frac{z}{\varepsilon+z}}\sqrt{\frac{z'}{\varepsilon+z'}}\,K_1(z|z')g_1(z)g_1(z') \tag{4·12}$$

where $g_1(z)$ is an arbitrary function having unit norm, i.e.

$$\int_0^1 dz\, g^2(z)=1.$$

Then a necessary condition for the existence of a solution of the dispersion relation (4·11) is

$$\int_0^1 dz\,\frac{z}{\varepsilon+z}K_1(z|z)>1 \quad \text{for } \varepsilon>1 \tag{4·13}$$

or

$$1<\int_0^1 dz\,\frac{z}{\varepsilon+z}\sum_l v_l\left[\frac{(l-|m|)!}{(l+|m|)!}\right](P_l^m(z))^2<\frac{1}{\varepsilon}\sum_{l=m}^{\infty}v_l\frac{1}{2l+1}$$

$$=\frac{1}{\varepsilon}\left\{\sqrt{2}\int_{-1}^1 dz\,\frac{1}{\sqrt{1-z}}v(p_F,p_F,z)-\sum_{l=0}^{m-1}\frac{v_l}{2l+1}\right\}. \tag{4·13′}$$

We can obtain a much more precise condition using the Kryloff-Weinstein theorem which gives the upper and the lower bounds to the true eigenvalue. (However, we shall not make use of this in this paper.) From Eq. (4·13) it is clear that if the effective interaction is sufficiently large a transverse excitation can exist. By assuming the potential to be Yukawa type, the explicit condition is given by

$$1<\frac{1}{\varepsilon}\frac{Mg^2}{4\mu} \tag{4·14}$$

or

$$1<\frac{M}{4\mu}\,g^2 \tag{4·14′}$$

(where g is the coupling constant and μ^{-1} is the range of the force).

This is the high density limit. More precisely, (4·14) is

$$1<\frac{Mg^2}{2\pi\mu}\left[\arctan\left(\frac{p_F}{\mu}\right)-\frac{\mu}{p_F}\ln\left(1+4\left(\frac{p_F}{\mu}\right)^2\right)\right].$$

§ 5. Discussion

In this note, we have studied the bound states of a particle hole pair. These bound states can be interpreted as collective excitations and, of course, the ordinary phonon type excitation is included. From § 3–ii), we can conclude that if the effective interaction is repulsive a longitudinal excitation is always possible and in the weak coupling limit the solution of the dispersion relation is similar to that in the zero range approximation. However, the condition for the existence of a transverse excitation is more restricted and the larger the value of m, the more severe become the restrictions.

Formally, the existence condition depends on the sign of the interaction. Of course, in the case of an attractive interaction, we should insure the stability of the Fermi sphere. That is, in this case we should examine whether or not bound states of two or more particles can exist. If the attractive interaction is so strong that bound states of two particles can exist, the Fermi distribution has to be changed drastically and then the basis of our discussion has to be modified. In the case of a repulsive interaction, however, we can consider that the Fermi distribution does not change drastically. It is worth-while to note that in the transverse excitations states are doubly degenerate ($\pm|m|$) and these two states can be interpreted as a right-handed and a left-handed polarization. The momentum dependence of the excitation spectra of transverse oscillations is the same as that of a longitudinal oscillation or phonon.

In this paper, though we assume that the effective interaction is independent of spin coordinates, the generalization of our discussion is not difficult. In general, if the interaction is dependent on spin coordinates, besides longitudinal and transverse excitations, there will exist spin waves.[2] Thus, in general, in the many fermion system we have three kinds of collective excitations, i.e. longitudinal, transverse and spin waves. (See ref. 2).)

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Appendix

Let us see the kernel $K_m(z|z')$ to be expressed more compactly. $K_m(z|z')$ has been introduced by the infinite series:

$$K_m(z|z')=\sum_{l=|m|}^{\infty}v_l(p_F,p_F)P_l^m(z)P_l^m(z')\frac{(l-|m|)!}{(l+|m|)!}. \qquad (A\cdot1)$$

Using the sum rule of Legendre Polynomials, we obtain the following equation,

$$K_m(z|z') = \sum_{l=0}^{\infty} v_l(p_F, p_F) \int_{-\pi}^{\pi} d\varphi \cos m\varphi \, P_l(zz' - \sqrt{1-z^2}\sqrt{1-z'^2}\cos\varphi). \quad (A\cdot 2)$$

On the other hand, from the definition of

$$v(\boldsymbol{p}, \boldsymbol{q}) = v(p, q, \cos\theta) = \sum v_l(p, q) P_l(\cos\theta)$$

where

$$\cos\theta = (\boldsymbol{pq})/pq,$$

we find that

$$v(p_F, p_F, \cos\theta) = v(p_F, p_F, zz' - \sqrt{1-z^2}\sqrt{1-z'^2}\cos\varphi)$$

$$= \sum_{l=0}^{\infty} v_l(p_F, p_F) P_l(zz' - \sqrt{1-z^2}\sqrt{1-z'^2}\cos\varphi) \quad (A\cdot 3)$$

where we ahve used the well-known relation

$$\cos\theta = \cos\theta\cos\theta' - \sin\theta\sin\theta'\cos\varphi.$$

Substituting (A·3) in (A·2), we obtain

$$K_m(z|z') = \frac{1}{4\pi} \int_{-\pi}^{\pi} d\varphi \cos m\varphi \, v(p_F, p_F, zz' - \sqrt{1-z^2}\sqrt{1-z'^2}\cos\varphi). \quad (A\cdot 4)$$

For example, let us apply the formula (A·4) to the Yukawa potential. In this case, $K_1(z|z')$ becomes

$$K_1(z|z') = \frac{1}{4\pi} \frac{g^2}{p_F^2} A \int_{-\pi}^{\pi} d\varphi \frac{\cos\varphi}{1 + (\mu/p_F)^2 - zz' + \sqrt{1-z^2}\sqrt{1-z'^2}\cos\varphi}$$

$$= \frac{1}{\pi} \left(\frac{g}{p_F}\right)^2 A \sqrt{1-z^2}\sqrt{1-z'^2} \int_0^1 d\zeta \frac{\zeta^2}{\sqrt{1-\zeta^2}} \frac{1}{(1 + (\mu/p_F)^2 - zz')^2 - (1-z^2)(1-z'^2)\zeta^2} \quad (A\cdot 5)$$

Therefore, if $(\mu/p_F)^2 \gg 1$, $K_1(z|z')$ becomes approximately

$$K_1(z|z') = \frac{1}{\pi} \left(\frac{g^2}{p_F^2}\right) A \sqrt{1-z^2}\sqrt{1-z'^2} \left(\frac{p_F}{\mu}\right)^2 \int_0^1 d\zeta \frac{\zeta^2}{\sqrt{1-\zeta^2}}$$

$$= \frac{1}{4} A \frac{g^2}{p_F^2} \sqrt{1-z^2}\sqrt{1-z'^2}.$$

In this approximation, we can easily obtain the dispersion relation in the repulsive force and the result is

$$1 = \frac{1}{4} A \frac{g^2}{\mu^2} \int_0^1 dz \frac{z(1-z^2)}{\varepsilon + z} = \frac{\pi M p_F}{4(2\pi)^3} \frac{g^2}{\mu^2} \int_0^1 dz \frac{z(1-z^2)}{\varepsilon + z}. \quad (A\cdot 6)$$

And if the following inequality holds,

$$\frac{\pi M p_F}{4(2\pi)^3}\frac{g^2}{\mu^2}\int_0^1 dz\,\frac{z(1-z^2)}{1+z}>1 \tag{A·7}$$

or

$$\frac{\pi M p_F}{24(2\pi)^3}\frac{g^2}{\mu^2}>1,$$

we have a solution of (A·6) for $\varepsilon>1$.

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Some Converging Examples of the Perturbation Series in the Quantum Field Theory

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Properties of the Hamiltonian operator of the quantized field is studied in the framework of the theory of Hilbert space. The fixed source theory and the boson-fermion interaction are mainly investigated in the cases of both discrete and continuous spectra. The total Hamiltonian operator is defined first in a domain dense in the Hilbert space, under the condition that the interaction form factor in the momentum space is "square integrable". Then it is shown that the total Hamiltonian as a self-adjoint operator can be determined in terms of the perturbation series for every finite value of the coupling constant if the boson mass is not zero and that it is the unique self-adjoint extension of the symmetric operator defined initially. For the boson-fermion interaction with continuous spectrum, some additional condition on the interaction form factor is needed. Further the perturbation series of the S-matrix element for scattering is discussed in the framework of the wave packet formulation.

§ 1. Introduction

In the quantum field theory the problem of the convergence of the perturbation series expressing the solution of the fundamental equation is of both practical and theoretical importance and has been studied by many authors from various points of view.

The difficulty of the diverging integral which appears in each term of the perturbation series was successfully removed by the method of renormalization. In this note we study the other important problem, that is, the summability of the power series, introducing from the beginning a so-called form factor in the interaction Hamiltonian and making each term of the series finite without using the renormalization technique.

Some attempts have been made in this direction. For scalar boson field $\phi(\boldsymbol{x})$ with self-interaction $\lambda\phi^3$, it was shown[1] that the Green function or the S-matrix can never be written in a power series of λ, because the number of the Feynman diagrams in the n-th order is roughly proportional to $n!$ and the lower bound of the absolute value of each matrix element corresponding to the Feynman diagram in that order is a^n with a being some positive number independent of n, and a fabulous cancellation of the contributions within the same order does not occur. For the boson field interacting with the fermion current, a considerable cancellation between the various terms seems to happen,[1] al-

though both the number and the magnitude of the contribution of the diagrams are comparable to those in the case of the boson self-interaction. Indeed, it was shown[2] that the U-matrix element describing the change of the state during the finite time interval t is bounded above by a function $A\exp(Bt^2)$, A and B being some positive numbers, if the system is enclosed in a large box and the interaction is cut off in the momentum space in such a way that only a finite number of degrees of freedom of the fermion field can interact with the boson field. On the other hand, for the meson field interacting with an extended and fixed nucleon, an explicit formula for the nucleon propagator was given,[3] which is valid even if the volume of the box is infinite. It is to be noted, however, that the existence of the propagator gives no direct assurance of the convergence of the power series of its Fourier transform or of the S-matrix element. Thus one sees that the problem of the convergence is solved partly but important subjects are still left unsettled.

The main aim in the present article is to clarify the properties of the Hamiltonian operator in Hilbert space, especially the self-adjointness[4] of this operator, and to examine the convergence of the perturbation expansion. For a given system of the quantized fields, we construct the Hilbert space in which each element represents the state of the system, and define a Hamiltonian operator first as a symmetric operator with its domain dense in the Hilbert space. Then we extend the operator appropriately so that it becomes self-adjoint. We examine whether this self-adjoint operator can be determined by the perturbation method and whether it is the unique self-adjoint extension of the symmetric operator defined initially. Our procedure has more or less different features from those adopted by the authors quoted above, since we study the Hamiltonian operator as an operator in Hilbert space, whereas they principally discussed the matrix elements of the operator. Owing to the character of our method, we may hope to know the exact and general properties of the spectrum of the Hamiltonian operator, but it may be questionable whether the conditions for its applicability would be too stringent for practical use. It will turn out, however, that our method can be applied at least to the several systems having the non-relativistic interaction form factors. Hence, the convergence of the perturbation series of the S-matrix element, too, can be discussed in the framework of the wave packet formulation.

We give in § 2 some known properties of linear operators and an important theorem concerning the regular perturbation.[5] §§ 3～5 are devoted to the discussion of the fields having discrete spectra. The fixed source theory (§ 3) and the interaction among fields (§ 4) are examined. Some examples will be given to see the validity of the perturbation method. In § 5 the analytic property of the resolvent is described and the eigenstate and the eigenvalue of the total Hamiltonian are derived. Next we introduce the particle representation[6] or the representation by means of the Fock space[7] of the state of the field (§ 6).

Then we study the fields with continuous spectra (§ 7) and examine the convergence of the perturbation expansion of the S-matrix element (§ 8). We summarize the results and discuss some remaining problems in § 9.

§ 2. Mathematical preliminaries

In this section, definitions* and known properties of linear operators in Hilbert space will be stated for the preparation of the succeeding discussions.

1. *Resolvent*[4]

Let T be a linear operator in Hilbert space $\mathfrak{H}$. The complex numbers which do not belong to the spectrum $\Sigma(T)$ of T constitute the resolvent set $\Lambda(T)$ of T and the resolvent

$$R(l)=(T-l)^{-1} \tag{2·1}$$

is defined for all $l\in\Lambda(T)$ and is a bounded linear oeprator with the domain everywhere dense in $\mathfrak{H}$. When T is a closed linear operator, $R(l)$ is a bounded operator with domain $\mathfrak{H}$ and is a regular function of $l\in\Lambda(T)$. Further, when T is self-adjoint, all non-real complex numbers belong to $\Lambda(T)$.

2. *Regular operator*[5]

Let T_t be a closed linear operator depending on a complex parameter t. T_t is called a regular function of t if $R_t(l)=(T_t-l)^{-1}$ is regular in the neighbourhood of $t=0$ for some fixed $l=l_0$, that is, if $R_t(l)$ can be expanded in a power series of t. If $R_t(l)$ is regular in t for $t=0$ and $l=l_0$, it is shown to be regular in t in some neighbourhood of $t=0$ for every $l\in\Lambda(T_0)$.

3. *Rellich-Kato's theorem*

In the following sections we consider symmetric operators of the type

$$h_\lambda=h_0+\lambda h, \tag{2·2}$$

where λ is a real parameter (coupling constant). We now wish to get self-adjoint operators H_λ regular in λ by extending h_λ appropriately. For this purpose the following theorem given by F. Rellich[8] and T. Kato[5),9] will be useful.

Theorem. Let operators H_0 and H be linear extensions of symmetric operators h_0 and h in (2·2), respectively, and let H_0 be self-adjoint. If $D(H)\supseteq D(H_0)$** and there are two non-negative constants a and b such that

$$\|Hf\|\leqq a\|f\|+b\|H_0f\| \tag{2·3}$$

for every $f\in D(H_0)$, then the resolvent $R_\lambda(l)$ of $H_\lambda=H_0+\lambda H$ with $D(H_\lambda)=D(H_0)$ has the following power series expansion in λ:

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* As to definitions and notations not stated explicitly in the text, see references 4) and 5).

** $D(T)$ is the domain of an operator T.

$$R_\lambda(l)=(H_\lambda-l)^{-1}=R_0(l)\sum_{n=0}^{\infty}(-\lambda)^n\{HR_0(l)\}^n,$$

where (2·4)

$$R_0(l)=(H_0-l)^{-1}.$$

The power series in (2·4) is convergent for λ and purely imaginary l such that

$$|\lambda|<\{b+(a/|l|)\}^{-1}. \tag{2·5}$$

Therefore H_λ is self-adjoint and regular in λ at least for

$$-1/b<\lambda<1/b. \tag{2·6}$$

Further, H_λ is the unique self-adjoint extension of h_λ under the condition (2·6), if h_0 is essentially self-adjoint (that is, if $\tilde{h}_0=H_0$).*

§ 3. Boson field in interaction with the fixed source

For simplicity we consider a neutral scalar meson field $\phi(\boldsymbol{x})$ in interaction with an extended nucleon fixed at the origin of the coordinate system. The whole system is supposed to be enclosed in a large cubic box with volume Ω. The periodic boundary conditions are assumed.

1. *Hilbert space*

The function of the coordinates $\boldsymbol{x}$ defined in the box can be represented by its Fourier coefficient for the momentum vector $\boldsymbol{k}_i$ $(i=0, 1, 2, 3, \cdots)$, where

$$\begin{aligned}&\boldsymbol{k}_i=(2\pi/\Omega^{1/3})\boldsymbol{m}_i,\\ &m_{ix}, m_{iy}, m_{iz}=0, \pm1, \pm2, \cdots,\end{aligned} \tag{3·1}$$

and $\boldsymbol{k}_i$'s are arranged in such a way that

$$i>j \quad \text{if} \quad |\boldsymbol{k}_i|^2>|\boldsymbol{k}_j|^2, \tag{3·2}$$

and for the same value of $|\boldsymbol{k}_i|^2$ the order is arbitrary (the number of $\boldsymbol{k}_i$'s which have the same $|\boldsymbol{k}_i|$ being finite).

The state of the system is represented by a vector in the Hilbert space, for which the complete orthonormal system is composed of the basic vectors $\psi(n\,;n_0, n_1, \cdots)$, where n_i $(i=0, 1, 2, \cdots)$ is the number of mesons in the i-th momentum state $\boldsymbol{k}_i$ and $n=\sum_{i=0}^{\infty}n_i$ is finite but may be arbitrarily large. Since in each $\psi(n\,;n_0, n_1, \cdots)$ n is finite, there is a minimum number i_M such that $n_i=0$ for $i\geqq i_M$ and we can enumerate the basic vectors one by one by putting

$$\chi_\rho=\psi(n\,;n_0, n_1, \cdots), \quad \rho=0, 1, 2, \cdots, \tag{3·3}$$

where, for example, $\rho<\rho'$ if $ni_M<n'i'_M$ and the labeling is arbitrary for the same

* $\tilde{T}$ denotes the closure of an operator T.

value of ni_M (the number of such ρ's being also finite). The basic vectors satisfy the following orthonormal relation:

$$(\chi_\rho, \chi_{\rho'}) = \delta_{\rho\rho'}. \tag{3·4}$$

Now every f in the Hilbert space is represented as

$$f = \sum_{\rho=0}^{\infty} \alpha_\rho \chi_\rho = \sum_{n=0}^{\infty} \sum_{\sum n_i = n} \alpha(n\,;\, n_0, n_1, \cdots)\psi(n\,;\, n_0, n_1, \cdots)$$

$$\equiv \sum_{(n)} \alpha(n\,;\, n_0, n_1, \cdots)\psi(n\,;\, n_0, n_1, \cdots), \tag{3·5}$$

where $\sum_{\sum n_i = n}$ means the sum over all possible n_i under the condition $\sum_{i=0}^{\infty} n_i = n$, and has the finite norm defined by

$$\|f\| = \left(\sum_{\rho=0}^{\infty} |\alpha_\rho|^2\right)^{1/2} = \left\{\sum_{(n)} |\alpha(n\,;\, n_0, n_1, \cdots)|^2\right\}^{1/2} < \infty. \tag{3·6}$$

The inner product of elements f and f' is given as follows:

$$(f, f') = \sum_{\rho=0}^{\infty} \alpha_\rho \alpha'^*_\rho. \tag{3·7}$$

The space defined in this way is evidently complete and separable, and will be denoted by $\mathfrak{H}$.

2. *Creation and annihilation operators of the meson*

Creation and annihilation operators of a meson with momentum $\boldsymbol{k}_i$ are denoted by a_i^* and a_i, respectively, and defined for every element of the complete orthonormal system as follows:

$$a_i^* \psi(n\,;\, n_0, \cdots, n_i, \cdots) = \sqrt{n_i+1}\,\psi(n+1\,;\, n_0, \cdots, n_i+1, \cdots),$$

and (3·8)

$$a_i \psi(n\,;\, n_0, \cdots, n_i, \cdots) = \sqrt{n_i}\,\psi(n-1\,;\, n_0, \cdots, n_i-1, \cdots).$$

3. *Hamiltonian operator*

By a_i^* and a_i we construct operators h_0 and h:

$$h_0 = \sum_{i=0}^{\infty} w_i a_i^* a_i,$$

$$h = \sum_{i=0}^{\infty} (V_i^* a_i + V_i a_i^*),$$

where

$$w_i^2 = |\boldsymbol{k}_i|^2 + \mu^2 \quad (\mu: \text{ meson mass}), \tag{3·9}$$

and

$$V_i = v_i / \sqrt{2w_i \Omega},$$

v_i being the Fourier coefficient of the nucleon form factor for the momentum $\boldsymbol{k}_i$.

For every $f\in\mathfrak{H}$ we write formally,

$$h_0 f=\sum_{(n)}\Big(\sum_{i=0}^{\infty} n_i w_i\Big)\alpha(n\,;\,n_0,\cdots)\psi(n\,;\,n_0,\cdots), \tag{3·10}$$

$$hf=\sum_{(n)}\Big\{\sum_{i=0}^{\infty}\sqrt{n_i+1}\,V_i^*\alpha(n+1\,;\,n_0,\cdots,n_i+1,\cdots)$$

$$+\sum_{i=0}^{\infty}\sqrt{n_i}\,V_i\alpha(n-1\,;\,n_0,\cdots,n_i-1,\cdots)\Big\}\psi(n\,;\,n_0,\cdots,n_i,\cdots). \tag{3·11}$$

$h_0\chi_\rho\in\mathfrak{H}$ for every χ_ρ, but $h\chi_\rho\in\mathfrak{H}$ *if* and *only if*

$$F_1^2\equiv\sum_{i=0}^{\infty}|V_i|^2=(1/\Omega)\sum_{i=0}^{\infty}|v_i|^2/2w_i<\infty. \tag{3·12}$$

Let D_1 be the totality of elements $f=\sum_{\rho=0}^{N}\alpha_\rho X_\rho$ with finite number N. D_1 is clearly dense in $\mathfrak{H}$. Then it is easily verified that h_0 and h are the symmetric operators defined in D_1 under the condition (3·12), namely, for every f, $g\in D_1$,

$$(h_0 f, g)=(f, h_0 g)\quad\text{and}\quad(hf, g)=(g, hf)$$

are valid.

Now an operator H_0 is defined by the same expression as (3·10) but for elements f such that

$$\|H_0 f\|^2=\sum_{(n)}\Big|\sum_{i=0}^{\infty}(n_i w_i)^2|\alpha(n\,;\,n_0,\cdots)|^2<\infty.$$

H_0 is shown to be self-adjoint and called the Hamiltonian operator of the free field. Since $(1+H_0)D(H_0)\supset(1+H_0)D_1$ is dense in $\mathfrak{H}$, $\tilde{h}_0=H_0$, i.e. h_0 is essentially self-adjoint.[9]

For every $f\in D(H_0)$ and a linear extension H of h, we have from (3·11)

$$\|Hf\|^2=\sum_{(n)}\Big|\sum_{i=0}^{\infty}\sqrt{n_i+1}\,V_i^*\alpha(n+1\,;\,n_0,\cdots,n_i+1,\cdots)$$

$$+\sum_{i=0}^{\infty}\sqrt{n_i}\,V_i\alpha(n-1\,;\,n_0,\cdots,n_i-1,\cdots)\Big|^2$$

$$\leqq\sum_{(n)}\Big\{\sum_{i=0}^{\infty}\sqrt{n_i+1}\,|V_i|\cdot|\alpha(n+1\,;\,n_0,\cdots,n_i+1,\cdots)|$$

$$+\sum_{i=0}^{\infty}\sqrt{n_i}\,|V_i|\cdot|\alpha(n-1\,;\,n_0,\cdots,n_i-1,\cdots)|\Big\}^2\quad(\equiv J_f^2). \tag{3·13}$$

It is easily seen by repeatedly applying Cauchy's inequality that

$$J_f^2\leqq 2\sum_{(n)}\Big[\Big\{\sum_{i=0}^{\infty}\sqrt{n_i+1}\,|V_i|\cdot|\alpha(n+1\,;\,n_0,\cdots,n_i+1,\cdots)|\Big\}^2$$

$$+\Big\{\sum_{i=0}^{\infty}\sqrt{n_i}\,|V_i|\cdot|\alpha(n-1\,;\,n_0,\cdots,n_i-1,\cdots)|\Big\}^2\Big]$$

$$\leqq 2\sum_{(n)}\Big[\Big(\sum_{i=0}^{\infty}|V_i|^2\Big)\Big\{\sum_{i=0}^{\infty}(n_i+1)|\alpha(n+1\,;\,n_0,\cdots,n_i+1,\cdots)|^2\Big\}$$

$$+\Big(\sum_{i=0}^{\infty}n_i\Big)\Big\{\sum_{i=0}^{\infty}|V_i|^2|\alpha(n-1\,;\,n_0,\cdots,n_i-1,\cdots)|^2\Big\}\Big]. \qquad (3\cdot14)$$

Replacing $n\pm1$ and $n_i\pm1$ by n and n_i respectively, we obtain

$$J_f^2\leqq 2F_1^2\sum_{(n)}\sum_{i=0}^{\infty}n_i|\alpha(n\,;\,n_0,\cdots,n_i,\cdots)|^2$$

$$+2\sum_{(n)}(n+1)\sum_{i=0}^{\infty}|V_i|^2|\alpha(n\,;\,n_0,\cdots,n_i,\cdots)|^2$$

$$=2F_1^2\sum_{(n)}(2n+1)|\alpha(n\,;\,n_0,\cdots,n_1,\cdots)|^2. \qquad (3\cdot15)$$

Now for $f\in D(H_0)$ we have

$$\|H_0f\|^2\geqq\mu^2\sum_{(n)}n^2|\alpha(n\,;\,n_0,\cdots)|^2,$$

since $w_i\geqq\mu$, and

$$\|f\|^2=\sum_{(n)}|\alpha(n\,;\,n_0,\cdots)|^2<\infty.$$

Therefore the following inequality holds,

$$\|Hf\|^2\leqq J_f^2\leqq a_\sigma^2\|f\|^2+b_\sigma^2\|H_0f\|^2,$$

or

$$\|Hf\|\leqq a_\sigma\|f\|+b_\sigma\|H_0f\|, \qquad (3\cdot16)$$

if we take numbers a_σ and b_σ under the restrictions

$$a_\sigma\geqq 4F_1\sigma \quad\text{and}\quad b_\sigma\geqq(4F_1/\mu\sigma), \qquad (3\cdot17)$$

with

$$\sigma\geqq 1.$$

As σ is not bounded above, we conclude from Rellich-Kato's theorem that for every real finite λ we have an operator H_λ which, if $\mu>0$, is defined in $D(H_0)$ as the unique self-adjoint extension of h_λ and regular in λ. H_λ thus obtained is called the total Hamiltonian operator of the field interacting with the source.

For any other example in the fixed source theory, that is, even if the boson has charge and/or spin or interacts with a finite number of sources, the results are essentially the same as in the neutral scalar case, since we then meet with a finite multiple of degrees of freedom of the boson field in comparison with the latter case.

§ 4. Interaction among fields

Since this section is devoted to clarifying under what condition the theorem of § 2–3 concerning the regularity of the Hamiltonian is applicable to an inter-

action among fields, some of the examples presented are merely artificial and have no correspondence to physical reality. The whole system is enclosed in a box as in the preceding section.

1. *Boson-fermion interaction*

$a_i \equiv a(\boldsymbol{k}_i)$ and $a_i^* \equiv a^*(\boldsymbol{k}_i)$ (b_i and b_i^*; c_i and c_i^*) are the annihilation and creation operators of the particle $A(B, C)$ with the momentum $\boldsymbol{k}_i$ (cf. Eqs. (3·1), (3·2)). A(B and C) indicates the boson (fermion) field, all of the fields being assumed to be neutral for simplicity. A Hilbert space $\mathfrak{H}$ is constructed from the following complete orthonormal system:

$$\chi_\rho = \psi(l\,;\, m\,;\, n\,;\, l_0, \cdots\,;\, m_0, \cdots\,;\, n_0, \cdots), \quad \rho = 0, 1, 2, \cdots, \tag{4·1}$$

where $l_i (=0, 1, 2, \cdots)$, m_i $(=0, 1)$ and n_i $(=0, 1)$ are the numbers of the particles A, B and C respectively in the state $\boldsymbol{k}_i$ and the numbers $l = \sum_{i=0}^{\infty} l_i$, $m = \sum_{i=0}^{\infty} m_i$ and $n = \sum_{i=0}^{\infty} n_i$ are all finite but may be arbitrarily large. Further,

$$\begin{aligned} &a_i \chi_\rho = 0 \quad \text{if} \quad l_i = 0, \\ &b_i \chi_\rho = 0 \quad \text{if} \quad m_j = 0, \\ &c_k \chi_\rho = 0 \quad \text{if} \quad n_k = 0, \\ &[a_i, a_j^*] = \delta_{ij}, \quad \{b_i, b_j^*\} = \delta_{ij}, \quad \{c_i, c_j^*\} = \delta_{ij}, \end{aligned} \tag{4·2}$$

and the other commutators or anti-commutators vanish.

We introduce the following operators:

$$\begin{aligned} &h_\lambda = h_0 + \lambda h, \\ &h_0 = \sum_{i=0}^{\infty} (w_i a_i^* a_i + W_{Bi} b_i^* b_i + W_{Ci} c_i^* c_i), \\ &h = \sum_{ijk=0}^{\infty} (V_{ijk} a^*(\boldsymbol{k}_i) + V_{ikj} a(-\boldsymbol{k}_i))\,(b^*(\boldsymbol{k}_j) + c(-\boldsymbol{k}_j))\,(c^*(\boldsymbol{k}_k) + b(-\boldsymbol{k}_k)), \end{aligned} \tag{4·3}$$

where

$$\begin{aligned} &w_i^2 = M_A^2 + |\boldsymbol{k}_i|^2, \quad W_{Bi}^2 = M_B^2 + |\boldsymbol{k}_i|^2, \quad W_{Ci}^2 = M_C^2 + |\boldsymbol{k}_i|^2, \\ &V_{ijk} \equiv V(\boldsymbol{k}_i, \boldsymbol{k}_j, \boldsymbol{k}_k) = V^*(-\boldsymbol{k}_i, -\boldsymbol{k}_j, -\boldsymbol{k}_k) \equiv V_{ij}\,\delta(\boldsymbol{k}_i, \boldsymbol{k}_j, \boldsymbol{k}_k). \end{aligned} \tag{4·4}$$

M_A, M_B, M_C are the masses of the particles A, B, C, and $\delta(\boldsymbol{k}_i, \boldsymbol{k}_j, \boldsymbol{k}_k)$ represents that the momentum conserves in this interaction: $\boldsymbol{k}_i + \boldsymbol{k}_j + \boldsymbol{k}_k = 0$. Now we define h_0 and h first for all χ_ρ's (the bases of $\mathfrak{H}$) and then consider their linear extensions. It is easily seen that $\tilde{h}_0 = H_0$ is a self-adjoint operator with the domain $D(H_0)$ which consists of all the elements f such that

$$f = \sum_{\rho=0}^{\infty} \alpha_\rho \chi_\rho = \sum_{(lmn)} \alpha(l\,;\, m\,;\, n\,;\, l_0, \cdots\,;\, m_0, \cdots\,;\, n_0, \cdots) \cdot \psi(l\,;\, m\,;\, n\,;\, \cdots) \in \mathfrak{H},$$

and

$$\|H_0 f\|^2=\sum_{(lmn)}\Big(\sum_{i=0}^{\infty} w_i l_i+W_{Bi}m_i+W_{Ci}n_i\Big)^2$$
$$\times|\alpha(l\,;\,m\,;\,n\,;\,l_0,\cdots;\,m_0,\cdots;\,n_0,\cdots)|^2<\infty.$$

h can be defined as a symmetric operator for all χ_ρ's *if* and *only if*

$$F_2^2\equiv\sum_{ijk}|V_{ijk}|^2<\infty. \tag{4·5}$$

A linear extension of h is now written formally as

$$H=\sum_{\nu=1}^{8}H^{(\nu)},$$

where

$$\begin{aligned} H^{(1)}&=\sum V_{ijk}a_i^*b_j^*c_k^*, \quad H^{(2)}=\sum V_{ijk}a_i^*b_j^*b_k,\\ H^{(3)}&=\sum V_{ijk}a_i^*c_jc_k^*, \quad H^{(4)}=\sum V_{ijk}a_i^*c_jb_k, \end{aligned} \tag{4·6}$$

and

$$H^{(\nu)}=H^{(\nu-4)*} \quad \text{for } \nu=5,\,6,\,7,\,8.$$

For every $f\in D(H_0)$, we have, for example,

$$\|H^{(1)}f\|^2\leqq\sum_{(lmn)}\Big\{\sum_{[ijk]}\sqrt{l_i}\,|V_{ijk}|\cdot|\alpha(l-1\,;\,m-1\,;\,n-1\,;$$
$$l_0,\cdots l_i-1,\cdots;\,m_0,\cdots,m_j-1,\cdots;\,n_0,\cdots,n_k-1,\cdots)|\Big\}^2, \tag{4·7}$$

where the summation $(\sum_{[ijk]})$ in curly brackets is carried out, fixing a set of numbers $\{l_i, m_j, n_k\ (i,j,k=0,1,2,\cdots)\}$, over all i, j, k for which $l_i\geqq 1$, $m_j=1$, $n_k=1$. Using Cauchy's inequality and (4·4)

$$\begin{aligned} \|H^{(1)}f\|^2&\leqq\sum_{(lmn)}\Big\{\sum_{[ijk]}\sqrt{l_i}\,|V_{ij}|\cdot|\alpha(l-1\,;\cdots)|\cdot\delta(\boldsymbol{k}_i,\boldsymbol{k}_j,\boldsymbol{k}_k)\Big\}^2\\ &\leqq\sum_{(lmn)}\Big\{\sum_{[ijk]}l_i\delta(\boldsymbol{k}_i,\boldsymbol{k}_j,\boldsymbol{k}_k)\Big\}\cdot\sum_{[rst]}\{|V_{rs}|^2\cdot|\alpha(l-1\,;\cdots)|^2\cdot\delta(\boldsymbol{k}_r,\boldsymbol{k}_s,\boldsymbol{k}_t)\}\\ &\leqq\sum_{(lmn)}lm\sum_{rst}|V_{rst}|^2|\alpha(l-1\,;\cdots)|^2. \end{aligned}$$

Replacing $l-1$, l_r-1, by l, l_r and so on, we get

$$\|H^{(1)}f\|^2\leqq\sum_{(lmn)}(l+1)(m+1)\Big\{\sum_{rst}|V_{rst}|^2\Big\}$$
$$\times|\alpha(l\,;\,m\,;\,n\,;\,l_0,\cdots;\,m_0,\cdots;\,n_0,\cdots)|^2,$$

i. e.

$$\|H^{(1)}f\|^2\leqq F_2^2\sum_{(lmn)}(l+1)(m+1)\cdot|\alpha(l\,;\,m\,;\,n\,;\,l_0,\cdots;\,m_0,\cdots;\,n_0,\cdots)|^2.$$

After the analogous calculation we obtain

$$\|H^{(\nu)}f\|^2\leqq F_2^2\sum_{(lmn)}(l+1)(m+n+1)$$
$$\times|\alpha(l\,;\,m\,;\,n\,;\,l_0,\cdots;\,m_0,\cdots;\,n_0,\cdots)|^2,\quad(\nu=1,\cdots,8). \tag{4·8}$$

Hence,

$$\|Hf\|^2 \leqq 8 \sum_{\nu=1}^{8} \|H^{(\nu)} f\|^2 \leqq 8^2 F_2^2 \sum_{(lmn)} (l+1)(m+n+1) |\alpha(l; m; n; l_i; m_j; n_k)|^2. \tag{4·9}$$

On the other hand it is seen that

$$\|f\|^2 = \sum_{(lmn)} |\alpha(l; m; n; l_i; m_j; n_k)|^2, \tag{4·10}$$

and

$$\|H_0 f\|^2 > \sum_{(lmn)} \{lM_A + (m^{4/3} + n^{4/3})\gamma\}^2 \cdot |\alpha(l; m; n; l_i; m_j; n_k)|^2, \tag{4·11}$$

where $\gamma < (3\pi/2)(\Omega\beta)^{-1/3}$, β being some positive constant. (4·11) is obtained in the following way: From (3·1), (3·2) $|\boldsymbol{k}_j| = 2\pi\Omega^{-1/3}|\boldsymbol{m}_j|$ and there is a positive finite constant β such that $\beta|\boldsymbol{m}_j|^3 \geqq j$ for all values of j because $(4/3)\times\pi|\boldsymbol{m}_j|^3 \sim j$ for large j. Thus we have

$$|\boldsymbol{k}_j| \geqq 2\pi\Omega^{-1/3}(j/\beta)^{1/3}$$

and

$$\sum_{j=0}^{\infty} m_j W_{Bj} = \sum_{j=0}^{\infty} m_j \sqrt{M_B^2 + |\boldsymbol{k}_j|^2} \geqq \sum_{j=0}^{\infty} m_j |\boldsymbol{k}_j|$$

$$\geqq 2\pi(\Omega\beta)^{-1/3} \sum_{j=0}^{\infty} m_j j^{1/3} \geqq (3\pi/2)(\Omega\beta)^{-1/3} m^{4/3}.$$

$\sum_{j=0}^{\infty} m_j W_{Cj}$ is also bounded below by a similar expression and $\sum_{j=0}^{\infty} l_j w_{Aj} \geqq lM_A$.

As γ may be positive so far as $\Omega < \infty$, we get the following relation from (4·9), (4·10) and (4·11), if $M_A > 0$:

$$\|Hf\| < a\|f\| + b\|H_0 f\| \quad \text{for every } f \in D(H_0),$$

where b can be taken as small as we like. Hence, if the boson mass is not zero, $\tilde{h}_\lambda = H_\lambda$ is self-adjoint in $D(H_0)$ and regular in λ for all real λ.

Finally, it is remarked that H conserves the difference between the numbers of two kinds of fermions $\sum_{j=0}^{\infty}(b_j^* b_j - c_j^* c_j)$ and accordingly it conserves the so-called "fermion number" if C is regarded as the anti-particle of B, and that, on the basis of Dirac's hole theory,[10] we have constructed the Hilbert space only by those basic vectors in which a finite number of fermions (particles and/or anti-particles) are present, and assumed a positive definite free Hamiltonian like (4·3).

2. *Boson-boson interaction*

If all the particles A, B and C of (4·3) are considered as bosons, the total Hamiltonian cannot be defined by the present method. To show this we consider here the simplest example of the boson-boson interaction, self-interaction of the boson field with only one degree of freedom.

The total Hamiltonian is written formally as

$$H_\lambda = H_0 + \lambda H, \quad H_0 = a^* a, \quad H = (a^* + a)^3,$$

with $a(a^*)$ the annihilation (creation) operator, and the complete orthonormal system of the Hilbert space $\mathfrak{H}$ is a set of χ_n $(n=0, 1, 2, \cdots)$ with n bosons.

Then $D(H_0)$ consists of all $f=\sum_{n=0}^{\infty} \alpha_n \chi_n$ such that

$$\|f\|^2 = \sum_{n=0}^{\infty} |\alpha_n|^2 < \infty \quad \text{and} \quad \|H_0 f\|^2 = \sum_{n=0}^{\infty} n^2 |\alpha_n|^2 < \infty.$$

H is certainly defined for the bases of $\mathfrak{H}$ and is symmetric. However, H cannot be defined for some elements of $D(H_0)$. In fact, for such an element as

$$f = \sum_{n=1}^{\infty} (1/n^2)\chi_n \in D(H_0),$$

$\|Hf\|^2$ does not exist. Therefore, the theorem of §§ 2~3 is not applicable to the present case. This is consistent with the result obtained by several authors.[1)]

Next we consider the following interaction among three kinds of bosons A, B and C,

$$h_\lambda = h_0 + \lambda h, \quad (h_0 \text{ having the same form as } (4\cdot3)),$$

$$h = \sum_{ijk} (V_{ijk} a^*(\boldsymbol{k}_i) + V_{ikj} a(-\boldsymbol{k}_i))\,(b^*(\boldsymbol{k}_j) + c^*(\boldsymbol{k}_j))\,(b(-\boldsymbol{k}_k) + c(-\boldsymbol{k}_k)).$$

This interaction conserves the sum of the numbers of the particles B and C, $\sum_i (b_i^* b_i + c_i^* c_i) \equiv \mathfrak{N}$. Hence, if we decompose the entire Hilbert space in a direct sum of subspaces in each of which $\mathfrak{N}$ is constant, we can define the total Hamiltonian H_λ in each subspace with finite $\mathfrak{N}$ in essentially the same way as in the fixed source theory with a finite number of sources. It is shown, however, that the power series expansion of $(H_\lambda - l)^{-1}$ as an operator in the entire Hilbert space can not be justified by the present method.

§ 5. Spectrum of the total Hamiltonian

The result of the preceding sections states that the resolvent $R_\lambda(l)$ is expressible in the form $(2\cdot4)$ and is an analytic function of λ for every real $\lambda=\lambda_0$ and some $l=l_0$. Since H_λ is self-adjoint, $R_\lambda(l)$ can be defined for all $l \in \Lambda(H_\lambda)$ as an analytic continuation from l_0. However, of course, $R_\lambda(l)$ thus defined can not necessarily be written as a convergent power series in λ as $(2\cdot4)$.

1. *Matrix element of the resolvent*

Now we consider an example of the matrix element of the resolvent for the neutral scalar meson:

$$\mathcal{G}(\lambda, l) \equiv (\psi_0, [H_\lambda - l]^{-1} \psi_0) = \sum_{N=0}^{\infty} |(\psi_0, \Psi_N)|^2 (E_N - l)^{-1}, \qquad (5\cdot1)$$

where ψ_0 is the no meson state and $\Psi_N (N=0, 1, 2, \cdots)$ is the eigenstate of H_λ with the eigenvalue E_N corresponding to the eigenstate ψ_N of H_0 with the

eigenvalue e_N. It is known[11] that Ψ_N and E_N can be explicitly calculated by a unitary transformation. We finally get

$$\mathcal{G}(\lambda, l)=Z[(\delta m-l)^{-1}+\sum_{n=1}^{\infty}(\lambda^{2n}/n!)\sum_{\alpha l i s=0}^{\infty}\{(\sum_{p=1}^{n}w_{i(p)}+\delta m-l)^{-1}\times\prod_{q=1}^{n}2|V_{i(q)}|^2/w_{i(q)}^2\}],$$

where

$$\delta m\equiv E_0=-\lambda^2\sum_{i=0}^{\infty}|V_i|^2/w_i,$$

$$Z\equiv|(\psi_0, \Psi_0)|^2=\exp(-\lambda^2\sum_{i=0}^{\infty}|V_i|^2/w_i^2). \tag{5·2}$$

As is easily seen, the right-hand side converges for every finite λ and for every finite complex number l except for the set of points:

$$\Sigma(H_\lambda)=\{\delta m(\lambda)+\sum_{i=0}^{\infty}n_i w_i\}\quad(n_i=0, 1, 2, \cdots),$$

(see Fig. 1).

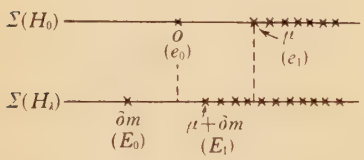

Fig. 1. $\Sigma(H_0)$ and $\Sigma(H_\lambda)$ in the l-plane. × denotes a discrete eigenvalue.

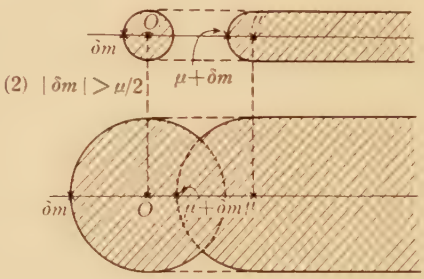

Fig. 2.

$\Sigma(H_\lambda)$ is exactly the spectrum of the total Hamiltonian H_λ. It should be noticed that $\mathcal{G}(\lambda, l)$ can not be expressed in a power-series of λ for l in a shaded domain of the l-plane in Fig. 2 since δm depends on λ, but it can be if we consider δm to be constant as in the convention of renormalization.

2. *Eigenvectors of the total Hamiltonian*

We denote by $\Psi_\rho(\chi_\rho)$ the eigenvector of $H_\lambda=H_0+\lambda H(H_0)$ with eigenvalue $E_\rho(e_\rho)$:

$$(H_\lambda-E_\rho)\Psi_\rho=0, \tag{5·3}$$

and

$$(H_0-e_\rho)\chi_\rho=0. \tag{5·4}$$

Here $E_\rho\to e_\rho$, $\Psi_\rho\to\chi_\rho$ for $\lambda\to 0$ since E_ρ and Ψ_ρ are analytic functions of λ.[5]

Next we introduce the projection operator P_ρ:

$$P_\rho \chi_{\rho'} = (1-\delta_{\rho\rho'})\chi_{\rho'} \quad (\rho, \rho' = 0, 1, 2, \cdots), \tag{5·5}$$

and define the symmetric operator

$$H_{\lambda\rho} = H_0 + \lambda P_\rho H P_\rho \quad (\rho = 0, 1, 2, \cdots), \tag{5·6}$$

which is evidently self-adjoint with domain $D(H_0)$ and regular for real λ.

If we put

$$\Psi(\rho, \lambda, E) = \{1 + (E - H_{\lambda\rho})^{-1}\lambda H\}\chi_\rho, \tag{5·7}$$

then by the power series expansion of the resolvent and by the analytic continuation with respect to E, we get

$$(H_\lambda - E)\Psi(\rho, \lambda, E) = \{e_\rho - E + \Sigma_\rho(\lambda, E)\}\chi_\rho \in \mathfrak{H} \tag{5·8}$$

for $E \in \Lambda(H_{\lambda\rho})$, where

$$\Sigma_\rho(\lambda, E) \equiv \lambda^2 \{H\chi_\rho, (E - H_{\lambda\rho})^{-1} H\chi_\rho\} = \sum_{n=0}^{\infty} |\gamma_{n\rho}|^2 (E - \varepsilon_n)^{-1} \tag{5·9}$$

with

$$\gamma_{n\rho} = (\phi_n, \lambda H\chi_\rho),$$

ε_n anh ϕ_n being the eigenvalues and the eigenvectors of $H_{\lambda\rho}$. The graph of $\Sigma_\rho(\lambda, E)$ as a function of E is shown in Fig. 3. If we denote by $E_\rho^{(n)}$ $(n = 0, 1, \cdots)$ the roots of the equation

$$E - e_\rho = \Sigma_\rho(\lambda, E), \tag{5·10}$$

then, since $E_\rho^{(n)} \in \Lambda(H_{\lambda\rho})$, (5·8) becomes

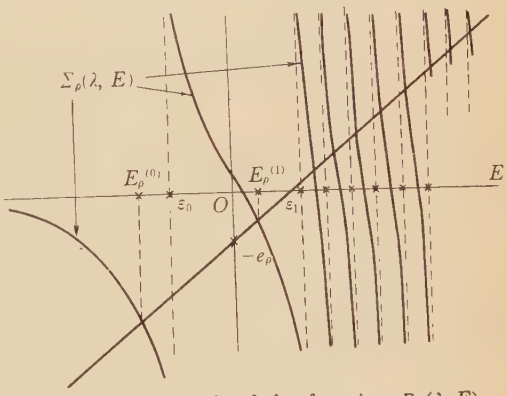

Fig. 3. The graph of the function $\Sigma_\rho(\lambda, E)$.

$$(H_\lambda - E_\rho^{(n)})\Psi(\rho, \lambda, E_\rho^{(n)}) = 0, \tag{5·11}$$

where

$$\Psi(\rho, \lambda, E_\rho^{(n)}) = \{1 + (E_\rho^{(n)} - H_{\lambda\rho})^{-1}\lambda H\}\chi_\rho.$$

$E_\rho^{(n)}$ and $\Psi(\rho, \lambda, E_\rho^{(n)})$ are the eigenvalues and the eigenvectors of H_λ respectively. If we formally expand the right-hand side of (5·10) and (5·11) in a power series of λ, we get the familiar formulae of the Brillouin-Wigner perturbation theory.[12]

§ 6. Particle representation of the field with continuous spectrum

In the preceding sections the system is considered to be enclosed in a large box, so that the spectrum of the total Hamiltonian as well as the free

Hamiltonian is discrete. The states of the system, there, are described by the occupation number representation[6] in which the basic vectors are specified by the number of particles in each discrete level. In this and the following sections we consider the case where the system is not enclosed in a box and the spectrum of the Hamiltonian is continuous.

Though the formulation of the occupation number representation is possible also in this case, it is greatly cumbersome as compared with that of the particle representation.[6] Therefore we here employ the particle representation or the representation by the Fock space.[7]

1. *Hilbert space*

We consider a denumerably infinite system of functions

$$\Psi \equiv \{\psi_n(s)\} \equiv \{\psi_0(s), \psi_1(s), \cdots, \psi_n(s), \cdots\}, \tag{6·1}$$

where $\psi_n(s) \equiv \psi_n(s_1, \cdots, s_n)$ is a complex-valued function of n real variables $s_1, \cdots, s_n$ defined in the domain $R_n\{a \leqq s_i \leqq b\ (i=1, 2, \cdots, n)$, a and b may be infinite$\}$ and $\psi_0(s) \equiv \psi_0$ is a complex number. We shall impose the following condition on Ψ:

$$\|\Psi\|^2 \equiv \sum_{n=0}^{\infty} \int |\psi_n(s)|^2 (ds)_n < \infty,$$

where

$$\int |\psi_0(s)|^2 (ds)_0 \equiv |\psi_0|^2, \tag{6·2}$$

$$\int |\psi_n(s)|^2 (ds)_n \equiv \int_{R_n} |\psi_n(s)|^2 ds_1 \cdots ds_n \quad \text{for } n > 0.$$

All Ψ's defined by (6·1) and (6·2) constitute a Hilbert space, which will be denoted by $\mathfrak{H}$. Ψ is called a state vector.

If all the functions except $\psi_n(s)$ vanish in Ψ, we say that the field is in a state which consists of an assembly of n particles. Particularly, the state represented by a vector $\{\psi_0 \neq 0, \psi_n(s) = 0\ (n \geqq 1)\}$ is called the vacuum state. $\psi_n(s)$ is called the n-th component of Ψ and the space composed of all elements of the form $\{\psi_m(s)\delta_{nm}\}$ is also a Hilbert space and specified by the symbol $\mathfrak{H}^n (\subset \mathfrak{H})$.

It should be noted that all the components of Ψ are the totally symmetric (anti-symmetric) functions of the arguments s_n's for the boson (fermion) field.

Now we consider the case in which $a = -\infty$ and $b = +\infty$. Let $D_1 \subset \mathfrak{H}$ be a set of all the systems of the functions of the form

$$f^{(N)} = \sum_{m=0}^{N} \Delta^m,$$

$$\Delta^m = \left\{ g_n(s)\delta_{nm} \equiv P_n(s) \exp\left\{ -\frac{1}{2}(s_1^2 + \cdots + s_n^2) \right\} \delta_{nm} \right\}, \tag{6·3}$$

where $P_n(s)$ is an arbitrary polynomial of $\{s_1, \cdots, s_n\}$ and N is an arbitrary but finite integer.

The set of all $\mathcal{A}^n$'s with n fixed is denoted by D_1^n. D_1^n is evidently linear and dense in $\mathfrak{H}^n$, since D_1^n contains all the elements of the Hermite orthogonal system which is complete in R_n. Therefore D_1 is a linear set dense in $\mathfrak{H}$.

In this section each s_n will be considered as the momentum vector $\boldsymbol{k}_n$ of each particle with components k_{ni} $(i=1, 2, 3$ and $-\infty < k_{ni} < +\infty)$. It should be remarked here, however, that we can pass from the momentum space $\{\boldsymbol{k}_n\}$ to the coordinate space $\{\boldsymbol{x}_n\}$ and vice versa by the Fourier transformation, namely, we have

$$\int |\psi_n(\boldsymbol{k})|^2 (d\boldsymbol{k})_n = \int |\phi_n(\boldsymbol{x})|^2 (d\boldsymbol{x})_n < \infty, \tag{6·4}$$

where $\phi_n(\boldsymbol{x})$ is the Fourier transform of $\psi_n(\boldsymbol{k})$ $(n \geqq 1)$ and $\psi_0(\boldsymbol{k}) = \phi_0(\boldsymbol{x})$, and thus the correspondence is one-to-one, linear and isometric. If an operator is defined in the momentum space, it can also be defined in the coordinate space.

Finally we describe the correspondence between the particle representation and the occupation number representation. This will be of supplementary use for the discussion in the next section. We take the orthonormal system $u_p(s)$ $(p=0, 1, 2, \cdots)$ which is complete in the domain $R_1\{a \leqq s \leqq b\}$,

$$\int_{R_1} u_p^*(s) u_q(s)\, ds = \delta_{pq} \quad (p, q=0, 1, 2, \cdots), \tag{6·5}$$

and expand every component $\psi_n(s)$ of $\Psi \in \mathfrak{H}$ as follows:

$$\psi_n(s) = \sum_{all\ p_i's=0}^{\infty} c(p_1, \cdots, p_n)\, u_{p_1}(s_1)\, u_{p_2}(s_2) \cdots u_{p_n}(s_n). \tag{6·6}$$

We define

$$d(n; n_0, n_1, \cdots, n_j, \cdots) = \mathrm{P}(n! / \prod_{j=0}^{\infty} n_j!)^{1/2}\, c(p_1, \cdots, p_n), \tag{6·7}$$

where

$$\sum_{j=0}^{\infty} n_j = n,$$

n_j indicating how many times j appears in $(p_1, \cdots, p_n)$. For the boson field $n_j=0, 1, 2, \cdots$ $(j=0, 1, 2, \cdots)$ and $\mathrm{P}=1$. For the fermion field $n_j=0$, 1 and $\mathrm{P}=+1$ or -1 according as $(p_1, p_2, \cdots, p_n)$ is an even or an odd permutation of its natural ordering. Then we have from (6·6) and (6·7)

$$\sum_{n=0}^{\infty} \sum_{\Sigma n_j = n} |d(n; n_0, n_1, \cdots)|^2 = \sum_{all\ p_i's=0}^{\infty} |c(p_1, \cdots, p_n)|^2 = \sum_{n=0}^{\infty} \int |\psi_n(s)|^2 (ds)_n < \infty. \tag{6·8}$$

We construct a Hilbert space $\mathfrak{R}$ in which each element is of the following form,

$$f=\sum_{n=0}^{\infty}\sum_{\Sigma n_j=n} d(n;n_0,\cdots,n_j,\cdots)\chi(n;n_0,\cdots,n_j,\cdots),$$

where χ's constitute the complete orthonormal system of $\mathfrak{R}$. This expression is quite similar to (3·5) in the case of discrete momentum but it should be remarked that the state j here is specified not by the momentum $\boldsymbol{k}_j$ but by the function $u_j(s)$ of some orthonormal system. Each element of $\mathfrak{R}$ corresponds to one of $\mathfrak{H}$ and conversely through (6·7). Thus $\mathfrak{R}$ is a kind of occupation number representation of $\mathfrak{H}$.

2. *Creation and annihilation operators*

We define for every $\Phi^n\in D_1^n\subset\mathfrak{H}^n$ two operators $A^-\{G^*\}$ and $A^+\{G\}$:

$$\begin{aligned}A^-\{G^*\}\Phi^n&=\int G^*(s)A^-(s)\,ds\,\Phi^n\\&\equiv\{0,0,\cdots,\sqrt{n}\int_{R_1}G^*(s_n)g_n(s_1,\cdots,s_n)\,ds_n,0,\cdots\}\in\mathfrak{H}^{n-1}\\&\qquad\qquad\text{for } n\geqq 1,\\&\equiv 0\qquad\qquad\text{for } n=0,\end{aligned}\tag{6·9$_-$}$$

$$\begin{aligned}A^+\{G\}\Phi^n&=\int G(s)A^+(s)\,ds\,\Phi^n\\&\equiv\{0,0,\cdots,\sqrt{n+1}\,\underset{\text{Asy}}{\text{Sy}}\{G(s_{n+1})g_n(s)\},0,\cdots\}\in\mathfrak{H}^{n+1}\\&\qquad\qquad\text{for } n=0,1,2,\cdots.\end{aligned}\tag{6·9$_+$}$$

0-th component of (6·9$_+$) is always zero. It is easily seen that (6·9) are justified if and only if the following inequality holds for the function $G(s)$:

$$\int_{R_1}|G(s)|^2\,ds<\infty.\tag{6·10}$$

For the boson (fermion) field, all $g_n(s)$'s are already the symmetric (anti-symmetric) functions and the symbol Sy (Asy) in (6·9$_+$) stands for the symmetrization (anti-symmetrization) between s_{n+1} and n variables $s_1,\cdots,s_n$. In general, $\text{Sy}f(s_1,\cdots,s_n)$ denotes the sum of the function $f(s_1,\cdots,s_n)$ over all permutations of $\{s_1,\cdots,s_n\}$ divided by $n!$. On the other hand, $\text{Asy}f_n(s)$ is the similar sum but each summand is multiplied by a factor $+1$ or -1 according as the permutation is even or odd.

Since the operator $A^+\{G\}$ or $A^-\{G^*\}$ transforms a state corresponding to n particles into one corresponding to $n+1$ or $n-1$ particles, $A^+\{G\}$ or $A^-\{G^*\}$ may be called a creation or an annihilation operator, respectively. It can be shown that the two operators $A^+\{G\}$ and $A^-\{G^*\}$ in D_1 are adjoint to each other[6] and can be defined for every element of $\mathfrak{H}^n(n=0,1,2,\cdots)$ but not necessarily for elements of $\mathfrak{H}$.

$A^+\{G\}$ and $A^-\{G^*\}$ are represented in $\mathfrak{R}$ as follows :

$$A^+\{G\}=\sum_{i=0}^{\infty}G_i a_i^*,\quad A^-\{G^*\}=\sum_{i=0}^{\infty}G_i^* a_i, \tag{6·11}$$

where

$$G_i=\int_{R_1}G(s)u_i^*(s)\,ds,\quad G(s)=\sum_{i=0}^{\infty}G_i u_i(s), \tag{6·12}$$

$$a_i\chi(n\,;\,n_0,\cdots,n_i,\cdots)=0,\quad \text{if } n_i=0,$$

and

$$[a_i, a_j^*]=\delta_{ij}\quad \text{for the boson field,} \tag{6·13}$$

$$\{a_i, a_j^*\}=\delta_{ij}\quad \text{for the fermion field.}$$

§ 7. Hamiltonian operator with continuous spectrum

We now examine two examples of the Hamiltonian operator with continuous spectrum.

1. *Scalar boson field interacting with a fixed source* (cf. § 3)

First h_0, h and h_λ with a real parameter λ are defined for every element $\Phi^n\in D_1^n\subset D_1$,

$$h_\lambda=h_0+\lambda h,$$

$$h_0\Phi^n\equiv\{0,\,0,\,\cdots,\,\sum_{r=1}^{n}w(\boldsymbol{k}_r)\,g_n(\boldsymbol{k}),\,0,\,\cdots\}\in\mathfrak{H}^n,$$

$$h\Phi^n\equiv(A^+\{V\}+A^-\{V^*\})\Phi^n$$

$$=\int\{V(\boldsymbol{k})A^+(\boldsymbol{k})+V^*(\boldsymbol{k})A^-(\boldsymbol{k})\}\,dk\,\Phi^n\in(\mathfrak{H}^{n-1}+\mathfrak{H}^{n+1}), \tag{7·1}$$

where $V(\boldsymbol{k})$ is subject to the condition like (6·10) and $w(\boldsymbol{k})=\sqrt{\mu^2+|\boldsymbol{k}|^2}$. h_0, h and h_λ are symmetric operators since D_1 is dense in $\mathfrak{H}$ and $A^+\{V\}$ is adjoint to $A^-\{V^*\}$ in D_1. Now it is easily seen that h_0 can be extended to H_0 with domain $D(H_0)\subset\mathfrak{H}$, for which every $\Psi\in D(H_0)$ satisfies

$$\|\Psi\|^2=\sum_{n=0}^{\infty}\int|\psi_n(\boldsymbol{k})|^2(d\boldsymbol{k})_n<\infty, \tag{7·2}$$

and

$$\|H_0\Psi\|^2=\sum_{n=0}^{\infty}\int\{\sum_{r=1}^{n}w(\boldsymbol{k}_r)\}^2|\psi_n(\boldsymbol{k})|^2(d\boldsymbol{k})_n<\infty. \tag{7·3}$$

H_0 thus defined is obviously self-adjoint. Since $(1+H_0)D_1$ is dense in $\mathfrak{H}$, it follows that $\tilde{h}_0=H_0$, hence h_0 is essentially self-adjoint.[9]

Next considering a linear extension H of h and estimating $\|H\Psi\|^2$ for every $\Psi\in D(H_0)$ by means of Schwarz's inequality, we can easily obtain a relation

$$\|H\Psi\|^2 \leqq 2\int |V(\boldsymbol{k})|^2 d\boldsymbol{k} \sum_{n=0}^{\infty} (2n+1) \int |\psi_n(\boldsymbol{k})|^2 (d\boldsymbol{k})_n, \tag{7·4}$$

which just corresponds (3·15) in the case of discrete spectrum. Hence, if $\mu > 0$, by the theorem of § 2, we have a unique self-adjoint extension H_λ of h_λ with the properties that $H_\lambda = \tilde{h}_\lambda$ and H_λ is defined in $D(H_0)$ and regular in λ for all real λ.

2. *Boson-fermion interaction* (cf. § 4)

We consider the interaction among the boson field A and the fermion fields B and C. Every element $\Psi \in \mathfrak{H}$ is represented by a set of functions in the momentum space $\psi_{lmn}(\boldsymbol{k})$, where l, m and n indicate the numbers of the particles A, B and C respectively, and $\psi_{lmn}(\boldsymbol{k}) \equiv \psi_{(p)}(\boldsymbol{k})$ is the function of $3p = 3(l+m+n)$ variables and Ψ is of course subject to the condition

$$\|\Psi\|^2 = \sum_{(p)=0}^{\infty} \int |\psi_{(p)}(\boldsymbol{k})|^2 (d\boldsymbol{k})_{(p)} = \sum_{(p)=0}^{\infty} \int |\phi_{(p)}(\boldsymbol{x})|^2 (d\boldsymbol{x})_{(p)} < \infty, \tag{7·5}$$

$\phi_{(p)}(\boldsymbol{x})$ being the Fourier transform of $\psi_{(p)}(\boldsymbol{k})$.

Here again the Hamiltonian operator $h_\lambda = h_0 + \lambda h$ is defined first in the domain D_1 dense in $\mathfrak{H}$, that is, for the assembly of all finite linear combinations of elements of the type

$$\Delta^{lmn} = \{0, 0, \cdots, g_{Al}(\boldsymbol{k}_A) g_{Bm}(\boldsymbol{k}_B) g_{Cn}(\boldsymbol{k}_C), 0, \cdots\}.$$

We define:

$$h_0 \Delta^{lmn} = \{0, 0, \cdots, \{\sum_{r=1}^{l} w(\boldsymbol{k}_{Ar}) + \sum_{r=1}^{m} W_B(\boldsymbol{k}_{Br}) + \sum_{r=1}^{n} W_C(\boldsymbol{k}_{Cr})\} g_{Al} g_{Bm} g_{Cn}, 0, \cdots\},$$

$$h\Delta^{lmn} = \int \{V(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) A^+(\boldsymbol{k}_1) + V(\boldsymbol{k}_1, \boldsymbol{k}_3, \boldsymbol{k}_2) A^-(-\boldsymbol{k}_1)\}$$
$$\times \{B^+(\boldsymbol{k}_2) + C^-(-\boldsymbol{k}_2)\} \{B^-(-\boldsymbol{k}_3) + C^+(\boldsymbol{k}_3)\} d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \Delta^{lmn}. \tag{7·6}$$

Unless otherwise stated, the notations should be understood as natural generalizations of those in §§ 4–1 and 7–1. Only it should be noted that $V(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ in (7·6), unlike V_{ijk} in (4·4), does not contain a factor expressing the momentum conservation.

The *necessary* and *sufficient* condition in order that h can be defined in D_1 is

$$F_3^2 \equiv \int |V(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)|^2 d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 = \int |U(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)|^2 d\boldsymbol{x}_1 d\boldsymbol{x}_2 d\boldsymbol{x}_3 < \infty, \tag{7·7}$$

where $U(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$ is the Fourier transform of $V(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$. h_0, h and h_λ are symmetric and $\tilde{h}_0 = H_0$ is self-adjoint.

Now we consider a linear extension H of h and wish to estimate $\|H\Psi\|$ for $\Psi \in D(H_0)$. We take, for instance,

$$H^{(1)}\Psi=\int V(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)A^+(\boldsymbol{k}_1)B^+(\boldsymbol{k}_2)C^+(\boldsymbol{k}_3)d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3\Psi,$$

$$\|H^{(1)}\Psi\|^2=\sum_{(p)=0}^{\infty}(l+1)(m+1)(n+1)\int|{}^{\mathrm{Sy}}_{\mathrm{Asy}}\{V(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\psi_{(p)}(\boldsymbol{k})\}|^2$$
$$\times d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3(d\boldsymbol{k})_{(p)}. \tag{7·8}$$

Then for each term under the summation the following inequality is valid:

$$(l+1)(m+1)(n+1)\int(d\boldsymbol{k})_{(p)}\int d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3|{}^{\mathrm{Sy}}_{\mathrm{Asy}}\{V(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\psi_{(p)}(\boldsymbol{k})\}|^2$$
$$=(l+1)(m+1)(n+1)\int(d\boldsymbol{x})_{(p)}\int d\boldsymbol{x}_1 d\boldsymbol{x}_2 d\boldsymbol{x}_3|{}^{\mathrm{Sy}}_{\mathrm{Asy}}\{U(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)\phi_{(p)}(\boldsymbol{x})\}|^2$$
$$\leqq F_4^2(l+1)\int|\phi_{(p)}(\boldsymbol{x})|^2(d\boldsymbol{x})_{(p)}=F_4^2(l+1)\int|\psi_{(p)}(\boldsymbol{k})|^2(d\boldsymbol{k})_{(p)}. \tag{7·9}$$

Here F_4 may be either

$$F_{4,K}\equiv\sum_{ijk=0}^{\infty}|V_{ijk}|,$$

or (7·10)

$$F_{4,x}\equiv\sum_{ijk=0}^{\infty}|U_{ijk}|,$$

where $V_{ijk}(U_{ijk})$ is the expansion coefficient of $V(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$ $(U(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3))$ in terms of the Hermite orthonormal system. To prove (7·9), we use the occupation number representation, as described in § 6, by taking the Hermite orthonormal system as the basic set. Then the integral (7·9) is written in a similar form as in § 4, but now i, j and k denoting the elements of the orthonormal system. Applying Cauchy's inequality, we have

$$(l+1)(m+1)(n+1)\int(d\boldsymbol{k})_{(p)}d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3|{}^{\mathrm{Sy}}_{\mathrm{Asy}}\{V(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\psi_{(p)}(\boldsymbol{k})\}|^2$$
$$\leqq\sum_{\substack{\Sigma l_i=l\\ \Sigma m_j=m\\ \Sigma n_k=n}}\Big\{\sum_{ijk}\sqrt{l_i+1}\,|V_{ijk}|\cdot|\alpha(l;m;n;l_i;m_j;n_k)|\Big\}^2\binom{m_j=0}{n_k=0}$$
$$\leqq\Big\{\sum_{ijk}(l_i+1)|V_{ijk}|\Big\}\sum_{\substack{\Sigma l_i=l\\ \Sigma m_j=m\\ \Sigma n_k=n}}\Big\{\sum_{ijk}|V_{ijk}|\cdot|\alpha(l;m;n;\cdots)|^2\Big\}\binom{m_j=0}{n_k=0}$$
$$\leqq(l+1)F_{4,K}^2\int|\psi_{(p)}(\boldsymbol{k})|^2(d\boldsymbol{k})_{(p)}.$$

In the coordinates space we obtain the similar result. Taking account of (7·5), we have (7·9).

Therefore

$$\|H\Psi\|^2 \leqq 8^2 F_4^2 \sum_{(p)=0}^{\infty} (l+1) \int |\psi_{(p)}(\boldsymbol{k})|^2 (d\boldsymbol{k})_{(p)}. \tag{7·11}$$

This shows that under the conditions $M_A > 0$ and

$$F_4 < \infty, \tag{7·12}$$

the theorem in § 2 can be applied and $\tilde{h}_\lambda = H_\lambda$ is self-adjoint with domain $D(H_0)$ and regular in λ for real finite λ.

§ 8. S-matrix

We shall for simplicity discuss only one process : the scattering of a neutral scalar meson by a fixed source (cf. § 7–1). The similar arguments are valid for the other cases which can be treated by the method described in §§ 6 and 7.

First, we give the usual definition[13] of the S-matrix element for the scattering of a meson in the momentum state $\boldsymbol{k}_j$ to the state $\boldsymbol{k}_i$:

$$S(\boldsymbol{k}_i, \boldsymbol{k}_j) = \sum_{n=0}^{\infty} \lambda^n S^{(n)}(\boldsymbol{k}_i, \boldsymbol{k}_j), \tag{8·1}$$

$$S^{(n)}(\boldsymbol{k}_i, \boldsymbol{k}_j) = (-i)^n \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n (A^+(\boldsymbol{k}_i)\Psi_0, [H(t_1)\cdots H(t_n)]A^+(\boldsymbol{k}_j)\Psi_0),$$

where Ψ_0 and $A^+(\boldsymbol{k}_i)\Psi_0$ indicate the no meson state and the one meson plane wave state having the momentum $\boldsymbol{k}_i$, respectively, and

$$H(t) = e^{iH_0 t} H e^{-iH_0 t}. \tag{8·2}$$

By the use of the adiabatic hypothesis and the square integrability of the form factor, (8·2) becomes

$$S^{(n)}(\boldsymbol{k}_i, \boldsymbol{k}_j) = \begin{cases} 2\pi i (-1)^n \delta(w(\boldsymbol{k}_i) - w(\boldsymbol{k}_j)) T^{(n)}(\boldsymbol{k}_i, \boldsymbol{k}_j, w(\boldsymbol{k}_j) + i\varepsilon), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \geqq 1, \\ \delta(\boldsymbol{k}_i - \boldsymbol{k}_j), \qquad\qquad\qquad\qquad\qquad\qquad n = 0, \end{cases} \tag{8·3}$$

where $+i\varepsilon$ stands for the boundary value above the singularity of an analytic function defined for $l \in \Lambda(H_0)$:

$$T^{(n)}(\boldsymbol{k}_i, \boldsymbol{k}_j, l) = (A^+(\boldsymbol{k}_i)\Psi_0, H\{[H_0 - l]^{-1} H\}^{n-1} A^+(\boldsymbol{k}_j)\Psi_0), \quad n \geqq 1. \tag{8·4}$$

Although the right-hand side of (8·4) brings the finite result, the calculation is not performed in the framework of Hilbert space because $A^+(\boldsymbol{k}_i)\Psi_0$ does not belong to $\mathfrak{H}$. Therefore, in order to apply the foregoing arguments to the power series expansion of the S-matrix element, we must give to $T^{(n)}(\boldsymbol{k}_i, \boldsymbol{k}_j, l)$ the meaning as a limit of a sequence of matrix elements of an operator in

Hilbert space. For this purpose the "wave packet" formulation will be most convenient.

Let $f(\boldsymbol{k}_i, N)$ $(N=1, 2, 3, \cdots)$ be a sequence of the one boson states called "wave packets" of momentum $\boldsymbol{k}_i$, and let their particle representation in the momentum space be written as

$$\begin{aligned} &f(\boldsymbol{k}_i, N)=\{0, \psi(\boldsymbol{k}\,;\boldsymbol{k}_i, N), 0, \cdots\}\in\mathfrak{H}, \quad N=1, 2, \cdots, \\ &\|f(\boldsymbol{k}_i, N)\|^2=\int|\psi(\boldsymbol{k}\,;\boldsymbol{k}_i, N)|^2 d\boldsymbol{k}<\infty, \int\psi(\boldsymbol{k}\,;\boldsymbol{k}_i, N)\,d\boldsymbol{k}=1, \end{aligned} \tag{8·5}$$

where $\psi(\boldsymbol{k}\,;\boldsymbol{k}_i, N)$ is supposed to vanish except in the domain $|\boldsymbol{k}-\boldsymbol{k}_i|<\delta_{(N)}$ and

$$\lim_{N\to\infty}\delta_{(N)}=0.$$

It is easily seen that for every $n\geqq 2$ (for $\boldsymbol{k}_i\neq\boldsymbol{k}_j$)

$$T^{(n)}(\boldsymbol{k}_i, \boldsymbol{k}_j, l)=\lim_{N, M\to\infty}\mathcal{R}^{(n)}\{f(\boldsymbol{k}_i, N), f(\boldsymbol{k}_j, M)\,;l\}, \quad l\in\Lambda(H_0), \tag{8·6}$$

where

$$\begin{aligned} \mathcal{R}^{(n)}\{f(\boldsymbol{k}_i, N), f(\boldsymbol{k}_i, M)\,;l\}\equiv(Hf(\boldsymbol{k}_i, N), [H_0-l]^{-1} \\ \times\{H[H_0-l]^{-1}\}^{n-2}Hf(\boldsymbol{k}_j, M)). \end{aligned} \tag{8·7}$$

We now consider the following power series in λ:

$$\begin{aligned} &\sum_{n=0}^{\infty}(-\lambda)^{n+2}\mathcal{R}^{(n+2)}\{f(\boldsymbol{k}_i, N), f(\boldsymbol{k}_j, M)\,;l\} \\ &\quad=\lambda^2(Hf(\boldsymbol{k}_i, N), [H_\lambda-l]^{-1}Hf(\boldsymbol{k}_j, M))\equiv\mathcal{R}\{f(\boldsymbol{k}_i, N), f(\boldsymbol{k}_j, M)\,;l\}, \end{aligned} \tag{8·8}$$

where N and M are finite but arbitrary and $Hf(\boldsymbol{k}_i, N)$, $Hf(\boldsymbol{k}_j, M)\in\mathfrak{H}$. Although the series converges only for the values of λ and l which satisfy the condition (2·5), $\mathcal{R}\{f(\boldsymbol{k}_i, N), f(\boldsymbol{k}_j, M)\,;l\}$ is an analytic function of $l\in\Lambda(H_\lambda)$ since it is a matrix element of the resolvent of the self-adjoint operator H_λ. Hence, taking account of (8·1), (8·3), (8·6) and (8·8), we define the S-matrix element for the present in the framework of the wave packet formulation:

$$\begin{aligned} \mathcal{S}\{f(\boldsymbol{k}_i, N), f(\boldsymbol{k}_j, M)\}\equiv 2\pi i\iint\delta(w(\boldsymbol{k})-w(\boldsymbol{k}'))\psi(\boldsymbol{k}\,;\boldsymbol{k}_i, N)\psi^*(\boldsymbol{k}'\,;\boldsymbol{k}_j, M)\,d\boldsymbol{k}d\boldsymbol{k}' \\ \times\mathcal{R}\{f(\boldsymbol{k}_i, N), f(\boldsymbol{k}_j, M)\,;l=w(\boldsymbol{k}_i)+i\varepsilon\} \quad (\text{for } \boldsymbol{k}_i\neq\boldsymbol{k}_j). \end{aligned} \tag{8·9}$$

The singularity of $\mathcal{R}\{f(\boldsymbol{k}_i, N), f(\boldsymbol{k}_j, M)\,;l\}$ appears on $\Sigma(H_\lambda)$ of the l-plane, while that of $\mathcal{R}^{(n)}\{f(\boldsymbol{k}_i, N), f(\boldsymbol{k}_j, M)\,;l\}$ or $T^{(n)}(\boldsymbol{k}_i, \boldsymbol{k}_j, l)$ appears on $\Sigma(H_0)$. If the series

$$\sum_{n=0}^{\infty}(-\lambda)^{n+2}T^{(n+2)}(\boldsymbol{k}_i, \boldsymbol{k}_j, l)\equiv T_\lambda(\boldsymbol{k}_i, \boldsymbol{k}_j, l) \tag{8·10}$$

converges for some values of λ and l and defines an analytic function $T_\lambda(\boldsymbol{k}_i, \boldsymbol{k}_j, l)$

for $l \in \Delta(H_\lambda)$ like (8·8), then $S(\boldsymbol{k}_i, \boldsymbol{k}_j)$ can be defined in terms of the boundary value $T_\lambda(\boldsymbol{k}_i, \boldsymbol{k}_j, l=w(\boldsymbol{k}_i)+i\varepsilon)$. But it is not clear whether this is the case or not.

When the mass renormalization is taken into account, we must modify the above expressions a little. Here the mass renormalization should be understood as the shift of the lower limit of the spectrum, since our formulation is not relativistically invariant and the mass of the meson is not corrected. We write

$$H_\lambda = H_0' + H_I,$$

where

$$H_0' = H_0 + \delta m(\lambda) \quad \text{and} \quad H_I = \lambda H - \delta m(\lambda),$$

$\delta m(\lambda)$ being the discrete eigenvalue of the total Hamiltonian and corresponding to the level shift of the "no meson" state. Then the S-matrix element (8·9) should be replaced by

$$\mathcal{S}_r\{f(\boldsymbol{k}_i, N), f(\boldsymbol{k}_j, M)\} \equiv 2\pi i \iint \delta(w(\boldsymbol{k}) - w(\boldsymbol{k}'))\varphi'(\boldsymbol{k}; \boldsymbol{k}_i, N)\varphi'^*(\boldsymbol{k}'; \boldsymbol{k}_j, M)\, d\boldsymbol{k} d\boldsymbol{k}'$$

$$\times \mathcal{R}_r\{f(\boldsymbol{k}_i, N), f(\boldsymbol{k}_j, M)\,;\, l = w(\boldsymbol{k}_i) + \delta m + i\varepsilon\} \quad (\text{for } \boldsymbol{k}_i \neq \boldsymbol{k}_j), \qquad (8\cdot 9')$$

where

$$\mathcal{R}_r\{f(\boldsymbol{k}_i, N), f(\boldsymbol{k}_j, M)\,;\, l\} \equiv (H_I f(\boldsymbol{k}_i, N), [H_\lambda - l]^{-1} H_I f(\boldsymbol{k}_j, M)). \qquad (8\cdot 11)$$

This definition of the mass renormalization is also valid for the other cases at least if λ is so small that crossing of the spectra does not occur.

§ 9. Summary and discussion

The main aim of the present article is to examine the self-adjointness of the Hamiltonian operator and the convergence of the perturbation series in the quantum field theory. We have mainly investigated the fixed source theory and the boson-fermion interaction in the framework of Hilbert space, and obtained the results which are summarized as follows.

(A) The total Hamiltonian operator is self-adjoint, that is, the eigenvalue problem is completely solvable for every finite value of the coupling constant for the fixed source theory and for the boson-fermion interaction under the condition like (6·10) and (7·12), respectively, if the boson mass is not zero.

(B) The total Hamiltonian is the unique self-adjoint extension of a linear operator defined in some domain dense in the Hilbert space.

(C) The regular perturbation is possible for the cases described above and the power series expansion of the resolvent completely determines the total Hamiltonian by the method of analytic continuation.

(D) In these cases, the S-matrix element which is considered in the framework of the wave packet formulation is expressed in terms of the matrix

element of the resolvent and can be determined by the perturbation expansion. It is not clear whether the matrix element of the resolvent has a finite limit when the wave packet becomes the plane wave.

The condition (3·12), (6·10) or (7·7), that is, the square integrability of the source or of the interaction form factor is essential for the present method to be applicable, since otherwise the interaction Hamiltonian can not be defined as the symmetric operator in the way described in the text. It remains, however, to examine whether we can discuss the Hamiltonian without these conditions, for example, by defining the total Hamiltonian in a somewhat different way from ours, or by taking the limit $\int |V(\boldsymbol{k})|^2 d\boldsymbol{k} \to \infty$, etc., in the results of the present article.

For the boson-boson interaction we have not been able to prove the convergence of the perturbation series. The Hamiltonian h_λ is, however, symmetric, so it might be hoped that the problem would be completely analyzed by some other method though the self-adjoint extension of h_λ could not be unique.

Our method would probably be applicable to the interaction among four fermion fields and others under some condition on the interaction form factor.

The mass renormalization has not been studied in detail in the present article but is important not only because of its physical meaning but also for the improvement of our results. In fact, the example of the resolvent in § 5 suggests that the convergence property of the power series will be improved by taking account of the mass renormalization. This becomes still more important to fix the lower bound of the spectrum if we wish to take the limit $\int |V(\boldsymbol{k})|^2 d\boldsymbol{k} \to \infty$.

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Note added in proof :

1.—After completion of the manuscript, the author received a paper in which a mathematically rigorous investigation is performed in the theory of a neutral scalar boson field in interaction with a given source : J. M. Cook, J. Math. Phys. **2** (1961), 33. The method to prove the self-adjointness of the total Hamiltonian would be essentially similar to ours and be applicable to the examples given in the present article.

2.—The convergence of the series on the left-hand side of Eq. (8·10) was not proved in the text. It can be shown, however, that the series converges to an analytic function $T_\lambda(\boldsymbol{k}_i, \boldsymbol{k}_f, l)$ of λ and l for the values which satisfy the condition (2·5) and $T_\lambda(\boldsymbol{k}_i, \boldsymbol{k}_f, l)$ is analytically continuable to all $l \in \Lambda(H_\lambda)$ for every real λ. The usual S-matrix element (8·1) is expressible by the formal perturbation expansion in λ of the boundary value $T_\lambda(\boldsymbol{k}_i, \boldsymbol{k}_f, l = w(\boldsymbol{k}_i) + i\epsilon)$. Thus one of the conclusions of the text (§9–(D)) can be improved.

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Magnetohydrostatic Equilibrium for Fluid Enclosed in an Axisymmetric Toroidal Surface

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The solution of the magnetohydrostatic equations for the ideally conducting fluid contained in an axisymmetric toroidal surface is investigated. The basic equations are reduced to a partial differential equation of the second order, of two variables and of elliptic type. This equation contains two arbitrary functions which give nonlinear terms. The Dirichlet problem for this equation has a solution according to Nagumo's theorem and the solution gives the equilibrium. The vacuum field sustaining the equilibrium is examined for the toroidal fluid with circular cross section.

§ 1. Introduction

In this note[1] we start from the equation of magnetohydrostatics for an ideally conducting fluid:

$$\begin{aligned} &\operatorname{grad} p = -\boldsymbol{B}\times\boldsymbol{j}, \\ &\boldsymbol{j} = (1/4\pi)\operatorname{rot}\boldsymbol{B}, \\ &\operatorname{div}\boldsymbol{B} = 0, \end{aligned} \tag{1}$$

where $\boldsymbol{B}$, $\boldsymbol{j}$ and p denote the magnetic field, the electric current density and the pressure of the fluid respectively. The pressure is assumed to be a scalar.

For the case, where the current flows exclusively at the surface of the fluid and the pressure is constant in the fluid, the condition for the existence of the solution of (1) in the outside of the surface was examined by Kippenhahn.[2] Especially for the axisymmetric system it was shown by Jörgens[3] that there exists such a solution that the magnetic field has no singularity in the outside of the fluid except at the axis of symmetry and at infinity if the surface is a toroid (by which we mean a topological torus) with circular cross section but the solution has the singularity if the cross section is, for instance, an ellipse.

In this note the axisymmetric toroidal fluid will be considered and the solution of (1) without the singularity which gives the sheet current at the surface will be investigated.[4] First, in § 2, Eqs. (1) are reduced to one equation, (4), among the pressure, the azimuthal component of the magnetic field and the flux (F) of the meridian component of the field. This equation which contains two arbitrary functions of F will be treated as a partial differential

equation of the function F.[5] Assuming that the surface of the fluid constitutes a magnetic surface, we set up a Dirichlet problem for the domain occupied by the fluid and verify the existence of the solution by applying Nagumo's theorem[6),7)] (§ 3). Since Eq. (4) contains arbitrary functions of F, it is not sufficient to give the boundary condition $F \equiv$ constant in order to get the unique solution of (1). This fact is consistent with the discussions given by Kruskal and Kulsrud[8)] and Grad and Rubin[9)] (see also reference 5)). In § 4, some examples of Eq. (4) which have the solutions of physical significance will be presented.

Next the vacuum field sustaining the toroidal fluid will be studied. Eqs. (1) become the following system of the equations:

$$\begin{aligned} &\text{rot } \boldsymbol{B}=0, \\ &\text{div } \boldsymbol{B}=0, \end{aligned} \tag{1'}$$

or Eq. (4) with $k(s, F)=0$. Thus we have the partial differential equation of the elliptic type with given boundary values F ($\equiv$constant) and with the derivative of F at the boundary in the direction normal to the boundary, which is determined by the solution in the inside of the boundary. Since the solution of the problem would not always exist except in the neighbourhood of the boundary, the domain of the existence of the solution should be determined, for example, by the method described in § 5. In § 6, a simple example of the vacuum field which has no singularity except at the axis of the symmetry and at infinity will be presented for fluid in a toroid with circular cross section.

§ 2. Magnetohydrostatic equation for the axisymmetric system

Since we treat in this note the axisymmetric system, all the physical quantities should depend only on the distance s from the z-axis and the distance z from the equatorial plane but not on the azimuthal angle φ.

The general axisymmetric magnetic field $\boldsymbol{B}$ and electric current $\boldsymbol{j}$ can be written in the following forms:[10)]

$$\begin{aligned} \boldsymbol{B}&=(B_s, B_\varphi, B_z)=(\partial F(s, z)/\partial z, T(s, z), -\partial F(s, z)/\partial s)/s, \\ \boldsymbol{j}&=(j_s, j_\varphi, j_z)=(-\partial T(s, z)/\partial z, GF(s, z), \partial T(s, z)/\partial s)/s, \end{aligned} \tag{2}$$

where B_s, B_φ and B_z (j_s, j_φ and j_z) denote the components of the vector $\boldsymbol{B}$($\boldsymbol{j}$) in the directions of increasing s, φ and z respectively and

$$G \equiv (\partial^2/\partial s^2)+(\partial^2/\partial z^2)-(1/s)(\partial/\partial s)=\Delta-(1/s)(\partial/\partial s). \tag{3}$$

The function $F(s, z)$ has the meaning that the line of the force in the meridian plane (s, z-plane) is described by the equation $F(s, z) \equiv$ constant. The function $T(s, z)$ has the analogous meaning for the line of the electric current.

It is known[10] that the system of Eqs. (1) for the axisymmetric system reduces to the following one equation among $p(F)$, $g(F)$ and $F(s, z)$ except for the trivial case in which $F(s, z) \equiv$ constant:

$$\left.\begin{aligned} &\Delta F(s, z) = h(s, F, \partial F/\partial s), \\ &\text{where} \\ &h(s, F, \partial F/\partial s) \equiv k(s, F) + (1/s)(\partial F/\partial s), \\ &k(s, F) \equiv -(1/2)g'(F) - 4\pi s^2 p'(F), \end{aligned}\right\} \tag{4}$$

$$g(F) \equiv \{T(F)\}^2 \geqq 0, \qquad p(F) \geqq 0. \tag{5}$$

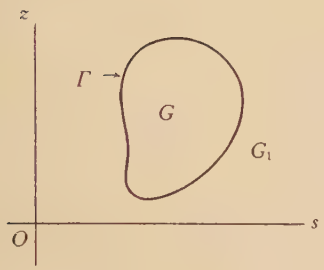

Fig. 1. The meridian cross section of the axisymmetric toroidal surface.

Here $g'(F)$ and $p'(F)$ are derivatives of arbitrary functions $g(F)$ and $p(F)$ of F respectively. These functional relations between F, p and T imply that the pressure is constant along the magnetic line of force, that is, on the magnetic surface and the electric current flows at that surface.

Now we consider an axisymmetric toroidal fluid with its meridian cross section Γ which does not touch on the z-axis, and determine the function $F(s, z)$ in the inside G and the outside G_1 of Γ for given functions $p(F)$ and $g(F)$. (See Fig. 1.)

Thus, we must solve the differential equation (4) of $F(s, z)$ under the boundary condition

$$F(s, z) \equiv \text{const.} = 0 \qquad \text{on } \Gamma. \tag{6}$$

Further $\text{grad} F \equiv (\partial F/\partial s, \partial F/\partial z)$ is a continuous function of s and z, provided that the sheet current does not flow at Γ. Then,

$$p(F) = p(0) = 0 \qquad \text{on } \Gamma.$$

§ 3. Existence of the solution of the Dirichlet problem

First we describe a theorem concerning the existence and the uniqueness of the solution of the semilinear elliptic differential equation of the second order.[6),7)]

Theorem. We consider the differential equation (4) in the domain G with its boundary Γ. Let G be a bounded domain satisfying Poincaré's condition* and put $\max_{\Gamma} s = s_M$, $\min_{\Gamma} s = s_m > 0$. Let $h(s, u, p)$ be a continuous function of

* We say that a bounded domain G satisfies Poincaré's condition, if every point of Γ can be touched from the outside by a vertex of a triangle.

u and have a finite first derivative with respect to u for finite u and $s_m \leqq s \leqq s_M$. Suppose that there exist functions $\bar{\omega}(s, z)$ and $\underline{\omega}(s, z)$ such that

$$\Delta\bar{\omega} < h(s, \bar{\omega}, \partial\bar{\omega}/\partial s), \quad \Delta\underline{\omega} > h(s, \underline{\omega}, \partial\underline{\omega}/\partial s),$$

$$|\bar{\omega}| < L, \quad |\underline{\omega}| < L \text{ and } \underline{\omega} < \bar{\omega} \quad \text{in } G,$$

and

$$|h(s, u, p)| < M + N|p|^{\gamma} \qquad (\gamma < 2),$$

for all the sets $\{s, z, u\}$ which satisfy the condition

$$\underline{\omega}(s, z) \leqq u \leqq \bar{\omega}(s, z) \qquad \text{for } \{s, z\} \in G,$$

L, M and N being some positive constants. Let $\psi(s, z)$ be a continuous function defined in $G+\Gamma$ such that

$$\underline{\omega}(s, z) \leqq \psi(s, z) \leqq \bar{\omega}(s, z) \quad \text{in } G,$$

then there exists a solution $F(s, z)$ of (4) in G, which assumes the boundary values $\psi(s, z)$ on Γ. There exists just one solution if the function $h(s, u, p)$ is an increasing function of u for $s_m \leqq s \leqq s_M$.

For the present problem we assume that $k(s, F)$ is bounded below for $F > 0$ and bounded above for $F < 0$, i.e.

$$k(s, F) > \bar{A}, \qquad \text{for } F > 0, \ s_m \leqq s \leqq s_M,$$

$$k(s, F) < \underline{A}, \qquad \text{for } F < 0, \ s_m \leqq s \leqq s_M,$$

and put

$$\bar{\omega}(s, z) = |\bar{A}| s_M s, \quad \underline{\omega}(s, z) = -|\underline{A}| s_M s,$$

$$\psi(s, z) = 0 \quad \text{on } \Gamma,$$

then, we can easily see that Eq. (4) has a solution in G with boundary condition (6). Further, if $k(s, F)$ is an increasing function of F for $s_m \leqq s \leqq s_M$, the solution is unique.

§ 4. Examples of the equation

Although in the preceding section the solution of (4) and (6) in G was shown to exist, it could have its physical meaning only if the condition (5) is satisfied. We give in this section some examples of the functions $p(u)$ and $g(u)$ such that $p(F) \geqq 0$ and $g(F) \geqq 0$ hold in $G+\Gamma$ for the solution $F(s, z)$ of (4) and (6).

1) $k(s, u) < 0$ (i.e. $(1/2)g'(u) + 4\pi s^2 p'(u) < 0$) for $u > u_1$, $s_m \leqq s \leqq s_M$; $p'(u) \leqq 0$ for $u \leqq 0$; $g(0) > g_1$; where u_1 is an arbitrary negative number and g_1 is some sufficiently large positive number.

If $k(s, u) < 0$ for $u > u_1$, the solution $F(s, z)$ of (4) and (6) has no maximum at the point of G where $F(s, z) > u_1$, since $\Delta F = k(s, F) > 0$ at the point (s, z)

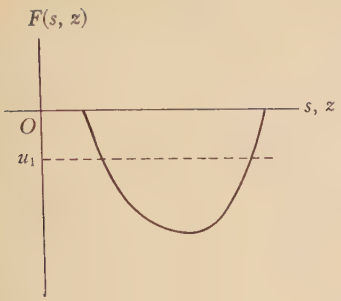

Fig. 2.

where $\partial F/\partial s=\partial F/\partial z=0$. As $F(s, z)=0$ on Γ, so $F(s, z)<0$ in G (cf. Fig.2). Hence a sufficient condition for (5) to hold is $p'(u)\leqq 0$ for $u\leqq 0$ and $g(0)>g_1$, where g_1 is some positive number such that $g(F)\geqq 0$ holds for the solution $F(s, z)$ of (4) and (6). In a similar way the following example can be justified.

2) $k(s, u)<0$ for $u<u_2$, $s_m\leqq s\leqq s_M$; $p'(u)\geqq 0$ for $u\geqq 0$; $g(0)>g_2$; where u_2 is an arbitrary positive number and g_2 is some sufficiently large positive number.

§ 5. Vacuum field

We examine in this section the differential equation (4) in the outside of Γ with the boundary conditions

$$F(s, z)\equiv 0 \qquad \text{on } \Gamma, \tag{6}$$

$$\partial F(s, z)/\partial n=\psi(s, z) \qquad \text{on } \Gamma, \tag{7}$$

where $\partial F/\partial n$ denotes the derivative of $F(s, z)$ in the direction normal to the boundary Γ and $\psi(s, z)$ is the derivative normal to Γ of the solution of (4) and (6) in § 3. It should be noticed that we must determine the region G_1 where the solution of (4), (6) and (7) exists, since the solution does not always exist in the outside of Γ.

Putting $p\equiv q\equiv 0$ in the outside of Γ and $u(s, z)=F(s, z)/\sqrt{s}$, we have from (4)

$$\{\Delta-(3/4s^2)\}u(s, z)=k(s, \sqrt{s}\,u)/\sqrt{s},$$

where

$$k(s, q)=\begin{cases} -(1/2)g'(q)-4\pi s^2 p'(q) & \text{in } G+\Gamma, \\ 0 & \text{in } G_1. \end{cases} \tag{8}$$

An elementary solution with singularity at (σ, ζ) of the differential equation

$$\Delta u-(3/4s^2)u=0 \tag{9}$$

is given as follows:[3]

$$u(s, z)=H(s, z\,;\,\sigma, \zeta)$$

$$=(\sqrt{s\sigma}/2\pi)\int_0^{\pi}\{\cos\theta/\sqrt{s^2+\sigma^2+(z-\zeta)^2-2s\sigma\cos\theta}\}\,d\theta. \tag{10}$$

We introduce the dipole coordinates (ξ, η) given by

$$s=a\sinh\eta/(\cosh\eta-\cos\xi),\quad z-b=a\sin\xi/(\cosh\eta-\cos\xi),$$
$$J(s,z)=\partial(s,z)/\partial(\xi,\eta)=a^2/(\cosh\eta-\cos\xi)^2=s^2/\sinh^2\eta. \tag{11}$$

The region $(\xi,\eta\,;\,0\leqq\xi<2\pi,\ 0\leqq\eta\leqq\infty)$ corresponds to the half-plane $(s,z\,;\,s\geqq 0)$ and the line $\eta=$ const. to the circle $C(\eta\,;a,b)$ including the point (a,b) which corresponds to $(\eta=+\infty)$. The line $(s=0)$ is transformed to the line $(\eta=0)$.

Then

$$H\{s(\xi,\eta),\ z(\xi,\eta)\,;\,\sigma(\xi',\eta'),\ \zeta(\xi',\eta')\}$$
$$=-\frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\frac{e^{in(\xi-\xi')}}{n^2-(1/4)}\begin{cases} q_n(\eta)p_n(\eta') & \text{for}\quad \eta\geqq\eta',\\ p_n(\eta)q_n(\eta') & \text{for}\quad \eta\leqq\eta',\end{cases} \tag{12}$$

where

$$p_n(\eta)=p_{-n}(\eta)=\sqrt{\sinh\eta}\,P^1_{n-1/2}(\cosh\eta),$$
$$q_n(\eta)=q_{-n}(\eta)=\sqrt{\sinh\eta}\,Q^1_{n-1/2}(\cosh\eta), \tag{13}$$

$P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$ being the associated Legendre functions of the first and the second kinds respectively. The asymptotic forms of $p_n(\eta)$ and $q_n(\eta)$ for $n\to\infty$ are as follows $(\eta\neq 0)$:

$$p_n(\eta)=\sqrt{|n|/2\pi}\ e^{|n|\eta}\{1+O(1/|n|)\},$$
$$q_n(\eta)=-\sqrt{\pi|n|/2}\ e^{-|n|\eta}\{1+O(1/|n|)\}. \tag{14}$$

If a real function $u(s,z)$ is a solution of (9) in the neighbourhood D of (a,b), $u(s,z)$ is expressible in a series at least in a circle contained in D, i.e.

$$u(s,z)=\sum_{n=-\infty}^{+\infty}a_n e^{in\xi}q_n(\eta),\qquad a_{-n}=a_n^*. \tag{15}$$

The right-hand side of (15) converges for $\eta>\eta_0$ and is a solution of (9), but it diverges for $\eta<\eta_0$, where

$$\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}=\exp(\eta_0). \tag{16}$$

Now it will be easily seen that the solution $u(s,z)$ of (8) satisfies the following equation in a circle $C(\eta_1\,;a,b)$ contained in Γ:

$$u(s,z)=-\iint_{\Gamma}H(s,z\,;\sigma,\zeta)k\{\sigma,\sqrt{\sigma}u(\sigma,\zeta)\}/\sqrt{\sigma}\,d\sigma d\zeta$$
$$+\sum_{n=-\infty}^{+\infty}a_n e^{in\xi}q_n(\eta), \tag{17}$$

where

$$a_n q_n(\eta_1) = (1/2\pi)\int_0^{2\pi} d\xi\ e^{-in\xi}\ u(s,\ z)$$

$$+(1/2\pi)\int_0^{2\pi} d\xi\ e^{-in\xi}\iint_G H(s, z\,;\, \sigma, \zeta)k\{\sigma, \sqrt{\sigma}u(\sigma, \zeta)\}/\sqrt{\sigma}d\sigma d\zeta. \qquad (18)$$

We determine the largest circle $C(\eta_2\,;\, a, b)$ $(\eta_2 \leqq \eta_1)$ in which the series in (17) with $\{a_n\}$ given by (18) is convergent, and construct the sum G' of $C(\eta_2\,;\, a, b)$ for all points $P(a, b) \in G$. Evidently there exists the solution of (4), (6) and (8) in $G' \supseteqq G$. If $G' \supset G$. we construct again the sum G'' of $C(\eta_2\,;\, a, b)$ for all points $P(a, b) \in G'$. Continuing this procedure, we may determine the region $G+\Gamma+G_1$, where the solution of (4), (6) and (8) exists.

§ 6. Example of the vacuum field

Since the right-hand side of (18) depends on the solution $u(s, z)$ in G, it is difficult except in some special cases to determine explicitly the domain of the existence of the solution (17), i.e. to estimate the limit given by (16) and (18).

We consider in this section a simple example such that Γ is a circle $C(\eta_0)$ with its center at $(s=a\coth\eta_0,\ z=0)$ and

$$p(q) = -Aq,\quad \{T(q)\}^2 = g(q) = T_0^2 - Bq,$$

i.e.
$$k(s, q) = -(1/2)g'(q) - 4\pi s^2 p'(q) = A + Bs^2 \quad \text{in } G,$$

A, B and T_0 being some real constants.

Then, from (6), (12) and (18), we get

$$\left.\begin{aligned} a_n q_n(\eta_0) &= -(1/2\pi)p_n(\eta_0)\{n^2-(1/4)\}^{-1}\int_{\eta_0}^{\infty} d\eta\ \gamma_n(\eta), \\ \text{where}\quad \gamma_n(\eta) &= \int_0^{2\pi} d\xi\{(A+B\sigma^2)/\sqrt{\sigma}\}e^{-in\xi}J(\xi, \eta)q_n(\eta), \\ \sigma &= a\sinh\eta/(\cosh\eta-\cos\xi). \end{aligned}\right\} \qquad (19)$$

To estimate the limit, $\overline{\lim_{n\to\infty}}\ \sqrt[n]{|\gamma_n(\eta)|}$ for $\eta \geqq \eta_0$, we put

$$w = e^{i\xi}.$$

Then we have, after elementary calculations,

$$\gamma_n(\eta) = \frac{a^{3/2}q_n(\eta)}{i\sqrt{\sinh\eta}}\int_{K(1)}\frac{dw}{w^{n+1}}\left[\frac{A+Ba^2\sinh\eta\{\cosh\eta-(1/2)(w+1/w)\}^{-2}}{\{\cosh\eta-(1/2)(w+1/w)\}^{3/2}}\right]. \qquad (20)$$

Here $K(r)$ denotes the circle of radius r with its center at the zero of the w-plane. Since the integrand of (20) has no singularity in the domain bounded by two circles $K(r_1)$ and $K(r_2)$ $(e^{-\eta} < r_1 < r_2 < e^{\eta})$, we can deform the path $K(1)$ to $K(e^{\eta-\varepsilon})$ (ε being an arbitrarily small positive constant) without changing the value of the integral and get

$$|\gamma_n(\eta)| \leqq |q_n(\eta)| e^{-n(\eta-\varepsilon)} M(\varepsilon) \quad \text{for } \eta \geqq \eta_0,$$

M being some positive constant independent of n and η. Hence, from (13), (14), (15) and (19), we have (for $n>0$)

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} \leqq e^{\varepsilon}.$$

As $\varepsilon>0$ may be arbitrarily small, we see that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} \leqq 1,$$

and the unique solution of (4), (6) and (8) exists in the finite half plane $(s>0)$.

In conclusion the author would like to express his sincere thanks to Prof. Y. Tanikawa and Dr. N. Mugibayashi for their kind interest and criticisms and Mr. Y. Hirasawa for his mathematical advices.

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On a Convergent Model of Quantum Field Theory with Indefinite Metric

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A convergent model of quantum field theory with indefinite metric is proposed to the case of an indirect interaction between a physical neutral scalar field and a physical spinor field, by introducing four kinds of unphysical spinor fields which play the roles of the intermediate states connecting these two of physical fields. Two of these unphysical fields are set to have negative anticommutators and consequently this fact makes the metric of our Hilbert space indefinite; nevertheless it is shown that the unitarity of the actual S-matrix holds strictly. Final results are such that every vertex in the usual local theory is exactly replaced with some kind of extended vertex in this model which prepares the sufficient convergency for all results. Although the extended vertex in this model becomes singular at the two momentum values depending on the masses of unphysical fields and the coupling constant, it is also shown that there remains a considerable wide degree of freedom to control the stable mass levels of the physical particles as suitably as possible. The definite results about this problem are left for the future.

§ 1. Introduction

In order to avoid the divergence difficulties inherent in the ordinary quantum field theory, it has been suggested by many authors to introduce the Hilbert space with indefinite metric. Especially, after the investigations of Källén[1] and Lehmann,[2] such a device has become still more plausible to get the sufficient convergency for the whole contents of the theory.

As is well known, however, other difficulties arise in the theory with indefinite metric: Since, in this problem, in order to obtain the negative propagators in the calculations of the S-matrix we must introduce some kinds of unphysical fields whose commutators or anticommutators have the opposite signs to those of usual ones, we are compelled to deal with the negative probabilities which are physically unacceptable.

Moreover, a serious difficulty still remains, even if we suitably exclude the states with negative norms in the initial and final states of the scattering and formulate the theory of observation only to the processes for physical particles; this is the breakdown of the unitarity condition for an actual S-matrix which is defined so as to describe physical processes only.

From such troublesome reasons, although many attempts to solve this problem have been proposed, no satisfactory theory has so far been found.

Recently, in this respect, Bogoliubov et al.[3] have suggested an interesting

formulation to keep the unitarity condition for the actual S-matrix, but it seems to be very artificial and far from the future true theory, since it only makes the structure of the actual S-matrix more complicated, making no reference to the essential relations between the unitarity and the convergency, so that the whole theory necessarily becomes to have curious non-local charactors, as if it were in a kind of feed-back circuit with a vacuum tube, though it is simply expressed as long as the scattering K-matrix is concerned.

On the other hand, the well-known regularization method of Feynman[4] and Pauli-Villars[5] is one of the skillful examples in the theory with indefinite metric, but it is only an expedient method to remove the divergent results from the current unsatisfactory theories and the unitarity condition of the actual S-matrix is evidently broken down at the stage of the finite masses of regularizing fields.

Moreover, although there have been several investigations[6),7)] about indefinite metric with the aim of the application to the Lee-model, they are not promising for the formulation of relativistic quantum field theory.

In view of these circumstances, it may be worthwhile to propose a kind of convergent local field theory with indefinite metric in which the unitarity condition of the actual S-matrix holds strictly. The aim of the present paper is to give a model of the convergent and unitary S-matrix theory to the case of the interaction between a physical neutral scalar field $A(x)$ and a physical spinor field $\psi(x)$, by introducing four kinds of unphysical spinor fields which play the roles of the intermediate states (or intermediate chains) connecting $A(x)$ and $\psi(x)$ without their direct interactions.

In introducing these four kinds of unphysical fields, we postulate about them in the interaction representation that they make pairs two by two in such ways that two fields in each pair have an equal mass (the mass of one pair is different from that of the other pair) and their own anticommutation relations are equal in absolute value but opposite to each other in sign. Then one of unphysical fields in each pair makes the metric of our Hilbert space indefinite and gives rise to the negative propagators in the calculation of the S-matrix. On the other hand, two pairs with different masses are essentially important in the present model in order to keep the actual S-matrix unitary and guarantee all results to be convergent. As far as we want to obtain the convergent results only for the transition amplitudes, it will be sufficient only to introduce a single pair of these unphysical fields; however, to keep the actual S-matrix to be unitary and furthermore to keep all results convergent, not only for the transition amplitudes but also for the expectation values of the suitably defined physical observables, which will probably contain the bi-linear terms of $\psi(x)$ and $\bar{\psi}(x)$, it will become clear in the following paragraph that such two pairs of unphysical fields are inevitable and at least four unphysical fields are necessary.

At first sight, six of physical and unphysical fields may seem to make the contents of the theory complicated further; however, as far as the graphical

contents for physical processes in the perturbation expansion is concerned, the essential difference between the ordinary theory and the present model is no more than that every vertex joined directly by the physical internal lines in the former is exactly replaced with the extended vertex in the latter in which the physical internal lines never join to one point but join to the end points of the intermediate chains composed of a number of unphysical propagators. Of course, it goes without saying that the above-mentioned extended vertex prepares the sufficient inverse powers of the momentum to the integrand in any integration for the convergency of the result and divergent result occurs nowhere.

Another feature of the present model is that the causality condition is not sacrificed on account of the local property of the starting Lagrangian, though the resultant extended vertex makes the contents of the theory equivalent to the case of a certain non-local interaction.

On the other hand, however, there exist some troublesome problems in this model. As mentioned above, although the extended vertex guarantees all results to be convergent, it becomes singular at the two momentum values depending on the masses of the unphysical fields and the coupling constant. This seems to give rise to the troublesome problems with respect to the stable mass levels of the physical particle described by $\psi(x)$; however, as will be shown in the last section, there remains a considerable wide degree of freedom to control these mass levels as suitably as possible; nevertheless it is difficult to give the definite and satisfactory results at once at this stage of the present investigation, since the simple solutions of the equation that determines such mass levels in question are not found. This problem is left for the future investigation.

At any rate, the existence itself of a convergent and unitary S-matrix will be of use for a step to the future theory, though the present model may be nothing more than a transient formulation by a trial approach.

§ 2. Lagrangian formalism

As mentioned tediously in the preceding section, to discuss the indirect interaction between a physical neutral scalar field $A(x)$ and a physical spinor field $\psi(x)$, the mechanical masses of which are μ and κ respectively, we introduce four kinds of unphysical spinor fields $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_1'(x)$, and $\varphi_2'(x)$. These four unphysical fields are set to make pairs two by two in such ways that $\varphi_1(x)$ and $\varphi_1'(x)$ ($\varphi_2(x)$ and $\varphi_2'(x)$) satisfy the same Dirac equation with equal mass M (m) and have opposite signs to each other about their own anticommutation relations in the interaction representation.

In addition to this requirement, we further assume that any two of distinct spinor fields anticommute with each other in the interaction representation. For convenience, the signs of anticommutators for $\varphi_1'(x)$ and $\varphi_2'(x)$ are set to be opposite to those of $\varphi_1(x)$, $\varphi_2(x)$and $\psi(x)$.

Moreover, throughout this paper, we do not use the metric operator η explicitly, but it is as a matter of course that the adjoint conjugation $\mathcal{Q}^\dagger$ of an arbitrary operator $\mathcal{Q}$ and the adjoint vector $\Phi^\dagger$ of an arbitrary state vector Φ always correspond to the hermitian conjugation $\mathcal{Q}^*$ and Φ^* in the usual case respectively.* Thus all norms of state vectors and all matrix elements and expectation values of any operator are defined with the implicit η. Real scalar field $A(x)$ is self-adjoint and physical observables will generally be also self-adjoint, instead of hermitian, in order that their expectation values may be real.

A starting point of our discussion is the following self-adjoint Lagrangian in the Heisenberg representation, which contains our requirements just mentioned above:

$$\begin{aligned}
&\mathcal{L}(x)=\mathcal{L}_0(x)+\mathcal{L}'(x),\ \mathcal{L}'(x)=\mathcal{L}_1(x)+\mathcal{L}_2(x)+\mathcal{L}_3(x),\\
&\mathcal{L}_0(x)=-\bar{\psi}(\gamma\partial+\kappa)\psi-\bar{\varphi}_1(\gamma\partial+M)\varphi_1-\bar{\varphi}_2(\gamma\partial+m)\varphi_2\\
&\qquad+\bar{\varphi}_1'(\gamma\partial+M)\varphi_1'+\bar{\varphi}_2'(\gamma\partial+m)\varphi_2'+(1/2)A(\Box-\mu^2)A,\\
&\mathcal{L}_1(x)=f(\bar{\varphi}_1+\bar{\varphi}_1')(\varphi_1+\varphi_1')A,\\
&\mathcal{L}_2(x)=g\bar{\psi}(\varphi_2-\varphi_2')+\textit{adjoint conj.},\\
&\mathcal{L}_3(x)=g'(\bar{\varphi}_1\varphi_2-\bar{\varphi}_1'\varphi_2')+\textit{adjoint conj.},
\end{aligned}\tag{1}$$

where $\bar{\psi}\equiv\psi^\dagger\gamma_4$, $\bar{\varphi}_1\equiv\varphi_1^\dagger\gamma_4$, and so on. The coupling constants f, g and g' are taken to be real.

In $\mathcal{L}_0$, the opposite signs for $\varphi_1'(x)$ and $\varphi_2'(x)$ terms to the other ones make the eigenvalues of free energies for these fields positive and definite, since the number operators of these fields are defined by $-a^\dagger a$ in order always to give the positive values.** It must be understood that $\mathcal{L}_0$ and $\mathcal{L}'$ in (1) could be antisymmetrized if necessary. For the sake of convenience, the mass renormalization of physical fields is omitted without loss of generality.

From the Lagrangian (1), the equations of motion for all fields are derived as

$$\begin{aligned}
&(\gamma\partial+\kappa)\psi=g(\varphi_2-\varphi_2'),\\
&(\gamma\partial+M)\varphi_1=g'\varphi_2+f(\varphi_1+\varphi_1')A,\\
&(\gamma\partial+m)\varphi_2=g'\varphi_1+g\psi,
\end{aligned}\tag{2}$$

* The adjoint conjugations $\mathcal{Q}^\dagger$ and $\Phi^\dagger$ are defined such that $\mathcal{Q}^\dagger\equiv\eta\mathcal{Q}^*\eta$ and $\Phi^\dagger\equiv\Phi^*\eta$ respectively. If it holds that $\mathcal{Q}=\mathcal{Q}^\dagger$, $\mathcal{Q}$ is called self-adjoint.

** The letter a represents an annihilation operator of φ_1' or φ_2' particle. Of course, the vacuum is defined by the state belonging to the minimum eigenvalue of $-a^\dagger a$. As to the detail of the η method, see, for example, references 8)—12).

$$(\gamma\partial+M)\varphi_1'=g'\varphi_2'-f(\varphi_1+\varphi_1')A,$$
$$(\gamma\partial+m)\varphi_2'=g'\varphi_1'+g\psi.$$

The commutation relation for $A(x)$ and the anticommutation relations for all spinor fields, in their conventional canonical forms, are also derived from the Lagrangian (1) :

$$\begin{aligned}&[A(\boldsymbol{r}, t),\ (\partial/\partial t)A(\boldsymbol{r}', t)]=i\delta(\boldsymbol{r}-\boldsymbol{r}'),\\&\{\psi_\alpha(\boldsymbol{r}, t),\ (\bar{\psi}(\boldsymbol{r}', t)\gamma_4)_\beta\}=\{\varphi_{i,\alpha}(\boldsymbol{r}, t),\ (\bar{\varphi}_i(\boldsymbol{r}', t)\gamma_4)_\beta\}\\&=-\{\varphi_{i,\alpha}'(\boldsymbol{r}, t),(\bar{\varphi}_i'(\boldsymbol{r}', t)\gamma_4)_\beta\}=\delta_{\alpha\beta}\delta(\boldsymbol{r}-\boldsymbol{r}'),\quad i=1, 2.\end{aligned}\tag{3}$$

In the above expression, we have omitted a number of vanishing and trivial commutators (and anticommutators). The negative anticommutators for φ_1' and φ_2' just satisfy our requirements and the reason why we get such results is again due to the positive signs for φ_1' and φ_2' terms in $\mathcal{L}_0$.

Moreover, in addition to Eqs. (3) our assumption is that

> *any two of distinct spinor fields anticommute with each other at the same instant of time.* (4)

The above statement are not derived from the Lagrangian only but it is the necessary condition for the consistency of the present theory.*

Now, the next step to go on our discussion is to transform the total system from the Heisenberg representation to the interaction representation, through a pseudo-unitary operator $S(t)$. Any state vector $\Phi(t)$ and any operator $\Omega(x)$ of the interaction representation are obtained by the equations

$$\Phi(t)=S(t)\boldsymbol{\Phi},\ \Omega(x)=S(t)\boldsymbol{\Omega}(x)S^\dagger(t),\tag{5}$$

where the bold face letters denote the state vector and the operator of the Heisenberg representation respectively. In the equation of motion for $S(t)$

$$i\frac{\partial}{\partial t}S(t)=H(t)S(t),\quad H(t)\equiv\int_{x_0=t}\mathcal{H}(x)d^3\boldsymbol{x},\tag{6}$$

it will be shown that the desirable transformation is obtained with $\mathcal{H}(x)$ chosen as

$$\begin{aligned}&\mathcal{H}(x)=\mathcal{H}_1(x)+\mathcal{H}_2(x)+\mathcal{H}_3(x),\\&\mathcal{H}_1(x)=-f(\bar{\varphi}_1+\bar{\varphi}_1')(\varphi_1+\varphi_1')A,\\&\mathcal{H}_2(x)=-g\bar{\psi}(\varphi_2-\varphi_2')+adjoint\ conj.,\\&\mathcal{H}_3(x)=-g'(\bar{\varphi}_1\varphi_2-\bar{\varphi}_1'\varphi_2')+adjoint\ conj.\end{aligned}\tag{7}$$

Of course, $\mathcal{H}(x)$ is self-adjoint and this fact guarantees our $S(t)$ to be

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* The statement (4) is necessary in order that the theory may be invariant under the space-time reflection, too. In this respect, for example, see references 13)—15).

pseudo-unitary, that is,

$$S^{\dagger}(t)=S^{-1}(t). \tag{8}$$

By the usual procedure using (2)–(8), we see at once that all fields of the interaction representation satisfy their free-field equations of motion and free-field commutation or anticommutation relations with their respective masses given in the free Lagrangian $\mathcal{L}_0$ in (1), that is,

$$\begin{aligned}&(\gamma\partial+\kappa)\psi=0, \quad (\Box-\mu^2)A=0,\\ &(\gamma\partial+M)\varphi_1=(\gamma\partial+M)\varphi_1'=0,\\ &(\gamma\partial+m)\varphi_2=(\gamma\partial+m)\varphi_2'=0,\end{aligned} \tag{9}$$

and

$$\begin{aligned}&\{\psi(x),\ \bar{\psi}(x')\}=\frac{1}{i}S(x-x';\kappa),\\ &\{\varphi_1(x),\bar{\varphi}_1(x')\}=-\{\varphi_1'(x),\bar{\varphi}_1'(x')\}=\frac{1}{i}S(x-x';M),\\ &\{\varphi_2(x),\bar{\varphi}_2(x')\}=-\{\varphi_2'(x),\bar{\varphi}_2'(x')\}=\frac{1}{i}S(x-x';m),\\ &[A(x),\ A(x')]=i\Delta(x-x';\mu).\end{aligned} \tag{10}$$

The S-matrix including all of physical and unphysical particles are also given as usual by

$$S\equiv S(\infty)=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int T[\mathcal{H}(x_1)\cdots\mathcal{H}(x_n)]dx_1\cdots dx_n \tag{11}$$

which is of course pseudo-unitary on account of the self-adjointness of $\mathcal{H}(x)$.

It is our next problem to show that the S in (11) is also unitary in the partial space composed of physical particle states only.

For this purpose, however, since it is more convenient to have the solutions only for the interactions $\mathcal{L}_2$ and $\mathcal{L}_3$ beforehand, we deal with this problem in the next section.

§ 3. Solutions for the interactions $\mathcal{L}_2$ and $\mathcal{L}_3$

In this section we deal with the case where only interactions $\mathcal{L}_2$ and $\mathcal{L}_3$ exist. This is a problem to be solved exactly without use of the perturbation method.

For the sake of simplicity, let us first consider the system in which $\varphi_1(x)$ and $\varphi_2(x)$ interact with each other only through the interaction $\mathcal{L}_3$. Then the equations of motion for them become

$$(\gamma\partial+M)\varphi_1=g'\varphi_2,$$

$$(\gamma\partial+m)\varphi_2=g'\varphi_1. \tag{12}$$

By simple calculations using the canonical anticommutation relations

$$\{\varphi_{1,\alpha}(\boldsymbol{r}, t), (\bar{\varphi}_1(\boldsymbol{r}', t)\gamma_4)_\beta\}=\{\varphi_{2,\alpha}(\boldsymbol{r}, t), (\bar{\varphi}_2(\boldsymbol{r}', t)\gamma_4)_\beta\}=\delta_{\alpha\beta}\delta(\boldsymbol{r}-\boldsymbol{r}'),$$

we obtain the relations between the vacuum expectation values of two field operators,*

$$\begin{aligned}&(\gamma\partial+M)(0^\dagger, T\varphi_1(x)\bar{\varphi}_1(x')0)=g'(0^\dagger, T\varphi_2(x)\bar{\varphi}_1(x')0)-i\delta(x-x'),\\&(\gamma\partial+m)(0^\dagger, T\varphi_2(x)\bar{\varphi}_1(x')0)=g'(0^\dagger, T\varphi_1(x)\bar{\varphi}_1(x')0),\end{aligned} \tag{13}$$

and

$$\begin{aligned}&(\gamma\partial+m)(0^\dagger, T\varphi_2(x)\bar{\varphi}_2(x')0)=g'(0^\dagger, T\varphi_1(x)\bar{\varphi}_2(x')0)-i\delta(x-x'),\\&(\gamma\partial+M)(0^\dagger, T\varphi_1(x)\bar{\varphi}_2(x')0)=g'(0^\dagger, T\varphi_2(x)\bar{\varphi}_2(x')0).\end{aligned} \tag{13'}$$

Going into the momentum representation such that

$$\begin{aligned}&(0^\dagger, T\varphi_i(x)\bar{\varphi}_j(x')0)=-\frac{1}{2}S_{ij}(x-x'),\\&S_{ij}(x-x')=-\frac{2i}{(2\pi)^4}\int S_{ij}(k)e^{ik(x-x')}\,dk, \qquad i, j=1, 2,\end{aligned} \tag{14}$$

the relations (13) and (13)′ reduce to

$$\begin{aligned}&(i\gamma k+M)S_{11}(k)=g'S_{21}(k)-1,\\&(i\gamma k+m)S_{21}(k)=g'S_{11}(k),\end{aligned} \tag{15}$$

and

$$\begin{aligned}&(i\gamma k+m)S_{22}(k)=g'S_{12}(k)-1,\\&(i\gamma k+M)S_{12}(k)=g'S_{22}(k),\end{aligned} \tag{15'}$$

which give the solutions

$$\begin{aligned}S_{11}(k)&=-\frac{(i\gamma k+m)}{(i\gamma k+M)(i\gamma k+m)-g'^2},\\S_{22}(k)&=-\frac{(i\gamma k+M)}{(i\gamma k+M)(i\gamma k+m)-g'^2},\\S_{12}(k)=S_{21}(k)&=-\frac{g'}{(i\gamma k+M)(i\gamma k+m)-g'^2}.\end{aligned} \tag{16}$$

Similarly for the system of φ_1' and φ_2' only, we obtain the same relations as (12) with their canonical anticommutation relations

$$\{\varphi_{1,\alpha}'(\boldsymbol{r}, t), (\bar{\varphi}_1'(\boldsymbol{r}', t)\gamma_4)_\beta\}=\{\varphi_{2,\alpha}'(\boldsymbol{r}, t), (\bar{\varphi}_2(\boldsymbol{r}', t)\gamma_4)_\beta\}=-\delta_{\alpha\beta}\delta(\boldsymbol{r}-\boldsymbol{r}').$$

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* The vacuum is taken as that of the total system given in (1).

Thus the solutions S'_{11}, S'_{22}, S'_{12} and S'_{21}, for them become

$$S'_{ij}(k) = -S_{ij}(k). \tag{17}$$

With the aid of (16) and (17), the contributions from the diagrams describing one-body processes (Fig. 1) are easily obtained to give the results

$$\begin{aligned} F_{11}(k) &= g'^2 S_{22}(k), \\ F_{22}(k) &= g'^2 S_{11}(k), \\ F_{12}(k) &= F_{21}(k) = g'^2 S_{12}(k) - g' = -\frac{g'(i\gamma k+M)(i\gamma k+m)}{(i\gamma k+M)(i\gamma k+m)-g'^2}, \end{aligned} \tag{18}$$

and

$$F'_{ij}(k) = -F_{ij}(k). \tag{18}'$$

In Fig. 1, each line for all of unphysical fields is described by the wavy line with each index. Hereafter, in this paper, all unphysical lines are denoted by the wavy lines whether they are internal or external.

Now, returning to the system where all spinor fields interact through the interactions $\mathcal{L}_2$ and $\mathcal{L}_3$, we again consider their one-body processes as shown in Fig. 2. In Fig. 2, the real lines denote the physical spinor lines for ψ.

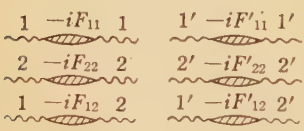

Fig. 1.

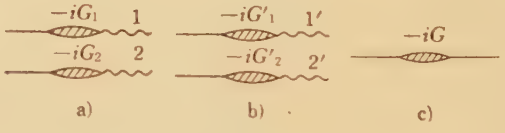

Fig. 2.

Suppose first that the G in Fig. 2, c) is constructed only through the interaction $\mathcal{L}_3$ except at the terminal points at both ends, then from the relations (18) it is clear that the G vanishes exactly, that is,

$$G(k) = g^2 S_{22}(k) + g^2 S'_{22}(k) = 0. \tag{19}$$

Thus the above expression holds also in the case where the G is constructed by both $\mathcal{L}_2$ and $\mathcal{L}_3$.

Moreover, Eq. (19) tells us that all the G's in Fig. 2, a) and b) are constructed through the interaction $\mathcal{L}_3$ only, except at their terminals. Then from (16) or (18), G's are easily obtained as follows:

$$\begin{aligned} G_1(k) &= g g' S_{22}(k) = -g S_m(k) F_{21}(k) = -\frac{g g'(i\gamma k+M)}{(i\gamma k+M)(i\gamma k+m)-g'^2}, \\ G_2(k) &= -g + g g' S_{21}(k) \\ &= -g - g S_m(k) F_{22}(k) = -\frac{g(i\gamma k+M)(i\gamma k+m)}{(i\gamma k+M)(i\gamma k+m)-g'^2}, \end{aligned} \tag{20}$$

and

$$G_1'(k) = -G_1(k), \quad G_2'(k) = -G_2(k), \tag{20}'$$

where

$$S_m(k) \equiv \frac{i\gamma k - m}{k^2 + m^2 - i\epsilon} = -\frac{1}{i\gamma k + m}.$$

These relations are clear in Fig. 3.

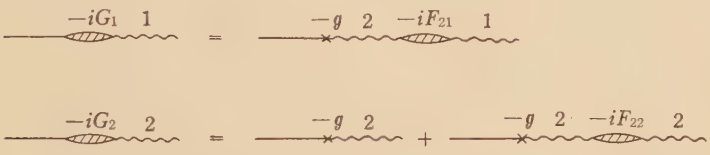

Fig. 3.

Eqs. (18), (18)′, (19), (20), and (20)′ are important for the discussions in the next section.

§ 4. Unitarity of the S-matrix and the convergency of the results

In this section we first show that the S in (11) is unitary in the partial space composed of physical particle states only, that is, we show

$$\tilde{S}^* = \tilde{S}^{-1}, \quad \tilde{S} \equiv pSp, \tag{21}$$

where p denotes the projection operator to physical states.

For this purpose, it is convenient to go into the usual graphical consideration in perturbation expansions.

With respect to S in (11), we can use the usual procedure to divide the T-products of interaction Hamiltonians in S into the combinations of the contractions and normal products of all fields in the application of Wick's theorem.

The difference from the usual theory, in this case, is no more than that there appear the negative propagators for φ_1' and φ_2' fields.

From the form of the interaction Hamiltonian (7), all kinds of vertices in any Feynman diagram are as given in Fig. 4. From Fig. 4, it goes without saying that there occurs no direct connection between ψ- and A-lines.

For the sake of convenience, each vertex in Fig. 4, a), b), and c) is called

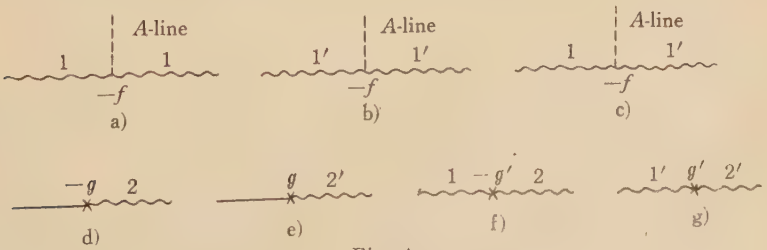

Fig. 4.

an "*f-vertex*" while each vertex in Fig. 4, d), e), f), and g) is called a "*g-vertex*".

Now let us consider the contributions from all possible diagrams connecting two *f-vertices* by way of *g-vertices* only. Then from (18) and (18)′, it is clear that the connection containing no ϕ-line in its composition vanishes exactly by the cancellation between F_{11} and F_{11}', that is,

$$f^2 S_M(k) F_{11}(k) S_M(k) + f^2 S_M'(k) F_{11}'(k) S_M'(k) = 0,$$

where

$$S_M(k) \equiv -S_M'(k) \equiv \frac{i\gamma k - M}{k^2 + M^2 - i\epsilon} = -\frac{1}{i\gamma k + M}.$$

This circumstance is also clear in the consideration that there always exist two kinds of diagrams in question in which the signs of propagators and coupling constants are all opposite to each other and the total number of the sign change is surely odd.

Thus, non-vanishing connections between two *f-vertices*, in which no *f-vertex* is included except at both ends, are restricted to the diagrams of such a type as shown in Fig. 5. In Fig. 5, it should be noted that there exists one

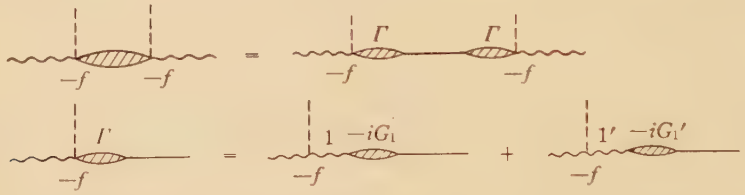

Fig. 5.

and only one ϕ-line between two *f-vertices* because of Eq. (19) in the preceding section. The Γ in Fig. 5 becomes from Eqs. (20) and (20)′

$$\Gamma(k) = 2 S_M(k) G_1(k) = \frac{2gg'}{(i\gamma k + M)(i\gamma k + m) - g'^2}. \tag{22}$$

After all, as far as the internal connections between any two of *f-vertices* are concerned, the graphical contents in the present model is just that every vertex in the usual theory is exactly replaced with the extended vertex given by the $\Gamma(k)$ in (22), and in spite of the appearing complexity of the interaction Hamiltonian (7), we can deal with this model rather simply.

We are now in a position to show that the unitarity condition (21) strictly holds. For this purpose, let us investigate the behaviours of unphysical external lines.

From Eqs. (18), (18)′, (20) and (20)′ it is clear that every real one-body process for unphysical particles through *g-vertices* only vanishes exactly, that is, for the processes containing no ϕ-line except their terminals it holds that

$$F_{11}(k)=F_{12}(k)=G_1(k)=0 \quad \text{for} \quad i\gamma k+M=0,$$
$$F_{22}(k)=F_{12}(k)=G_2(k)=0 \quad \text{for} \quad i\gamma k+m=0, \tag{23}$$

and for the processes containing a single ψ-line except their terminals (Fig. 6) it holds that

$$G_i(k)\frac{1}{i\gamma k+\kappa}G_j(k)=0 \quad \text{for} \quad i\gamma k+\alpha_i=0 \quad \text{or} \quad i\gamma k+\alpha_j=0, \tag{24}$$
$$i, j=1, 2, \quad \alpha_1=M, \ \alpha_2=m.$$

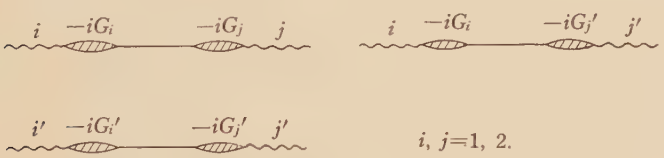

Fig. 6.

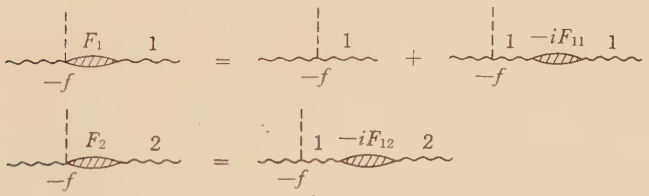

Fig. 7.

Moreover outputs (inputs) for unphysical lines from (to) any *f-vertex* (Fig. 7) are also obtained to give the vanishing results as follows: From Fig. 7, it holds that for φ_1 and φ_2

$$F_1(k)=1+S_M(k)F_{11}(k)=\frac{(i\gamma k+M)(i\gamma k+m)}{(i\gamma k+M)(i\gamma k+m)-g'^2},$$
$$F_2(k)=S_M(k)F_{12}(k)=\frac{g'(i\gamma k+m)}{(i\gamma k+M)(i\gamma k+m)-g'^2}, \tag{25}$$

and similarly for φ_1' and φ_2'

$$F_1'(k)=F_1(k), \quad F_2'(k)=F_2(k). \tag{25}'$$

Then

$$F_1(k)=F_1'(k)=0 \quad \text{for} \quad i\gamma k+M=0,$$
$$F_2(k)=F_2'(k)=0 \quad \text{for} \quad i\gamma k+m=0. \tag{26}$$

For the processes shown in Fig. 8 which are similar to Fig. 6, it holds that

$$S_M(k)G_1(k)\frac{1}{i\gamma k+\kappa}G_i(k)=0 \quad \text{for} \quad i\gamma k+\alpha_i=0. \tag{27}$$

Thus Eqs. (23), (24), (26) and (27) just show that no unphysical particle

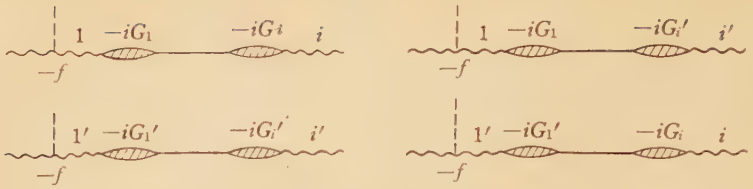

Fig. 8.

appears from the total S-matrix given by (11) as a real process and then Eq. (21) holds strictly without use of any further condition.

Thus, if we postulate that there is no unphysical particle in the initial states of the scattering, there never appears any unphysical particle in the final states of the scattering and we have been able to get a kind of desirable unitary S-matrix.

Turning to any physical process described by $\widetilde{S}$ in (21), it is now clear that there appears no divergent result in the present model because of the extended vertex given by $\Gamma(k)$ in (22) which behaves for the large momentum value as

$$\Gamma(k) \sim \frac{1}{k^2} \tag{28}$$

and prepares the sufficient convergency for all results. For example, the self-energy parts of φ and A fields and the closed vertex part are given by the diagrams shown in Fig. 9 in their lowest order calculations with respect to f.

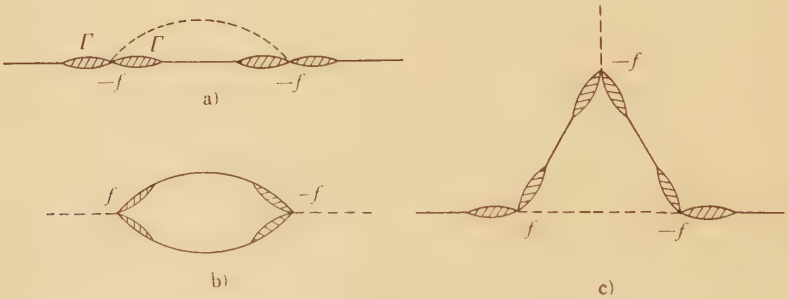

Fig. 9.

Moreover, if the present model is applicable to any cases of the current physics (e.g. quantum electrodynamics, meson theory, etc.) through some suitable modifications, we should further deal with the expectation values of many observables, which will probably contain the bi-linear forms of physical spinor fields in order to hold the analogous results as those of the usual ones. This fact shows that there are many diagrams of such a type as shown in Fig. 10, and in order to get the convergent results not only for the transition amplitudes but also for such expectation values, at least the asymptotic behaviour of $\Gamma(k)$ shown by (28)

is necessary.

Such good property of $\Gamma(k)$ comes from the total interaction Lagrangian $\mathcal{L}'$ in (1), and we have here understood that the unitarity of the S-matrix and the convergency of the theory are never independent of each other.

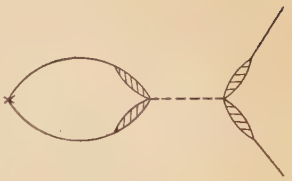

Fig. 10.

On the other hand, however, there exist some troublesome problems in the form of $\Gamma(k)$, which come from its singularities at the momentum values

$$(i\gamma k+M)(i\gamma k+m)-g'^2=0. \tag{29}$$

This fact seems to give rise to several new possibilities about the mass levels of the physical particle described by $\psi(x)$. The discussion about this problem will be given in the next section.

Finally, the present model has its local limit, when we take the limit

$$M\to\infty, \quad m\to\infty,$$

in (22), keeping the relation

$$g'^2+2gg'-Mm=0$$

which has always real solutions.

Thus in the application of the present model to any of the current physics, if it is possible, we will be able to obtain the consistent results with experiments for low energy problems, by choosing M and m suitably large under the condition such as the above expression.

§ 5. Discussions and further remarks

In this section we mainly discuss the possible variable ranges of the singular points of $\Gamma(k)$, on which the real stable mass levels of $\psi(x)$ depend.

First, suppose the mass operator of the physical spinor field $\psi(x)$, $\mathfrak{M}(-i\gamma k)$, which is obtained from the self-energy diagrams such as Fig. 9, a). Then the actual propagator $S'_F(k)$ for $\psi(x)$ including all higher order corrections must satisfy the equation

$$i(i\gamma k+\kappa+\mathfrak{M}(-i\gamma k))S'_F(k)=1. \tag{30}$$

Thus, the mass values of the stable particle described by $\psi(x)$ are given by the discrete roots of the equation

$$\alpha-\kappa-\mathfrak{M}(\alpha)=0 \tag{31}$$

in the domain

$$\alpha\leqq\kappa+\mu,$$

where we have assumed $\kappa>\mu$.

From the form of $\Gamma(k)$, $\mathfrak{M}(\alpha)$ in (31) takes the form

$$\mathfrak{M}(\alpha)=\frac{\rho(\alpha)}{(\alpha-\lambda_1)^2(\alpha-\lambda_2)^2} \tag{32}$$

where $\rho(\alpha)$ is some function of α and λ's are given by the equation

$$(\lambda-M)(\lambda-m)-g'^2=0$$

or

$$\begin{aligned}\lambda_1&=\frac{1}{2}(M+m)+\frac{1}{2}\sqrt{(M-m)^2+4g'^2}\,,\\ \lambda_2&=\frac{1}{2}(M+m)-\frac{1}{2}\sqrt{(M-m)^2+4g'^2}\,.\end{aligned} \tag{33}$$

Although the roots of Eq. (31) are not always real, it is conjectured for the case of the sufficient large values of λ_1 and λ_2 that there probably exist at least four real roots near the values of λ_1 and λ_2, except a root which is near by the mechanical mass κ and obtainable by the perturbation method, if the property of the function $\rho(\alpha)$ in (32) is suitably good, and probably these four roots do not belong to the stable mass levels of $\psi(x)$ on account of their large values (see Fig. 11).

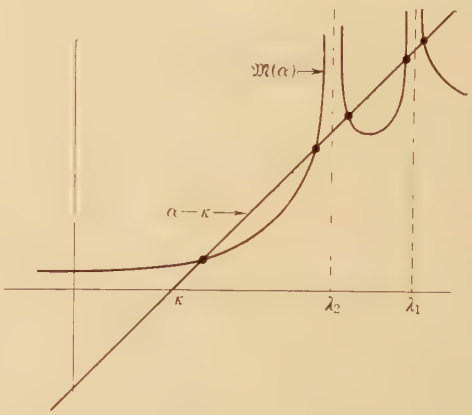

Fig. 11. $\rho(\alpha)$ is assumed to be nearly unity.

However, since it is difficult to calculate the function $\rho(\alpha)$ rigorously because of the interaction Lagrangian $\mathcal{L}_1$, it is impossible to give the definite and satisfactory results for the roots of (31) at once at this stage of the present investigation.

Moreover, in some case where the function $\rho(\alpha)$ has an undesirable behaviour, it may happen that several troublesome problems about the roots of (31) occur and consequently it may be rather convenient to have the suitable complex values of λ_1 and λ_2 to avoid the curious unphysical results, if any.

From these points of view and, furthermore, in view of the theoretical interest about the construction of the present model, we next investigate the possibilities of the modifications with respect to the type of the model and, consequently, discuss possible variable ranges of λ_1 and λ_2 which affect the properties of the roots of (31), by manipulating the form of the interaction Lagrangian $\mathcal{L}'$.

In the following, we confine ourselves to the discussion about this problem :

a) Case 1. Although it is clear in (33) that λ_2 can become negative too, it is further possible to get the negative values for both of λ_1 and λ_2.

For this purpose, we add the mass term interaction

$$\mathcal{L}_4 \equiv g_{11}(\bar{\varphi}_1\varphi_1 - \bar{\varphi}_1'\varphi_1') + g_{22}(\bar{\varphi}_2\varphi_2 - \bar{\varphi}_2'\varphi_2') \tag{34}$$

to the total interaction Lagrangian $\mathcal{L}'$ in (1). Then, it is shown through the same procedures as those used in the present paper so far that without any violations of the unitarity and the convergency we obtain

$$\Gamma(k) = \frac{2gg'}{(i\gamma k+\lambda_1')(i\gamma k+\lambda_2')}, \tag{35}$$

where λ_1' and λ_2' are roots of the equation

$$(\lambda - M + g_{11})(\lambda - m + g_{22}) - g'^2 = 0$$

or

$$\begin{aligned}\lambda_1' &= \frac{1}{2}(M+m-g_{11}-g_{22}) + \frac{1}{2}\sqrt{(M-m-g_{11}+g_{22})^2+4g'^2}, \\ \lambda_2' &= \frac{1}{2}(M+m-g_{11}-g_{22}) - \frac{1}{2}\sqrt{(M-m-g_{11}+g_{22})^2+4g'^2}.\end{aligned} \tag{36}$$

Then under the condition

$$M+m < g_{11}+g_{22},$$

we can get the negative λ_1' and λ_2'. Of course, although in this case the appearing masses of all unphysical fields become negative, when we renormalize the mass terms $\mathcal{L}_4$ into the free parts $\mathcal{L}_0$ in (1), this is not troublesome at all, if we retain the $\mathcal{L}_4$ in $\mathcal{L}'$, since it is sufficient enough for these unphysical fields that some suitable representations for them exist at least in the sense of mathematical manipulations.

b) Case 2. It is further possible to obtain the complex λ_1 and λ_2 in (32). It is necessary for this purpose that we set either φ_1 or φ_2 so as to have the negative anticommutator and consequently either φ_1' or φ_2' so as to have the usual anticommutator (the positive anticommutator).

For example, if we interchange the signs of φ_1 and φ_1' terms in $\mathcal{L}_0$ with each other, then the signs of their own anticommutators are all interchanged in all expressions so far, and the equations corresponding to (12) become

$$\begin{aligned}-(\gamma\partial + M)\varphi_1 &= g'\varphi_2, \\ (\gamma\partial + m)\varphi_2 &= g'\varphi_1.\end{aligned} \tag{37}$$

Solving the above equation and again tracing the same procedures as those given so far, we arrive at the result that

$$\Gamma(k) = \frac{2gg'}{(i\gamma k+\lambda_1'')(i\gamma k+\lambda_2'')} \tag{38}$$

where λ_1'' and λ_2'' are given as the roots of the equation

$$(\lambda-M)(\lambda-m)-g'^2=0$$

or

$$\begin{aligned}\lambda_1''&=\frac{1}{2}(M+m)+\frac{1}{2}\sqrt{(M-m)^2-4g'^2}\,,\\ \lambda_2''&=\frac{1}{2}(M+m)-\frac{1}{2}\sqrt{(M-m)^2-4g'^2}\,.\end{aligned}\tag{39}$$

Thus with the condition

$$|g'|>\frac{1}{2}|M-m|,$$

we have been able to obtain the complex λ_1'' and λ_2''.

Moreover, it goes without saying that the negative real parts of λ_1'' and λ_2'' would be obtained, by adding $\mathcal{L}_4'$ in (34) to the total Lagrangian of this case, to give the results

$$\begin{aligned}\lambda_1''&=\frac{1}{2}(M+m+g_{11}-g_{22})+\frac{1}{2}\sqrt{(M-m+g_{11}+g_{22})^2-4g'^2}\,,\\ \lambda_2''&=\frac{1}{2}(M+m+g_{11}-g_{22})-\frac{1}{2}\sqrt{(M-m+g_{11}+g_{22})^2-4g'^2}\,,\end{aligned}\tag{40}$$

with the conditions

$$g_{22}>M+m+g_{11},\quad |g'|>\frac{1}{2}(M-m+g_{11}+g_{22}).$$

It should be noted that the coupling constant g connecting $\psi'(x)$ and $\varphi_2(x)$ ($\psi'(x)$ and $\varphi_2'(x)$) does not enter at all in all expressions (30), (36), (39) and (40) and consequently we can obtain good approximations in perturbation expansions with respect to f by choosing the value of g suitably in all cases.

After all, returning to Eq. (31), it has been shown that there remains a considerable wide degree of freedom to decide the mass levels of the physical particle $\psi'(x)$ according to the form of $\rho(k)$ in (32). The future problems about the approaches such as the present model will be focused on the solutions of Eq. (31).

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Letters to the Editor

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Operator Equations in Two Field Theory Models*

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February 21, 1961

The field operators in two non-relativistic field theory models have already been shown to satisfy local field equations.[1] That is, the operators satisfy differential equations depending only on the operators and their derivatives at a single time. This is true even if some of the renormalization constants are infinite. In this paper we show the same result for two relativistic field theory models, unfortunately two-dimensional; however, the forms are very suggestive of a similar result for more realistic field theories.

The Lagrangians for the two theories ars as follows:

$$L=\psi^{*T}\sigma_y\left(\sigma_x\frac{\partial}{\partial x}+i\sigma_y\frac{\partial}{\partial t}+M\right)\psi - gJ^\mu\frac{\partial\phi}{\partial x^\mu}-\frac{1}{2}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x_\mu}-\frac{\widetilde{M}^2}{2}\phi^2, \tag{1}$$

$$L=\psi_1^{*T}\sigma_y\left(\sigma_x\frac{\partial}{\partial x}+i\sigma_y\frac{\partial}{\partial t}+M_1\right)\psi_1 + \psi_2^{*T}\sigma_y\left(\sigma_x\frac{\partial}{\partial x}+i\sigma_y\frac{\partial}{\partial t}+M_2\right)\psi_2 - \lambda J_1^\mu J_2^\nu\varepsilon_{\mu\nu}, \tag{2}$$

$J^\mu=i\psi^{*T}\sigma_y\gamma^\mu\psi$, $\gamma^1=\sigma_x$, $\gamma^0=i\sigma_y$,

$\varepsilon_{10}=-\varepsilon_{01}=1$, $\varepsilon_{11}=\varepsilon_{00}=0$.

The σ-matrices are the usual 2×2 Pauli spin matrices. The first model is the derivative coupling type (between a Boson and a Fermion field) that leads to no scattering in any dimension. All vacuum matrix elements are easily expressible, but in four dimensions the theory is non-renormalizable and the singularities too strong to be satisfactory. The second model leads to a non-trivial but soluble S-matrix.[2] Both models have infinite wave function renormalization. The results for the two models are similar and for model 1 can be checked trivially.

First we produce the expressions for the current operators:

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$$J^{\mu}(y)=\lim_{\substack{x\to y\\ spacelike}}\left\{J^{\mu}(x,y)-\langle 0|J^{\mu}(x,y)|0\rangle-\left(\frac{\langle 0|\psi^{*T}(x)\psi(y)|0\rangle}{\langle 0|\psi^{*T}(x)\psi(y)|0\rangle_{(0)}}\right)\langle 0|J^{\mu}(x,y)|0\rangle_{(0)}(i\eta)(y-x)^{\nu}\frac{\partial\phi}{\partial x^{\nu}}\right\}, \quad (3)$$

$$J_1^{\mu}(y)=\lim_{\substack{x\to y\\ spacelike}}\left\{J_1^{\mu}(x,y)-\langle 0|J_1^{\mu}(x,y)|0\rangle-\left(\frac{\langle 0|\psi_1^{*T}(x)\psi_1(y)|0\rangle}{\langle 0|\psi_1^{*T}(x)\psi_1(y)|0\rangle_{(0)}}\right)\langle 0|J_1{}^{\mu}(x,y)|0\rangle_{(0)}(i\lambda)(y-x)^{\alpha}\varepsilon_{\alpha\beta}J_2{}^{\beta}(y)\right\}, \quad (4)$$

$$J_2^{\mu}(y)=\lim_{\substack{x\to y\\ spacelike}}\left\{J_2^{\mu}(x,y)-\langle 0|J_2^{\mu}(x,y)|0\rangle-\left(\frac{\langle 0|\psi_2^{*T}(x)\psi_2(y)|0\rangle}{\langle 0|\psi_2^{*T}(x)\psi_2(y)|0\rangle_{(0)}}\right)\langle 0|J_2{}^{\mu}(x,y)|0\rangle_{(0)}(i\lambda)(y-x)^{\alpha}\varepsilon_{\beta\alpha}J_1{}^{\beta}(y)\right\}, \quad (5)$$

$$J^{\mu}(x,y)=i\psi^{*T}(x)\sigma_y\gamma^{\mu}\psi(y).$$

The subscript (0) indicates the free field value of the matrix element. Eqs. (4) and (5) are from reference 2). In all cases the equations represent the operator expressions for the current operator as conventionally defined. It is interesting to note that these expressions would all be zero evaluated in a vacuum expectation value to any finite order in perturbation theory.

The differential equations satisfied by the operators follow:

$$(\Box^2+\tilde{M}^2)\phi=0, \quad (6)$$

$$(\gamma^{\mu}\partial_{\mu}+M)\psi=ig\lim_{\substack{\varepsilon\to 0\\ spacelike}}\left\{\frac{\partial\phi(x+\varepsilon)}{\partial x^{\mu}}\gamma^{\mu}\psi(x)-\gamma^{\mu}\psi\times\lim_{\substack{y\to\infty\\ spacelike}}\left[\frac{\langle 0|(\partial\phi(x+\varepsilon)/\partial x^{\mu})\psi(x)\psi^{*T}(y)|0\rangle}{\langle 0|\psi(x)\psi^{*T}(y)|0\rangle}\right]\right\}, \quad (7)$$

$$(\gamma^{\mu}\partial_{\mu}+M_1)\psi_1=i\lambda\lim_{\substack{\varepsilon\to 0\\ spacelike}}\left\{J_2{}^{\mu}(x+\varepsilon)\varepsilon_{\nu\mu}\gamma^{\nu}\psi_1(x)-\gamma^{\nu}\psi_1\times\lim_{\substack{y\to\infty\\ spacelike}}\left[\frac{\langle 0|J_2{}^{\mu}(x+\varepsilon)\varepsilon_{\nu\mu}\psi_1(x)\psi_1{}^{*T}(y)|0\rangle}{\langle 0|\psi_1(x)\psi_1{}^{*T}(y)|0\rangle}\right]\right\}, \quad (8)$$

$$(\gamma^{\mu}\partial_{\mu}+M_2)\psi_2=i\lambda\lim_{\substack{\varepsilon\to 0\\ spacelike}}\left\{J_1{}^{\mu}(x+\varepsilon)\varepsilon_{\mu\nu}\gamma^{\nu}\psi_2(x)-\gamma^{\nu}\psi_2\times\lim_{\substack{y\to\infty\\ spacelike}}\left[\frac{\langle 0|J_1{}^{\mu}(x+\varepsilon)\varepsilon_{\mu\nu}\psi_2(x)\psi_2{}^{*T}(y)|0\rangle}{\langle 0|\psi_2(x)\psi_2{}^{*T}(y)|0\rangle}\right]\right\}. \quad (9)$$

It is seen that all the expressions (3) through (9) have a great similarity to the naive expressions one would get for finite renormalization. However, the differences would seem to be hard to guess for an arbitrary theory. In addition the expressions involve functions such as $\langle 0|\psi^{*T}(x)\psi(y)|0\rangle$ not

known until the theory is solved.

The interesting question is whether any similar results in realistic field theories could prove useful: either in formulating high energy approximations, or renormalizing presently non-renormalizable interactions, for example. In any case such expressions increase one's understanding of local field theory.

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A New Model for the High Energy Nuclear Interaction

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The data on a large number of jet showers recently observed in nuclear emulsions and emulsion chambers have revealed the existence of a wide variety in the angular distribution and the energy spectrum of secondary particles.[1),2),3),4)] From an analysis of this wide variety of jet showers, one can find striking properties of the multiple meson production, which, consequently, suggest a new model which seems to indicate a fundamental feature of the elementary particles.

Fig. 1 illustrates several typical examples of the log-tan θ plots of the angular distributions of jet showers observed by the Bristol group[1)] and the Chicago group.[2),3)] Let us look for some kind of unit constituting the wide variety of the angular distribution observed. In the event $0+6p$ observed by the Bristol group, one notices that all shower particles are grouped in a narrow region of the log-tan θ plot; it suggests a strong correlation of secondary particles. Let us assume this grouping of secondary particles found in the event $0+6p$ as the typical example of the units which construct the angular distribution observed. Then, the distribution found in the other events in Fig. 1 is now interpreted as a composition of several groups of shower particles. More quantitatively, we define the unit according to the following criterion, a group of secondary particles distributing in a narrow region of about a factor of $3\sim5$ in the log-tan θ plot of the angular distribution. Fig. 1 shows identification of the unit made in this way.

On the basis of this grouping of secondary particles of the jet showers, various features of the unit are studied.* There is a close resemblance between the prong number distribution of the unit and of P-$\bar{P}$ annihilation events,[5)] though the unit has a slightly longer tail in the distribution. The angular distribution of the shower particles in the rest system of the unit shows roughly the dipole-type radiation, $\sin^2\theta^* \, d\cos\theta^*$ in the examination

* For the experimental data adopted for our analysis and the detailed results, see S. Hasegawa. INSJ No. 38. (1961) (Tokyo University).

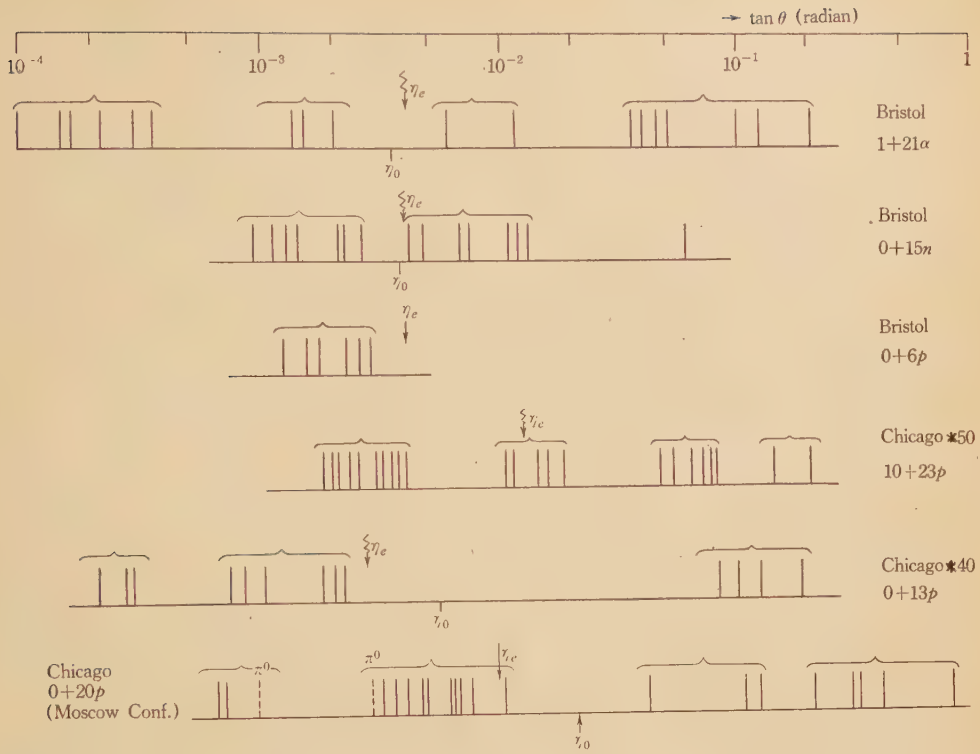

Fig. 1. Some interesting examples of the angular distribution which suggest the existence of the "quantum".
η_e: Half angle determined with Castagonoli's method by each author.
η_0: Determination of the half-angle from the hypothesis of the symmetrical emission of the "quantum".

by the method of F-plot,[6] where θ^* is the angle in the rest system of the unit; this shows the polarization character of the unit with respect to the incident direction. One can also find that the distribution of the transverse momentum of secondary pions[7),8] in jet showers is similar to the momentum spectrum of P-$\bar{P}$ annihilation events.[5]

The above considerations suggest the "quantum" hypothesis of the unit in log-tan θ plot. An energy estimate for the mean internal energy of the "quantum" can be made by using the mean value of the transverse momentum observed, 0.3 Bev/c, and the mean prong number of a "quantum". The mean prong number in a "quantum" is found as 4.7. Correcting for missing π^0-mesons, we estimate that the mean number of emitted π-mesons from a "quantum" is about 6. Therefore, the mean internal energy of the "quantum" is about $2Mc^2$—M is the nucleon rest mass. Thus the "quantum" may be considered to be composed of a baryon and an anti-baryon. The experimental evidence that about 20% of the shower particles in jet showers are not pions[8] shows that this

baryon is not always a nucleon but sometimes a hyperon.

Thus, we can consider the following mechanism of the explosive meson production; at the moment of a violent collision of high energy, the emission of several polarized "quanta" takes place. This "quantum" can be identified as a composite of a fundamental baryon and an anti-baryon. Later the "quantum" will turn into several pions and possibly to K-mesons.

For the further study of the production mechanism of the "quantum" at a violent collision, one finds important information from the Lorentz factor of the "quantum". The symmetrical emission of "quanta" in forward and backward directions referring to the C.M. system is found for most of the experimental data. The overall distribution of Lorentz-factor of the "quantum" in the C.M. system, $\tilde{\gamma}_q$, is expressed approximately as $1/\tilde{\gamma}_q$.

This relation is consistent with our knowledge that the angular distribution of secondary particles averaged over all jet showers is $d\cos\tilde{\theta}/\sin\tilde{\theta}$,[8),9)] where $\tilde{\theta}$ is the emission angle in the C.M. system. The fine structure of the distribution of Lorentz-factor of the "quantum" will reveal some interesting unknown dynamics of the elementary particles.

Lastly, we examine the critical energy at which the emission of the polarized "quantum" takes place. A rough estimate shows that this critical energy is about $\gamma_c \sim 5$. We remember that the change of normalized multiplicity of secondary particles occurs at $\gamma_c \sim 8$, as studied by Kaneko and Okazaki.[10)] Detailed discussions of this new model will be published succesively.

In conclusion, the author wishes to express his sincere thanks to Prof. Y. Fujimoto, Prof. G. Takeda, Prof. J. Nishimura, Dr. K. Niu and Dr. K. Yokoi for their helpful discussions and criticisms throughout this work. Especially, the author is much indebted to Dr. K. Yokoi for his various contributions to this work.

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Hard Core and Shape Parameter in the Effective Range Expansion

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May 9, 1961

The effective range expansion for the $^1S_0$ nuclear scattering phase shift reads

$$k \cot {}^1\delta_0 = -\frac{1}{a} + \frac{1}{2} r_e k^2 - P r_e^3 k^4 + \cdots . \quad (1)$$

The shape parameter P has recently received some attention in connection with the hard core hypothesis in the singlet even parity nucleon-nucleon interaction. Gammel and Thaler[1] have stated that a potential with a hard core of radius greater than 0.3 f, which seems to be required in fitting the $^1\delta_0$ at 310 Mev,[2] necessarily has a negative P. It follows then that the hard core hypothesis must be rejected if the experimental results show that P is positive, which is likely to be the case.[1]

Gammel and Thaler's statement is based on the work by Preston and Preston.[3] The latter authors consider four conventional purely phenomenological potential models, square well, Gaussian, exponential and Yukawa. It is not clear whether the statement holds for more realistic potential models. Indeed, Perring and Phillips[4] disprove Gammel and Thaler's statement by presenting several examples of potentials with the hard core radii ≥ 0.4 f and still have positive P. In the present short note we wish to add one more example to Perring and Phillips' and discuss its significance.

The singlet even parity potential we have considered is

$$V(x) = -(g^2/4\pi)\mu(e^{-x}/x) \times [1 + 10(e^{-x}/x) + 8(e^{-x}/x)^2], \quad (2)$$

with the hard core radius $r_0 = 0.4769$ f, $g^2/4\pi = 0.08$ and x = internucleon distance measured in the pion Compton wavelength $\mu^{-1} = 1.415$ f. The potential (2) has been successfully applied in a recent work on the proton-proton scattering below 310 Mev.[5]* The zero energy scattering parameters given by (2) are**

$$a = -18.41 \text{ f}, \quad r_e = 2.853 \text{ f}. \quad (3)$$

We have calculated $^1\delta_0$ over the energy range 2.5 to 6.0 Mev and analysed according to (1) and (3). Only the first three terms of the expansion (1) have been kept. The results are shown in Table I. It is clear that the potential (2) has a small but positive P in spite of its large hard core radius. This is in agreement with Perring and Phillips.[4] Table I also shows

* A weak potential proportional to $\boldsymbol{L}^2$ has been added to (2) in reference 5). Such a potential has no effect on the $^1S_0$ scattering and hence is not considered in the present note.

** No account has been taken of the Coulomb effect in the present calculation. The scattering length given here is slightly smaller than that published in reference 5). The difference is due to the round-off error in the published value of r_0. The scattering length is extremely sensitive to r_0. For the potential (2) the change of 0.001 f in r_0 produces the change of 2.2 f in a.

that the fourth term in the expansion (1) is positive. The phase shifts at higher energies predicted by (2) are given in reference 5).

Table I. The $^1S_0$ phase shift and shape parameter calculated from (2).

E (Mev)	$^1\delta_0$	P
2.482	1.0613	0.0158
3.310	1.0650	0.0158
3.748	1.0640	0.0156
4.147	1.0620	0.0153
4.561	1.0591	0.0150
4.976	1.0557	0.0147
5.137	1.0543	0.0145
5.391	1.0519	0.0143
6.000	1.0457	0.0137

It is clear now that the hard core hypothesis is not necessarily inconsistent with $P>0$. The hard core hypothesis appears to be the only way to reconcile zero energy scattering parameters with the behaviour of $^1\delta_0$ at higher energies unless strong energy dependence of the potential is invoked. If $P>0$ is proved experimentally, then, what must be rejected is not the hard core hypothesis but the four purely phenomenological models considered by Preston and Preston.[3] The Gammel-Thaler potential[6] belongs to this category.

Another, more important, point is that all potential models considered in reference 4) and the present work have the form: one pion exchange potential + short ranged singular phenomenological potential. In both cases the phenomenological parts are in qualitative agreement with the pion theoretical predictions.[7),8)] Further, the phenomenological parts of the potentials contain, in all examples, only two adjustable parameters—the same number of parameters as in four potential models considered by Preston and Preston.[3] It seems to us, then, that $P>0$, if proved experimentally, may be taken as another support for the pion theory of nuclear forces.

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Errata

Dynamical Theory of Ultrasonic Attenuation in Metals

Noboru TAKIMOTO

Prog. Theor. Phys. **25** (1961), 327

Page	Line	
332	Eq. (3·7)	$\delta_0(\kappa)$ and $\delta_\nu(\kappa)$ should read $\delta_0(-\kappa)$ and $\delta_\nu(-\kappa)$ respectively.
338	10	$\alpha_t=\dfrac{3\pi m q_t^2 u_t^2}{8M\omega v_0}$ should read $\alpha_t=\dfrac{3\pi m q_t^4 u_t^4}{8M\omega^3 v_0}$.
341	foot note	"p. 13" should read "p. 339".
343	14	The right hand side of Eq. (7·8b) should be multiplied by ω.
344	4	The left hand side should be multiplied by -1.
	Eq. (7·10)	The right hand side should be multiplied by -1.
	20	"(5·6a)" should read "(5·6b)".
348	16	"(4·1)" should read "(4·7)".
349	Eq. (AI·12)	The last expression should read $3f_0(\partial^4 E_K/\partial K_y^2\partial K_z^2)$.
350	8 and 11	$K_{tf}(\kappa, \omega)$ in the equations should be multiplied by -1.
	Eq. (AII·6)	The equation should be corrected to $$\omega(\kappa)=\omega_p^2\left\{1+\frac{9}{5}\left(\frac{\kappa}{q_e}\right)^2-\frac{492}{175}\left(\frac{\kappa}{q_e}\right)^4+L^4\kappa^4+\cdots\right\}.$$

JOURNAL OF THE PHYSICAL SOCIETY OF JAPAN
Volume 16, Number 6, June 1961

CONTENTS

Page

Elementary Processes Related to the Catalytic Action of μ^- for the p-d Fusion Reaction ..... Yukio MIZUNO 1043

Inelastic Scattering of Deuterons from C^{12}, Mg^{24} and Ni^{60} ..... Yoshihide ISHIZAKI 1056

Anomalies in the Scattering of Alpha Particles by Carbon ..... Takashi MIKUMO 1066

He^3-Alpha Scattering ..... Ren CHIBA, Homer E. CONZETT, Haruhiko MORINAGA, Naoshi MUTSURO, Katsufusa SHODA and Motoharu KIMURA 1077

Angular Correlation of the Mo^{95} Gamma-Gamma Cascades ..... Atsushi AOKI 1086

Angular Distribution of (α, p) Reactions in Some Light Nuclei ..... Michiya KONDO, Takashi YAMAZAKI and Shotaro YAMABE 1091

A Cloud Chamber Study of High Energy Nuclear Interactions of Cosmic Ray Particles (I)—A Possible Evidence of Successive Collisions in Carbon Nuclei ..... Isao KITA 1099

A Cloud Chamber Study of High Energy Nuclear Interactions of Cosmic Ray Particles. (II)—Results of Single Nucleon-Nucleon Interactions .. Goro FUJIOKA 1107

Effects of s-d Exchange Interaction and Temperature-Independent Paramagnetism on Susceptibilities of Nickel, Palladium, and Platinum ..... Masao SHIMIZU 1114

Magnetic Properties of Intermetallic Compounds in Manganese-Tin System : $Mn_{3.67}Sn$, $Mn_{1.77}Sn$, and $MnSn_2$ ..... Kō YASUKŌCHI, Kazuo KANEMATSU and Tetuo OHOYAMA 1123

The Magnetic Investigation of Mn^{++}, Co^{++}, Ni^{++} and Cu^{++} Ammine and Anhydrate Salts at Low Temperatures ..... Takashi WATANABE 1131

Ultrasonic Attenuation in Tin Single Crystals at Low Temperatures .. Kazuo KAMIGAKI 1141

The Thermoelectric Power of Magnesium-Cadmium Alloys ..... Seiichiro NOGUCHI 1145

Resonance Transfer of Ionization Energy in Semiconductors ..... Shigeharu KOSHINO and Toshihiko ANDO 1151

Ferroelectricity in (K, NH_4), (K, Rb) and (K, Tl) Ferrocyanide Mixed Crystals .... Shigeru WAKU, Kunio MASUNO and Toshisuke TANAKA 1158

The Superstructures of Iron Selenide Fe_7Se_8 ..... Atsushi OKAZAKI 1162

Ultrasonic Attenuation in Magnetite at Low Temperatures ..... Kazuo KAMIGAKI 1170

Crystal Structures and Electrical Properties of Alkali Antimonides ..... Jun-ichi CHIKAWA, Shunji IMAMURA, Kenzo TANAKA and Makoto SHIOJIRI 1175

Weak Ferromagnetism in β-$NaFeO_2$ ..... Hiroshi WATANABE and Masao FUKASE 1181

The Crystal Structure of Chloranil and Bromanil ..... Ikuhiko UEDA 1185

Study on Precipitation Phenomena in Thin Films of Supersaturated Gold-Nickel Alloy ..... Yasushige FUKANO 1195

A Direct Determination of the Crystal Structure of Ethylenediammonium Sulphate ..... Kiichi SAKURAI 1205

Study of Metallic Carbides by Electron Diffraction Part IV. Cobalt Carbides ..... Sigemaro NAGAKURA 1213

Study on the Internal Motions of Some Acryl- and Methacryl-group High Polymers by Pulsed NMR Method ..... Takao KAWAI 1220

Mechanical Dispersions and Transition Phenomena in Semicrystalline Polymers .... Yasaku WADA 1226

Microwave Spectrum of Formaldehyde III. Vibration-Rotation Interaction ..... Takeshi OKA and Yonezo MORINO 1235

Double Quantum Transition in Electron Spin Resonance of Diphenyl-Picryl Hydrazyl at Low Magnetic Fields ..... Tsuneo HASHI 1243

On the Pendent Drop, I ..... Ko TAMADA and Yoshio SHIBAOKA 1249

(continued on back page)

SHORT NOTES

Observation of Dislocations in Germanium Single Crystal ....Masakazu SHOJI, Shoji TAUCHI, Tomokuni MITSUISHI and Masami TOMONO 1253
Anomalous Temperature Dependence of Lattice Constants of Ferroelectric Sodium Nitrite..... Sadao HOSHINO and Iwao SHIBUYA 1254
Moving Striation in Microwave Discharge Plasma..... ..Mikio TAKEYAMA, Tanehiro NAKAU, Shōzaburo YAMASAKI and Takeshi ISA 1255
Electron Spin Resonance of Manganese in $MgCl_2$ ....Kuniya FUKUDA, Hiroaki MATSUMOTO, Toru TAKAGI and Yoichi UCHIDA 1256
The New Magnetic Phase in Manganese-Aluminium-Nickel System..... .....Ichiro TSUBOYA and Makoto SUGIHARA 1257
Effect of Current Injection on Properties of Cadmium Sulfide Crystals..... ..... Hiroji MITSUHASHI 1258
Absorption and Reflection Spectra of Afterglow Plasma .....Shigetoshi TANAKA 1259
The Base Resistance of the Transistor in the Saturated Region.....Osamu NAKAHARA 1260
On the Secondary Tunnelling Phenomena in a *p-n* Junction of Gallium Arsenide .. ..Akikazu SHIBATA 1261
The Optical Absorption of Localized Exciton in KCl-KI Crystal .....Kaizo NAKAMURA, Kuniya FUKUDA, Riso KATO, Atsuo MATSUI and Yoichi UCHIDA 1262
Refractive Index Difference of Dilute Aqueous Solution of Polyvinylpyrolidon..... ..... Yasuhiro MIYAKE and Jiro FURUICHI 1263
The Electron-Coupled Nuclear Spin Interactions in Ionic Crystals by the Overlap Model.....Tatsuo SHIMIZU 1264
Magnetic Properties of B8 Type Compounds in Manganese-Tin System .... Mitsuru ASANUMA 1265
On the Decomposition of the Ferromagnetic Phase of the Heusler Alloy, Cu_2MnAl ..... Ren'iti KIMURA, Tetuo OHOYAMA and Keizo ENDO 1266
Microwave Measurement of Fast Time Varying Plasmas.....Susumu TAKEDA 1267
Hot Electron Behavior at High Pressures.....Louis GOLD 1269

Errata

On the Viscosity of Irradiated Polymers. III. Effect of Branching ..Kanji KATSUURA 1270

VOLUME CONTENTS, NOS. 1~6 i
AUTHOR INDEX TO VOL. 16, NOS. 1~6 ix

Progress of Theoretical Physics, Vol. 26, No. 2, August 1961

Fixation of Physical Space-Time Coordinates and Equation of Motion of Two-Body Problem

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The consistency of the canonical reduction of gravitational field coupled with point particles is investigated. It is shown that the Hamiltonian, obtained by eliminating constraints and imposing coordinate condition, gives exactly the same equations of motion as obtained by Einstein, Infeld and Hoffmann. Furthermore, the derivation of the approximate (up to the fourth order of velocity of particle) equation of motion is presented without tedious calculation.

§ 1. Introduction

The theory of relativity differs from any other usual field theory in that the energy momentum tensor vanishes identically by virtue of the differential constraint equations. Physically, the origin of this phenomenon is related to the general covariance of the theory. That is, the generator of translations with respect to the arbitrary time coordinate is indeed expected to vanish since any apparent "time" translation can be removed by a mere relabeling. This does not mean that the physical energy and momentum of the gravitational field is zero. Real motion is to be expressed only in terms of physically meaningful space-time coordinates which must be functional of the metric tensor. Recently, in order to obtain the correct non-vanishing energy-momentum vector, Arnowitt, Deser and Misner[1] (hereafter referred to as ADM) have put the general relativity into a canonical form. They started a series of investigation with the Palatini Lagrangian and (3+1) dimensional form. If the constraint variables are eliminated from the Lagrangian by means of the constraint equation, there results a new structure which depends upon four dynamical variables and four undetermined quantities. The specification of these arbitrary quantities as the coordinates corresponds to the definition of a coordinate frame. In this form, a new non-vanishing Hamiltonian density, which will represent the physical energy density of the system, arises in the Lagrangian. At first sight, the formalism of ADM seems to impair the general covariance of the theory. However, this is not the case, since it can be seen that the action of general relativity is in the "already parameterized" form where a theory is invariant under an arbitrary re-parametrization. By using this canonical form, ADM have

investigated several interesting problems such as the criteria for gravitational radiation, classical self-energy question, etc. However, the problem of the equation of motion (Einstein, Infeld and Hoffmann analysis[2]) has not been investigated from their viewpoint. This should be one of the major topics of interest in the classical theory.

The equations of the two-body problem in general relativity were derived long ago by Einstein and his co-workers (hereafter referred to as EIH) from the assumption that the bodies can be regarded as singularities of the gravitational field. They used the field equation in empty space,

$$ {}^4R_{\mu\nu}-\frac{1}{2}{}^4g_{\mu\nu}{}^4R=0, \tag{1·1} $$

instead of

$$ {}^4R_{\mu\nu}-\frac{1}{2}{}^4g_{\mu\nu}{}^4R=-\frac{1}{2}T_{\mu\nu}, \tag{1·2} $$

where $T_{\mu\nu}$ is the energy-momentum tensor of matter.* They obtained the well-known equation (EIH Eq.) of the two-body problem within the approximation up to fourth order term of the velocity of particle,

$$ \begin{aligned} \ddot{\underset{1}{x}}^i=&Gm_2\left\{\left[1+\dot{\underset{1}{x}}^s\dot{\underset{1}{x}}^s+\frac{3}{2}\dot{\underset{2}{x}}^s\dot{\underset{2}{x}}^s-4\dot{\underset{1}{x}}^s\dot{\underset{2}{x}}^s-4Gm_2/r-5Gm_1/r\right](1/r)_{,i}\right. \\ &\left.+\left[4\dot{\underset{1}{x}}^s(\dot{\underset{2}{x}}^s-\dot{\underset{1}{x}}^s)+3\dot{\underset{1}{x}}^i\dot{\underset{2}{x}}^s-4\dot{\underset{2}{x}}^i\dot{\underset{2}{x}}^s\right](1/r)_{,s}+\frac{1}{2}\dot{\underset{2}{x}}^s\dot{\underset{2}{x}}^r r_{,irs}\right\}, \end{aligned} \tag{1·3} $$

where the index a written as m and x is used to distinguish between the particles, $r=\{(\overset{s}{x}-\overset{s}{x})(\overset{s}{x}-\overset{s}{x})\}^{1/2}$, $\varphi_{,i}=\partial\varphi/\partial\overset{i}{x}$, $\dot{\overset{i}{x}}=d\overset{i}{x}/dt$ and $\overset{a}{m}$ is the constant rest mass of a'th particle. On the other hand, Fock and Papapetrou[3] have deduced the same equation from Eq. (1·2) using definite expressions for $T_{\mu\nu}$.

By using Eq. (1·2) with $T_{\mu\nu}$ proportional to the three-dimensional δ function, Infeld has linked the two methods.[4] He started with the integral of the equation $(\sqrt{-{}^4g}\,T^{\mu\nu})_{;\nu}=0$ over the three-dimensional region surrounding the first singularity, and found a Lagrangian

$$ \begin{aligned} -L=&-\sum_{a=1}^{2}\left\{\frac{1}{2}\overset{a}{m}\dot{\overset{i}{x}}\dot{\overset{i}{x}}+\frac{1}{8}\overset{a}{m}(\dot{\overset{i}{x}}\dot{\overset{i}{x}})^2\right\}+Gm_1m_2\left\{-1/r+\frac{1}{2}G(m_1+m_2)/r^2\right. \\ &\left.-\frac{3}{2}\sum_a\dot{\overset{i}{x}}\dot{\overset{i}{x}}/r+4\dot{\underset{1}{x}}^i\dot{\underset{2}{x}}^i/r-\frac{1}{2}r_{,ij}\dot{\underset{1}{x}}^i\dot{\underset{2}{x}}^j\right\}, \end{aligned} \tag{1·4} $$

* Latin indices run from 1 to 3, Greek from 0 to 3, and $x^0=t$. All tensors and covariant operations are three-dimensional unless specified such as ${}^4g_{\mu\nu}$. Ordinary differentiation is denoted by comma. The vertical bar "|" indicates three-dimensional covariant differentiation, while the semicolon four-dimensional one. We use units such that $\kappa=16\pi Gc^{-4}=1=c$ where G is Newton's gravitational constant, and summation convention for all repeated tensor indices.

which yields the EIH Eq. (1·3) by varying with respect to $\underset{1}{x^i}$. Though his method is less tedious than the original EIH one, it is necessary to impose the following superfluous condition in order to ensure the internal consistency:

$$\overline{(\partial^4 g_{\mu\nu}/\partial x^i)} = \partial^4 \overline{g_{\mu\nu}}/\partial \underset{1}{x^i}, \quad \partial^4 \overline{g_{\mu\nu}}/\partial \underset{1}{\dot{x}^i} = 0 \tag{1·5}$$

where

$$\overline{\varphi} = \int d^3x \delta(\boldsymbol{x} - \underset{1}{\boldsymbol{x}}) \varphi. \tag{1·6}$$

In the above equation and in the following discussion we narrow the definition of δ function, following Infeld,[4] so that Eq. (1·6) remains true even if φ has a singularity, and that $\overline{\varphi}$ represents only regular part of the function of the $\underset{a}{x^i}$, $\underset{a}{\dot{x}^i}$ and $\underset{a}{\ddot{x}^i}$.

Since these features are extremely complicated, it seems that there is still room for an alternative simplified derivation. Following the line of Gupta's approach to gravitation, the present author[5] and independently Corinaldesi[6] have obtained in a straightforward way exactly the same equation as that of EIH. Though Corinaldesi treated every component of the gravitational field symmetrically, the author showed that only the potential, obtained by eliminating the longitudinal and scalar parts of gravitational field with the aid of the supplementary condition, contributes to the EIH Eq.

Thus, the fact that the transverse traceless part of the gravitational field plays no role in the EIH approximation method shows that the equation of two-body problem may be obtained reasonably by the Arnowitt, Deser and Misner formalism where a precise determination of the independent dynamical modes is arrived at. Since the Lagrangian densities in the usual methods[3),4)] are determined so as to deduce the EIH Eq. derived from the equation $(\sqrt{-{}^4g}\,T^{\mu\nu})_{,\nu} = 0$, we cannot give explicit expressions for the Lagrangian densities from the beginning. Therefore, if we start with the Lagrangian density

$$L = -m(\overline{ds}/dt) = -m\underset{1}{(}\overline{-{}^4g_{\mu\nu}}\,\underset{1}{\dot{x}^\mu}\,\underset{1}{\dot{x}^\nu})^{1/2} \quad (\underset{1}{\dot{x}^0} = 1), \tag{1·7}$$

it is necessary to impose the superfluous condition (1·5). This may be due to the fact that the Hamiltonian does not exist in the usual formalism of general relativity. It will be shown in this paper that the canonical method of ADM is very useful from a practical point of view, and further that Infeld's condition (1·5) is needless, because our Hamiltonian is not directly related with

$$H = \underset{1}{p_i}\,\underset{1}{\dot{x}^i} - \overline{L}, \tag{1·7′}$$

where $\overline{L}$ is given by (1·7).

In § 2, it will be shown that the equation of motion derived by the reduced

canonical method is identical with that of EIH irrespective of approximation method. It should be valuable for the practical use to show the approximation method explicitly, as done by EIH. Since the feature of our method based on the reduced canonical formalism is different from that of the usual one, we present an approximation procedure in § 3. The Hamiltonian is obtained explicitly by approximation and the EIH Eq. (1·3) is derived without tedious calculation. Some remarks on the quantization of physical space-time coordinates are given in § 4.

§ 2. Consistency of the canoical reduction

The question of consistency of the canonical reduction of general relativity (obtained by eliminating constraints and also imposing coordinate conditions in the action or generator) has been examined by ADM.[7] That is, it has been shown that the equations of motion obtained from the "reduced" formalism agree with the original Einstein equations. But their demonstration is confined only to the uncoupled gravitational field. We are interested in the dynamics of point particles coupled with the gravitational field. In the first place, we shall briefly sketch the relevant parts of ADM's canonical formalism for the coupled system. We start with the total action of the system:

$$I=\int \sqrt{-{}^4g}\,{}^4g^{\mu\nu}\,{}^4R_{\mu\nu}({}^4\Gamma^{\alpha}_{\beta\gamma})\,d^4x+\int L_m\,d^4x, \tag{2·1}$$

where

$$L_m=\sum_a \int ds\left\{p_{a\mu}(s)[dx^{\mu}_a(x)/ds]-\frac{1}{2}\,\lambda_a(s)[p_{a\mu}(s)p_{a\nu}(s)\,{}^4g^{\mu\nu}(x)+m_a^2]\right\}\delta^4(x-x_a(s)).$$

By putting Lagrangian multiplier $\lambda_a(s)=m_a^{-1}$ and eliminating arbitrary parameter s by performing the indicated s integration ADM have shown that Eq. (2·1) can be reduced to the following form (ADM–V):

$$I=\int d^4x\left\{\pi^{ij}\partial_t g_{ij}+\sum_a [p_{ai}dx^i_a(t)/dt]\delta(\boldsymbol{x}-\boldsymbol{x}_a(t))+N^{\mu}R_{\mu}\right\}, \tag{2·2}$$

where

$$N^0\equiv(-g^{00})^{-1/2},\quad N_i\equiv {}^4g_{0i},$$

$$R_0\equiv g^{-1/2}\left(gR+\frac{1}{2}\,\pi^2-\pi^{ij}\pi_{ij}\right)-\sum_a (p_{ai}p^i_a+m_a^2)^{1/2}\delta(\boldsymbol{x}-\boldsymbol{x}_a(t)),$$

$$R_i\equiv 2\pi^j_{i|j}+\sum_a p_{ai}\delta(\boldsymbol{x}-\boldsymbol{x}_a(t)),$$

$$\pi^{ij}=(-{}^4g)^{1/2}({}^4\Gamma^0_{mn}-g_{mn}\,{}^4\Gamma^0_{pq}g^{pq})g^{im}g^{jn},\quad \pi=g_{ij}\pi^{ij},$$

and R means the three-dimensional curvature scalar formed from g_{ij} and $g^{ij}(={}^4g^{ij}+N^iN^j/(N^0)^2)$. Here, in the action (2·1) the divergence terms are

discarded. This is essential to the establishment of consistency. Varying Eq. (2·2) with respect to N^μ, we obtain the constraint equations:

$$R_0 \equiv g^{-1/2}\left(gR+\frac{1}{2}\pi^2-\pi_{ij}\pi^{ij}\right)-\sum_a (p_i p_a^i+m_a^2)^{1/2}\delta(\boldsymbol{x}-\boldsymbol{x}_a(t))=0, \quad (2\cdot3a)$$

$$R_i \equiv 2\pi_i^j{}_{|j}+\sum_a p_{ai}\delta(\boldsymbol{x}-\boldsymbol{x}_a(t))=0. \quad (2\cdot3b)$$

We now make use of an orthogonal decomposition of g_{ij} and π^{ij} according to the following scheme:*

$$g_{ij}=-\delta_{ij}+g_{ij}^{TT}+g_{ij}^{T}+(g_{i,j}+g_{j,i}), \quad (2\cdot4a)$$

$$\pi^{ij}=\pi^{ijTT}+\pi^{ijT}+(\pi^i{}_{,j}+\pi^j{}_{,i}). \quad (2\cdot4b)$$

Here g_{ij}^{TT} and π^{ijTT} mean the transverse traceless part of g_{ij} and π^{ij}, respectively, and

$$g_{ij}^{T}=\frac{1}{2}\left(g^T\delta_{ij}-\frac{1}{\Delta}g^T{}_{,ij}\right), \quad (2\cdot5a)$$

$$g^T=g_{ii}-\frac{1}{\Delta}g_{ij,ij}-3, \quad (2\cdot5b)$$

$$g_i=\frac{1}{\Delta}\left(g_{ij,j}-\frac{1}{2}\Delta^{-1}g_{kl,kli}\right), \quad (2\cdot5c)$$

where $1/\Delta$ is the inverse of the Laplacian operator $\Delta=\partial^2/\partial x^i\partial x^i$ with appropriate boundary conditions. Also π^{ijT}, etc., have the similar forms though the term -3 on the right-hand side of (2·5b) does not appear.

The total generator (obtained from variations at the end point)

$$G=\sum_a p_{ai}(t)\delta x_a^i(t)+\int d^3x\,\pi^{ij}\delta g_{ij}, \quad (2\cdot6)$$

in which the constraint equations (2·3) are taken into account, becomes

$$G=\sum p_{ai}(t)\delta x_a^i(t)+\int d^3x\left\{\pi^{ijTT}\delta g_{ij}^{TT}-(-\Delta g^T)\,\delta\left[-\frac{1}{2}\,\frac{1}{\Delta}\left(\pi^T+\frac{1}{\Delta}\pi^{kl}{}_{,kl}\right)\right]\right.$$
$$\left.+(-2\pi^{ij}{}_{,j})\,\delta\left[g_i-\frac{1}{4}\,\frac{1}{\Delta}g^T{}_{,i}\right]\right\}, \quad (2\cdot6')$$

on inserting the orthogonal decomposition of g_{ij} and π^{ij}.

The Π_μ defined by

$$\Pi_0=\Delta g^T, \quad \Pi_i=-2\pi^{ij}{}_{,j} \quad (2\cdot7)$$

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* We use a different decomposition from that of ADM. The negative sign of the first term on the right-hand side of Eq. (2·4a) plays an important role in our discussions.

can be expressed in terms of the canonical variables g_{ij}^{TT}, π^{ijTT}, $\underset{a}{x^i}(t)$ and $\underset{a}{p_i}(t)$ by solving the constraint Eqs. (2·3) :

$$\Pi_0=T_0^0[g_{ij}^{TT}, \pi^{ijTT}, \underset{a}{x^i}(t), \underset{a}{p_i}(t)]\equiv -H, \tag{2·8a}$$

$$\Pi_i=T_i^0[g_{ij}^{TT}, \pi^{ijTT}, \underset{a}{x^i}(t), \underset{a}{p_i}(t)]. \tag{2·8b}$$

We shall now fix the frame of the physical space-time coordinates. For this purpose, we impose the following coordinate conditions (ADM–V) :

$$x^0\equiv t=-\frac{1}{2}\frac{1}{\Delta}\left(\pi^T+\frac{1}{\Delta}\pi^{kl}{}_{,kl}\right)=-\frac{1}{2}\frac{1}{\Delta}\pi^{tt}, \tag{2·9a}$$

$$x^i=g_i-\frac{1}{4}\frac{1}{\Delta}g^T{}_{,i}. \tag{2·9b}$$

Now, the generator can be written in canonical form :

$$G=\sum_a \underset{a}{p_i}(t)\delta\underset{a}{x^i}(t)+\int d^3x[\pi^{ijTT}\delta g_{ij}^{TT}+T_\mu{}^0\delta x^\mu]. \tag{2·10}$$

Thus, H is the Hamiltonian density of the entire coupled system and T_i^0 is the total momentum density.

Returning now to the action integral, we see that Eq. (2·2) can be reduced to

$$I_R=\int d^4x\left\{\sum_a \underset{a}{p_i}[d\underset{a}{x^i}(t)/dt]\delta(\boldsymbol{x}-\underset{a}{\boldsymbol{x}}(t))+\pi^{ijTT}\partial_0 g_{ij}^{TT}-H\right\}, \tag{2·11}$$

by eliminating constraints (2·3) and imposing coordinate conditions (2·9). On the other hand, in our coordinate system (2·9), the metric g_{ij} takes on the form

$$g_{ij}=g_{ij}^{TT}+\left(1+\frac{1}{2}g^T\right)\delta_{ij}, \tag{2·12}$$

with the boundary condition $g^T\to 0$ as $r\to\infty$.

Now that we have recapitulated the theory of ADM, we shall show that the equations of motion obtained from this "reduced" formalism agree with the equations of motion obtained by EIH.

Varying the original action integral (2·2) with respect to $\underset{a}{p_i}$, $\underset{a}{x^i}$, we obtain the equations of motion

$$d\underset{a}{x^i}(t)/dt=-\int d^4y\, N^\mu(y)\delta R_\mu(y)/\delta\underset{a}{p_i}(t) \tag{2·13a}$$

$$=\int d^3y\left\{N^0\underset{a}{p^i}(\underset{a}{p_s}\underset{a}{p^s}+m^2)^{-1/2}-N^i\right\}\delta(\boldsymbol{y}-\underset{a}{\boldsymbol{x}}(t)), \tag{2·14a}$$

and

$$dp_{a\,i}(t)/dt = +\int d^4y\, N^{\mu}(y)\,\delta R_{\mu}(y)/\delta x^{i}_{a}(t) \tag{2·13b}$$

$$= -\int d^3y \left\{ (p_{a\,s}p^{s}_{a}+m^2_{a})^{1/2} N^{0}_{,i} + \frac{1}{2} p_{a\,l}p_{a\,m} g^{lm}_{,i} (p_{a\,l}p^{l}_{a}+m^2_{a})^{-1/2} N^{0} \right.$$

$$\left. - p_{a\,l} N^{l}_{,i} \right\} \delta(y - x_{a}(t)). \tag{2·14b}$$

In the Appendix, we show that Eqs. (2·14) yield

$$\ddot{x}^{i}_{a} + \overline{{}^4\Gamma^{i}_{\mu\nu}}\,\dot{x}^{\mu}_{a}\dot{x}^{\nu}_{a} - \frac{d}{dt}(\log \bar{L})\,\dot{x}^{i}_{a} = 0, \tag{2·15}$$

where

$$\bar{L} = -m_{a}\,\overline{ds}/dt = -m_{a}(-\overline{{}^4g_{\mu\nu}\dot{x}^{\mu}_{a}\dot{x}^{\nu}_{a}})^{1/2}, \tag{2·16}$$

$$\overline{{}^4\Gamma^{i}_{\mu\nu}} = \overline{{}^4g^{i\lambda}\,{}^4\Gamma_{\lambda,\mu\nu}}.$$

The definition of $\bar{\varphi}$ is already given by (1·6) :*

$$\bar{\varphi} = \int d^3x\, \varphi\, \delta(x - x_{a}(t)).$$

As mentioned already in § 1, our three-dimensional δ function have properties that the x^{k}_{a} are introduced for x^{k} and that the singularity of φ are ignored. The latter property distinguishes our function from the usual one, and makes it difficult to guarantee the existence of such δ function. If we admit ADM's argument about the existence of a compensating effect of the gravitational field on the divergent self-energies due to other fields, it may be possible to admit the ignoring of the singular part, although higher and higher infinities will arise in a perturbation analysis.

From the analysis of Infeld, it is easily seen that Eq. (2·15) is nothing but the EIH Eq. derived from

$$(dt/\overline{ds})\int d^3x (\sqrt{-{}^4g}\,T^{\mu\nu})_{;\nu}\,\delta(x - x_{a}) = 0, \tag{2·17}$$

with

$$\sqrt{-{}^4g}\,T^{\mu\nu} = \sum_{a} M_{a}\dot{x}^{\mu}_{a}\dot{x}^{\nu}_{a}\,\delta(x - x_{a})$$

in which

$$M/m = dt/\overline{ds}.$$

On the other hand, we shall now extend ADM's argument about the consistency of the canonical reduction of uncoupled gravitational field to the case

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* In this paper, we assume $\overline{\varphi\psi} = \overline{\varphi}\,\overline{\psi}$. This will be true if φ and ψ are expressed in terms of odd powers of r. See reference 4).

of point particle coupled with gravitational field. Using the functional Taylor series about the point $\Pi_\mu = T_\mu^0$

$$R_\mu(y) = \int d^4z [\Pi_\nu(z) - T_\nu^0(z)][\delta R_\mu(y)/\delta \Pi_\nu(z)]_{\Pi=T} + \cdots ,$$

and the relation

$$\int d^4y\, N^\mu(y)[\delta R_\mu(y)/\delta \Pi_\nu(z)] = -\delta_{\nu 0},$$

which can be obtained from

$$\left[-\frac{1}{2}(1/\Delta)\left(\pi^T + \frac{1}{\Delta}\pi^{kl}{}_{,kl}\right)\right]_{,0} = -\int d^4y\, N^\mu(y)[\delta R_\mu(y)/\delta \Pi_0(x)]$$

by imposing the coordinate condition (2·9a), we can reduce Eq. (2·13) to

$$\begin{aligned} \underset{a}{\dot{x}^i} &= \partial \boldsymbol{H}/\partial \underset{a}{p_i}, \\ \underset{a}{\dot{p}_i} &= -\partial \boldsymbol{H}/\partial \underset{a}{x^i}, \end{aligned} \tag{2·18}$$

with

$$\boldsymbol{H} = \int d^3x H.$$

As is easily seen, Eqs. (2·18) are simply the Hamiltonian equations obtained from the reduced action integral (2·11).

Thus we have shown that the equation of motion of point particles obtained from the reduced canonical formalism is equivalent to the EIH Eq. Our Hamiltonian density H for each particle has quite a different form from

$$H' = \underset{a}{p_i}\,\underset{a}{\dot{x}^i} - \bar{L} = (\underset{a}{p_i}\underset{a}{p^i} + m^2)^{1/2}\bar{N}^0 - \underset{a}{p_i}\bar{N}^i,$$

where $\bar{L}$ is defined by Eq. (2·16) and $\underset{a}{p_i} = \partial \bar{L}/\partial \underset{a}{\dot{x}^i}$. Therefore we need not impose the conditions (1·5) by which Infeld deduced the EIH Eq. from $\bar{L}$.

§ 3. Approximate form of the Hamiltonian

We shall now get the explicit expression for the Hamiltonian density $H = -\Delta g^T$ by using the approximation method which consists in disregarding the terms beyond the fourth order of the velocities of particles, as EIH did.

We shall first rewrite Eq. (2·3a). The relevant part of $g^{1/2}R$ is:

$$g^{1/2}R = -\Delta g^T + \frac{1}{2}g_{ln,m}g_{lm,n} - \frac{1}{4}g_{mn,l}g_{mn,l} - \frac{1}{2}g_{ll,n}g_{mn,m} + \frac{1}{4}g_{ll,n}g_{ll,n}. \tag{3·1}$$

It should be noticed that in our coordinate system g_{ij} can be expressed in the form (2·12), and that the transverse traceless parts g_{ij}^{TT}, π^{ijTT} can be neglected,

because they have no effect on the EIH approximation as mentioned in § 1. Hence Eq. (3·1) can be written as

$$g^{1/2}R=-\Delta g^T-\frac{1}{8}g^T\cdot\Delta g^T, \tag{3·1'}$$

where divergence terms are discarded.

For our purposes, the differentiated forms of the coordinate conditions (2·9) are more relevant. If we express the right-hand side in terms of π^{ij} and g_{ij}, we find

$$\pi^{ii}=0, \tag{3·2a}$$

$$\Delta g_{ij,j}-\frac{1}{4}g_{kl,kli}-\frac{1}{4}\Delta g_{jj,i}=0. \tag{3·2b}$$

Accordingly, in our approximation the term π^2 in the constraint Eq. (2·3a) is reduced to

$$\pi^2=0. \tag{3·3}$$

And, also,

$$g^{-1/2}\pi^{ij}\pi_{ij}\cong\pi^{ij}\pi_{ij}=\pi^{ij}\pi^{ijT}+2\pi^{ij}\pi^{j}{}_{,i}=\frac{1}{2}\pi^{ij}{}_{,j}\frac{1}{\Delta^2}\pi^{kl}{}_{,kli}-2\pi^{ij}{}_{,j}\pi^{i}. \tag{3·4}$$

Thus our $H=-\Delta g^T$ is determined, in our approximation, from the following simultaneous equations:

$$-\Delta g^T=\frac{1}{8}g^T\cdot\Delta g^T-2\pi^{ij}{}_{,j}\pi^{i}+\frac{1}{2}\pi^{ij}{}_{,j}\frac{1}{\Delta^2}\pi^{kl}{}_{,kli}+\sum_a(p_{ai}p_a^i+m_a^2)^{1/2}\delta(\boldsymbol{x}-\boldsymbol{x}_a), \tag{3·5a}$$

$$-2\pi^{ij}{}_{,j}=-2(\Delta\pi^i+\pi^j{}_{,ij})=\sum_a p_a^i\delta(\boldsymbol{x}-\boldsymbol{x}_a). \tag{3·5b}$$

The last term on the right-hand side of (3·5a) has a form

$$\begin{aligned}(p_{ai}p_a^i+m_a^2)^{1/2}&=m_a\left[1+p_{ai}p_{ai}\left(1-\frac{1}{2}g^T\right)/m_a^2\right]^{1/2}\\&=m_a\left\{1+\frac{1}{2}p_{ai}p_{ai}\left(1-\frac{1}{2}g^T\right)/m_a^2-\frac{1}{8}(p_{ai}p_{ai})^2/m_a^4+\cdots\right\}.\end{aligned} \tag{3·6}$$

Let us expand g^T in a power series of the velocity $\boldsymbol{v}=\dot{\boldsymbol{x}}$ of particle:

$$g^T={}_2g^T+{}_4g^T+\cdots \tag{3·7}$$

where the indices written as left subscripts indicate the order of v absorbed by the g^T's.* The lowest order approximation of Eq. (3·5a) is

$$-\Delta_2 g^T=\sum_a m_a\delta(\boldsymbol{x}-\boldsymbol{x}_a). \tag{3·8}$$

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* $\dot{\boldsymbol{x}}$ is of the order "one" and $\ddot{\boldsymbol{x}}$ (hence $1/r$) of the order "two".

A solution of this equation is

$$ {}_2g^T = \sum_a m_a/4\pi r_a, \tag{3·9} $$

with

$$ r_a = \{(x^k - x_a^k)(x^k - x_a^k)\}^{1/2}. $$

On inserting Eq. (3·9) into the first term on the right-hand side of Eq. (3·5a) and adopting the second term of Eq. (3·6) for the factor $(p_a^t p_a^t + m_a^2)^{1/2}$ of the last term in Eq. (3·5a), we have for ${}_4g^T$

$$ -\Delta {}_4g^T = -(m_1 m_2/32\pi)\{\delta(\boldsymbol{x}-\boldsymbol{x}_1)/r_2 + \delta(\boldsymbol{x}-\boldsymbol{x}_2)/r_1\} + \sum_a p_a^t p_a^t \delta(\boldsymbol{x}-\boldsymbol{x}_a)/2m_a. \tag{3·10} $$

Here the term corresponding to the self-energy is discarded. The solution of Eq. (3·10) is

$$ {}_4g^T = \sum_a \{-m_1 m_2/128\pi^2 r_a r_a + p_a^t p_a^t/8\pi m_a r_a\}, \tag{3·11} $$

with

$$ r_a = \{(x_1^k - x_2^k)(x_1^k - x_2^k)\}^{1/2}. $$

Therefore, we have

$$ \begin{aligned} \frac{1}{8}\int g^T \cdot \Delta g^T d^3x &= \frac{1}{8}\int d^3x \{{}_2g^T \cdot \Delta {}_2g^T + {}_4g^T \cdot \Delta {}_2g^T + {}_2g^T \cdot \Delta {}_4g^T\} \\ &= -m_1 m_2/16\pi r + m_1 m_2 (m_1 + m_2)/2(16\pi)^2 r^2 \\ &\quad - \{(m_2/m_1) p_1^t p_1^t + (m_1/m_2) p_2^t p_2^t\}/2(16\pi) r. \end{aligned} \tag{3·12} $$

The lowest order solution (which is of order v^3) of Eq. (3·5b) for π^t is

$$ \pi^t = -\sum_a (p_a^k/2)\left(\delta^{jk}/\Delta - \frac{1}{2}\partial_j\partial_k/\Delta^2\right)\delta(\boldsymbol{x}-\boldsymbol{x}_a) = \sum_a \{p_a^j/8\pi r_a - p_a^k r_{a,jk}/32\pi\}, \tag{3·13} $$

where we have used the relation

$$ (1/\Delta)\delta(\boldsymbol{x}-\boldsymbol{x}_a) = -r_a/8\pi. $$

On the other hand, we have in our approximation

$$ (1/\Delta)\pi^{kl}{}_{,kl} = \sum_a p_a^l r_{a,l}/16\pi. \tag{3·14} $$

Therefore,

$$ \int d^3x \left[-2\pi^{ij}{}_{,j}\pi^i + \frac{1}{2}\pi^{ij}{}_{,j}\frac{1}{\Delta}\pi^{kl}{}_{,kl}\right] = 4p_1^i p_2^i/16\pi r - 3p_1^i p_2^j r_{,ij}/32\pi. \tag{3·15} $$

Collecting the results (3·12), (3·15) and (3·6), we find

$$\boldsymbol{H}=\int H d^3x=\sum_a \{p_i p_i/2m-(p_i p_i)^2/8m^3\}-Gmm/r+G^2 mm(m+m)/2r$$
$$-\frac{3}{2}G\{(m/m)p_i p_i+(m/m)p_i p_i\}/r+4Gp_i p_i/r-\frac{3}{2}Gp_i p_j r_{,ij}. \quad (3\cdot16)$$

From this Hamiltonian, the Poisson bracket equation of motion follows. If we express this equation in terms of $\boldsymbol{x}$, $\dot{\boldsymbol{x}}$ and $\ddot{\boldsymbol{x}}$, the EIH Eq. (1·3) may be obtained. In addition to this direct method, now we show that our Hamiltonian is nothing but the one obtained from Infeld's Lagrangian (1·4).

From Eq. (2·14a), p_a^i is written as

$$p_i=(1/\overline{N^0})(p_i p^l+m^2)^{1/2}\overline{g_{ij}}(\dot{x}^j+\overline{N^j}). \quad (3\cdot17)$$

We shall now express N^μ in terms of x_a^i, and p_{ai}. The field equation obtained from varying Eq. (2·2) with respect to π^{ij} reads

$$g_{ij,0}=2N^0 g^{-1/2}\left(\pi_{ij}-\frac{1}{2}g_{ij}\pi\right)+N_{i|j}+N_{j|i}. \quad (3\cdot18)$$

The coordinate conditions (3·2) must be compatible with this equation. In our approximation procedure, we can put on the right-hand side of Eq. (3·18)

$$N^0=g=1, \quad \pi=0, \quad N_{i|j}=N_{i,j}.$$

If we substitute Eq. (3·18) in (3·2b), it follows that

$$\Delta N_i=-2\left(\pi^{ij}{}_{,j}-\frac{1}{4}(1/\Delta)\pi^{kl}{}_{,kli}\right), \quad (3\cdot19)$$

which has the solution

$$N_i=\sum_a \{-p_{ai}/4\pi r+p_j r_{,ij}/32\pi\}. \quad (3\cdot20)$$

Since it is sufficient to use only the static part of N^0 in our approximation, we adopt the result of ADM (ADM–Va (3·16a)) for N^0:

$$N^0=1-\sum_a m/16\pi r. \quad (3\cdot21)$$

Thus, up to the fourth order of velocity v,

$$\sum_a p_i p_i/2m=\frac{1}{2}\sum_a m\{\dot{x}^i\dot{x}^i+g^T\dot{x}^i\dot{x}^i\dot{x}^i+\dot{x}^i\dot{x}^i/(\overline{N^0})^2+2\dot{x}^i\overline{N^i}+(\dot{x}^i\dot{x}^i)^2\}$$
$$=\frac{1}{2}\sum_a m\{\dot{x}^i\dot{x}^i+(\dot{x}^i\dot{x}^i)^2+6G\dot{x}^i\dot{x}^i/r\}-8Gmm\left\{\dot{x}^i\dot{x}^i/r-\frac{1}{8}\dot{x}^i\dot{x}^j r_{,ij}\right\}. \quad (3\cdot22)$$

In our approximation procedure, it does not affect our result to replace p_{ai} by $m\dot{x}^i$ in the other terms of Eq. (3·16). Finally, we have

$$H=\sum_a\left\{\frac{1}{2}m_a\dot{x}_a^i\dot{x}_a^i+\frac{3}{8}m_a(\dot{x}_a^i\dot{x}_a^i)^2\right\}+Gm_1m_2\left\{-1/r+\frac{1}{2}G(m_1+m_2)/r^2\right.$$
$$\left.+\frac{3}{2}\sum_a\dot{x}_a^i\dot{x}_a^i/r-4\dot{x}_1^i\dot{x}_2^i/r-\frac{1}{2}\dot{x}_1^i\dot{x}_2^jr_{,ij}\right\}. \tag{3·23}$$

Accordingly,

$$L=\sum_a p_{ia}\dot{x}_a^i-H,$$

is nothing but the Infeld's Lagrangian (1·4).

Thus, we have shown that our equation of motion of particles, derived from the reduced canonical formalism, is identical with the original EIH Eq.

§ 4. Concluding remarks

In the preceding sections we have seen that the reduced Hamiltonian

$$H=\int \Pi_0 d^3x, \tag{4·1}$$

derived from the original Lagrangian by inserting the solutions of the constraint equations (2·8) and the coordinate condition (2·9), gives the equation of motion of two-body problem proposed by EIH.

In the last section we have obtained the Hamiltonian which contains the terms up to the fourth order of velocity of particle. It has also been shown that this Hamiltonian is identical with that of Infeld. Our formalism will be generalized to the higher order approximation. Since we hardly see any physical meaning in developing the calculations one further step, such a general formulation will be of little practical value. Further, we have not known anything about the convergency of approximation procedure. These are the reasons why we have not entered into the formal generalization of our approximation procedure beyond the fourth order.

Finally, we shall discuss some problems related with the quantization of space-time coordinates. In connection with the divergence difficulty encountered in the quantum field theory, some investigators have conjectured that it may be removed by quantizing the space-time coordinates which are treated as *c*-number parameters in the conventional field theory. In spite of such attempts at the quantization of space-time coordinates, it seems to us that neither guiding principles nor promising procedures have been discovered. In the canonical formalism of gravitational field developed in this paper, our physical space coordinates are identified with some components of metric tensor. Therefore, it will be natural to quantize physical space-time coordinates when the gravitational field is subjected to quantization. We can see this from the analogy of quantum electrodynamics where the scalar and longitudinal parts of electromagnetic four potential mix with the transverse part when a Lorentz transformation

is made. In quantum electrodynamics, therefore, in order to explicitly exhibit Lorentz invariancy it was desirable to quantize all components of four potential symmetrically. Similarly, for the purpose of exhibiting the covariancy manifestly it will be natural to treat the physical space-time coordinates as dynamical operators.

In order to treat the physical space-time coordinates as dynamical variables it is necessary to deny the coordinate condition (2·9). In so far as this condition holds, the dynamical equation for physical space-time coordinates does not follow from the action integral (2·11). Similar situation has already appeared in the quantum electrodynamics whose Lagrangian density is

$$L_{EM}=\frac{1}{2}\{A_\nu F^{\mu\nu}{}_{,\mu}-A_{\nu,\mu}F^{\mu\nu}\}+\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{4·2}$$

from which dynamical equation for A_0 cannot be obtained. Now we put

$$\begin{aligned} X^0(x)&=-\frac{1}{2}\frac{1}{\Delta}\pi^{ii}\ , \\ X^i(x)&=g_i(x)-\frac{1}{4}\frac{1}{\Delta}g^T{}_{,i}, \end{aligned} \tag{4·3}$$

instead of Eq. (2·9). Hence the action integral (2·2) can be written as

$$I'=\int d^4x\{L_M(x)+\pi^{ijTT}\partial_t g_{ij}^{TT}+\Pi_\mu(x)\partial_t X^\mu(x)+N^\mu R_\mu\}, \tag{4·4}$$

where L_M is the Lagrangian density for matter fields. Thus we can treat and then quantize $X^\mu(x)$ on an equal footing as g_{ij}^{TT}. So far we have mentioned only the advantage of denying the coordinate condition (2·9). However, we must touch upon a disadvantageous point. The canonical theory of general relativity does not directly determine N^μ. As has been shown in § 3, the consistency of the coordinate condition (2·9) and the gravitational equation has played a role of determining N^μ. This is due to the following reason. When gravitational field is not coupled with the matter field, the dynamical equation for X^μ derived from (4·4),

$$\frac{\partial}{\partial t}X^\mu(x)=-\int d^4y\,N^\lambda(y)[\delta R_\lambda(y)/\delta\Pi_\mu(x)],$$

can be written in the lowest order approximation

$$\eta^\mu\equiv N^\mu-\frac{\partial X^\mu}{\partial t}=0. \tag{4·5}$$

Since both N^μ and X^μ are undetermined, we cannot regard Eq. (4·5) as a true dynamical equation. Therefore, unless we have not taken account of the coordinate condition, we must either regard Eq. (4·5) as a supplementary condi-

tion* or assign some suitable values such as (3·20) and (3·21) to N^μ from the beginning.

In the above we have discussed only the formal point of quantizing physical space-time coordinates. There still remains several important problems to be solved, such as the meaning of measurements in the quantized space-time, and the dependency of field variable on the quantized physical space-time coordinates. Especially, unless the meaning of measurements is clarified, our approach will be criticized as being merely a formal approach. However, we stress here that the problem of quantizing the physical space-time coordinates can be neither separated from, nor discussed independently of, the quantum theory of gravitational field.

Appendix

Derivation of Eq. (2·15)

We start with Eqs. (2·14a) and (2·14b). For the sake of simplicity we here pay attention to one particular particle. Accordingly, we drop the index a.

$$p_i=(1/\overline{N^0})\,(p_ip_j\overline{g^{ij}}+m^2)^{1/2}\overline{g_{ij}}(\dot{x}^j+\overline{N^j}), \tag{A·1}$$

$$-\dot{p}_i=\int d^3x\left\{N^0_{,i}(p_ip^i+m^2)^{1/2}+\frac{1}{2}N^0(p_ip^i+m^2)^{-1/2}p_ip_m\mathcal{G}^{im}_{,i}-p_lN^l_{,i}\right\}. \tag{A·2}$$

We define $\overline{L}$ by

$$-\overline{L}/m^2=(\overline{ds}/dt)/m=\overline{N^0}(p_ip^i+m^2)^{-1/2}, \tag{A·3}$$

where $\overline{ds}=(-\overline{g_{\mu\nu}}dx^\mu dx^\nu)^{1/2}$. The insertion of (A·1) into the left-hand side of (A·2) yields

$$\begin{aligned}-\dot{p}_i&=-\frac{d}{dt}\left\{-(m^2/\overline{L})\overline{g_{ij}}(\dot{x}^j+\overline{N^j})\right\}\\&=(m^2/\overline{L})\left\{\overline{L}\frac{d}{dt}(1/\overline{L})\overline{g_{ij}}\dot{x}^j+\overline{g_{ij,k}}\dot{x}^k\dot{x}^j+\overline{g_{ij,0}}\dot{x}^j+\overline{g_{ij}}\ddot{x}^j\right.\\&\qquad\left.+L\frac{d}{dt}(1/\overline{L})\overline{g_{i0}}+\overline{g_{i0,0}}+\overline{g_{i0,k}}\dot{x}^k\right\}.\end{aligned} \tag{A·4}$$

Here we have used the formula

* Recently, I. Watanabe have attempted to quantize the space-time coordinates starting with the parametrized formalism for the system of ordinary matter fields (Prog. Theor. Phys. **24** (1960), 465). He used also Eq. (4·5) as a supplementary condition and added $g_{\mu\nu}\eta^\mu\eta^\nu$ to the original Lagrangian density on analogy of quantum electrodynamics where $(A^\mu_{,\mu})^2$ has been added to Eq. (4·2). In his formulation, contrary to our case, gravitational field did not play any role. The meaning of Eq. (4·5) may be clarified only from the viewpoint of our formalism.

$$\frac{d}{dt}\overline{\varphi}=\overline{\varphi_{,0}}+\overline{\varphi_{,s}}\overset{s}{\dot{x}}.$$

On the other hand, the terms on the right-hand side of (A·2) are written as

$$(p_i p^i+m^2)^{1/2}N^0_{,i}=(m^2/L)\left\{\frac{1}{2}(N^0)^4 g^{0m}g^{0n}g_{mn,i}-(N^0)^2 g^{0m}g_{0m,i}+\frac{1}{2}g_{00,i}\right\},$$

$$\frac{1}{2}N^0(p_l p^l+m^2)^{-1/2}p_l p^n g_{mn}g^{lm}{}_{,i}=\frac{1}{2}(L/m^2)p^m p^n g_{mn,i}, \tag{A·5}$$

$$-p_l N^l_{,i}=-p^m g_{0m,i}+p^l N^m g_{lm,i}.$$

Taking into account Eq. (3·18) and the relation

$$-\frac{d}{dt}\log\overline{L}=(N^0)^{-1}\{\overline{K_{ij}}\dot{x}^i\dot{x}^j-2\dot{x}^i(\overline{N^0_{,i}}-\overline{K_{ij}N^j})-\overline{N^0_{,0}}-\overline{N^0_{,i}N^i}+\overline{N^i N^j K_{ij}}\} \tag{A·6}$$

with $K_{ij}=-N^0\,{}^4\Gamma^0_{ij}$, which corresponds to

$$\frac{d}{d\overline{s}}(dt/d\overline{s})=-\overline{{}^4\Gamma^0_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}(dt/d\overline{s})^2,$$

we can rewrite (A·2) in a form

$$\ddot{x}^i-\frac{d}{dt}(\log\overline{L})\dot{x}^i+\overline{\Gamma^i_{jk}}\dot{x}^i\dot{x}^j-2\overline{N^0K_j{}^i}\dot{x}^j+2\overline{N^0(N^i/N^0)_{|j}}\dot{x}^j+2\overline{N^iN^jK_{jk}/N^0}\dot{x}^k$$

$$+\overline{N^0g^{ij}(N_j/N^0)_{,0}}-\overline{N^0_{,j}N^jN^i/N^0}+\overline{N^iN^jN^kK_{jk}/N^0}+\frac{1}{2}\overline{g^{ij}\{(N^0)^2-N^kN_k\}_{,j}}=0. \tag{A·7}$$

With the aid of the relations:

$${}^4\Gamma^i_{j0}=N^0\{-K_j{}^i+(N^i/N^0)_{|j}+N^iN^kK_{jk}/(N^0)^2\},$$

$${}^4\Gamma^i_{00}=N^0g^{ij}(N_j/N^0)_{,0}+\frac{1}{2}g^{ij}\{(N^0)^2-N^kN_k\}_{,j}-N^0_{,j}N^iN^j/N^0+N^iN^jN^kK_{jk}/N^0, \tag{A·8}$$

Eq.(A·7) can be written as

$$\ddot{x}^i+\overline{{}^4\Gamma^i_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}-\frac{d}{dt}(\log\overline{L})\dot{x}^i=0$$

which is nothing but Eq.(2·15).

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Note added in proof: After the manuscript of the present paper was submitted to the editors, the author had a chance to see Infeld-Plebasński's book "*Motion and Relativity*" (Pergamon Press, 1961) in which their recent works on the problem of motion in gravitational theory are collected. The present author is informed by this book that recently Infeld found the reasonable Lagrangian for two-body problem without imposing the superfluous condition (1·5). While his treatment is confined only to the Lagrangian formalism, the method in the present paper is based on the Hamiltonian formalism which is useful to treat the dynamical problems and to represent the energy of system.

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Generalized Theory of Condensing Systems†

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The theory in the previous paper II, which has dealt with the two-phase separation in condensing systems on the basis of the "cluster" concept and by means of a mathematically rigorous method, is generalized to the case of systems with *volume-dependent* cluster integrals.

The cluster integrals are assumed to satisfy several mathematical conditions, which are considered to represent the essential features of the real systems. One of these conditions is concerned with the effects of the volume dependence of the cluster integrals composed of a comparatively small number of molecules. Another essential condition is that the contribution, per molecule, to a cluster integral in which the number l of the constituent molecules is infinitely large and of the same order as the volume V of the system, should be definite, finite, and independent of the volume per molecule, V/l, in a certain range.

By means of a function $l^*(N)$ which satisfies certain conditions, the sizes of clusters are classified into "large" $[l>l^*(N)]$ and "small" $[l\leqq l^*(N)]$. The "small" cluster part of the system in consideration and that of the "(0)-system" (i.e. a system with volume-independent cluster integrals) are proved to be practically equivalent. Then, after some discussions of the "large" clusters, it is proved that, in a certain range of the specific volume, the isotherm is horizontal and a "huge" cluster (i.e. a cluster of size comparable to the volume of the system) coexists with the saturated set of "small" clusters. This corresponds to the liquid-vapour coexistence range. And it is proved that, in the gas range, the system is thermodynamically equivalent to the (0)-system and has practically no "large" clusters.

A remark is given on the analytical properties of the singularity representing the condensation. It is noted that the present theory applies to both the cases of "analytical" singularity and of "non-analytical" singularity.

A non-rigorous treatment which leads to the same results as those in the text, is given in the Appendix.

Contents

§ 1. Introduction; the G-system.—Assumptions (G·1), (G·2), (G·3), (G·4), (G·5), (G·6); Lemmas 1a, 1b, 1c.
§ 2. A few lemmas on the volume dependence of the cluster integrals.—Lemmas 2a, 2b, 2c.
§ 3. Discrimination between "large" and "small" clusters (in the G- and (0)-systems).—Lemma 3.
§ 4. Relation between the "small" clusters in the G-system and those in the (0)-system.—Lemma 4.
§ 5. "Large" clusters in the G-system.—Lemma 5.

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† The contents of this paper were read at the meeting of the Physical Society of Japan held at Tokyo College of Science on April 2, 1959 [cf. the abstract book of the meeting, **2**, p. 413].

§ 6. Maximum term in the G-system.—Lemma 6.
§ 7. The Γ-system; its relation with the G-system.—Lemma 7.
§ 8. Appearance of a "huge" cluster (i.e. the two-phase separation) in the G-system.—Theorems 1, 1a; Corollaries 1, 2.
§ 9. Gaseous state of the G-system.—Theorems 2, 2a; Corollary.
§ 10. Concluding remarks.

Appendix A: Non-rigorous treatment.
§ A1. Introduction.
§ A2. Method of treatment.
§ A3. Remarks and conclusions.

Appendix B: Comparison of notations.

§ 1. Introduction; the G-system

§ 1·1. In the present paper, a generalization is given of the cluster theory of condensing systems developed in the author's previous paper[2] (which will be referred to as II).

§ 1·2. The configurational partition function $\mathit{\Omega}_N$ of a system of N interacting particles (each of mass m) in a volume V at a temperature T, is given by

$$\mathit{\Omega}_N(V,T)=\lambda^{3N}Q_N(V,T)=\frac{1}{N!}\int_V\cdots\int_V W_N\{N\}\,d\{N\}\;; \tag{1}^{\dagger}$$

where Q_N is the partition function of the system and $\lambda=h\cdot(2\pi mkT)^{-1/2}$; and where, for classical systems,

$$W_N\{N\}=\exp(-U_N\{N\}/kT), \tag{2}$$

$U_N\{N\}$ being the total potential energy; and, for quantum mechanical systems,

$$W_N\{N\}=N!\,\lambda^{3N}\sum_n \varphi_n^*\{N\}\,e^{-E_n/kT}\varphi_n\{N\}, \tag{3}$$

$\varphi_n\{N\}$ being the nth energy-eigenfunction (with eigenvalue E_n).

The cluster integrals[3),4)] b_l (composed of l particles) are defined by

$$b_l(V,T)=\frac{1}{l!\,V}\int_V\cdots\int_V S_l\{l\}\,d\{l\}\;; \tag{4}^{\dagger\dagger}$$

where, for classical systems with two-body forces,

$$S_l\{l\}=\sum_{l\geq i>j\geq 1}\prod f_{ij} \tag{5}$$

sum over all products consistent with cluster

$$f_{ij}=f(r_{ij})=\exp\{-u(r_{ij})/kT\}-1, \tag{6}$$

$u(r_{ij})$ being the pair potential, $[U_N\{N\}=\sum_{i>j}\sum u(r_{ij})]$; and, for general systems (including quantum mechanical systems),

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† $\{N\}\equiv(\boldsymbol{r}_1,\boldsymbol{r}_2\cdots,\boldsymbol{r}_N)$; $d\{N\}\equiv d\boldsymbol{r}_1\,d\boldsymbol{r}_2\cdots d\boldsymbol{r}_N$.

†† $b_1\equiv 1$.

$$S_l\{l\} = \sum (-)^{\kappa-1}(\kappa-1)\,! \prod_{i=1}^{\kappa} W_{\nu}\{\nu_i\} \tag{7}$$

sum over all sets $\{\{\nu_1\}, \{\nu_2\}, \cdots, \{\nu_\kappa\}\}$ (for various values of κ) of unconnected sets $\{\nu_1\}, \{\nu_2\}, \cdots, \{\nu_\kappa\}$ whose union is the set $\{l\}$.

Then we have the following expansion due to Ursell and Mayer *et al.*[3,4]

$$\Omega_N(V, T) = \sum_{m_l \geqq 0} \Big(\sum_{l=1}^{N} l m_l = N \Big) \prod_{l=1}^{N} \frac{\{V b_l(V, T)\}^{m_l}}{m_l!} \tag{8}$$

sum over all sets $\{m_1, m_2, \cdots, m_N\}$ of non-negative integers m_l satisfying the condition stated in the parentheses.

§ 1·3. In II,[2] starting from 1·2(8),* we discussed a condensing system (classical or quantum mechanical) [in which the total number of molecules N tends to infinity with $v = V/N$ (volume per molecule) fixed] especially in the state of vapour-liquid coexistence:—we proved, by a mathematically rigorous method, the appearance of the liquid phase, i.e. the separation of the gaseous and liquid phases. The theory in II was based on the following assumptions about the cluster integrals $b_l(V)$:

(a) *One may use the limiting value*** $b_l^{(0)} = \lim_{V\to\infty} b_l(V)$ *in place of* $b_l(V)$ *;— in other words, one may neglect the volume dependence of the cluster integrals.*

(b) $b_l^{(0)} > 0$ *for every* l.

(c) $\lim_{l\to\infty} (b_l^{(0)})^{1/l} = b_0^{(0)}$ *exists and* $0 < b_0^{(0)} < +\infty$.

(d) *The series* $\sum_{l=1}^{\infty} l b_l^{(0)} (b_0^{(0)})^{-l}$ *is convergent.*

§ 1·4. In the present paper, we adopt instead of (a)—(d) the following assumptions (G·1)—(G·6) which are more general than (a)—(d).

(G·1) $|b_l(V) - b_l^{(0)}| \leqq c_0 \Theta(l) l^{\rho}/V^{\sigma}$ *whenever* $l^r/V^s \leqq c_1$; *where* $b_l^{(0)}$ *is independent of* V; *and where* c_0, c_1, ρ, r, σ, *and* s *are constants, independent of* l *and* V, *such that* $c_0 \geqq 0$, $c_1 > 0$, $\rho \geqq 0$, $r \geqq 0$, $\sigma > 0$, $s > 0$; *and where* $\Theta(l)$ *is a function (of* l *but not of* V*) which is* $\geqq 0$ *for every* l *and for which there exists a constant* $G(>0)$, *independent of* l *and* V, *such that* $\{\Theta(l)\}^{1/l} \leqq G$ *for every* l.

(G·2) $b_l(V) > 0$ *whenever* $V_0 > \max\{V_0, l v_0\}$, ***where* $V_0(\geqq 0)$ *and* v_0 $(\geqq 0)$ *are constants independent of* l *and* V.

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* Formula (8) of §1·2 is referred to as 1·2(8).

** Or better, $b_l^{(0)}$ [$= b_l^{(0)}(T)$] is defined[1] as

$$b_l^{(0)} = (1/l\,!) \int_{\infty} \cdots \int_{\infty} S_l\{l\} d\{l-1\}.$$

The lth particle being fixed at a point, the integration, with respect to each of the other $l-1$ particles, is extended over the whole infinite space (with no boundary).

We must assume the potentials between the molecules to tend sufficiently rapidly to zero as the molecules separate in space, in order that $b_l^{(0)}$ may be definite and finite.

*** $\max\{\alpha, \beta\}$ means the larger of α and β.

(G·3) *As* $l\to\infty$, $\{b_l(lu)\}^{1/l}$ *converges to* $b_0^{\dagger}$ *uniformly in every interval interior to the interval* $u_1<u<+\infty$,* *where* $b_0^{\dagger}$ *and* u_1 *are constants, independent of* l *and* u *(and* V*), such that* $0<b_0^{\dagger}<+\infty$ *and* $v_0\leqq u_1<+\infty$.

(G·4) *Given any* $\epsilon>0$, *there exist an integer* $l'=l'(\epsilon)[>0]$ *and a real number* $V'=V'(\epsilon)[\geqq V_0]$, *depending on* ϵ, *such that* $\{b_l(V)\}^{1/l}<b_0^{\dagger}+\epsilon$ *whenever* $l\geqq l'$ *and* $V>\max\{V', lu_0\}$, *where* u_0 *is a constant, independent of* l *and* V, *such that* $u_1\geqq u_0\geqq v_0$.

(G·5) *The series* $\sum_{l=1}^{\infty} l b_l^{(0)}(b_0^{\dagger})^{-l}$ *is convergent and is* $<u_0^{-1}$.

(G·6) $\lim_{l\to\infty}(b_l^{(0)})^{1/l}=b_0^{(0)}$ *exists and* $0<b_0^{(0)}<+\infty$.

§ 1·401. *(Remark)* In the statements of (G·1)—(G·6), and throughout the present paper, the temperature T is considered to be fixed at a certain value. In general, the constants c_0, c_1, ρ, r, σ, s, G, V_0, v_0, $b_0^{\dagger}$, u_1, u_0, and $b_0^{(0)}$ [and also $g^{(0)}$, G_0 in § 3·1] may depend on T; and $b_l^{(0)}$ and $\theta(l)$ [and also $\theta(l)$ defined in § 2·1] may depend on T besides on l; and l' and V' may depend on T besides on ϵ.

§ 1·402. Here we shall give a few lemmas resulting immediately from the assumptions in § 1·4.

Lemma 1a. $b_l^{(0)}=\lim_{V\to\infty} b_l(V)$ *with* l *fixed.*

[This follows from (G·1).]

Lemma 1b. $b_l^{(0)}>0$ *for every* l.

[This follows from Lemma 1a and (G·2).]

Lemma 1c. $\lim_{l\to\infty}(b_l^{(0)})^{1/l}\leqq b_0^{\dagger}$.

[If in (G·4) we make $V\to\infty$ with l fixed, then we have by Lemma 1a

$$\lim_{V\to\infty}\{b_l(V)\}^{1/l}=(b_l^{(0)})^{1/l}\leqq b_0^{\dagger}+\epsilon \tag{1}$$

for $l\geqq l'$ (i.e. for almost** every l). Here $\epsilon>0$ is arbitrary. Hence

$$\limsup_{l\to\infty}(b_l^{(0)})^{1/l}\leqq b_0^{\dagger}. \tag{2}$$

From this and (G·6) we have the lemma.]

§ 1·403. *(Remark)* If we put $V=Nv$ and regard $\{b_l(Nv)\}^{1/l}$ (for all positive integers l and N such that $Nv>\max\{V_0, lu_0\}$) as a double sequence, then (G·4) means that

$$\limsup_{\substack{l\to\infty\\ N\to\infty}}\{b_l(Nv)\}^{1/l}\leqq b_0^{\dagger}. \tag{1}***$$

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* "As $l\to\infty$, $f_l(u)$ converges uniformly in every interval interior to the interval (u_1, u_2)" is often expressed as: "As $l\to\infty$, $f_l(u)$ converges *uniformly in the wider sense* in the interval (u_1, u_2)".

** By "almost every (any) l" we mean "every (any) value of l except at most a finite number of values of l," —in other words, "every (any) sufficiently large value of l".

*** For the definition of *the greatest limit* (lim sup) *of a double sequence*, see a mathematical book.

From this, if we take (G·3) into account, we have

$$\limsup_{\substack{l\to\infty \\ N\to\infty}} \{b_l(Nv)\}^{1/l}=b_0^{\dagger}. \qquad (2)$$

§ 1·41. The set of generalized assumptions (G·1)—(G·6) will be called the "G-condition", while the old set of assumptions (a)—(d) will be called the "(0)-condition". A system satisfying the G-condition may be called a "G-system".

§ 1·411. It is easy to see that the (0)-condition is a sufficient (but not necessary) condition for the G-condition.

In fact, we can show that the (0)-condition is a special case of the G-condition. First, (G·1), with $c_0=0$ and $c_1=+\infty$, is reduced to (a); and then (G·2), with $v_0=V_0=0$, becomes (b) [(b) is also given by Lemma 1b]. Then, (G·3), considered together with (a), leads to (c) if one writes $b_0^{(0)}$ in place of $b_0^{\dagger}$; and (G·4), considered together with (a), is consistent with (c). Then, (G·5), with $b_0^{(0)}$ in place of $b_0^{\dagger}$ and with $u_0=0$, becomes (d). Lastly, (G·6) is nothing but (c).

§ 1·412. It should be remarked that the G-condition does not demand the neglect of the volume dependence of the cluster integrals. Namely, in general, the G-system, which we shall treat in the present paper, is a system for which the cluster integrals depend on the volume V, while the (0)-system, which satisfies the (0)-condition and was treated in II, is a system for which the cluster integrals are independent of the volume V.

§ 1·42. At this point we should mention the relation between the G-system and the real system. Indeed the G-system has been purely mathematically defined as any system which satisfies the G-condition; (it does not matter whether the system is classical or quantum mechanical). But the G-condition may be considered as a mathematical representation of the essential features of the cluster integrals of the physical real systems (classical or quantum mechanical) having the ordinary potential energy function, though we shall leave it to another paper to make the rigorous proof that the G-condition is satisfied by the real systems having some usual type of interaction forces.

§ 1·421. The assumption (G·1) shows to what extent the cluster integrals for comparatively small clusters can deviate from $b_l^{(0)}$'s by virtue of their dependence on V. [Note that we may take c_1 and s/r as small as we please, and c_0, ρ/σ, and G as large as we please.]

As will be seen in the present paper, an assumption of the form (G·1) is essential for us to be able to find a border-line $l^*(N)$ (between "large" and "small" clusters) which satisfies the conditions 1°—5° in Lemma 3 and which therefore validates Lemma 6 of the paper I[1] and Lemma 4 of the present paper, both of which are necessary in order that the limiting thermo-

dynamical functions for the gaseous state (i.e. for $v \geqq v_{\text{cond}}$)* of the G-system may be equal to the corresponding functions of the (0)-system [see Theorem 2(i)].

Here we must remark that (G·1) is a generalization of the following condition: $|b_l(V) - b_l^{(0)}| \leqq 8\, r_0 \Theta(l) l / V^{1/3}$ (whenever $l/V^{1/3} \leqq 1/(2r_0)$), which has, in Lemma 1 of I, been proved to hold for the classical real systems with two-body forces of finite range r_0.

§ 1·422. The assumption (G·2) means that, for a cluster in a volume large enough** (i.e. $V > V_0$) [note that we may take V_0 as large as we please], the cluster integral should be positive except when the volume per molecule, V/l, is sufficiently small ($V/l \leqq v_0$).

For the reason stated in the footnote on p. 658 of II, the positiveness of the cluster integrals is essential to the validity of the physical interpretation of the clusters (and, therefore, to the proof of the two-phase separation in terms of a "huge" cluster).

(G·2) may be valid for sufficiently low temperatures, i.e. for all temperatures lower than some temperature which may be a little below the critical temperature, because*** at such temperatures the positive part† ($r_{ij} > a$) of the function $f(r_{ij})$ in the integrand of the cluster integral plays a predominant role†† except when the volume per molecule, V/l, is small enough to make the average distance between each pair of molecules so small that the negative part††† ($r_{ij} < a$) of $f(r_{ij})$ becomes relatively effective in the integration over the volume V. [The behaviour of the cluster integral for sufficiently small V/l may in general be very irregular; it may be positive or negative according as the number of effective negative factors (f-bonds) is even or odd.]

§ 1·423. The assumption (G·3) means that the contribution, per molecule, to a cluster integral in which the number of the constituent molecules is infinitely large and of the same order as the volume V, should be definite, finite, positive, and independent of the volume per molecule, V/l, in a certain range.

As will be seen in the present paper, this is essential to the appearance of the liquid phase coexisting with the saturated vapour and to the existence of a horizontal part of the isotherm.

It is to be noted here that (G·3) never requires the independence on V, in a certain range, of the cluster integrals themselves, which is true of the

* The subscript "cond" stands for "condensation".

** Probably, of macroscopic size.

*** Here stated specifically for the case of classical systems with two-body forces. [See 1.2(5) and 1.2(6)].

† Due to the intermolecular attraction.

†† In the range $V/l > v_0$, in which the positive part is predominant in integration, the cluster integral will be steadily increasing with V, on account of the monotonous increase of the integration domain.

††† Due to the intermolecular repulsion.

(0)-system, but only requires the independence on V/l, in a certain range, of the *limit* of $\{b_l(V)\}^{1/l}$ for l and V infinitely large and of the same order; the latter requirement is very moderate as compared with the former, and may be much more easily satisfied by the real systems; in fact, there is good reason to believe that the deviation of $\{b_l(lu)\}^{1/l}$ from the constant $b_0^\dagger$, in a certain range, is at most of the order of magnitude of the surface effects, which will vanish in the infinite system.

§ 1·424. The assumption (G·4) means that, for l and V large enough but not necessarily of the same order, the contribution, per molecule, to a cluster integral should not practically exceed the constant $b_0^\dagger$ defined in (G·3). This is essential as an assumption auxiliary to (G·3), and may be valid in view of (G·3) if the functional relation between b_l and V is not so extraordinary. [For some usual physical cases we may assume that $u_0=v_0$; but we have distinguished $u_0(\geqq v_0)$ from v_0 in order to make the statements of our assumptions more general].

§ 1·425. The assumption (G·5) is necessary in order that the condensation may occur at a finite density, [see also the first footnote (‡) in § 8·1]. This is analogous to the assumption (d) in II.

§ 1·426. The assumption (G·6) is nothing but the assumption (c) in the paper II [see also Eq. (II) on p. 359 in the paper I], which is considered to be valid for the (0)-system.

§ 1·427. Thus the G-system, which is far more general than the (0)-system, may be said to be a mathematical model representing the essential features of the real system.

§ 1·5. In the present paper the (0)-system, whose structure is relatively simple, will be used as a theoretical medium; the G-system will often be discussed by comparison with the (0)-system. The symbols with superscript (0) concern the (0)-system, while the symbols for the G-system have no superscript.

§ 1·6. The contents of §§ 2—10 may be summarized as follows:

[§ 2.] Some discussions are given of the volume dependence of the cluster integrals for comparatively small clusters (Lemmas 2a, 2b, 2c).

[§ 3.] A function $l^*(N)$ satisfying certain conditions is proved to exist (Lemma 3), and is used to discriminate between "small" clusters ($l\leqq l^*(N)$) and "large" clusters ($l>l^*(N)$) in both the G-system and the (0)-system. Then the infinite sequences of G-systems and of (0)-systems ($N=1, 2, \cdots$ to ∞) for any fixed v are considered, and the partition function of each system is expressed in terms of the functions containing "small" clusters only and the functions containing "large" clusters only.

[§ 4.] It is proved that, as far as the "small" clusters are concerned, there is no difference between the G-system and the (0)-system from a thermodynamical viewpoint (Lemma 4).

[§ 5.] Some properties of the "large" clusters in the G-system are discussed (Lemma 5).

[§ 6.] A discussion is given of the maximum term in the partition function of the G-system (Lemma 6).

[§ 7.] An assumptive system—the Γ-system—which is relatively simple from a mathematical viewpoint, is introduced as a second theoretical medium, and its relation with the G-system is discussed (Lemma 7).

[§ 8.] Making use of the results of §§ 2—7, it is proved that, in the G-system, for $v^* < v < v_{cond}$, a "huge" cluster (representing the liquid phase) exists in thermodynamical equilibrium and the isotherm is horizontal (Theorems 1, 1a; Corollaries 1, 2). This corresponds to the range of two-phase coexistence.

[§ 9.] It is proved that, for $v \geqq v_{cond}$, the G-system is thermodynamically equivalent to the (0)-system, and no "large" clusters exist in thermodynamical equilibrium (Theorems 2, 2a; Corollary). This corresponds to the gas range.

[§ 10.] A remark is given on the analytical properties of the singularity representing the condensation.

§ 1·7. At the end of this introduction we must remark that the present paper gives a generalized theory which has both the character of the theory in II[2] treating the two-phase separation for the (0)-system and the character of the theory in Ia and I[1] treating the condensation phenomena (especially the analytical properties of the singularities) for imperfect gases with volume-dependent cluster integrals.

§ 2. A few lemmas on the volume dependence of the cluster integrals

§ 2·0. In this section, for the convenience of our later arguments, we make further discussions of the volume dependence of the cluster integrals for comparatively small clusters, on the basis of the assumption (G·1).

§ 2·1. The following lemma follows from (G·1) and Lemma 1b.

Lemma 2a. *If*

$$l^r / V^s \leqq c_1 \tag{1}$$

and

$$c_0 \theta(l) l^\rho / V^\sigma \leqq \alpha, \tag{2}$$

where

$$\theta(l) = \theta(l) / b_l^{(0)}, \tag{3}$$

and where c_0, c_1, ρ, r, σ, s, and $\theta(l)$ are given in (G·1), *and where α is a constant such that $0 < \alpha < 1$, then we have*

(i) $0 < 1-\alpha \leqq b_l(V)/b_l^{(0)} \leqq 1+\alpha < 2$

(ii) $$\left| \frac{1}{l} \ln \frac{b_l(V)}{b_l^{(0)}} \right| \leqq c_0 \theta(l) l^{\rho-1} \cdot \frac{1}{V^\sigma (1-\alpha)}.$$

〈*Proof*〉—Put

$$\eta(l, V)=b_l(V)/b_l^{(0)}-1. \tag{4}$$

Then, by (G·1) and Lemma 1b, we have from (1)

$$|\eta(l, V)|\leqq c_0\theta(l)l^\rho/V^\sigma. \tag{5}$$

Hence by (2)

$$|\eta(l, V)|\leqq\alpha<1, \tag{6}$$

whence (i) is obvious. Since $\ln(1+\eta)=\sum_{n=1}^{\infty}(-)^{n-1}\eta^n/n(-1<\eta\leqq 1)$, we have by (6) and (5)

$$|\ln\{1+\eta(l, V)\}|\leqq\sum_{n=1}^{\infty}|\eta(l, V)|^n\leqq\frac{c_0\theta(l)l^\rho/V^\sigma}{1-c_0\theta(l)l^\rho/V^\sigma}, \tag{7}$$

which proves (ii), by (2).

§ 2·2. We define a function $\bar{\theta}(l)$ of $l(=1, 2, 3, \cdots)$ as

$$\bar{\theta}(l)=\max_{1\leqq l'\leqq l}\theta(l'), \tag{1}$$

where $\theta(l)$ is defined by 2·1(3).

§ 2·3. Then we have the following lemma.

Lemma 2b.

(i) *$\bar{\theta}(l)$ is a monotonically increasing function of l; hence $\lim_{l\to\infty}\bar{\theta}(l)$ exists and is finite or $+\infty$.*

(ii) (a) *$\bar{\theta}(l)\geqq\theta(l')$ for $l>l'\geqq 1$.*

(b) *$\bar{\theta}(l)\geqq\theta(l)$ for $l\geqq 1$.*

(iii) (a) $\lim_{l\to\infty}\bar{\theta}(l)=$ l. u. b. $\theta(l)$.

(b) l. u. b. $_{l\geqq l_0}\{\theta(l)/\bar{\theta}(l)\}=1$,

where l_0 is the least l for which $\bar{\theta}(l)>0$.

[For the proof, see Appendix 1 of I.[1)]]

§ 2·4. By Lemmas 2a and 2b we have:

Lemma 2c. *If we take a positive integer l^* such that*

$$l^{*r}/V^s\leqq c_1 \tag{1}$$

and

$$c_0\bar{\theta}(l^*)l^{*\rho}/V^\sigma\leqq\alpha(<1), \tag{2}$$

then we have for every $l\leqq l^*$

(i) $0<1-\alpha\leqq b_l(V)/b_l^{(0)}\leqq 1+\alpha<2$

(ii) $\left|\frac{1}{l}\ln\frac{b_l(V)}{b_l^{(0)}}\right|\leqq c_0\bar{\theta}(l^*)\max\{l^{*\rho-1}, 1\}\cdot\frac{1}{V^\sigma(1-\alpha)}$.

《*Proof*》—By (1) we have

$$l^r/V^s \leqq c_1 \quad \text{for every} \quad l \leqq l^*. \tag{3}$$

By Lemma 2b(ii), we have

$$\theta(l) \leqq \bar{\theta}(l^*) \quad \text{for every} \quad l \leqq l^*. \tag{4}$$

Hence by (2)

$$c_0\theta(l)l^\rho/V^\sigma \leqq \alpha(<1) \quad \text{for every} \quad l \leqq l^*. \tag{5}$$

Applying Lemma 2a in view of (3) and (5), we obtain (i) and then (ii), by (4).

§ 3. Discrimination between "large" and "small" clusters (in the G- and (0)-systems)

§ 3·0. In this section we make discrmination between "large" and "small" clusters. First (§§ 3·1 & 3·2) we prove the existence of a function $l^*(N)$ satisfying certain conditions, which is to be used as a *border-line* between "large" and "small". Then (§ 3·3), using $l^*(N)$, we define the "large" cluster part and the "small" cluster part of the partition functions of the G- and (0)-systems. (This definition will play an important role in the present theory.) Finally (§ 3·4), we discuss the relation among the $l^*(N)$'s of I, II, and the present paper.

§ 3·1. With the use of (G·1), (G·6), and Lemmas 1b and 2b, we shall now establish the following lemma, which is analogous to Lemma 4 of I[1] and to the statement on pp. 661—662 in II.[2]

Lemma 3. *For any $v(>0)$, there exists a function $l^*(N)$ of $N(=1, 2, 3, \cdots)$ which satisfies the following six conditions simultaneously:*

1° $l^*(N)$ *assumes a positive integral value for every* N;

2° $\{l^*(N)\}^r/N^s v^s \leqq c_1$ *for almost every* N;

3° *there exists a constant* $\alpha(0<\alpha<1)$ *such that*
$$c_0\bar{\theta}(l^*(N))\{l^*(N)\}^\rho/N^\sigma v^\sigma \leqq \alpha \text{ for almost every } N;$$

4° *there exists a number* $K(>0)$, *independent of* N, *such that*
$$(\ln N)/l^*(N) \leqq K \text{ for every } N;$$

5° $\lim_{N\to\infty} \bar{\theta}(l^*(N)) \max\{(l^*(N))^{\rho-1}, 1\}/N^\sigma = 0$;

6° $\lim_{N\to\infty} l^*(N)/N^{1/2} = 0$;

where c_0, c_1, ρ, r, σ, and s are given in (G·1) *and $\bar{\theta}$ is defined in* § 2·2.

〈*Proof*〉—*Case 1*: the case in which the sequence $\{\theta(l)\}_{l=1}^{\infty}$ is bounded. In this case the sequence $\{\bar{\theta}(l)\}_{l=1}^{\infty}$ is bounded, by Lemma 2b(iii). Hence, for example, put

$$l^*(N)=[N^\nu]+1, \tag{1}^\dagger$$

where ν is a constant, independent of N, such that

$$0<\nu<\min\{s/r,\ \sigma/\rho,\ 1/2\}. \tag{2}$$

Then the conditions 1°—6° are satisfied.

Case 2: the case in which the sequence $\{\theta(l)\}_{l=1}^{\infty}$ is unbounded. In this case the sequence $\{\bar{\theta}(l)\}_{l=1}^{\infty}$ is unbounded, by Lemma 2b(iii).

By the assumption (G·1) we have for every l

$$0\leqq(\theta(l))^{1/l}\leqq G, \tag{3}$$

where G is a positive constant independent of l. In virtue of Lemma 1b and of (G·6), we have

$$\text{g. l. b.}_{l\geqq 1}\ (b_l^{(0)})^{1/l}>0\,; \tag{4}$$

that is, we have for every l

$$(b_l^{(0)})^{1/l}\geqq g^{(0)}>0, \tag{5}$$

where $g^{(0)}$ is a positive constant independent of l. From 2·1(3) we have by (3) and (5)

$$0\leqq(\theta(l))^{1/l}\leqq G/g^{(0)}[\equiv G_0,\ \text{say}] \tag{6}$$

for all l, where G_0 is a positive constant independent of l. That is, the sequence $\{(\theta(l))^{1/l}\}_{l=1}^{\infty}$ is bounded. And here

$$G_0>1; \tag{7}$$

for, if we suppose that $G_0\leqq 1$, we have by (6)

$$0\leqq\theta(l)\leqq G_0^l\leqq 1, \tag{8}$$

contrary to the assumption that the sequence $\{\theta(l)\}_{l=1}^{\infty}$ is unbounded.

By 2·2(1), for every l we can find at least one positive integer l' such that $l'\leqq l$ and $\theta(l')=\bar{\theta}(l)$. Hence, for every l such that $\bar{\theta}(l)>1$, we can find at least one positive integer l' such that

$$(\bar{\theta}(l))^{1/l}\leqq(\bar{\theta}(l))^{1/l'}=(\theta(l'))^{1/l'}. \tag{9}$$

And, for every l such that $\bar{\theta}(l)\leqq 1$, we have

$$(\bar{\theta}(l))^{1/l}\leqq 1. \tag{10}$$

From (9), (10), (6), and (7), we have for all l

$$0\leqq(\bar{\theta}(l))^{1/l}\leqq\max\{G_0,\ 1\}=G_0(>1). \tag{11}$$

That is, the sequence $\{(\bar{\theta}(l))^{1/l}\}_{l=1}^{\infty}$ is bounded.

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† Throughout the present paper, $[x]$ means the largest integer not exceeding x, except that square brackets following a bold-faced letter have another meaning, e.g $\boldsymbol{i}[N-N_2]$.

Now put

$$l^*(N)=[\kappa \ln N]+1, \tag{12}$$

where κ is a constant, independent of N, such that

$$0<\kappa<\sigma/(\ln G_0). \tag{13}$$

Then the condition 2°, as well as 1°, is satisfied; for there exists an integer $n_0(>0)$ such that

$$([\kappa \ln N]+1)^r/N^s \leqq c_1 v^s \quad (\text{for every } N \geqq n_0), \tag{14}$$

since the left-hand side tends to zero as $N\to\infty$ ($r\geqq 0$, $s>0$). Next, the condition 3° is satisfied: By (11) and (7) we have for all N

$$0 \leqq \bar{\theta}([\kappa \ln N]+1)/N^\sigma \leqq G_0^{[\kappa \ln N]+1}/N^\sigma \leqq G_0 N^{\kappa \ln G_0-\sigma}, \tag{15}$$

and by (13) we have

$$\lim_{N\to\infty}([\kappa \ln N]+1)^\rho/N^{\sigma-\kappa \ln G_0}=0 \quad (\rho \geqq 0,\ \sigma-\kappa \ln G_0>0). \tag{16}$$

By (15) and (16) we have

$$\lim_{N\to\infty}\bar{\theta}([\kappa \ln N]+1)\,([\kappa \ln N]+1)^\rho/N^\sigma=0, \tag{17}$$

whence there exists an integer $n_1(>0)$ such that

$$\bar{\theta}([\kappa \ln N]+1)\,([\kappa \ln N]+1)^\rho/N^\sigma \leqq \alpha v^\sigma/c_0 \quad (\text{for every } N \geqq n_1). \tag{18}$$

Next, the condition 4° is satisfied:

$$(\ln N)/([\kappa \ln N]+1)<1/\kappa \quad (\text{for all } N). \tag{19}$$

Next, the condition 5° is satisfied: We have for all N

$$\begin{aligned} 0 &\leqq \bar{\theta}([\kappa \ln N]+1)\max\{([\kappa \ln N]+1)^{\rho-1},\ 1\}/N^\sigma \\ &\leqq \bar{\theta}([\kappa \ln N]+1)\,([\kappa \ln N]+1)^\rho/N^\sigma, \end{aligned} \tag{20}$$

whence, by (17),

$$\lim_{N\to\infty}\bar{\theta}([\kappa \ln N]+1)\max\{([\kappa \ln N]+1)^{\rho-1},\ 1\}/N^\sigma=0. \tag{21}$$

Lastly, it is obvious that the condition 6° is satisfied. Thus the proof is completed.

§ **3·2.** In general there exist many functions $l^*(N)$ having the above properties. But we take *any one*† of these $l^*(N)$'s, and denote it by $l^*(N)$, the same symbol as above.

† It can be proved that all our results in the following are independent of which one of these $l^*(N)$'s has been taken.

We shall use this $l^*(N)$ as the *border-line* between "large" and "small" clusters [large $l>l^*(N)$; small $l\leqq l^*(N)$] in both the G-system and the (0)-system, just as in I and II.

§ 3·3. If we put $V=Nv$ (so that v is the volume per molecule), then [from 1·2(8)] the configurational partition functions of the G-system and the (0)-system are given by

$$\Omega_N=\sum_{m_l\geqq 0}\Big(\sum_{l=1}^{N} lm_l=N\Big)\prod_{l=1}^{N}\frac{\{Nvb_l(Nv)\}^{m_l}}{m_l!}, \tag{1}$$

$$\Omega_N^{(0)}=\sum_{m_l\geqq 0}\Big(\sum_{l=1}^{N} lm_l=N\Big)\prod_{l=1}^{N}\frac{(Nvb_l^{(0)})^{m_l}}{m_l!}, \tag{2}$$

respectively.

Then we consider the *infinite sequences* of Ω_N and $\Omega_N^{(0)}$, $N=1, 2, 3, \cdots$, to ∞, (and, therefore, those of systems defined by Ω_N and $\Omega_N^{(0)}$) for any fixed value of $v(>0)$ (of course, for a given temperature and for given intermolecular forces).

By the condition 6° of Lemma 4 we have for almost every N

$$l^*(N)<N. \tag{3}$$

Hence, with the use of $l^*(N)$, we may give the following expressions to Ω_N and $\Omega_N^{(0)}$ for almost every N.

$$\Omega_N=\sum_{\boldsymbol{i}[N]}\varphi_N(\boldsymbol{i}[N])+\sum_{\boldsymbol{j}[N]}\chi_N(\boldsymbol{j}[N]) \tag{4}$$

$$=\sum_{\boldsymbol{i}[N]}\varphi_N(\boldsymbol{i}[N])+\sum_{N_2=l^*(N)+1}^{N}\sum_{\boldsymbol{i}[N-N_2]}\varphi_N(\boldsymbol{i}[N-N_2])\sum_{\boldsymbol{k}[N_2]}\Phi_N(\boldsymbol{k}[N_2]), \tag{5}$$

$$\Omega_N^{(0)}=\sum_{\boldsymbol{i}[N]}\varphi_N^{(0)}(\boldsymbol{i}[N])+\sum_{\boldsymbol{j}[N]}\chi_N^{(0)}(\boldsymbol{j}[N]) \tag{6}$$

$$=\sum_{\boldsymbol{i}[N]}\varphi_N^{(0)}(\boldsymbol{i}[N])+\sum_{N_2=l^*(N)+1}^{N}\sum_{\boldsymbol{i}[N-N_2]}\varphi_N^{(0)}(\boldsymbol{i}[N-N_2])\sum_{\boldsymbol{k}[N_2]}\Phi_N^{(0)}(\boldsymbol{k}[N_2]). \tag{7}$$

Here we have defined

$$\chi_N(\boldsymbol{j}[N])\equiv\prod_{l=1}^{N}\frac{\{Nvb_l(Nv)\}^{m_l}}{m_l!}, \tag{8}$$

$$\chi_N^{(0)}(\boldsymbol{j}[N])\equiv\prod_{l=1}^{N}\frac{(Nvb_l^{(0)})^{m_l}}{m_l!}, \tag{9}$$

where $\boldsymbol{j}[N]$ represents a set $\{m_1, m_2, \cdots, m_N\}$ of non-negative integers m_l such that

$$\text{1°}\quad\sum_{l=1}^{N} lm_l=N, \tag{10}$$

$$\text{2°}\quad m_l\geqq 1 \text{ for at least one } l \text{ such that } N\geqq l\geqq l^*(N)+1;$$

and the m_l's given on the right-hand side of (8) [and (9)] are the set of m_l

represented by $\boldsymbol{j}[N]$ on the left-hand side. The symbol $\sum_{\boldsymbol{j}[N]}$ indicates summation over all sets $\boldsymbol{j}[N]$ for a given N. And we have defined

$$\varphi_N(\boldsymbol{i}[M]) \equiv \prod_{l=1}^{l^*(N)} \frac{\{Nvb_l(Nv)\}^{m_l}}{m_l!}, \tag{11}$$

$$\varphi_N^{(0)}(\boldsymbol{i}[M]) \equiv \prod_{l=1}^{l^*(N)} \frac{(Nvb_l^{(0)})^{m_l}}{m_l!}, \tag{12}$$

where $\boldsymbol{i}[M]$ represents a set $\{m_1, m_2, \cdots, m_{l^*(N)}\}$ of non-negative integers m_l such that

$$\sum_{l=1}^{l^*(N)} lm_l = M, \tag{13}$$

(where M is a non-negative integer); and the m_l's given on the right-hand side of (11) [and (12)] are the set of m_l represented by $\boldsymbol{i}[M]$ on the left-hand side. The symbol $\sum_{\boldsymbol{i}[M]}$ indicates summation over all sets $\boldsymbol{i}[M]$ for a given M. [In (5) and (7), we have put $M=N$ and $M=N-N_2$.] Especially for $M=0$ we have

$$\varphi_N(\boldsymbol{i}[0]) \equiv 1, \quad \varphi_N^{(0)}(\boldsymbol{i}[0]) \equiv 1, \quad \boldsymbol{i}[0] \equiv \{0, 0, \cdots, 0\}. \tag{14}$$

We have also defined

$$\Phi_N(\boldsymbol{k}[N_2]) \equiv \prod_{l=l^*(N)+1}^{N} \frac{\{Nvb_l(Nv)\}^{m_l}}{m_l!}, \tag{15}$$

$$\Phi_N^{(0)}(\boldsymbol{k}[N_2]) \equiv \prod_{l=l^*(N)+1}^{N} \frac{(Nvb_l^{(0)})^{m_l}}{m_l!}, \tag{16}$$

where $\boldsymbol{k}[N_2]$ represents a set $\{m_{l^*(N)+1}, m_{l^*(N)+2}, \cdots, m_N\}$ of non-negative integers m_l such that

$$\sum_{l=l^*(N)+1}^{N} lm_l = N_2, \tag{17}$$

(where N_2 is an integer such that $l^*(N)+1 \leqq N_2 \leqq N$ or $N_2=0$); and the m_l's given on the right-hand side of (15) [and (16)] are the set of m_l represented by $\boldsymbol{k}[N_2]$ on the left-hand side. The symbol $\sum_{\boldsymbol{k}[N_2]}$ indicates summation over all sets $\boldsymbol{k}[N_2]$ for a given N_2. Especially for $N_2=0$ we have

$$\Phi_N(\boldsymbol{k}[0]) \equiv 1, \quad \Phi_N^{(0)}(\boldsymbol{k}[0]) \equiv 1, \quad \boldsymbol{k}[0] \equiv \{0, 0, \cdots, 0\}. \tag{18}$$

§ 3·31. Here, as in I, we have divided each of Ω_N and $\Omega_N^{(0)}$ (for almost every N for any fixed T and v) into the part each term of which contains "small" clusters only and the part each term of which contains at least one "large" cluster [see 3·3(4) and 3·3(6)]. Moreover, because the present paper involves discussions of the condensation range (in which the latter part may not be negligible but may play an important role), we have given to the latter part a more detailed expression [see 3·3(5) and 3·3(7)] in terms of the functions (φ_N's,

$\varphi_N^{(0)}$'s) containing "small" clusters only and the functions (Φ_N's, $\Phi_N^{(0)}$'s) containing "large" clusters only, as we did for the (0)-system in II.

§ 3·4. At this point we may give a remark on the relation among the function $l^*(N)$ in I, that in II, and that in the present paper.

$l^*(N)$ in I has been so chosen that, if N is large enough, the effects of the volume dependence of the cluster integrals upon the "small" cluster part are as small as we please [Lemma 5 of I] and at the same time the entire partition function in the gas range may, with an error as small as we please, be represented by the "small" cluster part alone [Lemma 6 of I]; while $l^*(N)$ in II has been so chosen that, if N is large enough, the "small" clusters may, with errors as small as we please, be treated by means of the Stirling approximation and the continuum approximation $(\partial/\partial m_l)$ and at the same time the "large" clusters may, with errors as small as we please, be treated by means of the concept of a "huge" cluster.

Now, $l^*(N)$ in the present paper has both the character of $l^*(N)$ in I and that in II, because the present paper attempts to prove the appearance of *a "huge" cluster* in a system with *volume-dependent* cluster integrals. In fact, the conditions 1°—5° in Lemma 3 are analogous to the conditions 1°—5° in Lemma 4 of I, and the conditions 1°, 6°, and 4° in Lemma 3 are identical with the conditions 1°, 2°, and 3° on p. 662 of II.

§ 4. Relation between the "small" clusters in the G-system and those in the (0)-system

§ 4·1. By Lemmas 1b, 2c, and 3 we obtain the following lemma, which states the thermodynamical equivalence between a set of "small" clusters in the G-system and that in the (0)-system and will be employed in the later arguments.

Lemma 4. *For any $v>0$, the following are true:*

(i) *We have for almost every N*

$$\varphi_N(\boldsymbol{i}[N-N_2])>0 \tag{1}$$

for every $\boldsymbol{i}[N-N_2]$ for every N_2 ($N_2=0$ or $l^*(N)+1\leqq N_2\leqq N$).

(ii) *For any given $\epsilon>0$, we have for almost every N*

$$\left|\frac{1}{N}\ln\varphi_N(\boldsymbol{i}[N-N_2])-\frac{1}{N}\ln\varphi_N^{(0)}(\boldsymbol{i}[N-N_2])\right|<\epsilon \tag{2}$$

for every $\boldsymbol{i}[N-N_2]$ for every N_2 ($N_2=0$ or $l^*(N)+1\leqq N_2\leqq N$).

⟨Proof⟩—First, we note that, by Lemma 1b, we have

$$\varphi_N^{(0)}(\boldsymbol{i}[N-N_2])>0 \tag{3}$$

for every $\boldsymbol{i}[N-N_2]$. Now, in view of the conditions 2° and 3° (of Lemma 3) satisfied by $l^*(N)$, we can apply Lemma 2c, putting $V=N\nu$. Then, for almost every N, we have

$$0<1-\alpha\leqq b_l(Nv)/b_l^{(0)}\leqq 1+\alpha<2 \quad (l\leqq l^*(N)), \tag{4}$$

whence, by Lemma 1b,

$$b_l(Nv)>0 \quad (l\leqq l^*(N)). \tag{5}$$

Hence (i) is proved. Again, by Lemma 2c we have for almost every N

$$\left|\frac{1}{l}\ln\frac{b_l(Nv)}{b_l^{(0)}}\right|\leqq c_0\bar{\theta}(l^*(N))\max\{(l^*(N))^{\rho-1},1\}\frac{1}{N^\sigma v^\sigma(1-\alpha)} \quad (l\leqq l^*(N)). \tag{6}$$

Hence we have the following for every $\boldsymbol{i}[N-N_2]$ for every N_2 ($N_2=0$ or $l^*(N)+1\leqq N\leqq N_2$) for almost every N.

$$\left|\frac{1}{N}\ln\varphi_N(\boldsymbol{i}[N-N_2])-\frac{1}{N}\ln\varphi_N^{(0)}(\boldsymbol{i}[N-N_2])\right|\leqq\frac{1}{N}\sum_{l=1}^{l^*(N)}m_l\left|\ln\frac{b_l(Nv)}{b_l^{(0)}}\right|$$

$$\leqq c_0\bar{\theta}(l^*(N))\max\{(l^*(N))^{\rho-1},1\}\frac{1}{N^\sigma v^\sigma(1-\alpha)}, \tag{7}$$

since

$$\sum_{l=1}^{l^*(N)}lm_l=N-N_2\leqq N \text{ (for every } \boldsymbol{i}[N-N_2]). \tag{8}$$

By (7) and by the condition 5° (of Lemma 3) satisfied by $l^*(N)$, we prove (ii).

§ **4·2.** This lemma states the asymptotical equivalence (in the limit of infinite systems) between a term φ_N (containing "small" clusters only) for the G-system and the corresponding term $\varphi_N^{(0)}$ for the (0)-system.

§ **4·3.** The special case of this lemma, in which $N_2=0$, leads to Lemma 5 in I.

§ 5. "Large" clusters in the G-system

§ **5·1.** The following lemma, which is a consequence of (G·2), (G·4), and Lemma 3, teaches us some properties of the "large" clusters in the G-system and will be useful in the proof of the appearance of the liquid phase (§ 8).

Lemma 5. *If*

$$v>u_0 \text{ [where } u_0 \text{ is given in (G·4)]}, \tag{1}$$

then the following are true:

(i) *For any given* $\epsilon>0$, *we have for almost every* N

$$\frac{1}{N}\ln\sum_{k[N_2]}\Phi_N(\boldsymbol{k}[N_2])-\frac{N_2}{N}\ln b_0^\dagger<\epsilon \tag{2}$$

for every $N_2(l^*(N)+1\leqq N_2\leqq N)$.

(ii) *For any given* $\epsilon>0$, *we have for almost every* N

$$\left|\frac{1}{N}\ln\sum_{\boldsymbol{k}[N_2]}\Phi_N(\boldsymbol{k}[N_2])-\frac{N_2}{N}\ln b_0^{\dagger}\right|-\left|\frac{1}{N}\ln b_{N_2}(Nv)-\frac{N_2}{N}\ln b_0^{\dagger}\right|<\epsilon \tag{3}$$

for every $N_2(l^*(N)+1\leqq N_2\leqq N)$.

〈*Proof*〉—From (1) we have

$$V[=Nv]>Nu_0\geqq lu_0\geqq lv_0 \tag{4}$$

for every N and $l(\leqq N)$; and, obviously, we have

$$V[=Nv]>V_0 \text{ (for almost every } N), \tag{5}$$

where V_0 is given in (G·2). Thus by (G·2) we have

$$b_l(Nv)>0 \text{ (for every } l\leqq N \text{ for almost every } N). \tag{6}$$

Now let

$$F_N(\boldsymbol{k}[N_2])\equiv\prod_{l=l^*(N)+1}^{N}\{b_l(Nv)\}^{m_l}, \tag{7}$$

$$G_N(\boldsymbol{k}[N_2])\equiv\prod_{l=l^*(N)+1}^{N}\frac{(Nv)^{m_l}}{m_l!}. \tag{8}$$

Then we have

$$\Phi_N(\boldsymbol{k}[N_2])=F_N(\boldsymbol{k}[N_2])G_N(\boldsymbol{k}[N_2]); \tag{9}$$

and by (6) we have for almost every N

$$\begin{aligned}\max_{\boldsymbol{k}[N_2]}F_N(\boldsymbol{k}[N_2])\min_{\boldsymbol{k}[N_2]}G_N(\boldsymbol{k}[N_2])&\leqq\sum_{\boldsymbol{k}[N_2]}\Phi_N(\boldsymbol{k}[N_2])\\&\leqq q(N,N_2)\max_{\boldsymbol{k}[N_2]}F_N(\boldsymbol{k}[N_2])\max_{\boldsymbol{k}[N_2]}G_N(\boldsymbol{k}[N_2]),\end{aligned} \tag{10}$$

where $q(N, N_2)$ is the number of terms in the sum $\sum_{k[N_2]}$. If we let P_N be the *partitio numerorum* of N, we have

$$q(N, N_2)<P_N \tag{11}$$

for every N and N_2; and we have asymptotically

$$\ln P_N\sim\pi(2N/3)^{1/2} \tag{12}$$

as $N\to\infty$. Hence, for any given $\epsilon>0$, we have for almost every N

$$0\leqq\frac{1}{N}\ln q(N, N_2)<\epsilon \tag{13}$$

for every $N_2(l^*(N)+1\leqq N_2\leqq N)$.

Now, by the condition 4° of Lemma 3,

$$\lim_{N\to\infty}l^*(N)=+\infty. \tag{14}$$

Hence, for almost every N, we have

$$l^*(N) \geqq l' \qquad (15)$$

and

$$V[=Nv] \geqq V', \qquad (16)$$

where V' and l' (corresponding to any $\epsilon>0$) are given in (G·4). In virtue of (G·4), it follows from (15), (16), and (4) that, for any given $\epsilon>0$,

$$\ln F_N(\boldsymbol{k}[N_2]) < \ln \prod_{l=l^*(N)+1}^{N} \{(b_0^\dagger+\epsilon)^l\}^{m_l} = N_2 \ln(b_0^\dagger+\epsilon) < N_2 \ln b_0^\dagger + N_2\,\epsilon/b_0^\dagger \qquad (17)$$

for every $\boldsymbol{k}[N_2]$ for every $N_2(l^*(N)+1 \leqq N_2 \leqq N)$ for almost every N. Consequently, we have for almost every N

$$\ln b_{N_2}(Nv) \leqq \ln \max_{\boldsymbol{k}[N_2]} F_N(\boldsymbol{k}[N_2]) < N_2 \ln b_0^\dagger + N_2\,\epsilon/b_0^\dagger \qquad (18)$$

for every $N_2(l^*(N)+1 \leqq N_2 \leqq N)$.

Now, according to the Stirling theorem, we have

$$\ln m_l! = m_l(\ln m_l - 1) + \frac{1}{2}\ln m_l + \gamma(m_l) \qquad (19)$$

for every positive integer m_l, where $\gamma(m_l)$ is bounded; that is, there exists a positive absolute‡ constant K_1 such that

$$|\gamma(m_l)| < K_1 \quad \text{(for every } m_l\text{)}. \qquad (20)$$

Hence by (8)

$$\ln G_N(\boldsymbol{k}[N_2]) = \sum_{l=l^*(N)+1}^{N}{}' \left\{ m_l(1+\ln v) - \frac{1}{2}\ln m_l - \gamma(m_l) + m_l \ln(N/m_l) \right\}, \qquad (21)$$

where the prime denotes that the terms in which $m_l=0$ are omitted.‡‡ In virtue of (14), we can put $\lambda(N)=l^*(N)+1$ and $\nu=N_2$ in Lemma 1 of II. Thus for any given $\epsilon>0$ we have the following relations for every $\boldsymbol{k}[N_2]$ for every $N_2(l^*(N)+1 \leqq N_2 \leqq N)$ for almost every N.

$$0 < \frac{1}{N} \sum_{l=l^*(N)+1}^{N_2}{}' m_l < \frac{\epsilon}{4|1+\ln v|} \qquad (22a)$$

$$0 \leqq \frac{1}{N} \sum_{l=l^*(N)+1}^{N_2}{}' \ln m_l < \frac{\epsilon}{2} \qquad (22b)$$

$$0 < \frac{1}{N} \sum_{l=l^*(N)+1}^{N_2}{}' 1 < \frac{\epsilon}{4K_1} \qquad (22c)$$

‡ An *absolute* constant means a numerical constant independent of any physical quantity and of any property of the system.

‡‡ Throughout the present paper, the prime of Σ' has the same meaning as here, unless otherwise stated.

$$0 \leqq \frac{1}{N} \sum_{l=l^*(N)+1}^{N_2}{}' m_l \ln \frac{N}{m_l} < \frac{\epsilon}{4}. \tag{22d}$$

From (21), by (20) and (22) we have for any given $\epsilon > 0$

$$\left| \frac{1}{N} \ln G_N(\boldsymbol{k}[N_2]) \right| < \epsilon \tag{23}$$

for every $\boldsymbol{k}[N_2]$ for every $N_2 (l^*(N)+1 \leqq N_2 \leqq N)$ for almost every N. Hence

$$\left| \frac{1}{N} \ln \max_{k[N_2]} G_N(\boldsymbol{k}[N_2]) \right| < \epsilon, \quad \left| \frac{1}{N} \ln \min_{k[N_2]} G_N(\boldsymbol{k}[N_2]) \right| < \epsilon \tag{24}$$

for every $N_2 (l^*(N)+1 \leqq N_2 \leqq N)$ for almost every N.

By (10), (13), (18), and (24), we have for almost every N

$$\frac{1}{N} \ln b_{N_2}(Nv) - \frac{N_2}{N} \ln b_0^{\dagger} - \epsilon < \frac{1}{N} \ln \sum_{k[N_2]} \Phi_N(\boldsymbol{k}[N_2]) - \frac{N_2}{N} \ln b_0^{\dagger} < \left(2 + \frac{1}{b_0^{\dagger}}\right) \epsilon \tag{25}$$

for every $N_2 (l^*(N)+1 \leqq N_2 \leqq N)$. If, in (25), we replace ϵ by $\epsilon/(2+1/b_0^{\dagger})$, the lemma is proved. [If $\epsilon > \epsilon' > 0$ and $b - \epsilon' < a < \epsilon$, then $|a| < \max\{|b-\epsilon'|, \epsilon\} \leqq |b| + \epsilon$.]

§ 5·2. Lemma 5 (i) shows that $(1/N) \ln \sum \Phi_N(\boldsymbol{k}[N_2])$ [which contains "large" clusters only] does not practically exceed $(N_2/N) \ln b_0^{\dagger}$ if N is large enough; and Lemma 5 (ii) shows that the difference between the above two quantities does not practically exceed the difference between $(1/N) \ln b_{N_2}(Nv)$ [which corresponds to one "large" cluster composed of N_2 molecules] and $(N_2/N) \ln b_0^{\dagger}$ if N is large enough.

§ 6. Maximum term in the G-system

§ 6·1. In virtue of (G·2) we have now the following lemma, which is concerned with the maximum term in the partition function of the G-system and will often be used in the proofs of the theorems in §§ 8 and 9.

Lemma 6. *If*

$$v > v_0 \ [\textit{where } v_0 \textit{ is given in } (\mathrm{G}\cdot 2)], \tag{1}$$

then, for any given $\epsilon > 0$, *we have for almost every* N

$$\left| \frac{1}{N} \ln \Omega_N - \frac{1}{N} \max_{N_2, \boldsymbol{i}[N-N_2]} \ln \tau_N(N_2, \boldsymbol{i}[N-N_2]) \right| < \epsilon, \tag{2}$$

where

$$\tau_N(N_2, \boldsymbol{i}[N-N_2]) \equiv \varphi_N(\boldsymbol{i}[N-N_2]) \sum_{k[N_2]} \Phi_N(\boldsymbol{k}[N_2]), \tag{3}$$

in which $N_2 = 0$ *or* $l^*(N)+1 \leqq N_2 \leqq N$.

[For the proof, note that by (1) we have for every N and $l\ (\leqq N)$

$$V = Nv > Nv_0 \geqq lv_0, \tag{4}$$

hence by 5·1(5) and (G·2) we have 5·1(6); thus $\tau_N(N_2, \boldsymbol{i}[N-N_2])>0$ for every $\boldsymbol{i}$ and N_2, for almost every N. See the proof of Lemma 2 of II.]

§ **6·2.** This lemma shows that, if we define the terms τ_N [each of which is determined by the total number (N_2) of molecules composing "large" clusters and the manner ($\boldsymbol{i}[N-N_2]$) of grouping the remaining molecules ($N-N_2$) into "small" clusters], then, for N large enough, $(1/N)\ln \Omega_N$ may, with an error as small as we please, be replaced by the largest of $(1/N)\ln \tau_N$.

§ 7. The Γ-system; its relation with the G-system

§ **7·1.** By Lemmas 3, 4, and 5 we obtain the following lemma, which deals with an assumptive simple system, called the "Γ-system", and will be useful in the proofs of the theorems in §§ 8 and 9.

Lemma 7. *If*

$$v>u_0 \text{ [where } u_0 \text{ is given in (G·4)]}, \tag{1}$$

then the following are true:

(i) *For any given* $\epsilon>0$, *we have for almost every* N

$$\frac{1}{N}\ln\tau_N(N_2, \boldsymbol{i}[N-N_2]) - \frac{1}{N}\ln\tau_N^{\Gamma}(N_2, \boldsymbol{i}[N-N_2]) < \epsilon \tag{2}$$

for every $\boldsymbol{i}[N-N_2]$ *for every* $N_2(l^*(N)+1 \leqq N_2 \leqq N$ *or* $N_2=0)$, *where*

$$\ln\tau_N^{\Gamma}(N_2, \boldsymbol{i}[N-N_2]) \equiv \sum_{l=1}^{l^*(N)}{}' m_l\{\ln(Nvb_l^{(0)}) - \ln m_l + 1\} + N_2 \ln b_0^{\dagger}. \tag{3}$$

(ii) *For any given* $\epsilon>0$, *we have for almost every* N

$$\left|\frac{1}{N}\ln\tau_N(N_2, \boldsymbol{i}[N-N_2]) - \frac{1}{N}\ln\tau_N^{\Gamma}(N_2, \boldsymbol{i}[N-N_2])\right| - \left|\frac{1}{N}\ln b_{N_2}(Nv) - \frac{N_2}{N}\ln b_0^{\dagger}\right| < \epsilon \tag{4}$$

for every $\boldsymbol{i}[N-N_2]$ *for every* $N_2(l^*(N)+1 \leqq N_2 \leqq N$ *or* $N_2=0)$. *Here we agree:* $b_{N_2}(Nv)\equiv 1$ *for* $N_2=0$.

〈*Proof*〉—Using the Stirling theorem [5·1(19)], we have

$$\left|\frac{1}{N}\ln\varphi_N^{(0)}(\boldsymbol{i}[N-N_2]) - \frac{1}{N}\sum_{l=1}^{l^*(N)}{}' m_l\{\ln(Nvb_l^{(0)}) - \ln m_l + 1\}\right| \leqq \frac{1}{2N}\sum_{l=1}^{l^*(N)}{}' \ln m_l + \frac{1}{N}\sum_{l=1}^{l^*(N)} |\gamma(m_l)|. \tag{5}$$

Since

$$m_l \leqq N \quad (l=1, 2, \cdots, l^*(N)), \tag{6}$$

we have

$$\frac{1}{2N}\sum_{l=1}^{l^*(N)}{}' \ln m_l \leqq \frac{l^*(N)}{2N}\ln N. \tag{7}$$

By 5·1(20) we have

$$\frac{1}{N}\sum_{l=1}^{l^*(N)}{}' |\gamma(m_l)| \leqq \frac{l^*(N)}{N}K_1, \tag{8}$$

where K_1 is a positive absolute constant. By the condition 6° of Lemma 3, for any given $\epsilon>0$ we have for almost every N

$$0<l^*(N)(\ln N)/N<\epsilon \tag{9}$$

$$0<l^*(N)/N<\epsilon, \tag{10}$$

since we have for almost every N

$$0<l^*(N)/N<l^*(N)(\ln N)/N<l^*(N)/N^{1/2}. \tag{11}$$

Now we have by (3) and 6·1(3)

$$\begin{aligned}\frac{1}{N}\ln \tau_N(N_2, \boldsymbol{i}[N-N_2]) - \frac{1}{N}\ln \tau_N^{\Gamma}(N_2, \boldsymbol{i}[N-N_2])\\ =\frac{1}{N}\ln \varphi_N(\boldsymbol{i}[N-N_2]) - \frac{1}{N}\sum_{l=1}^{l^*(N)}{}' m_l\{\ln(Nvb_l^{(0)}) - \ln m_l + 1\}\\ +\frac{1}{N}\ln \sum_{k[N_2]} \Phi_N(\boldsymbol{k}[N_2]) - \frac{N_2}{N}\ln b_0^{\dagger}.\end{aligned} \tag{12}$$

In virtue of (5), (7), (8), (9) [in which ϵ is replaced by $\epsilon/2$], (10) [in which ϵ is replaced by $\epsilon/(4K_1)$], and (12), the lemma is proved if we use Lemma 4 (ii) [in which ϵ is replaced by $\epsilon/4$] and if we employ (as we can in virtue of (1)) Lemma 5 [in which ϵ is replaced by $\epsilon/4$]. [When $N_2=0$, employ 3·3 (18).]

§ 7·2. In this lemma we have introduced an assumptive system—the Γ-system, which is defined by the relatively simple expression 7·1(3) for τ_N—, and have established some relations between the G-system and the Γ-system. The Γ-system will be used as a theoretical medium in the proofs of the theorems in §§ 8 and 9.

§ 7·3. It may be noted that the Δ-system introduced in Lemma 3 of II corresponds to the special case of the Γ-system, in which $b_0^{\dagger}=b_0^{(0)}$.

§ 8. Appearance of a "huge" cluster (i. e. the two-phase separation) in the G-system.

§ 8·1. With the aid of the lemmas so far given, and on the basis of the assumptions given in § 1·4, we shall now prove the following theorems (and corollaries) concerning the appearance of the liquid phase in the G-system and the existence of a horizontal part of the isotherm for the G-system. (The physical interpretation of the theorems will be given in § 8·2.)

Theorem 1. *Let*

$$z_{\text{cord}} \equiv (b_0^{\dagger})^{-1}, \tag{1}$$

$$v_{\text{cond}} \equiv \Big(\sum_{l=1}^{\infty} l b_l^{(0)} z_{\text{cond}}^l\Big)^{-1}, \tag{2}$$

$$v^* \equiv \max\{u_0,\ (u_1^{-1}+v_{\text{cond}}^{-1})^{-1}\}. \tag{3}$$

Then, if

$$v^* < v < v_{\text{cond}}, \tag{4}\ddagger$$

the following are true:

(i) *For any given $\epsilon>0$, there exists, for almost any N, at least one set $\{N_2, \boldsymbol{i}[N-N_2]\}$ for which*

$$\left| \frac{1}{N} \ln \tau_N(N_2, \boldsymbol{i}[N-N_2]) - \frac{1}{N} \max_{N_2,\, \boldsymbol{i}[N-N_2]} \ln \tau_N(N_2, (\boldsymbol{i}[N-N_2]) \right| < \epsilon \qquad (\alpha)$$

and

$$\left| \frac{N_2}{N} - \Big(1 - \sum_{l=1}^{\infty} l v b_l^{(0)} z_{\text{cond}}^l\Big) \right| < \epsilon \qquad (\beta)$$

are simultaneously satisfied.

(ii) *The statement* (i) *still holds if we replace* (α) *by*

$$\left| \frac{1}{N} \ln \tau_N(N_2, \boldsymbol{i}[N-N_2]) - \frac{1}{N} \ln \Omega_N \right| < \epsilon. \qquad (\alpha')$$

(iii) *The statement* (i) *still holds if we replace* (α) *by*

$$\left| \frac{1}{N} \ln \tau_N(N_2, \boldsymbol{i}[N-N_2]) - \Big(\sum_{l=1}^{\infty} v b_l^{(0)} z_{\text{cond}}^l - \ln z_{\text{cond}}\Big) \right| < \epsilon. \qquad (\alpha'')$$

(iv) *The statements* (i), (ii), *and* (iii) *still hold if we replace the first term* $(1/N) \ln \tau_N(N_2, \boldsymbol{i}[N-N_2])$ *in* (α), (α'), *and* (α'') *by* $(1/N) \ln \tau_N^{\Gamma}(N_2, \boldsymbol{i}[N-N_2])$.

(v) *For any sufficiently small $\epsilon>0$, we have for almost any N*

$$\left| \frac{1}{N} \ln \sum_{k[N_2]} \Phi_N(\boldsymbol{k}[N_2]) - \frac{1}{N} \ln b_{N_2}(Nv) \right| < \epsilon \qquad (\gamma)$$

and

$$\left| \frac{1}{N} \ln \varphi_N(\boldsymbol{i}[N-N_2]) - \left(\sum_{l=1}^{l^*(N)} v b_l^{(0)} z_{\text{cond}}^l - \frac{v}{v_{\text{cond}}} \ln z_{\text{cond}} \right) \right| < c\epsilon \qquad (\delta)$$

for every set $\{N_2, \boldsymbol{i}[N-N_2]\}$ which satisfies both (α) *and* (β). *(Here c is a constant independent of ϵ, N, and v.)*

⟨Proof of (i)*⟩*—We now consider $\max'_{N_2,\, \boldsymbol{i}[N-N_2]} \ln \tau_N^{\Gamma}(N_2, \boldsymbol{i}[N-N_2])$, where the prime denotes that (when the maximum is taken) $N_2, m_1, m_2, \cdots, m_{l^*(N)}$ are as-

‡ By (G·5), $v_{\text{cond}} > u_0$; hence $v_{\text{cond}} > v^*$.

sumed to be continuous variables which take real non-negative values and, of course, satisfy the condition

$$\sum_{l=1}^{l^*(N)} lm_l + N_2 = N. \tag{5}$$

Between this max′ and the (actual) max, there is the following obvious relation:

$$\max'_{N_2, \boldsymbol{i}[N-N_2]} \ln \tau_N^r(N_2, \boldsymbol{i}[N-N_2]) \geqq \max_{N_2, \boldsymbol{i}[N-N_2]} \ln \tau_N^r(N_2, \boldsymbol{i}[N-N_2]). \tag{6}$$

Using the undetermined multiplier $\ln z$, we see that the set (of real numbers) $\{N_2, m_1, m_2, \cdots, m_{l^*(N)}\}$ which gives max′ must satisfy the equations

$$\frac{\partial}{\partial m_l} \{\ln \tau_N^r(N_2, \boldsymbol{i}[N-N_2]) + (\sum_{l=1}^{l^*(N)} lm_l + N_2) \ln z\} = 0 \quad (l=1, 2, \cdots, l^*(N)) \tag{7a}$$

and

$$\frac{\partial}{\partial N_2} \{\ln \tau_N^r(N_2, \boldsymbol{i}[N-N_2]) + (\sum_{l=1}^{l^*(N)} lm_l + N_2) \ln z\} = 0. \tag{7b}$$

By 7·1(3) we obtain, from (7a),

$$m_l = Nvb_l^{(0)} z^l \quad (l=1, 2, \cdots, l^*(N)) \tag{8a}$$

and, from (7b),

$$z = (b_0^\dagger)^{-1} [\equiv z_{\text{cond}}, \text{ by (1)}]. \tag{8b}$$

Hence by (5) we have, for the set giving max′,

$$m_l = \overline{\overline{m}}_l \equiv Nvb_l^{(0)} z_{\text{cond}}^l \quad (l=1, 2, \cdots, l^*(N)) \tag{9a}$$

$$N_2 = \overline{\overline{N}}_2 \equiv N - \sum_{l=1}^{l^*(N)} l\overline{\overline{m}}_l \equiv N - N\sum_{l=1}^{l^*(N)} lvb_l^{(0)} z_{\text{cond}}^l. \tag{9b}$$

In virtue of Lemma 1b, we have

$$[1/v_{\text{cond}} \equiv] \sum_{l=1}^{\infty} lb_l^{(0)} z_{\text{cond}}^l > \sum_{l=1}^{l^*(N)} lb_l^{(0)} z_{\text{cond}}^l. \tag{10}$$

By this we have from (9b)

$$\overline{\overline{N}}_2 > N(1 - v/v_{\text{cond}}) > 0 \tag{11}$$

[by (4)]; and from (9a) we have

$$\overline{\overline{m}}_l > 0 \quad (l=1, 2, \cdots, l^*(N)). \tag{12}$$

These are consistent with the assumption that $N_2, m_1, m_2, \cdots, m_{l^*(N)}$ are real non-negative numbers. Therefore

$$\max'_{N_2, \boldsymbol{i}[N-N_2]} \ln \tau_N^r(N_2, \boldsymbol{i}[N-N_2]) = N\sum_{l=1}^{l^*(N)} vb_l^{(0)} z_{\text{cond}}^l - \overline{\overline{N}}_1 \ln z_{\text{cond}} + \overline{\overline{N}}_2 \ln b_0^\dagger, \tag{13}$$

where

$$\overline{\overline{N}}_1 \equiv \sum_{l=1}^{l^*(N)} l\overline{\overline{m}}_l \equiv N \sum_{l=1}^{l^*(N)} l v b_l^{(0)} z_{\text{cond}}^l, \tag{14}$$

that is,

$$\overline{\overline{N}}_1 + \overline{\overline{N}}_2 = N. \tag{15}$$

By (1) and (15), (13) becomes

$$\max'_{N_2, \boldsymbol{i}[N-N_2]} \ln \tau_N^r(N_2, \boldsymbol{i}[N-N_2]) = N \sum_{l=1}^{l^*(N)} v b_l^{(0)} z_{\text{cond}}^l - N \ln z_{\text{cond}}. \tag{16}$$

Now, for each N, we take the set (of integers) $\{\hat{N}_2, \hat{m}_1, \hat{m}_2, \cdots, \hat{m}_{l^*(N)}\}$:

$$\hat{m}_l \equiv [\overline{\overline{m}}_l] \equiv [N v b_l^{(0)} z_{\text{cond}}^l] \quad (l=1, 2, \cdots, l^*(N)) \tag{17a}$$

$$\hat{N}_2 \equiv N - \sum_{l=1}^{l^*(N)} l\hat{m}_l \equiv N - \sum_{l=1}^{l^*(N)} l[N v b_l^{(0)} z_{\text{cond}}^l]. \tag{17b}$$

Then we have

$$\sum_{l=1}^{l^*(N)} l\hat{m}_l + \hat{N}_2 = N, \tag{18}$$

and, by (9b), (11), and the condition 6° of Lemma 3, we have for almost every N

$$\hat{N}_2 \geqq \overline{\overline{N}}_2 > l^*(N) \tag{19}$$

and, by (12), we have

$$\hat{m}_l \geqq 0 \quad (l=1, 2, \cdots, l^*(N)). \tag{20}$$

By (18), (19), and (20), for almost every N this set $\{\hat{N}_2, \hat{m}_l\}$ satisfies the conditions of $\{N_2, \boldsymbol{i}[N-N_2]\}$ (that is, this set is one of the sets $\{N_2, \boldsymbol{i}[N-N_2]\}$). Hence for almost every N we shall denote this set by $\{\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]\}$.

Then, introducing this set into 7·1(3) for almost every N, and using (16) and (1), we have

$$\frac{1}{N} \ln \tau_N^r(\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]) - \frac{1}{N} \max'_{N_2, \boldsymbol{i}[N-N_2]} \ln \tau_N^r(N_2, \boldsymbol{i}[N-N_2]) = A(N) + B(N), \tag{21}$$

where

$$A(N) = -\frac{1}{N} \sum_{l=1}^{l^*(N)}{}' [N v b_l^{(0)} z_{\text{cond}}^l] \ln ([N v b_l^{(0)} z_{\text{cond}}^l] / N v b_l^{(0)} z_{\text{cond}}^l) \tag{22}$$

(where the prime denotes that the terms in which $[N v b_l^{(0)} z_{\text{cond}}^l]=0$, i. e. $N v b_l^{(0)} z_{\text{cond}}^l < 1$, are omitted) and

$$B(N) = -\frac{1}{N} \sum_{l=1}^{l^*(N)} (N v b_l^{(0)} z_{\text{cond}}^l - [N v b_l^{(0)} z_{\text{cond}}^l]). \tag{23}$$

Hence we have [by the lemma in the footnote‡]

‡ **Lemma.** *The function* $[x] \ln ([x]/x)$ *is bounded in the half-open interval* $1 \leqq x < +\infty$. [For the proof, see p. 672 of II.]

$$|A(N)| \leqq \frac{1}{N} \sum_{l=1}^{l^*(N)} K_0 \leqq K_0 \frac{l^*(N)}{N} \tag{24}$$

(where K_0 is a positive absolute constant), and we have

$$|B(N)| < l^*(N)/N. \tag{25}$$

From (21), (24), and (25), using 7·1(10) [in which ϵ is replaced by $\epsilon/(6K_0)$ and by $\epsilon/6$], we have for any given $\epsilon > 0$

$$\left| \frac{1}{N} \ln \tau_N^r(\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]) - \frac{1}{N} \max'_{N_2, \boldsymbol{i}[N-N_2]} \ln \tau_N^r(N_2, \boldsymbol{i}[N-N_2]) \right| < \frac{\epsilon}{3} \tag{26}$$

for almost every N.

It is obvious that, for almost every N,

$$\max_{N_2, \boldsymbol{i}[N-N_2]} \ln \tau_N^r(N_2, \boldsymbol{i}[N-N_2]) \geqq \ln \tau_N^r(\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]). \tag{27}$$

From (6), (26), and (27), we have for any given $\epsilon > 0$

$$\left| \frac{1}{N} \ln \tau_N^r(\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]) - \frac{1}{N} \max_{N_2, \boldsymbol{i}[N-N_2]} \ln \tau_N^r(N_2, \boldsymbol{i}[N-N_2]) \right| < \frac{\epsilon}{3} \tag{28}$$

for almost every N.

Now by (17b) we have

$$\frac{\hat{N}_2}{N} - (1 - \sum_{l=1}^{\infty} l v b_l^{(0)} z_{\mathrm{cond}}^l) = \frac{1}{N} \sum_{l=1}^{l^*(N)} l (N v b_l^{(0)} z_{\mathrm{cond}}^l - [N v b_l^{(0)} z_{\mathrm{cond}}^l]) + \sum_{l=l^*(N)+1}^{\infty} l v b_l^{(0)} z_{\mathrm{cond}}^l. \tag{29}$$

Hence

$$0 < \frac{\hat{N}_2}{N} - (1 - \sum_{l=1}^{\infty} l v b_l^{(0)} z_{\mathrm{cond}}^l) < \frac{1}{N} \sum_{l=1}^{l^*(N)} l + \sum_{l=l^*(N)+1}^{\infty} l v b_l^{(0)} z_{\mathrm{cond}}^l. \tag{30}$$

By the condition 6° of Lemma 3 we have for any given $\epsilon > 0$

$$\left[0 < \frac{1}{N} \sum_{l=1}^{l^*(N)} l = \right] \frac{1}{2N} l^*(N)(l^*(N)+1) < \frac{2}{3}\epsilon \tag{31}$$‡

for almost every N.

Since by (G·5) the series $\sum_{l=1}^{\infty} l b_l^{(0)} z_{\mathrm{cond}}^l$ is convergent, there exists an integer $M(\epsilon) > 0$, corresponding to any given $\epsilon > 0$, such that

$$\left[0 < \sum_{l=l^*+1}^{\infty} v b_l^{(0)} z_{\mathrm{cond}}^l < \right] \sum_{l=l^*+1}^{\infty} l v b_l^{(0)} z_{\mathrm{cond}}^l < \frac{\epsilon}{3} \text{ whenever } l^* \geqq M(\epsilon). \tag{32}$$

By the condition 4° of Lemma 3 we have $\lim_{N \to \infty} l^*(N) = +\infty$; hence

$$l^*(N) \geqq M(\epsilon) \tag{33}$$

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‡ Note that $l^*(N)(l^*(N)+1)/2N \leqq \{l^*(N)/N^{1/2}\}^2$ for every N.

for almost every N. Hence by (32) we have

$$\left[0<\sum_{l=l^{\sharp}(N)+1}^{\infty} vb_l^{(0)} z_{\mathrm{cond}}^l<\right]\sum_{l=l^{\sharp}(N)+1}^{\infty} lvb_l^{(0)} z_{\mathrm{cond}}^l<\frac{\epsilon}{3} \tag{34}$$

for almost every N.

By (30), (31), and (34), we have for any given $\epsilon>0$

$$\left|\frac{\hat{N}_2}{N}-\left(1-\sum_{l=1}^{\infty} lvb_l^{(0)} z_{\mathrm{cond}}^l\right)\right|<\epsilon \tag{35}$$

for almost every N.

Now by (G·3), if any $\epsilon>0$ is given, then in *any* interval ($u'\leqq u\leqq u''$, say) interior to the interval $u_1<u<+\infty$, there exists a positive integer $M_0(\epsilon; u', u'')$ [$>V_0/u'$], independent of u, such that

$$\left|\frac{1}{l}\ln b_l(lu)-\ln b_0^{\dagger}\right|<\epsilon \text{ whenever } l\geqq M_0(\epsilon; u', u''). \tag{36}$$

Now put

$$\hat{N}_2 u=Nv. \tag{37}$$

Then, if we take (as we can in virtue of (4) and (3)) any δ such that

$$0<\delta<\min\{u_1^{-1}+v_{\mathrm{cond}}^{-1}-v^{-1},\ v^{-1}-v_{\mathrm{cond}}^{-1}\}, \tag{38}$$

then, by (35) [where we choose $\epsilon=v\delta$] and (37), there exists a positive integer $N_0(\delta)$ such that

$$u'<u<u'' \text{ whenever } N\geqq N_0(\delta), \tag{39}$$

where

$$u'\equiv(v^{-1}-v_{\mathrm{cond}}^{-1}+\delta)^{-1}>u_1 \tag{39a}$$

$$u''\equiv(v^{-1}-v_{\mathrm{cond}}^{-1}-\delta)^{-1}<+\infty. \tag{39b}$$

Thus u is in an interval (u', u'') interior to the open interval $(u_1, +\infty)$, whenever $N\geqq N_0(\delta)$. By (37) and (39),

$$\hat{N}_2 u''>Nv \text{ whenever } N\geqq N_0(\delta). \tag{40}$$

Hence, for any given integer $M(>0)$, we have

$$\hat{N}_2>M \text{ whenever } N\geqq\max\{N_0(\delta),\ Mu''/v\}. \tag{41}$$

By (36) [with $l=\hat{N}_2$], (37), (39), and (41), we have for any given $\epsilon>0$

$$\left|\frac{1}{\hat{N}_2}\ln b_{\hat{N}_2}(Nv)-\ln b_0^{\dagger}\right|<\epsilon \text{ whenever } N\geqq\max\{N_0(\delta),\ M_0(\epsilon; u', u'')u''/v\}. \tag{42}$$

Thus (since $\hat{N}_2\leqq N$) we have for almost every N

$$\left|\frac{1}{N}\ln b_{\hat{N}_2}(Nv)-\frac{\hat{N}_2}{N}\ln b_0^{\dagger}\right|<\epsilon. \tag{43}$$

Now, from (4) and (3) we have

$$v > u_0 [\geqq v_0]. \tag{44}$$

If we apply (as we can in virtue of (44)) Lemma 7(ii) [in which ϵ is replaced by $\epsilon/6$] to the set $\{\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]\}$, we have for any given $\epsilon > 0$

$$\left| \frac{1}{N} \ln \tau_N(\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]) - \frac{1}{N} \ln \tau_N^r(\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]) \right| - \left| \frac{1}{N} \ln b_{\hat{N}_2}(Nv) - \frac{\hat{N}_2}{N} \ln b_0^{\dagger} \right| < \frac{\epsilon}{6} \tag{45}$$

for almost every N. By this and (43) [in which ϵ is replaced by $\epsilon/6$] we have for any given $\epsilon > 0$

$$\left| \frac{1}{N} \ln \tau_N(\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]) - \frac{1}{N} \ln \tau_N^r(\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]) \right| < \frac{\epsilon}{3} \tag{46}$$

for almost every N.

Now let $\{\bar{N}_2, \bar{\boldsymbol{i}}[N-\bar{N}_2]\}$ be the set for which $\tau_N(N_2, \boldsymbol{i}[N-N_2])$ is largest. Then obviously

$$\tau_N(\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]) \leqq \tau_N(\bar{N}_2, \bar{\boldsymbol{i}}[N-\bar{N}_2]). \tag{47}$$

If we apply (as we can in virtue of (44)) Lemma 7(i) [in which ϵ is replaced by $\epsilon/3$] to the set $\{\bar{N}_2, \bar{\boldsymbol{i}}[N-\bar{N}_2]\}$, we have for any given $\epsilon > 0$

$$\frac{1}{N} \ln \tau_N(\bar{N}_2, \bar{\boldsymbol{i}}[N-\bar{N}_2]) - \frac{1}{N} \ln \tau_N^r(\bar{N}_2, \bar{\boldsymbol{i}}[N-\bar{N}_2]) < \frac{\epsilon}{3} \tag{48}$$

for almost every N. Let $\{\bar{N}_2^r, \bar{\boldsymbol{i}}^r[N-\bar{N}_2^r]\}$ be the set for which $\tau_N^r(N_2, \boldsymbol{i}[N-N_2])$ is largest. Then obviously

$$\tau_N^r(\bar{N}_2, \bar{\boldsymbol{i}}[N-\bar{N}_2]) \leqq \tau_N^r(\bar{N}_2^r, \bar{\boldsymbol{i}}^r[N-\bar{N}_2^r]). \tag{49}$$

By (28), (46), (47), (48), and (49), we have for any given $\epsilon > 0$

$$\left| \frac{1}{N} \ln \tau_N(\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]) - \frac{1}{N} \max_{N_2, \boldsymbol{i}[N-N_2]} \ln \tau_N(N_2, \boldsymbol{i}[N-N_2]) \right| < \epsilon \tag{50}$$

for almost every N. [For, if $|\hat{b}-b_2| < \epsilon/3$, $|\hat{a}-\hat{b}| < \epsilon/3$, $\hat{a} \leqq a_1$, $a_1 - b_1 < \epsilon/3$, and $b_1 \leqq b_2$, then $0 \geqq \hat{a}-a_1 \geqq (\hat{a}-\hat{b}) + (\hat{b}-b_2) - (a_1-b_1) > -\epsilon/3-\epsilon/3-\epsilon/3 = -\epsilon$.]

From (50) and (35) we see that, for any given $\epsilon > 0$, the set $\{\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]\}$, for almost any N, satisfies both (α) and (β). Thus the proof of (i) is completed.

〈*Proof of* (ii)〉—From (50) [in which ϵ is replaced by $\epsilon/2$], if we use (as we can in virtue of (44)) Lemma 6 [in which ϵ is replaced by $\epsilon/2$], we see that the set $\{\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]\}$, for almost every N, satisfies (α'). Thus (ii) is proved.

〈*Proof of* (iii)〉—From (16) and (34), we have for any given $\epsilon > 0$

$$\left|\frac{1}{N}\max'_{N_2,\,\boldsymbol{i}[N-N_2]}\ln\tau^r_N(N_2,\boldsymbol{i}[N-N_2])-\left(\sum_{l=1}^{\infty}vb_l^{(0)}z_{\mathrm{cond}}^l-\ln z_{\mathrm{cond}}\right)\right|$$

$$\left[=\sum_{l=l^*(N)+1}^{\infty}vb_l^{(0)}z_{\mathrm{cond}}^l\right]<\frac{\epsilon}{3} \tag{51}$$

for almost every N. From (26), (46), and (51), we see that the set $\{\hat{N}_2, \hat{\boldsymbol{i}}[N-\hat{N}_2]\}$, for almost every N, satisfies (α''). Thus (iii) is proved.

〈*Proof of* (iv)〉—Use (46).

〈*Proof of* (v)〉—We shall now prove that, if any sufficiently small‡ $\epsilon>0$ is given, then, for almost any N, every set $\{N_2, \boldsymbol{i}[N-N_2]\}$ which satisfies (α) and (β), satisfies (γ) and (δ). From (β), we have, for every such N_2 for almost any N,

$$\left|\frac{1}{N}\ln b_{N_2}(Nv)-\frac{N_2}{N}\ln b_0^{\dagger}\right|<\frac{\epsilon}{3}, \tag{52}$$

by means of the same argument (with $N_2, \boldsymbol{i}$ in place of $\hat{N}_2, \hat{\boldsymbol{i}}$) as in (35)—(43).‡‡ Hence by Lemma 5(ii) [in which ϵ is replaced by $\epsilon/3$], we have for every such N_2 for almost any N,

$$\left|\frac{1}{N}\ln\sum_{\boldsymbol{k}[N_2]}\Phi_N(\boldsymbol{k}[N_2])-\frac{N_2}{N}\ln b_0^{\dagger}\right|<\frac{2\epsilon}{3}. \tag{53}$$

Hence by (52) we obtain (γ).

Next, eliminating $\hat{N}_2, \hat{\boldsymbol{i}}$ from (26), (46), and (50) [in which ϵ is replaced by $\epsilon/3$], we have for almost every N

$$\frac{1}{N}\max\ln\tau_N(N_2,\boldsymbol{i}[N-N_2])-\frac{1}{N}\max'\ln\tau^r_N(N_2,\boldsymbol{i}[N-N_2])<\epsilon. \tag{54}$$

Using this and (16), from (α) we have for almost any N

$$\left|\frac{1}{N}\ln\tau_N(N_2,\boldsymbol{i}[N-N_2])-\left(\sum_{l=1}^{l^*(N)}vb_l^{(0)}z_{\mathrm{cond}}^l-\ln z_{\mathrm{cond}}\right)\right|<2\epsilon \tag{55}$$

for every set $\{N_2, \boldsymbol{i}[N-N_2]\}$ which satisfies (α) and (β). By 6·1(3), (53), (55), and (β), we obtain (δ) if we take $c\geqq 3+|\ln b_0^{\dagger}|$. This completes the proof.

Theorem 1a *If* $v^*<v<v_{\mathrm{cond}}$, *then* $\lim_{N\to\infty}(1/N)\ln\Omega_N(Nv,T)$ *exists and*

$$\lim_{N\to\infty}\frac{1}{N}\ln\Omega_N(Nv,T)=\sum_{l=1}^{\infty}vb_l^{(0)}z_{\mathrm{cond}}^l-\ln z_{\mathrm{cond}}. \tag{56}$$

〈*Proof*〉—Use (50), 6·1(2), (26), (46) and (51).

Corollary 1. *If* $v^*<v<v_{\mathrm{cond}}$ *then for any given* $\epsilon>0$, *there exists, for almost any* N, *at least one* N_2 *for which*

‡ This condition means: $\epsilon\leqq v\delta$ in (β). [See (38).]

‡‡ In (36), ϵ is replaced by $\epsilon/3$.

$$\left|\frac{1}{N}\ln\sum_{\boldsymbol{i}[N-N_2]}\tau_N(N_2,\boldsymbol{i}[N-N_2])-\frac{1}{N}\ln\Omega_N\right|<\epsilon \tag{57}$$

and (β) *are simultaneously satisfied.*

〈*Proof*〉—By Theorem 1(ii) there exists, for almost any N, at least one set $\{N_2, \boldsymbol{i}[N-N_2]\}$ for which (α') and (β) are simultaneously satisfied. Since this N_2 satisfies (α'), it satisfies (57) in virtue of the relation:

$$\tau_N(N_2,\boldsymbol{i}[N-N_2])\leqq\sum_{\boldsymbol{i}[N-N_2]}\tau_N(N_2,\boldsymbol{i}[N-N_2])\leqq\Omega_N \tag{58}$$

which holds for almost every N since by (44), 5·1(5), and (G·2) we have 5·1(6). Thus the corollary is proved.

Corollary 2. *If* $v^*<v<v_{\text{cond}}$, *we have for any given* $\epsilon>0$

$$\left|\frac{1}{N}\ln\sum_{\boldsymbol{j}[N]}\chi_N(\boldsymbol{j}[N])-\frac{1}{N}\ln\Omega_N\right|<\epsilon \tag{59}$$

for almost every N.

〈*Proof*〉—By Theorem 1 (ii) there exists, for almost any N, at least one set $\{N_2, \boldsymbol{i}[N-N_2]\}$ for which (α') and (β) are simultaneously satisfied. For almost every N we have $N_2>0$ by (β) (since $v<v_{\text{cond}}$). Hence $\tau_N(N_2, \boldsymbol{i}[N-N_2])$ given by this set $\{N_2, \boldsymbol{i}[N-N_2]\}$ is a part of the sum $\sum_{\boldsymbol{j}[N]}\chi_N(\boldsymbol{j}[N])$. Therefore, since by (44), 5·1(5), and (G·2) we have 5·1(6), we obtain for almost every N

$$\tau_N(N_2,\boldsymbol{i}[N-N_2])\leqq\sum_{\boldsymbol{j}[N]}\chi_N(\boldsymbol{j}[N])\leqq\Omega_N. \tag{60}$$

By this and (α'), we have (59). Hence the corollary.

§ **8·2.** *Physical meaning of the theorems and the corollaries*

§ **8·21.** Theorem 1 shows that for the G-system in the interval $v^*<v<v_{\text{cond}}$, if N is large enough, there exists a set $\{N_2, \boldsymbol{i}[N-N_2]\}$ for which $\tau_N(N_2, \boldsymbol{i}[N-N_2])$ is practically maximum [see 8·1(α)]—i.e. $\tau_N(N_2, \boldsymbol{i}[N-N_2])$ represents practically the entire partition function Ω_N[see 8·1(α')]—and for which, at the same time, N_2/N is as near as we please to a finite positive number $(1-v/v_{\text{cond}})$ (independent of N but dependent on v and T) [see 8·1(β)]. Such an N_2 is called "huge".† In physical words, such a set $\{N_2, \boldsymbol{i}[N-N_2]\}$ gives the equilibrium state and such an N_2 is comparable to N or is of a macroscopic magnitude. For our equilibrium-state term τ_N, we have

$$\ln\tau_N(N_2,\boldsymbol{i}[N-N_2])=\ln\varphi_N(\boldsymbol{i}[N-N_2])+\ln\sum_{\boldsymbol{k}[N_2]}\Phi_N(\boldsymbol{k}[N_2]). \tag{1}$$

Here by 8·1(γ), if N is large enough, $\ln\Sigma\Phi_N(\boldsymbol{k}[N_2])$ [which represents the situations where a "huge" number (N_2) of molecules are grouped into "large" clusters in various manners] may, with an error as small as we please, be re-

† If a number is of the same order of infinity as $N(\to\infty)$, it is called "huge". Every "huge" number is "large", the converse being not true.

placed by $\ln b_{N_2}(V)$ [which represents the situation where the molecules together compose only one "huge" cluster];—in other words, in a macroscopic G-system, the N_2 molecules, which are destined to form "large" clusters in an arbitrary manner, should be expected to form one "huge" cluster. Thus we have rigorously deduced that, in thermodynamical equilibrium, whenever $v^*<v<v_{\text{cond}}$, there exists one "huge" (or macroscopic) cluster† $[\ln b_{N_2}(V)]$; it should (according to the physical interpretation of a cluster) be regarded as representing the liquid phase which has appeared by the two-phase separation in the condensation process and which coexists with the saturated vapour—represented by a set of "small" clusters: $\ln \varphi_N$ $(\boldsymbol{i}[N-N_2])$ in (1) [see 8·1(δ)]—in the condensation range. [We thus see that $1/v_{\text{cond}}$ represents the density of the saturated vapour, and also we see from 8·1(β) that the "huge" cluster steadily grows large as v decreases in the interval $v_{\text{cond}}>v>v^*$.]

§ **8·22.** By Theorem 1a we can know all the limiting thermodynamical functions (for $N\to\infty$ with $v=V/N$ fixed) of the G-system for $v^*<v<v_{\text{cond}}$. The argument (based on the Weierstrass theorem about uniform convergence and double sequence)†† given in the footnote on p. 382 of I is also true in the present case; hence we have for the pressure p_∞ of the G-system

$$p_\infty/kT=-(1/kT)\lim\{\partial A/\partial V\}^{\dagger\dagger\dagger}$$
$$=\lim\{(\partial/\partial v)(\ln \Omega_N)/N\}=(\partial/\partial v)\lim\{(\ln \Omega_N)/N\}, \qquad (1)$$

where lim is taken for $N\to\infty$ (and $V\to\infty$) with $v=V/N$ fixed. Thus by Theorem 1a we have (for $v^*<v<v_{\text{cond}}$)

$$p_\infty/kT=\sum_{l=1}^{\infty} b_l^{(0)} z_{\text{cond}}^l \ (\equiv p_{\text{cond}}/kT, \text{ say}), \qquad (2)$$

which is independent of v; that is, the isothermal p-v curve of the G-system is horizontal‡ for $v^*<v<v_{\text{cond}}$. And we have for the Gibbs free energy G of the G-system

$$\lim(G/N)=\lim\{(A+pV)/N\}=kT\ln(\lambda^3 z_{\text{cond}}) \qquad (3)$$

(for $v^*<v<v_{\text{cond}}$); hence the activity‡‡ of the G-system for $v^*<v<v_{\text{cond}}$ is z_{cond}, and is independent of v.

§ **8·23.** Corollary 1 shows that τ_N in Theorem 1 may be replaced by the sum of τ_N over $\boldsymbol{i}[N-N_2]$. That is, here we have given the equilibrium state

† It does not exist for $v\geqq v_{\text{cond}}$ [see § 9·22].

†† A detailed account of this argument showing the commutability of the operations lim and $(\partial/\partial v)$ in (1), is given in another place by the present author and Nakazawa.[5]

††† Here A is the Helmholtz free energy, given by:

$$A=NkT\ln\lambda^3-kT\ln\Omega_N, \text{ where } \lambda=h\cdot(2\pi mkT)^{-1/2}.$$

‡ It is not horizontal for $v\geqq v_{\text{cond}}$ [see § 9·23].

‡‡ The activity z (which is sometimes called "fugacity" or "active number density") is defined as: $z=a/\lambda^3$, where $a[\equiv\exp\{G/(NkT)\}]$ is the absolute activity.

by determining only the number N_2 of molecules composing the liquid phase, with no specification of the manner $\boldsymbol{i}[N-N_2]$ of grouping $N-N_2$ molecules into "small" clusters (i.e. of the detailed state of the vapour phase).

§ **8·24.** Corollary 2 shows that, when Ω_N is expressed as 3·3(4), Ω_N may practically be represented by the second part alone.

§ 9. Gaseous state of the G-system

§ **9·1.** We shall now prove the following theorems (and corollary) concerning the thermodynamical properties of the G-system in gaseous state. (The physical interpretation of the theorems will be given in § 9·2.)

Theorem 2. *If*

$$v \geqq v_{\text{cond}} \text{ [cf. 8·1(2)]}, \tag{1}$$

the following are true:

(i) $\lim_{N\to\infty} (1/N) \ln \Omega_N(Nv, T) = \lim_{N\to\infty} (1/N) \ln \Omega_N^{(0)}(Nv, T)$.

(ii) *For any given* $\epsilon > 0$, *there exists, for almost any* N, *at least one set* $\boldsymbol{i}[N]$ *for which*

$$\left| \frac{1}{N} \ln \varphi_N(\boldsymbol{i}[N]) - \frac{1}{N} \max_{N_2, \boldsymbol{i}[N-N_2]} \ln \tau_N(N_2, \boldsymbol{i}[N-N_2]) \right| < \epsilon \tag{α}$$

is satisfied.

(iii) *The statement* (ii) *still holds if we replace* (α) *by*

$$\left| \frac{1}{N} \ln \varphi_N(\boldsymbol{i}[N]) - \frac{1}{N} \ln \Omega_N \right| < \epsilon. \tag{α'}$$

(iv) *The statement* (ii) *still holds if we replace* (α) *by*

$$\left| \frac{1}{N} \ln \varphi_N(\boldsymbol{i}[N]) - \left(\sum_{l=1}^{\infty} v b_l^{(0)} z^l - \ln z \right) \right| < \epsilon, \tag{α''}$$

where z *is the least real positive root of the equation*

$$\sum_{l=1}^{\infty} l v b_l^{(0)} z^l = 1. \tag{2}$$

(v) *The statements* (ii), (iii), *and* (iv) *still hold if we replace* $\varphi_N(\boldsymbol{i}[N])$ *by* $\varphi_N^{\Gamma}(\boldsymbol{i}[N])[\equiv \tau_N^{\Gamma}(0, \boldsymbol{i}[N])]$.

《*Proof of* (i)》—To prove this, we shall use the following theorem† established in I:

If $v \geqq v_s[\equiv (\sum_{l=1}^{\infty} l b_l^{(0)} z_s^l)^{-1}$, *where* $z_s \equiv (b_0^{(0)})^{-1}]$,†† *the necessary and sufficient condition that* $\lim_{N\to\infty} (1/N) \ln \Omega_N(Nv, T)$ *should exist and*

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† Theorems [I] and [I′] in the paper I.

†† v_s is the volume per molecule of the saturated vapour for the (0)-system; and z_s is the activity of the (0)-system at the condensation point (and, therefore, in the condensation range).

$$\lim_{N\to\infty}\frac{1}{N}\ln\Omega_N(Nv,T)=\lim_{N\to\infty}\frac{1}{N}\ln\Omega_N^{(0)}(Nv,T)$$

is that:

$$\limsup_{\substack{N\in E_{(+)}\\ N\to\infty}}\frac{1}{N}\{\ln\sum_{j[N]}\chi_N(j[N])-\ln\sum_{i[N]}\varphi_N^{(0)}(i[N])\}\leqq 0 \tag{A}$$

when the set $E_{(+)}=E\{N(\text{positive integers});\ \sum_{j[N]}\chi_N(j[N])>0\}$ *is an infinite set, and*

$$\ln\left\{1+\frac{\sum_{j[N]}\chi_N(j[N])}{\sum_{i[N]}\varphi_N(i[N])}\right\}^{-1}=o(N)\quad \textit{for } N\in E_{(-)} \tag{B}$$

when the set $E_{(-)}=E\{N$ (positive integers) ; $\sum_{j[N]}\chi_N(j[N])\leqq 0\}$ *is an infinite set.*

In proving this theorem in I, we have used Lemmas 5 and 6 of I, which are valid also in the case of the present paper;† hence we may apply this theorem to the proof of the present theorem 2 (for $v\geqq v_{\text{cond}}$), since by Lemmas 1b and 1c we have

$$v_{\text{cond}}\geqq v_s. \tag{3}$$

Now, by Lemma 6 of I, we have for $v\geqq v_s$ (and, thus, for $v\geqq v_{\text{cond}}$)

$$\lim_{N\to\infty}\frac{1}{N}\ln\sum_{i[N]}\varphi_N^{(0)}(i[N])=\lim_{N\to\infty}\frac{1}{N}\ln\Omega_N^{(0)}\Big[=\sum_{l=1}^{\infty}vb_l^{(0)}z^l-\ln z\Big].^{\dagger\dagger} \tag{4}$$

That is, for any given $\epsilon>0$ we have for almost every N

$$\left|\frac{1}{N}\ln\sum_{i[N]}\varphi_N^{(0)}(i[N])-\Big(\sum_{l=1}^{\infty}vb_l^{(0)}z^l-\ln z\Big)\right|<\frac{\epsilon}{3}. \tag{5}$$

From (G·5) and (1), we have

$$v>u_0[\geqq v_0], \tag{6}$$

hence by 5·1(5) and (G·2) we have 5·1(6); thus

$$\Omega_N\geqq\sum_{j[N]}\chi_N(j[N]) \tag{7}$$

for almost every N.

We now employ the symbol max′ defined in the proof of Theorem 1. Then, using (5), (7), Lemma 6‡ [in which ϵ is replaced by $\epsilon/3$], 8·1(48),‡ 8·1(49), and 8·1(6), we have for any given $\epsilon>0$

† Lemma 5 of I follows also from Lemma 4 of the present paper; and Lemma 6 of I, concerning the (0)-system only, holds also in the present case in virtue of Lemma 1b and (G·6) of the present paper.

†† z is determined by (2), and represents the activity of the (0)-system for $v\geqq v_s$.

‡ Lemma 6 and 8·1(48) can be used in virtue of (6).

$$\frac{1}{N}\{\ln\sum_{\boldsymbol{j}[N]}\chi_N(\boldsymbol{j}[N])-\ln\sum_{\boldsymbol{i}[N]}\varphi_N^{(0)}(\boldsymbol{i}[N])\}$$

$$<\frac{1}{N}\max'_{N_2,\boldsymbol{i}[N-N_2]}\ln\tau_N^{\Gamma}(N_2,\boldsymbol{i}[N-N_2])-(\sum_{l=1}^{\infty}vb_l^{(0)}z^l-\ln z)+\epsilon \tag{8}$$

for almost every N.

Equation 8·1(9b) may be written as

$$\overline{\overline{N}}_2=N(1-v/v_{\text{cond}}+\sum_{l=l^*(N)+1}^{\infty}lvb_l^{(0)}z_{\text{cond}}^l). \tag{9}$$

Now if $v>v_{\text{cond}}$, then by 8·1(34) [in which we take $\epsilon/3$ smaller than $v/v_{\text{cond}}-1$] we have

$$\overline{\overline{N}}_2<0 \tag{10}$$

for almost every N. Hence, for $\max'_{N_2,\boldsymbol{i}[N-N_2]}\ln\tau_N^{\Gamma}(N_2,\boldsymbol{i}[N-N_2])$ in which N_2, m_1, m_2, $\cdots$, $m_{l^*(N)}$ are real and non-negative and satisfy the condition $\sum lm_l+N_2=N$, we have for almost every N

$$\max'_{N_2,\boldsymbol{i}[N-N_2]}\ln\tau_N^{\Gamma}(N_2,\boldsymbol{i}[N-N_2])=\max'_{\boldsymbol{i}[N]}\ln\tau_N^{\Gamma}(0,\boldsymbol{i}[N]). \tag{11}$$

With the use of the undetermined multiplier $\ln\zeta$, the set $\boldsymbol{i}[N]$ giving max′ must satisfy

$$\frac{\partial}{\partial m_l}\{\ln\tau_N^{\Gamma}(0,\boldsymbol{i}[N])+\sum_{l=1}^{l^*(N)}lm_l\ln\zeta\}=0\quad(l=1,2,\cdots,l^*(N)). \tag{12}$$

From this we have

$$m_l=Nvb_l^{(0)}\zeta^l\quad(l=1,2,\cdots,l^*(N)), \tag{13}$$

where $\zeta[=\zeta(N)]$ is determined by

$$\sum_{l=1}^{l^*(N)}lvb_l^{(0)}\zeta^l=1. \tag{14}$$

Thus

$$\max'_{\boldsymbol{i}[N]}\ln\tau_N^{\Gamma}(0,\boldsymbol{i}[N])=N\sum_{l=1}^{l^*(N)}vb_l^{(0)}\zeta^l-N\ln\zeta. \tag{15}$$

By (11) and (15) we have for almost every N

$$\frac{1}{N}\max'_{N_2,\boldsymbol{i}[N-N_2]}\ln\tau_N^{\Gamma}(N_2,\boldsymbol{i}[N-N_2])-(\sum_{l=1}^{\infty}vb_l^{(0)}z^l-\ln z)$$

$$=\sum_{l=1}^{l^*(N)}vb_l^{(0)}(\zeta^l-z^l)-\sum_{l=l^*(N)+1}^{\infty}vb_l^{(0)}z^l-(\ln\zeta-\ln z)$$

$$<\sum_{l=1}^{l^*(N)}lvb_l^{(0)}(\zeta^l-z^l)=\sum_{l=l^*(N)+1}^{\infty}lvb_l^{(0)}z^l. \tag{16}$$

Here we have used Lemma 1b and the fact that, by (2) and (14), $\zeta>z$ for

every N. Now by (8) [in which ϵ is replaced by $(2/3)\epsilon$] and (16) and 8·1(34), we have for any given $\epsilon>0$

$$\frac{1}{N}\{\ln\sum_{\boldsymbol{j}[N]}\chi_N(\boldsymbol{j}[N])-\ln\sum_{\boldsymbol{i}[N]}\varphi_N^{(0)}(\boldsymbol{i}[N])\}<\epsilon \tag{17}$$

for almost every N. Hence the condition (A) is satisfied.

Now if $v=v_{\text{cond}}$, then (9) becomes

$$\overline{\overline{N}}_2=N\sum_{l=l^*(N)+1}^{\infty} lvb_l^{(0)}z_{\text{cond}}^l>0. \tag{18}$$

Hence 8·1(16) holds. Thus

$$\frac{1}{N}\max'_{N_2,\,\boldsymbol{i}[N-N_2]}\ln\tau_N^r(N_2,\,\boldsymbol{i}[N-N_2])-(\sum_{l=1}^{\infty}vb_l^{(0)}z^l-\ln z)=-\sum_{l=l^*(N)+1}^{\infty}vb_l^{(0)}z_{\text{cond}}^l<0. \tag{19}$$

Here we have used the fact that $z=z_{\text{cond}}$ if $v=v_{\text{cond}}$. By (8) and (19), we have (17) for almost every N. Hence, also in this case, the condition (A) is satisfied. The set $\boldsymbol{E}_{(-)}$ is at most a finite set, since by (6), 5·1(5), and (G·2) we have 5·1(6). Thus the proof of (i) is completed.

〈*Proof of* (ii), (iii), *and* (iv)〉—Now it is seen from the proof of Lemma 6 of I that if $v[\gtreqless v_{\text{cond}}]\gtreqless v_s$, then for any given $\epsilon>0$ there exists, for almost every N, at least one set $\boldsymbol{i}[N]$ for which

$$\left|\frac{1}{N}\ln\varphi_N^{(0)}(\boldsymbol{i}[N])-(\sum_{l=1}^{\infty}vb_l^{(0)}z^l-\ln z)\right|<\frac{\epsilon}{2}. \tag{20}$$

By Lemma 4(ii) [with $N_2=0$], we have for this $\boldsymbol{i}[N]$

$$\left|\frac{1}{N}\ln\varphi_N(\boldsymbol{i}[N])-\frac{1}{N}\ln\varphi_N^{(0)}(\boldsymbol{i}[N])\right|<\frac{\epsilon}{2} \tag{21}$$

for almost every N. By (20) and (21) this $\boldsymbol{i}[N]$ satisfies (α''). By (i) of the present theorem, we have for almost every N

$$\left|\frac{1}{N}\ln\Omega_N-(\sum_{l=1}^{\infty}vb_l^{(0)}z^l-\ln z)\right|<\frac{\epsilon}{2}. \tag{22}$$

By (α'') [in which ϵ is replaced by $\epsilon/2$] and (22), we have (α'). By Lemma 6† [in which ϵ is replaced by $\epsilon/2$] and (α') [in which ϵ is replaced by $\epsilon/2$], we have (α).

〈*Proof of* (v)〉—Use Lemma 7(ii)† [with $N_2=0$].

Theorem 2a. *If* $v\gtreqless v_{\text{cond}}$, *we have*

$$\lim_{N\to\infty}\frac{1}{N}\ln\Omega_N(Nv,\,T)=\sum_{l=1}^{\infty}vb_l^{(0)}z^l-\ln z. \tag{23}$$

† Lemmas 6 and 7 can be used in virtue of (6).

〈*Proof*〉—Use Theorem 2(i) or (22).

Corollary. *If* $v \geqq v_{\text{cond}}$, *we have for any given* $\epsilon > 0$

$$\left| \frac{1}{N} \ln \sum_{\boldsymbol{i}[N]} \varphi_N(\boldsymbol{i}[N]) - \frac{1}{N} \ln \Omega_N \right| < \epsilon \tag{24}$$

for almost every N.

〈*Proof*〉—Note that, by (6), 5·1(5), and (G·2),

$$\varphi_N(\boldsymbol{i}[N]) \leqq \sum_{\boldsymbol{i}[N]} \varphi_N(\boldsymbol{i}[N]) \leqq \Omega_N$$

for almost every N, and use Theorem 2(ii).

§ **9·2.** *Physical meaning of the theorems and the corollary*

§ **9·21.** Theorem 2(i) shows that, when $v \geqq v_{\text{cond}}$, the limiting thermodynamical functions (for $N \to \infty$ with $v = V/N$ fixed) of the G-system are equal to those of the (0)-system.

§ **9·22.** Theorem 2(ii) shows that for the G-system in the range $v \geqq v_{\text{cond}}$, if N is large enough, there is a set $\boldsymbol{i}[N]$ for which $\varphi_N(\boldsymbol{i}[N])$ is practically the largest τ_N [i.e. $\varphi_N(\boldsymbol{i}[N])$ represents practically the entire partition function Ω_N]. Thus such a set $\boldsymbol{i}[N]$, which of course contains no "large" cluster, gives the equilibrium state; hence, in thermodynamical equilibrium, there are only "small" clusters in the G-system for $v \geqq v_{\text{cond}}$. This corresponds to the gaseous state.

§ **9·23.** By Theorem 2a we can know all the limiting thermodynamical functions (for $N \to \infty$ with $v = V/N$ fixed) of the G-system for $v \geqq v_{\text{cond}}$. Since the argument (using the Weierstrass theorem on uniform convergence and double sequence)* given in the footnote on p. 382 of I holds also in the present case [cf. 8·22(1)], it is shown (by Theorem 2a) that the pressure p_∞ of the G-system for $v \geqq v_{\text{cond}}$ is given by

$$p_\infty / kT = \sum_{l=1}^{\infty} b_l^{(0)} z^l, \tag{1}$$

and hence it follows that the activity of the G-system for $v \geqq v_{\text{cond}}$ is z. Thus in virtue of 9·1(2) the specific volume v of the G-system is given by

$$1/v = \sum_{l=1}^{\infty} l b_l^{(0)} z^l \tag{2}$$

as a function of the activity z of the G-system, provided that $v \geqq v_{\text{cond}}$.

§**9·24.** Corollary shows that, if N is large enough, Ω_N may practically be represented by the "small" cluster part [in 3·3(4)] alone.

§ 10. Concluding remarks

§ **10·1.** The most essential results of the present paper are as follows:

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* See the first footnote †† in § 8·22.

We have mathematically discussed the thermodynamical behaviour of the G-system, defined in § 1, and especially we have proved the *two-phase separation* in the G-system; that is, we have demonstrated that, as soon as the density of the system exceeds $1/v_{\text{cond}}$ given by

$$1/v_{\text{cond}}=\sum_{l=1}^{\infty} l b_l^{(0)} z_{\text{cond}}^l \quad [\text{where } z_{\text{cond}}=(b_0^{\dagger})^{-1}], \tag{1}$$

the logarithm of the "*huge*" *cluster integral*, which is given by

$$\ln b_{\bar{N}_2} \simeq \ln \sum \bar{\Phi} \simeq \bar{N}_2 \ln b_0^{\dagger} \tag{2}*$$

[where

$$\bar{N}_2=N-\bar{N}_1=N-\sum l V b_l^{(0)} z_{\text{cond}}^l], \tag{3}*$$

appears as a term of the logarithm of the partition function, namely

$$\ln \Omega \simeq \max \ln \tau \simeq \ln \bar{\tau}=\ln \bar{\varphi}+\ln \sum \bar{\Phi} \simeq \ln \bar{\varphi}+\ln b_{\bar{N}_2}, \tag{4}*$$

and $\ln b_{\bar{N}_2}$ describes the liquid phase which coexists with the saturated vapour represented by

$$\ln \bar{\varphi} \simeq V \sum b_l^{(0)} z_{\text{cond}}^l - \bar{N}_1 \ln z_{\text{cond}}. \tag{5}*$$

§ **10·2.** *Analytical properties of the singularity:* Now let R be the radius of convergence of the power series 9·23(1) and 9·23(2). Then according to the Cauchy-Hadamard theorem we have

$$1/R=\lim_{l\to\infty} \sup (b_l^{(0)})^{1/l}. \tag{1}$$

If we use z_s to denote the first singularity (on the real positive axis of z) of the analytic functions defined by 9·23(1) and 9·23(2), then by Lemma 1b we have

$$z_s=R[=(b_0^{(0)})^{-1}, \text{ by (G·6)}]. \tag{2}$$

Hence [by Lemma 1c], between z_s and z_{cond} [≡activity at the condensation point of the G-system, cf. 8·1(1)], there is only the following relation:

$$z_s \geqq z_{\text{cond}}. \tag{3}$$

Thus, about the problem of the analytical properties of the singularity representing the condensation of the G-system, two different conclusions are deduced according as $z_s=z_{\text{cond}}$ or $z_s>z_{\text{cond}}$ (see Fig. 1):

Case 1: $z_s=z_{\text{cond}}$. In this case the condensation point for the G-system

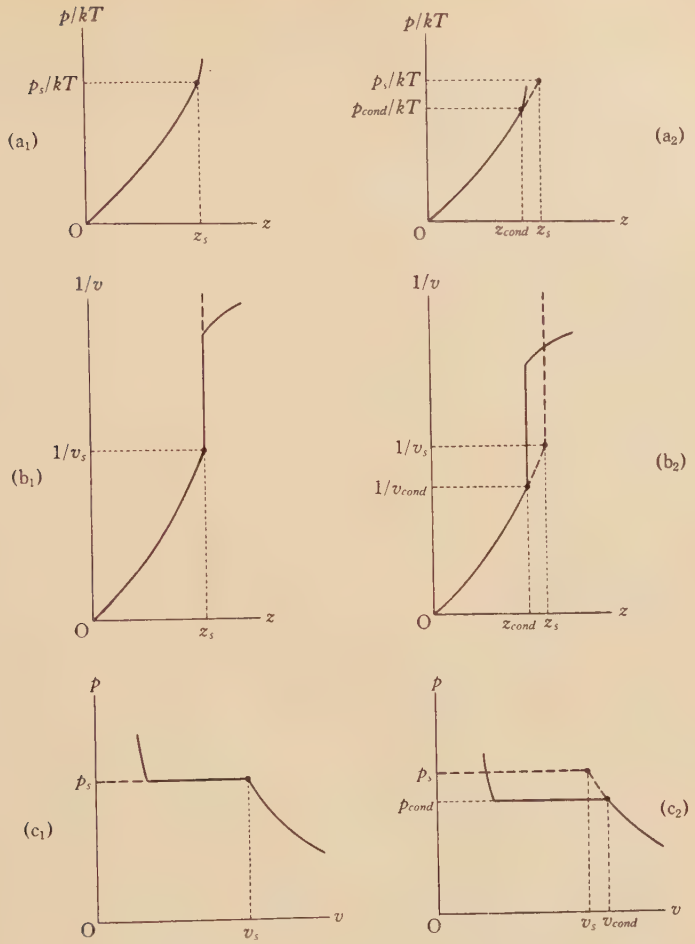
* Here we use concise notations and expressions, and we write b_l for $b_l(V)$. The bar means the "equilibrium state" (cf. § 8·2).

For (2), see 8·1(γ) and 8·1(43).

$\bar{N}_2$ given by (3) is practically equal to N_2 in 8·1 (β), $\bar{\bar{N}}_2$ in 8·1(9b), and $\hat{N}_2$ in 8·1(17b).

For (4), see 6·1(2), 8·1(α), 8·1(α'), and 8·21(1).

For (5), see 8·1(δ).



Case 1. Case 2.

Fig. 1. [Analytical behaviour of the condensing systems at a sufficiently low temperature.] *Case 1* [(a_1), (b_1), (c_1)]; *Case 2* [(a_2), (b_2), (c_2)].

The solid curves (or lines) represent the isotherms of the G-system. The broken curves (or lines) represent those parts of the isotherms of the (0)-system which have deviated from the isotherms of the G-system.

corresponds to the first "analytical" singularity z_s (on the real positive axis of z) of the gaseous functions p/kT [9·23(1)] and $1/v$ [9·23(2)], the word "analytical"* being used in the sense defined in Ia and I[1]. Thus the G-system and the (0)-system begin to condense at the same density $1/v_{\text{cond}}=1/v_s=\sum_{l=1}^{\infty} l b_l^{(0)} z_s^{\,l}$.

* See p. 550 of Ia; p. 377 of I. An "analytical" singularity means a singularity defined in the theory of functions of a complex variable; i.e. a singularity which an analytic function has; e.g. a pole, a branch-point, an essential singularity, etc. Any other kind of singularity is called "non-analytical"; e.g. a point of intersection of two different analytic functions, and a point of intersection of two different branches of one analytic function [Fig. 2 of Ia].

Case 2: $z_s > z_{cond}$. In this case the condensation point for the G-system corresponds to the "non-analytical" singularity $z_{cond}[<z_s]$ (on the real positive axis of z) of the gaseous functions p/kT and $1/v$, the word "non-analytical" being used in the sense defined in Ia and I. Thus, when the density exceeds $1/v_{cond}$, the behaviour of the G-system deviates from that of the (0)-system; the latter continues to be in gaseous state up to the density $\sum_{l=1}^{\infty} l b_l^{(0)} z_s^l [\equiv 1/v_s$, say], which (if it is finite) represents the density at the starting point of the condensation of the (0)-system [see II[2)]].

§ **10·201.** The above cases 1 and 2 correspond to the cases of § 3·12 and § 3·13 in I, respectively.

§ **10·202.** The case in which $z_{cond} > z_s$, is impossible in principle; for z_s is an "analytical" singularity of an analytic function, so that the function cannot be *smoothly* (in an analytical sense) continued beyond z_s [cf. p. 386 of I]. In the present paper this case is, in fact, avoided in virtue of (G. 4).

§ **10·21.** It has not been a purpose of the present paper to discuss the problem as to which of the cases 1 and 2 [§ 10·2] is realized in the physical real systems. About this problem we argued in detail in Ia and I. It should be emphasized that the present paper gives a generalized theory of two-phase separation which is comprehensive of the two cases 1 and 2; that is to say, we have been able to discuss the problem of two-phase separation independently of the problem of the analytical properties of the singularity. Thus, at least in this sense, our G-system may also be said to be more general than the real system itself. In another paper we shall give a more detailed comparison between the G-system and the real system.

§ **10·3.** For the G-system we have restricted the validity of (G·2), (G·3), and (G·4) within the specified ranges, and, as a result, the horizontality of the isotherm has been proved [§ 8·22] for (0<) $v^* < v < v_{cond}$ instead of for $0 < v < v_{cond}$ [note that if v_0, u_1, and u_0 are all zero, as is the case with the (0)-system, then we have $v^* = 0$]. Thus it is possible for the G-system to have a non-horizontal part of the isotherm for sufficiently small v [which corresponds to the range of pure liquid phase] owing to the invalidity of (G·2) and/or (G·3) and/or (G·4) for such values of v. For the G-system, however, the detailed behaviours of the cluster integrals for small volumes have not been described, so that the thermodynamical properties of the G-system for sufficiently small v have not been discussed. In another paper, we shall have an opportunity to enter into a detailed discussion of this problem.

§ **10·4.** At the end of the present paper, we must add that the theory given in § A5 of II (concerning the expression of the "huge" cluster integral in terms of the irreducible cluster integrals β_k) and its rigorous mathematical proof (whose description we have omitted in II) can also be generalized by the same arguments as those in the present paper.

Some applications of the present theory to quantum mechanical many-body systems, such as liquid helium, will be given in another place.

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Appendix A

Non-rigorous treatment

§ A1. Introduction

In this appendix we shall give a non-rigorous method of treatment which leads to the same results as those obtained by the rigorous method in the text.

In this appendix, just as in Appendix A of II,[2] we first classify the sizes of the clusters (i.e. the numbers l of molecules composing the clusters) into three kinds,† *viz.*, "large" (i.e. of the same order of magnitude as N[≡the total number of molecules in the system], where N is assumed to be very large; $N\sim10^{23}$ say), "small" (i.e. of the same order of magnitude as 1), and "intermediate" (i.e. neither "large" nor "small"), and then treat the clusters by different suitable methods according as their sizes are "large" or not.

As mentioned in Appendix A (§ A7) of II, the above classification is rather ambiguous from the pure mathematical viewpoint; that is, it is not based on any definite border-lines such as the function $l^*(N)$ in the text. However, this non-rigorous treatment is shown to give the same results as those in the text, hence it may be utilized (especially in applications to physical problems) as a convenient method which is rather simple (though not rigorous) in arguments and is rigorous in results.

We shall denote "small" l, "large" l, and l "of intermediate magnitude", respectively, by $l\sim1$, $l\sim N$, and $1\ll l\ll N$. The classification is also applied to any other quantity q; $q\sim1$, $q\sim N$, and $1\ll q\ll N$. [Note that, for example, by $l\ll N$ we shall mean l such that $l\sim1$ or $1\ll l\ll N$.]

Because of the relation

$$\sum_{l=1}^{N} lm_l=N \tag{1}$$

(where m_l is the number of clusters of size l), we have:

$$m_l\sim1 \text{ (if not zero) for } l\sim N, \tag{2}$$

† The definitions of the words "large" and "small" in this appendix are different from those in the text [cf. § 3·2]; "large" in this appendix may be said to be equivalent to "huge" in the text, though the former is only roughly defined.

and

$$m_l \gg 1 \text{ or } m_l \sim 1 \text{ (if not zero) for } l \ll N. \tag{3}$$

Hereafter we shall write L for "large" l, and simply write l for other l.

As stated in Appendix A of II, the *Stirling approximation* ($\ln m_l! = m_l \ln m_l - m_l$) and the *continuum approximation* (in which one assumes m_l to be continuous and performs the differentiation with respect to m_l) are considered to be valid* for $m_l \gg 1$ and invalid for $m_l \sim 1$. Hence, because of (2), to a set of "large" clusters these approximations cannot be applied but a new method of treatment should be applied, while, because of (3), to a set of clusters which are not "large", these approximations may be applicable, with the exception of the case: $m_l \sim 1$ for $l \ll N$, which case, however, is negligible since $lm_l \ll Lm_L$.

We may regard this applicability of the approximations as a standard of distinction between $l(\ll N)$ and $L(\sim N)$. This standard is adopted in Appendix A of II, and is expressed rigorously by the conditions 1°, 4°, and 6° in Lemma 3 of the text of the present paper (that is, the conditions 1°, 2°, and 3° in II). In the present paper, however, the volume dependence of the cluster integrals is taken into account so that one more standard of distinction between l and L is required; that is, we must consider a distinction such that $b_l(l \ll N)$ are almost volume-independent and b_L are volume-dependent in some specified manner. This standard is expressed rigorously by the conditions 1°, 2°, 3°, 4°, and 5° in Lemma 3 together with the assumptions (G·1) and (G·3). Consequently, in the problem of the present paper we should adopt an appropriate standard consistent with the above two standards. Such a standard is expressed rigorously by the conditions 1°–6° in Lemma 4. [See also §3·4 of the text.]

The treatment in this appendix starts from a rather ambiguous distinction between l and L, so that the problems as to the delicate difference and mathematical consistency between the above-mentioned standards are not so carefully discussed as in the text, and these standards are simply assumed to give distinctions which are roughly the same. Thus the assumptions (G·1), (G·2), (G·3), and (G·4) may be replaced by the following less rigorous expressions, respectively:

(g·1) If $l \ll N$, $b_l(V)$ may be replaced by $b_l^{(0)}$.

(g·2) $b_L(V) > 0$ (if $V/L > v_0$), $b_l(V) > 0$ (for $l \ll N$).**

(g·3) $\ln b_L(V) \fallingdotseq L \ln b_0^{\dagger}$ (if $V/L > u_1 [\geqq u_0 \geqq v_0]$).

(g·4) $\begin{cases} \ln b_L(V) \lesssim L \ln b_0^{\dagger} & \text{(if } V/L > u_0 [\geqq v_0]) \\ \ln b_l(V) \lesssim l \ln b_0^{\dagger} & \text{(if } 1 \ll l \ll N). \end{cases}$

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* A set of very large integers m's may be regarded as almost continuous, for there is a difference Δm between integers such that $\Delta m \ll m$, and thus one can take a differential $dm = \Delta m$. This is not the case with small integers. [See p. 680 of II].

** By (g·1) and (g·2) we have $b_l^{(0)} > 0$.

Here the equality symbol $X \eqsim Y$, defined in II [see p. 682 of II], means that X and Y are both "large" and the difference between X and Y is $\ll N$; and the inequality symbol $X \lesssim Y$, which is newly introduced in the preesnt paper, means that Y is "large" and X does not exceed Y or exceeds Y at most by an amount which is $\ll N$; and the inequality symbol $A \lesssim B$ means that A does not exceed B or exceeds B at most by an amount which is small compared with B; (thus $\leqq$ corresponds to the special case of $\lesssim$, in which B is "large".)

The assumptions (G·5) and (G·6) will be adopted in the unaltered forms, though the latter will not be explicitly used in this appendix.

(G·5) The series $\sum_{l=1}^{\infty} l b_l^{(0)} (b_0^{\dagger})^{-l}$ is convergent and is $< u_0^{-1}$.

(G·6) $\lim_{l \to \infty} (b_l^{(0)})^{1/l} = b_0^{(0)}$, $0 < b_0^{(0)} < +\infty$.

§ A2. Method of treatment*

Let N_1 be the total number of those molecules which compose "small" clusters and clusters "of intermediate size", and let N_2 be the total number of those molecules which compose "large" clusters. Then we have

$$\sum_{l=1}^{l \ll N} l m_l = N_1, \quad \sum_{L \sim N}^{L \simeq N} L m_L = N_2, \quad N_1 + N_2 = N. \tag{1}$$

Here we may note that N_2 is "large" or else zero, while N_1 may have any kind of magnitude.

Consequently we may write 1·2(8) [with $V = Nv$] of the text as

$$\begin{aligned} \Omega_N &= \sum_{m_l, m_L} \Big(\sum_{l=1}^{l \ll N} l m_l + \sum_{L \sim N}^{L \simeq N} L m_L = N \Big) \prod_{l=1}^{l \ll N} \frac{\{Nv b_l(Nv)\}^{m_l}}{m_l!} \prod_{L \sim N}^{L \simeq N} \frac{\{Nv b_L(Nv)\}^{m_L}}{m_L!} \\ &= \sum_{N_2, m_l} \Big(\sum_{l=1}^{l \ll N} l m_l + N_2 = N \Big) \prod_{l=1}^{l \ll N} \frac{\{Nv b_l(Nv)\}^{m_l}}{m_l!} \sum_{m_L} \Big(\sum_{L \sim N}^{L \simeq N} L m_L = N_2 \Big) \prod_{L \sim N}^{L \simeq N} \frac{\{Nv b_L(Nv)\}^{m_L}}{m_L!}. \end{aligned} \tag{2}$$

Now, if $Nv/N_2 > u_1$ (whence $Nv/L > u_1$ since $L \leqq N_2$), then by (g·3) the logarithm of any term in $\sum_{m_L}$ in (2) becomes

$$\ln \prod_{L \sim N}^{L \simeq N} \frac{\{Nv b_L(Nv)\}^{m_L}}{m_L!} \eqsim \sum_{L \sim N}^{L \simeq N} L m_L \ln b_0^{\dagger} = N_2 \ln b_0^{\dagger}, \tag{3}$$

and in particular the logarithm of the term for which $m_{N_2} = 1$ and $m_L = 0$ $(L \neq N_2)$ becomes

$$\ln \{Nv b_{N_2}(Nv)\} \eqsim \ln b_{N_2}(Nv) \eqsim N_2 \ln b_0^{\dagger}. \tag{4}$$

In deriving (3) and (4), we have used the fact that $m_L \sim 1$, $\sum_{L \sim N}^{L \simeq N} 1 \sim 1$, $\ln(Nv) \ll N$.

---

* The number of a formula in this section is the same as that of the corresponding formula in § A3 of II,[2] with the only exceptions of (5) and (6).

If $v>u_0$ (whence $Nv/L>u_0$ since $L\leqq N$), then by (g·4) we have for any term in $\sum_{m_L}$ for any $N_2(\leqq N)$

$$\ln \prod_{L\sim N}^{L=N} \frac{\{Nvb_L(Nv)\}^{m_L}}{m_L!} \lesssim \sum_{L\sim N}^{L=N} Lm_L \ln b_0^{\dagger} = N_2 \ln b_0^{\dagger}, \tag{5}$$

$$\ln\{Nvb_{N_2}(Nv)\} \lesssim N_2 \ln b_0^{\dagger}. \tag{6}$$

Then we have by (3) and (4)

$$\ln \sum_{m_L} \Big(\sum_{L\sim N}^{L=N} Lm_L = N_2\Big) \prod_{L\sim N}^{L=N} \frac{\{Nvb_L(Nv)\}^{m_L}}{m_L!} \eqq \ln\{Nvb_{N_2}(Nv)\}$$

$$\eqq \ln b_{N_2}(Nv) \eqq N_2 \ln b_0^{\dagger}, \tag{7}*$$

provided that $Nv/N_2>u_1$; and we have by (5) and (6)

$$\ln \sum_{m_L} \Big(\sum_{L\sim N}^{L=N} Lm_L = N_2\Big) \prod_{L\sim N}^{L=N} \frac{\{Nvb_L(Nv)\}^{m_L}}{m_L!} \lesssim N_2 \ln b_0^{\dagger}, \tag{7a}$$

provided that $v>u_0$. For, the logarithm of the total number of terms in $\sum_{m_L}$ is $\ll N$.

Now, for a set of clusters which are not "large", we have

$$\ln \prod_{l=1}^{l\leqq N} \frac{\{Nvb_l(Nv)\}^{m_l}}{m_l!} \eqq \sum_{l=1}^{l\leqq N} m_l\{\ln(Nvb_l^{(0)}) - \ln m_l + 1\}. \tag{8}$$

Here we have used (g·1) and applied the Stirling approximation to m_l (but not to $m_L[\sim 1]$).

We now write (2) in the following form:

$$\Omega_N = \sum_{N_2, m_l} \Big(\sum_{l=1}^{l\leqq N} lm_l + N_2 = N\Big) T(N_2, m_l). \tag{9}$$

Then we have by (7) and (8)

$$\ln T(N_2, m_l) \eqq \sum_{l=1}^{l\leqq N} m_l\{\ln(Nvb_l^{(0)}) - \ln m_l + 1\} + \ln b_{N_2}(Nv)$$

$$\eqq \ln T'(N_2, m_l), \tag{10}$$

provided that $Nv/N_2>u_1$; and we have by (7a) and (8)

$$\ln T(N_2, m_l) \lesssim \ln T'(N_2, m_l), \tag{10a}$$

provided that $v>u_0$. Here we have defined:

$$\ln T'(N_2, m_l) \equiv \sum_{l=1}^{l\leqq N} m_l\{\ln(Nvb_l^{(0)}) - \ln m_l + 1\} + N_2 \ln b_0^{\dagger}. \tag{10'}$$

---

* The range $(Nv>N_2u_1)$ of validity of (7) can, in fact, be extended to $Nv>N_2u_0$, because even in this extended range the logarithms of the largest terms in $\sum_{m_L}$ in (7) are $\eqq N_2 \ln b_0^{\dagger}$. In this way the horizontal part of the isotherm will be extended. The same is true of the rigorous arguments in the text. About this point we shall give detailed arguments in another paper.

Since the logarithm of the total number of terms in $\sum_{N_2, m_l}$ in (9) is $\ll N$, we have by (g·2) that if $v > v_0$,

$$\ln \Omega_N \fallingdotseq \mathrm{Max} \ln T(N_2, m_l). \tag{11}$$

We now calculate the maximum of $\ln T'(N_2, m_l)$ [(10′)] under the condition:

$$\sum_{l=1}^{l \ll N} l m_l + N_2 = N \tag{12}$$

and $m_l \geqq 0$ and $N_2 \geqq 0$. Using the undetermined multiplier $\ln z$, we obtain*

$$z = (b_0^{\dagger})^{-1} \quad (\equiv z_{\mathrm{cond}}, \text{ say}) \tag{13}$$

from $(\partial/\partial N_2)\{\ln T'(N_2, m_l) + (\sum_{l=1}^{l \ll N} l m_l + N_2) \ln z\} = 0$, and obtain

$$m_l = \bar{m}_l = N v b_l^{(0)} z^l = N v b_l^{(0)} z_{\mathrm{cond}}^l \quad (l = 1, 2, 3, \cdots, \ll N) \tag{14}**$$

from $(\partial/\partial m_l)\{\text{〃}\} = 0$ $(l = 1, 2, 3, \cdots, \ll N)$. Thus the values of N_1 and N_2 for the maximum are

$$\bar{N}_1 = \sum_{l=1}^{l \ll N} l \bar{m}_l = N \sum_{l=1}^{l \ll N} l v b_l^{(0)} z_{\mathrm{cond}}^l \tag{15}$$

and

$$\bar{N}_2 = N - N \sum_{l=1}^{l \ll N} l v b_l^{(0)} z_{\mathrm{cond}}^l, \tag{16}$$

respectively.

Then in order that the solution just now obtained may have a meaning, it is necessary that $\sum_{l=1}^{l \ll N} l v b_l^{(0)} z_{\mathrm{cond}}^l$ should be definite and finite and $\sum_{l=1}^{l \ll N} l v b_l^{(0)} z_{\mathrm{cond}}^l \leqq 1$ (since N_2 should not be negative). In order that this may be so, firstly, the series $\sum_{l=1}^{\infty} l b_l^{(0)} z_{\mathrm{cond}}^l$ should be convergent (since some of the l's "of intermediate magnitude" are very large compared with 1), unless v is infinitesimal. This convergence is ensured by (G·5). In this case we have $\sum_{l=1}^{l \ll N} l b_l^{(0)} z_{\mathrm{cond}}^l = \sum_{l=1}^{l \ll N} l b_l^{(0)} z_{\mathrm{cond}}^l$ (with a negligible error), and $\sum_{l=1}^{\infty} b_l^{(0)} z_{\mathrm{cond}}^l$ is convergent and equal to $\sum_{l=1}^{l \ll N} b_l^{(0)} z_{\mathrm{cond}}^l$ (with a negligible error). Secondly, the following should hold:

$$1/v \geqq \sum_{l=1}^{\infty} l b_l^{(0)} z_{\mathrm{cond}}^l \quad (\equiv 1/v_{\mathrm{cond}}, \text{ say}). \tag{17}$$

Now suppose that $N v / \bar{N}_2 > u_1$ and $v > u_0$ and (17) hold simultaneously; that is,

---

* The application of the continuum approximation to N_2 and m_l is valid since N_2 is "large" and $m_l \gg 1$.

** From this it is seen that m_l is "large" for "small" l.

$$v_{\text{cond}} \geqq v > \max\{u_0,\ (u_1^{-1}+v_{\text{cond}}^{-1})^{-1}\}\ (\equiv v^*,\ \text{say}).^{\ddagger} \tag{17a}$$

Then by (10) we have $\ln T(\bar{N}_2, \bar{m}_l) \rightleftharpoons \ln T'(\bar{N}_2, \bar{m}_l)$; hence by (10a) we have $\ln T(\bar{N}_2, \bar{m}_l) \gtrsim \ln T(N_2, m_l)$ for any N_2 and $\{m_l\}$ [since $\ln T'(\bar{N}_2, \bar{m}_l) = \text{Max}\ln T'(N_2, m_l) \geqq \ln T'(N_2, m_l)$]. Thus

$$\text{Max}\ln T(N_2, m_l) \rightleftharpoons \ln T(\bar{N}_2, \bar{m}_l) \rightleftharpoons \ln T'(\bar{N}_2, \bar{m}_l). \tag{18a}$$

Hence by (11), (10), and (14) we obtain, under the condition (17a),

$$\ln \Omega_N \rightleftharpoons N\sum_{l=1}^{l\ll N} v b_l^{(0)} z_{\text{cond}}^l - \bar{N}_1 \ln z_{\text{cond}} + \ln b_{\bar{N}_2}(Nv) \tag{18}$$

with (15) and (16). Here we may (with a negligible error) replace the upper limit $l \ll N$ of the sum by $l=\infty$, since some of the l's "of intermediate magnitude" are very large compared with 1. Thus (18) becomes, with the use of $z_{\text{cond}} \equiv (b_0^{\dagger})^{-1}$ and (4),

$$\ln \Omega_N \rightleftharpoons N\sum_{l=1}^{\infty} v b_l^{(0)} z_{\text{cond}}^l - N\ln z_{\text{cond}}. \tag{19}$$

Hence the pressure p of the system is given by

$$p/kT = (\partial/\partial V)\ln \Omega_N = \sum_{l=1}^{\infty} b_l^{(0)} z_{\text{cond}}^l\ (\equiv p_{\text{cond}}/kT,\ \text{say}), \tag{20}$$

which is independent of v. That is, the isothermal p-v curve is horizontal in this range $(v^* < v \leqq v_{\text{cond}})$. The Gibbs free energy G is given by

$$G = A + pV \rightleftharpoons NkT\ln(\lambda^3 z_{\text{cond}}). \tag{21}$$

Hence z_{cond} is the activity of the system, which is independent of v. This corresponds to the condensation range.

On the other hand, if v is so given that

$$1/v < \sum_{l=1}^{\infty} l b_l^{(0)} z_{\text{cond}}^l\ (\equiv 1/v_{\text{cond}}), \tag{22}^{\ddagger\ddagger}$$

then we have $\bar{N}_2 < 0$ in (16), and so we should have $N_2 = 0$ for the largest value of $\ln T'(N_2, m_l)$ that has a meaning. In this case we have

$$\ln \Omega_N \rightleftharpoons \text{Max}\ln T(0, m_l) \rightleftharpoons \text{Max}\ln T'(0, m_l)\ [\equiv \ln T'(0, \bar{m}_l),\ \text{say}] \tag{23}$$

with the condition

$$\sum_{l=1}^{l\ll N} l m_l = N; \tag{24}$$

for, by (8), (10), and (10a), we have, for any $N_2 (0 \leqq N_2 \leqq N)$ and $\{m_l\}$,

$$\ln T(0, \bar{m}_l) \rightleftharpoons \ln T'(0, \bar{m}_l) \geqq \ln T'(N_2, m_l) \gtrsim \ln T(N_2, m_l).$$

Using the undetermined multiplier $\ln z$, we obtain[‡‡‡] from $(\partial/\partial m_l)\{\ln T'(0, m_l)$

‡ By (G·5) we have: $v_{\text{cond}} > v^*$.

‡‡ Note that $v_{\text{cond}} > v^* \geqq u_0$.

‡‡‡ The continuum approximation is applied to $m_l (\gg 1)$.

$$+\Big(\sum_{l=1}^{l\leqslant N} lm_l\Big)\ln z\Big\}=0$$

$$m_l=\bar{m}_l=Nvb_l^{(0)}z^l \quad (l=1, 2, 3, \cdots, \leqslant N), \tag{25}$$

where z is determined by

$$N\sum_{l=1}^{l\leqslant N} lvb_l^{(0)}z^l=N. \tag{26}$$

Hence, unless $v\to 0$, we have

$$[1/v=]\sum_{l=1}^{l\leqslant N} lb_l^{(0)}z^l=\sum_{l=1}^{\infty} lb_l^{(0)}z^l=\text{convergent} \tag{26'}$$

and

$$\sum_{l=1}^{l\leqslant N} b_l^{(0)}z^l=\sum_{l=1}^{\infty} b_l^{(0)}z^l=\text{convergent.}$$

Thus from (23) and (25) we obtain

$$\ln \Omega_N \fallingdotseq N\sum_{l=1}^{\infty} vb_l^{(0)}z^l-N\ln z \tag{27}$$

and

$$p/kT=\sum_{l=1}^{\infty} b_l^{(0)}z^l \tag{28}$$

and

$$G\fallingdotseq NkT\ln(\lambda^3 z), \tag{29}$$

which shows that z is the activity of the system. This corresponds to the gaseous state.

The statements, printed in small type on pp. 686–687 of II,[2] concerning a more intelligible method for calculating the maximum term, hold also in the present case if we make the following replacements:

First, we replace "Now we have, by (A3·9) and (A3·10), ……, subject to the condition $\sum_l lm_l+N_2=N$." [p. 686, lines 10 ff.] by the following:

Now we have, by (9) and (11),

$$\Omega_N=\sum_{N_2, m_l}\Big(\sum_l lm_l+N_2=N\Big)T(N_2, m_l) \tag{30}$$

and (if $v>v_0$)

$$\ln \Omega_N=\text{Max}\ln T(N_2, m_l). \tag{30a}$$

Defining

$$\ln T'(N_2, m_l)\equiv\sum_l m_l\{\ln(Nvb_l^{(0)})-\ln m_l+1\}+N_2\ln b_0^{\dagger}, \tag{31}$$

we shall now calculate

$$\text{Max}\ln T'(N_2, m_l)\ [\equiv\ln T'(\bar{N}_2, \bar{m}_l), \text{ say}] \tag{32}$$

subject to the condition

$$\sum_l lm_l+N_2=N. \tag{33}$$

Next, in all the formulas (A3·34)–(A3·52), in all sentences, and in Figs. 1, 2, and 3, we replace T, b_l, b_0, z_s, and v_s by T', $b_l^{(0)}$, $b_0^{\dagger}$, z_{cond}, and v_{cond}, respectively.

Next, we replace "thus we have" before Eq. (A3·49) by the following:

thus, if $v_{\text{cond}} > v > \max\{u_0, (u_1^{-1}+v_{\text{cond}}^{-1})^{-1}\}$ ($\equiv v^*$, say), then we have by (18a) and (30a)

Finally, after Eq. (A3·52), we insert the following:

For, by (8), (10), and (10a), we have

$$\ln T(0, m_l^{\dagger}(0)) \fallingdotseq \ln T'(0, m_l^{\dagger}(0)) \geqq \ln T'(N_2, m_l) \fallingdotseq \ln T(N_2, m_l).$$

§ A3. Remarks and conclusions

By (g·1) and the second part of (g·4), we have:

$$\ln b_l^{(0)} \lesssim l \ln b_0^{\dagger} \quad (\text{for } 1 \ll l \ll N). \tag{1}$$

Hence, using (G·6), we have:

$$\lim_{l\to\infty} (b_l^{(0)})^{1/l} \leqq b_0^{\dagger}, \tag{2}$$

which is nothing but Lemma 1c in the text. Thus the statements in § 10·2 of the text, concerning the analytical properties of the singularity representing the condensation, are also true of the treatment in this appendix; here it should be noted that the power series 9·23(1) and 9·23(2) in the text correspond to A2(26′) and A2(28) in this appendix.

The physical conclusions deduced from § A2 are the following:

If $v > v_{\text{cond}}$, we obtain the activity z from A2(26), and $\ln \Omega_N$ from A2(27); and the system in thermodynamical equilibrium consists of a "large" number of "small" and "medium"‡ clusters, whose size distribution is given by A2(25); and, in determining the pressure, each cluster plays the role played by one molecule of a perfect gas;—(the gaseous state). If $v^* < v < v_{\text{cond}}$, the activity ($z_{\text{cond}}$) is independent of v, and $\ln \Omega_N$ is obtained from A2(18) or A2(19); and in thermodynamical equilibrium the $\bar{N}_1$ molecules [A2(15)] compose a "large" number of "small" and "medium" clusters, whose size distribution is given by A2(14), and the remaining $\bar{N}_2$ molecules [A2(16)] compose one "large" cluster ($\ln b_{\bar{N}_2}$). When v decreases, $\bar{N}_1/V$ is constant and equal to $1/v_{\text{cond}}$ and $\bar{N}_2$ (also $\bar{N}_2/V$) increases; ($\bar{N}_2=0$ when $v=v_{\text{cond}}$),

$$N/V=\bar{N}_1/V+\bar{N}_2/V \text{ or } 1/v=1/v_1+1/v_2 \quad (v_1\equiv v_{\text{cond}}). \tag{3}$$

The former part is considered as the saturated vapour. The pressure of the system takes the constant value p_{cond}[A2(20)] (the saturated vapour pressure)

‡ The adjective "medium" is an abbreviated expression for "of intermediate size (magnitude)".

which arises from the saturated set of " small " and " medium " clusters [A2(14), A2(15)], each cluster playing the role played by one molecule of a perfect gas; and the " large " cluster $\ln b_{\overline{N}2}$ makes no contribution to the pressure. This " large " cluster is considered to be the condensed phase which has appeared by two-phase separation;—(the condensation range).

Appendix B

Comparison of notations

paper I		paper II	this paper	
real system	(0)-system	(0)-system	(0)-system	G-system
$b_l^{\dagger}(V, T)$	$b_l^{(0)}(T)$	b_l	$b_l^{(0)}$	$b_l(V)$
—	$b_0^{(0)}(T)$	b_0	$b_0^{(0)}$	$b_0^{\dagger}$
$\Omega_N^{\dagger}(Nv, T)$	$\Omega_N^{(0)}(Nv, T)$	Ω_N	$\Omega_N^{(0)}$	Ω_N
$\tilde{T}_N^{\dagger}(i\|Nv, T)$	$\tilde{T}_N^{(0)}(i\|Nv, T)$	$\varphi_N(\boldsymbol{i}[N])$	$\varphi_N^{(0)}(\boldsymbol{i}[N])$	$\varphi_N(\boldsymbol{i}[N])$
$\psi_N^{\dagger}(Nv, T)$	$\psi_N^{(0)}(Nv, T)$	$\sum_{i[N]}\varphi_N(\boldsymbol{i}[N])$	$\sum_{i[N]}\varphi_N^{(0)}(\boldsymbol{i}[N])$	$\sum_{i[N]}\varphi_N(\boldsymbol{i}[N])$
$\tilde{\tilde{T}}_N^{\dagger}(j\|Nv, T)$	$\tilde{\tilde{T}}_N^{(0)}(j\|Nv, T)$	$\chi_N(\boldsymbol{j}[N])$	$\chi_N^{(0)}(\boldsymbol{j}[N])$	$\chi_N(\boldsymbol{j}[N])$
$\Psi_N^{\dagger}(Nv, T)$	$\Psi_N^{(0)}(Nv, T)$	$\sum_{j[N]}\chi_N(\boldsymbol{j}[N])$	$\sum_{j[N]}\chi_N^{(0)}(\boldsymbol{j}[N])$	$\sum_{j[N]}\chi_N(\boldsymbol{j}[N])$
—	—	$\Phi_N(\boldsymbol{k}[N])$	$\Phi_N^{(0)}(\boldsymbol{k}[N])$	$\Phi_N(\boldsymbol{k}[N])$

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K. Ikeda, Prog. Theor. Phys. **16** (1956), 341 (containing the complete proofs of the lemmas and theorems given in Ia); to be referred to as I.
2) K. Ikeda, Prog. Theor. Phys. **11** (1954), 336.
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4) J. E. Mayer, J. Chem. Phys. **5** (1937), 67.
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M. Born and K. Fuchs, Proc. Roy. Soc. **A 166** (1938), 391.
B. Kahn and G. E. Uhlenbeck, Physica **5** (1938), 399.
5) K. Ikeda and T. Nakazawa, a lecture at the meeting of the Physical Society of Japan held at Nagoya University on October 18, 1960 [cf. the abstract book of the meeting, **5**, p. 1].

Errata

Cluster Theory of Condensing Systems

Kazuyosi IKEDA

Prog. Theor. Phys. **19** (1958), 653–697

p. 653, abstract (line 10);	"larger"	should read	"large"
p. 655, line 16;	$v<v_s$	〃	$v>v_s$
p. 657, line 5;	e. i.	〃	i. e.
p. 666, (3·19);	$(1/N)\ln\Sigma$	〃	$|(1/N)\ln\Sigma$
p. 671, line 16;	$\{N_2, \hat{m}_1, \hat{m}_2, \cdots \hat{m}_{l^*(N)}\}$	〃	$\{\hat{N}_2, \hat{m}_1, \hat{m}_2, \cdots \hat{m}_{l^*(N)}\}$
p. 671, (5·15b);	N_2	〃	$\hat{N}_2$
p. 671, (5·16);	N_2	〃	$\hat{N}_2$
p. 671, line 3 from bottom;	$\{N_2, \hat{m}_l\}$	〃	$\{\hat{N}_2, \hat{m}_l\}$
p. 672, (5·20);	$\{(N-N_2)/N\}$	〃	$\{(N-\hat{N}_2)/N\}$
p. 674, line 12;	*If* $v\leqq v_s$	〃	*If* $v<v_s$
p. 693, line 4;	$N_2=0$	〃	$\bar{N}_2=0$

On the Theory of Condensation

Kazuyosi IKEDA

Prog. Theor. Phys. **16** (1956), 341–388

Addition to the errata given at page 698 of Prog. Theor. Phys. **19** (1958).

p. 348, line 10;	twenty-six	should read	eight
p. 348, line 14;	the last one vanishes	〃	Δ_0, Δ_1, Δ_2 vanish
p. 363, footnote (line 1);	$\ln(N+2)$	〃	$\ln(N+3)$
p. 367, line 22;	$\omega^\dagger(Nv, T)$	〃	$\omega^\dagger(v, T)$
p. 387, line 4 from bottom;	$\mu>0$	〃	$\mu\geqq 1$

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Nuclear Finite Size Effects on the Internal Conversion in the μ-Mesonic He^3 Atom

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The internal conversion processes in the μ-mesonic He^3 atom following nuclear fusion reactions by μ-mesons at low temperature are analyzed, where the finite size effects of the He^3 nucleus play an important role, so that the μ-mesonic irregular solutions outside the nucleus are considered. The reaction rates calculated by us can give reasonable agreements with experimental results by Ashmore et al. and Alvarez et al.

§ 1. Introduction

From the practical point of view, the *dead-end* problem is an important process in nuclear catalytic fusion reactions by μ-mesons at low temperature. In other words, concerning the μ-meson moving around the nucleus which has finished nuclear fusion reactions, it is an essential problem to compare the following two probabilities:

a) The probability that a μ-meson completes its life by natural decay.

b) The probability that a μ-meson is ejected by an internal conversion in the μ-mesonic He^3 atom and is used again as a catalyzer.

It is the purpose of our investigation to find the reality of the internal conversion process of the μ-mesonic He^3 atom exactly and to compare with the experimental results by Alvarez et al.[1] and Ashmore et al.,[2] though certain semi-qualitative estimations have already been given.[3)–6)]

In our theoretical standpoint the following assumptions are proposed:

i) The μ-meson is considered as the Dirac particle inside and outside the nucleus.

ii) Even for a light nucleus, such as He^3, nuclear finite size effects should be considered as far as the μ-meson is concerned. Therefore the irregular as well as the regular solutions in the Coulomb field are introduced outside the nucleus.

iii) If the internal conversion takes place as the electric monopole ($E0$) transition, the nuclear wave functions as well as the μ-mesonic wave functions inside the nucleus play an important role.

The γ-rays emitted in the p-d fusion reactions may be regarded as the magnetic dipole ($M1$) radiation based upon the analogy of the mirror-nuclei reaction:[7] $n+d\rightarrow H^3$. The $M1$ internal conversion coefficient, as mentioned in the next section, induces no interesting conclusion.

On the other hand, we give the general treatment in § 3 for the probability that the μ-meson is ejected by the $E0$ transition in the μ-mesonic He^3 atom, and we should emphasize the importance of the nuclear structure effects. Further, the μ-mesonic wave functions inside the nucleus are given in § 4, and then these functions should be connected with the extra-nuclear solutions composed of the irregular as well as the regular solutions.

In the last section the ejection probability of the μ-meson is calculated, and is compared with the experimental results, where our theoretical calculations prove to be justified.

§ 2. The magnetic dipole conversion

In the first place it will be shown that the radiative capture process

$$p+d\rightarrow He^3+\gamma$$

can be treated in the same way as the thermal neutron capture process

$$n+d\rightarrow H^3+\gamma.$$

The wave function of the ground state of the μ-mesonic p-d molecular ion may be approximated by[5]

$$\sigma_{vib}\sim\sqrt{\frac{\eta}{2\pi}}\,x\exp\left[-\frac{1}{2}\lambda(x)\right] \qquad (2\cdot1)$$

as the distance between the proton and the deuteron approaches to zero, where $\eta=M\hbar\omega/m_\mu$ (M being the reduced mass of the p-d system, $\hbar$ the Planck constant divided by 2π, ω the proper vibrational frequency, and m_μ the rest mass of the μ-meson), and x represents the inter-nuclear distance, finally $\lambda(x)$ means the barrier penetration factor determined by the W.K.B. method.

Then the relative velocity v between the proton and the deuteron for $x\sim0$ becomes

$$v\sim\frac{1}{M}\sqrt{\frac{\eta}{2\pi}}\exp\left[-\frac{1}{2}\lambda(x)\right]=6.2\times10^4\ \mathrm{cm/sec}, \qquad (2\cdot2)$$

since $\lambda(x)\sim6.0$. Therefore this process turns out to be treated based upon the analogy of the thermal neutron capture. Thus the initial state of the nuclear system may be considered as $S^+_{3/2}$ or $S^+_{1/2}$. In the former case the $M1$

and $E2$ transitions, and in the latter case the $M1$ and $E0$ transitions are possible. Since it is well known that the $E2$ transition probability is negligible in comparison with the $M1$ transition probability in the n-d reaction,[6] we first attend to the $M1$ internal conversion based upon nuclear structure effect.

Now we point out the following facts:

i) The energy of the emitted γ ray is about 5.5 Mev, so its wave number k is 0.047 in μ-mesonic relativistic units.*

ii) The velocity of the conversion μ-meson is about one third of the light velocity.

iii) The γ-ray energy 5.5 Mev is much larger than the binding energy ($\sim$10 kev) of the μ-mesonic atom.

These facts indicate that Drell's formula[8] can almost hold. In consequence the $M1$ conversion coefficient becomes 2.8×10^{-4}. The theory of nuclear structure effect by Weneser-Church[9] can extended to the μ-mesonic case ($S_{1/2}$ state), and the $M1$ conversion coefficient is given by

$$\beta_1^K=\beta_1^K(S_{1/2})\left\{1+\lambda\frac{3}{10\pi}\frac{R}{k}(f_i^0g_f^0+f_f^0g_i^0)\sqrt{\frac{2\pi^3e^2}{k}}\sqrt{\beta_1^K(S_{1/2})}\right\}^2, \qquad (2\cdot3)$$

where $\beta_1^K(S_{1/2})$ is the $M1$ conversion coefficient without nuclear structure effect, R the nuclear radius, λ the ratio of nuclear matrix elements, and f_i^0, f_f^0, g_i^0 and g_f^0 the first expansion coefficients of the radial parts of the Dirac wave functions, where suffices i and f mean initial and final states, respectively. The ratio λ of nuclear matrix elements in the μ-mesonic case is of the same order of magnitude as in the electronic case ($\sim$10). Thus the order of magnitude of the $M1$ conversion coefficient can be little altered by the nuclear structure effect, so we can give no interesting conclusion of the $M1$ conversion process. Therefore the $E0$ conversion will be considered in the following section.

§ 3. General formula for the electric monopole transition

We shall give the general formula for the ejection probability of the μ-meson by the $E0$ transition. In this case the perturbing Hamiltonian in μ-mesonic relativistic units is given by

$$H'=-\sum_p\frac{\alpha}{|\boldsymbol{r}_p-\boldsymbol{r}_\mu|}, \qquad (3\cdot1)$$

where α is the Sommerfeld fine structure constant, and $\boldsymbol{r}_p$ and $\boldsymbol{r}_\mu$ represent the position vectors of a proton and a μ-meson, respectively. Then the matrix element of the $E0$ transition is expressed by the following formula:

---

* In the μ-mesonic relativistic units (μ.r.u.) we take $m_\mu=c=\hbar=1$.

$$\langle i|H'(L=0)|f\rangle \equiv \langle H'\rangle$$

$$= -\alpha\left\{\int d\tau_n \int_0^{r_p} d\tau_\mu \tilde{\phi}_f \tilde{\psi}_f \frac{1}{r_p} \phi_i \psi_i + \int d\tau_n \int_{r_p}^{\infty} d\tau_\mu \tilde{\phi}_f \tilde{\psi}_f \frac{1}{r_\mu} \phi_i \psi_i\right\}$$

$$= -\alpha\left\{\int d\tau_n \int_0^{r_p} d\tau_\mu \tilde{\phi}_f \tilde{\psi}_f \left(\frac{1}{r_p} - \frac{1}{r_\mu}\right) \phi_i \psi_i + \int d\tau_n \int_0^{\infty} d\tau_\mu \tilde{\phi}_f \tilde{\psi}_f \frac{1}{r_\mu} \phi_i \psi_i\right\} \tag{3·2}$$

where i and f represent all quantum numbers of the initial and final states, respectively, and ϕ means the nuclear wave function, and ψ the μ-mesonic wave function.

The second term of the right-hand side in (3·2) can be separated into the product of two parts concerning the μ-mesonic and nuclear coordinates, and it vanishes due to the orthogonality of the initial and final states. Then (3·2) can be written as

$$\langle H'\rangle = -\alpha \int d\tau_n \int_0^{r_p} d\tau_\mu \tilde{\phi}_f \tilde{\psi}_f \left(\frac{1}{r_p} - \frac{1}{r_\mu}\right) \phi_i \psi_i$$

$$= -\alpha \int d\tau_n \tilde{\phi}_f(\boldsymbol{r}_p) \left[\int_0^{r_p} d\tau_\mu \tilde{\psi}_f(\boldsymbol{r}_\mu) \left(\frac{1}{r_p} - \frac{1}{r_\mu}\right) \psi_i(\boldsymbol{r}_\mu)\right] \phi_i(\boldsymbol{r}_p). \tag{3·3}$$

The integral with respect to the μ-mesonic coordinates is obviously limited in the finite nucleus. Therefore we need only the intra-nuclear part of the μ-mesonic wave function. Then we shall find the solutions of the Dirac equation in the uniform charge distribution by the series-expansion method, and determine the normalization constants by the continuity condition of the internal and external solutions. The matrix element can be obtained from (3·3) by making use of these wave functions. The μ-mesonic wave function may be referred to the coordinate system whose origin is the center of gravity of two protons. We need only the part of the nuclear wave function concerning two protons in contrast with the case of interaction with radiation field.

§ 4. Solutions of the Dirac equation inside the nucleus

The potential function for the μ-meson in the uniform charge distribution of the finite nucleus is given by

$$V(r) = -\frac{3}{2}\frac{\alpha Z}{R} + \frac{\alpha Z}{2R}\left(\frac{r}{R}\right)^2, \tag{4·1}$$

where Z is the nuclear charge, and R the nuclear radius. The solutions of the Dirac equation in this potential field can be given as follows:[10]

$$\begin{aligned} F_\kappa^{\,i}(r) &= \frac{\kappa}{|\kappa|} A_\kappa \sum_{n=0}^{\infty} (-1)^n f_n(\kappa) \left(\frac{r}{R}\right)^{1+\bar{l}+2n}, \\ G_\kappa^{\,i}(r) &= A_\kappa \sum_{n=0}^{\infty} (-1)^n g_n(\kappa) \left(\frac{r}{R}\right)^{1+\bar{l}+2n}, \end{aligned} \tag{4·2}$$

where

$$A_\kappa = \begin{cases} F_\kappa(R) / \sum_{n=0}^{\infty} (-1)^n f_n(\kappa) & \text{for } \kappa > 0\ , \\ G_\kappa(R) / \sum_{n=0}^{\infty} (-1)^n g_n(\kappa) & \text{for } \kappa < 0 \end{cases}$$

are the normalization constants, and l, j and κ are conventional notations:

$$l = j + \frac{1}{2}\frac{\kappa}{|\kappa|}, \quad j = |\kappa| - \frac{1}{2}, \quad \bar{l} \equiv l(-\kappa).$$

The coefficients $f_n(\kappa)$ and $g_n(\kappa)$ are all positive, and the following conditions hold:

$$f_0(\kappa) = 1 \quad \text{for } \kappa > 0,$$

$$g_0(\kappa) = 1 \quad \text{for } \kappa < 0.$$

The remaining coefficients are determined by the recurrence formula:

$$\left.\begin{aligned} 2n f_n(\kappa) &= \alpha_1 g_{n-1}(\kappa) + \beta g_{n-2}(\kappa) \\ (2n + 2|\kappa| + 1) g_n(\kappa) &= \alpha_2 f_n(\kappa) + \beta f_{n-1}(\kappa) \end{aligned}\right\} \quad \text{for } \kappa > 0, \tag{4·3a}$$

$$\left.\begin{aligned} 2n g_n(\kappa) &= \alpha_2 f_{n-1}(\kappa) + \beta f_{n-2}(\kappa) \\ (2n + 2|\kappa| + 1) f_n(\kappa) &= \alpha_1 g_n(\kappa) + \beta g_{n-1}(\kappa) \end{aligned}\right\} \quad \text{for } \kappa < 0, \tag{4·3b}$$

where

$$\begin{aligned} \alpha_1 &= \frac{3}{2}\alpha Z + R(W - 1), \\ \alpha_2 &= \frac{3}{2}\alpha Z + R(W + 1), \\ \beta &= -\frac{1}{2}\alpha Z \end{aligned} \tag{4·4}$$

and W is the μ-mesonic total energy.

So far as the K-conversion is concerned, we take $\kappa = -1$ in the above formula. Then the expansion coefficients are given in the following:

$$
\left.\begin{aligned}
f_0&=\frac{\alpha_1}{3},\\
f_1&=\frac{\alpha_1^2\alpha_2}{30}+\frac{\beta}{5},\\
f_2&=\frac{\alpha_1^3\alpha_2^2}{840}+\frac{\alpha_1^2\beta}{84}+\left(\frac{1}{140}+\frac{1}{42}\right)\alpha_1\alpha_2\beta,\\
f_3&=\frac{\alpha_1^4\alpha_2^3}{45360}+\left(\frac{1}{4536}+\frac{1}{1620}\right)\alpha_1^3\alpha_2\beta+\left(\frac{1}{7560}+\frac{1}{2268}+\frac{1}{1080}\right)\alpha_1^2\alpha_2^2\beta\\
&\quad+\left(\frac{1}{270}+\frac{1}{108}\right)\alpha_1\beta^2+\frac{\alpha_2\beta^2}{180},\\
&\cdots\cdots\cdots\cdots,
\end{aligned}\right\}\qquad(4\cdot5a)
$$

$$
\left.\begin{aligned}
g_0&=1,\\
g_1&=\frac{\alpha_1\alpha_2}{6},\\
g_2&=\frac{\alpha_1^2\alpha_2^2}{120}+\frac{\alpha_1\beta}{12}+\frac{\alpha_2\beta}{20},\\
g_3&=\frac{\alpha_1^3\alpha_2^3}{5040}+\left(\frac{1}{504}+\frac{1}{180}\right)\alpha_1^2\alpha_2\beta+\left(\frac{1}{840}+\frac{1}{252}\right)\alpha_1\alpha_2^2\beta+\frac{\beta^2}{30},\\
&\cdots\cdots\cdots\cdots,
\end{aligned}\right\}\qquad(4\cdot5b)
$$

where the nuclear radius R of He^3 is taken as 1.6 times the π-meson Compton wave-length,[11] which corresponds to $R\sim1.2$ μ.r.u.

Next we shall find the normalization constants. As κ is equal to -1, A_κ can be determined by the expression

$$A_\kappa=G_\kappa(R)/\sum_{n=0}^{\infty}(-1)^n g_n(\kappa),$$

where the denominator can be easily obtained

$$\sum_{n=0}^{\infty}(-1)^n g_n(\kappa)\cong\begin{cases}0.992 & \text{for the bound state}\\ 0.969 & \text{for the continuous states,}\end{cases}$$

and the numerator is expressed by

$$G_\kappa(R)=\frac{G_\kappa(R)}{G_\kappa^{c}(R)}G_\kappa^{c}(R).$$

The ratio $G_\kappa(R)/G_\kappa^{c}(R)$ can be determined by the continuity condition on the nuclear surface as follows:

$$\frac{G_\kappa}{G_\kappa^c}=\left(\frac{F_\kappa^c}{G_\kappa^c}-\frac{\bar{F}_\kappa^c}{\bar{G}_\kappa^c}\right)\Big/\left(\frac{F_\kappa^i}{G_\kappa^i}-\frac{\bar{F}_\kappa^c}{\bar{G}_\kappa^c}\right) \tag{4·6a}$$

for the bound state, and

$$\frac{G_\kappa}{G_\kappa^c}=\{1-H\cos(\delta-\bar{\delta})\}\left(\frac{F_\kappa^c}{G_\kappa^c}-\frac{\bar{F}_\kappa^c}{\bar{G}_\kappa^c}\right)\Big/\left(\frac{F_\kappa^i}{G_\kappa^i}-\frac{\bar{F}_\kappa^c}{\bar{G}_\kappa^c}\right) \tag{4·6b}$$

for the continuous states, where G_κ^c and F_κ^c represent regular solutions in the Coulomb field, $\bar{G}_\kappa^c$ and $\bar{F}_\kappa^c$ the irregular solutions, δ and $\bar{\delta}$ the phase shifts of approximate regular and irregular solutions, respectively, and H is given by

$$H=\frac{G_\kappa^c}{\bar{G}_\kappa^c}\left(\frac{F_\kappa^c}{G_\kappa^c}-\frac{F_\kappa^i}{\bar{G}_\kappa^i}\right)\Big/\left(\frac{F_\kappa^i}{G_\kappa^i}-\frac{\bar{F}_\kappa^c}{\bar{G}_\kappa^c}\right).$$

Coulomb wave functions on the nuclear surface are given as follows:

i) For the bound state:

$$\left.\begin{matrix}F_\kappa^c(R)\\ G_\kappa^c(R)\end{matrix}\right\}=-\frac{\sqrt{\Gamma(2\gamma+n'+1)}}{\Gamma(2\gamma+1)\sqrt{n'!}}\sqrt{\frac{1\mp\varepsilon}{4N(N-\kappa)}}\sqrt{\frac{2Z}{Na_\mu}}e^{-ZR/Na_\mu}\left(\frac{2ZR}{Na_\mu}\right)^\gamma$$
$$\times\left[n'F\left(-n'+1,\,2\gamma+1,\,\frac{2ZR}{Na_\mu}\right)\pm(N-\kappa)F\left(-n',\,2\gamma+1,\,\frac{2ZR}{Na_\mu}\right)\right] \tag{4·7}$$

where

$$\left.\begin{aligned}&n=n'+k,\quad k=|\kappa|=j+\frac{1}{2},\\ &\varepsilon=W/W_0=[1+\alpha^2Z^2/(n'+\gamma)^2]^{-1/2},\\ &\gamma=\sqrt{k^2-\alpha^2Z^2},\quad N=\sqrt{n^2-2n'(k-\sqrt{k^2-\alpha^2Z^2})}\end{aligned}\right\} \tag{4·7′}$$

are conventional notations, and a_μ is the μ-mesonic Bohr radius.

ii) For continuous states[12]

$$\left.\begin{matrix}F_\kappa^c(R)\\ G_\kappa^c(R)\end{matrix}\right\}=\frac{(1\mp W)^{1/2}(2pR)^\gamma e^{\alpha\pi ZW/2p}|\Gamma(\gamma+i\alpha ZW/p)|}{2(\pi p)^{1/2}\cdot\Gamma(2\gamma+1)}$$
$$\times[e^{-ipR+i\eta}(\gamma+i\alpha ZW/p)F(\gamma+1+i\alpha ZW/p,\,2\gamma+1;\,2ipR)$$
$$\mp\text{complex conjugate}], \tag{4·8}$$

where

$$\left.\begin{aligned}&e^{2i\eta}=-(\kappa-i\alpha Z/p)/(\gamma+i\alpha ZW/p),\\ &F(a,b;x)=\frac{\Gamma(b)}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(b+n)}\frac{x^n}{n!}.\end{aligned}\right\} \tag{4·8′}$$

The irregular solutions can be obtained by replacing γ with $-\gamma$ in (4·7) and (4·8). Particularly in the bound state we should replace γ and n' by $-\gamma$ and $n'+2\gamma$, respectively. As the results of numerical calculations for the μ-mesonic wave functions, we get

$$F^{c}_{-1}(R)/G^{c}_{-1}(R)=-7.28\times10^{-3},$$
$$\bar{F}^{c}_{-1}(R)/\bar{G}^{c}_{-1}(R)=-4.65\times10^{-3},$$
$$F^{t}_{-1}(R)/G^{t}_{-1}(R)=-5.78\times10^{-3}$$

for the bound state, where $A_{-1}=9.81\times10^{-2}$, and

$$F^{c}_{-1}(R)/G^{c}_{-1}(R)=-2.64\times10^{-2},$$
$$\bar{F}^{c}_{-1}(R)/\bar{G}^{c}_{-1}(R)=-2.37\times10^{-2},$$
$$F^{t}_{-1}(R)/G^{t}_{-1}(R)=-2.50\times10^{-2}$$

for the continuous states, where $A_{-1}=1.30$.

§ 5. Ejection probability of μ-mesons and comparison with experiments

In the matrix element of the $E0$ transition given by (3·3), the bracket part is written as

$$\int_0^{r_p} d\tau_\mu \tilde{\psi}_f(\boldsymbol{r}_\mu)\left(\frac{1}{r_p}-\frac{1}{r_\mu}\right)\psi_i(\boldsymbol{r}_\mu)=I-II$$
$$=\frac{1}{r_p}\int_0^{r_p} d\tau_\mu \tilde{\psi}_f(\boldsymbol{r}_\mu)\,\psi_i(\boldsymbol{r}_\mu)-\int_0^{r_p} d\tau_\mu \tilde{\psi}_f(\boldsymbol{r}_\mu)\frac{1}{r_\mu}\psi_i(\boldsymbol{r}_\mu), \tag{5·1}$$

where

$$I=\frac{1}{r_p}\int_0^{r_p} dr_\mu\{F^{(0)}_{-1}(r_\mu)F^{(0)}_{-1}(r_\mu)+G^{(0)}_{-1}(r_\mu)G^{(0)}_{-1}(r_\mu)\}, \tag{5·2}$$

$$II=\int_0^{r_p} dr_\mu\frac{1}{r_\mu}\{F^{(0)}_{-1}(r_\mu)F^{(0)}_{-1}(r_\mu)+G^{(0)}_{-1}(r_\mu)G^{(0)}_{-1}(r_\mu)\}. \tag{5·3}$$

By transforming to relative coordinates and using the wave functions in the preceding section, the matrix element (3·3) is reduced to

$$\langle H'\rangle=a_0\alpha\int d\tau_n\tilde{\phi}_f(\boldsymbol{r})\left\{4.17\cdot10^{-2}\left(\frac{r}{R}\right)^2-1.24\cdot10^{-4}\left(\frac{r}{R}\right)^4\right.$$
$$\left.+9.0\cdot10^{-7}\left(\frac{r}{R}\right)^6-\cdots\right\}\phi_i(\boldsymbol{r}) \tag{5·4}$$

where $a_0=1.28\times10^{-2}$ and $\boldsymbol{r}$ is the relative vector between two protons.

Then it is noted that the transition probability should be multiplied by one third, since the nuclear fusion reactions take place through the state with the total spin 1/2. The nuclear wave function of the initial state may be chosen as

$$\phi_i=\psi(0)\cdot(1-\lambda_d/r), \tag{5·5}$$

where $\psi(0)$ is the μ-mesonic p-d ion wave function at $r=0$, and λ_d the de Broglie wave-length of relative motion between the proton and the deuteron. Now assuming the nuclear charge of the final state as that distributed on the nuclear surface (by the Sliv approximation, which corresponds to $r=R$ in Eq. (5·4)), we obtain

$$\frac{2\pi}{3}|\langle H'\rangle|^2=1.7\times10^4 \text{ sec.}^{-1} \tag{5·6}$$

Let λ_1 be the nuclear fusion reaction rate $v\sigma\phi^2(0)$, λ_2 be the ejection rate of μ-mesons by the conversion process, and τ_μ be the lifetime of the μ-meson, then the results of our evaluations are given by Table I which includes experimental results and estimated results by other authors. Our calculated values seem to give fairly good agreement for two kinds of experimental results.

Table I

λ_2 (sec^{-1}) 10^{-5}	λ_2/λ_1	$\lambda_2/(\tau_\mu^{-1}+\lambda_2)$	Reference
4.7	0.78	0.5	(3)
0.93	0.17	0.17	(4)
—	0.14	—	(5)
1.7	0.17	0.27	(6)
0.17	0.069	0.035	ours
—	0.066±0.014[2)]	0.03[1)]	exp. values

§ 6. Concluding remarks

While the *dead-end* process is, as pointed out previously, one of the obstacles for the practical problem, we may be satisfied with the fact that it is possible to explain the experimental results by the $E0$ transition. Internal conversion experiments concerning usual atoms have recently been improved up to the stage in which nuclear structure effects can be taken into account. In μ-mesonic cases nuclear structure effects become more important even for light nuclei. As the three-body problem is, however, still in the formal stage,[13)] we cannot help phenomenologically comparing with experimental results by using a simple approximation in order to verify the order of magnitude.

The approximation of the surface distribution of the nuclear charge in the ground state seems to be inconsistent with the fact that we used the potential due to the uniform distribution in the case of determining the μ-mesonic wave function. But it may be permitted by the reason that the μ-mesonic wave function is not so sensitive to details of the nuclear charge distribution. On the other hand, Eq. (5·4) is very sensitive to the nuclear wave functions. Then,

for example, if we assume the exponential type wave function[14] or the uniform nuclear charge distribution in the final state, the calculated value of the $E0$ transition probability becomes larger than experimental values by an order of magnitude, though there is some arbitrariness even in the nuclear wave function of the initial state.

In conclusion these phenomena would provide usefull probe for the nuclear structure, since the matrix element for the $E0$ transition is sensitive to the nuclear wave functions, while the nuclear finite size effect would have a substantial feature for the μ-mesonic function. Hence it seems to be necessary for revealing these facts that such processes including other μ-mesonic atoms are studied thoroughly.

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Symmetry of the Strong Interaction in the Four-Dimensional Isotopic-Spin Space and in the Three-Dimensional κ-Space

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Without using strangeness a variant theory of higher symmetry is proposed for the strong interactions of baryons. We classify baryons into two groups: $(p\,n\,\Xi^0\,\Xi^-)$ called the p-group and $(\Sigma^+\Sigma^0\Sigma^-\Lambda)$ called the q-group in terms of a new quantum number $\kappa_3=\frac{1}{2}$ for the p-group and $\kappa_3=-\frac{1}{2}$ for the q-group. p and q make a spinor in the three-dimensional κ-space. Each group is, in turn, considered to represent a vector in the four-dimensional isotopic spin space, called N-space, which can be reduced to define two sets of spin matrices τ and ζ, the sum of the third components of which are related to the electric charge of baryons: $Q_e=\tau_3+\zeta_3$. In order to give an interaction Hamiltonian symmetric in κ-space and also in N-space, which are orthogonal to each other, we introduce sixteen kinds of mesons with spin 0 which mediate the strong interactions of the Yukawa type among baryons. Under the requirement of rotational invariance for the interaction Hamiltonian in the N-κ space, the character of mesons is naturally defined. It is shown that pions correspond to the component of $\kappa_3=0$, $\kappa=0$; strangeness-two mesons to $\kappa_3=0$, $\kappa=1$; and kaons to $\kappa_3=\pm1$, $\kappa=1$. We introduce two kinds of kaons, one called K-kaons exclusively coupling nucleons with the Σ- and Λ-particles and the other called L-kaons coupling the Ξ-particle with the Σ- and Λ-particles. It is shown that our scheme covers the current theory of higher symmetry and the strangeness rule, with the additional selection rule $\Delta\kappa_3=0$ which enables us to forbid the processes: $\Xi^-+p\rightarrow\Lambda+\Lambda$, $D^-+p+p\rightarrow\Sigma^++\Lambda$, etc. A number of experiments in hyperon-meson physics are, therefore, proposed which permit a selection between our scheme and the current theory.

§ 1. Introduction

More than thirty kinds of particles have already been confirmed and at present we can show a list of particles exibiting a puzzling mass pattern. It is believed that the conservation of strangeness and the symmetry of charge independence have been well established for the strong interactions between baryons and mesons. But theory of the strong interaction has not reached the final answer, in explaining the scattering problems of kaons, pions and hyperons. It may, therefore, be necessary to investigate the nature of the interactions in

a more unified way before studying the problems of the fundamental fields and the internal structure of particles.

We may obtain inference about the kinematical aspect of the structure of particles in terms of the symmetry properties of the interactions. At this point one may ask the question: Can we define the charge space independently of those in the Lorentz space? Several authors have directly connected the charge space to the Lorentz space. Such approaches will lead us, however, to a very complicated situation even at the first stage of the formulation. In order to obtain a general perspective, we shall develop a variant way of approach.

First, let us assume that symmetry in the charge space can be dealt with independently of the properties in the Lorentz space. Next, we shall take into account all possibilities for their properties in the Lorentz space independently of those of the charge space. Lastly, we shall pick up the solution, if any, which can be fitted in with the available evidences. Of course, in such a prescription, the original symmetry in the charge space can not be maintained in general. We call the symmetry thus violated "fragile symmetry".

We use fragile symmetry to classify the interactions phenomenologically. It has, however, nothing in common with the theory of symmetry in which the existence of a particular type of interactions with ideal symmetry and the violation of it in terms of the other interactions with less complete symmetry would be responsible for the mass spectrum of particles. On the other hand, we start by assuming that ideal symmetry should exist in the charge space for all strong interactions. Since it has been revealed that such complete symmetry is incapable of explaining the experiments, we search for its violation by utilizing the possibilities for the parities of mesons and the Dirac operators in the Lorentz space. It is impressive to find out that the ideal symmetry is not realized in nature. Our phenomenological approach may, in turn, be useful in obtaining a clue for the deeper understanding of the interactions which should reflect the internal structure which should depend on the charge space as well as on the Lorentz space.

§ 2. Fundamental idea

As is well known, we formulate the interaction among particles in terms of their properties of transmutation. It may be instructive to remind the symmetrical meson theory.[1] Here proton and neutron represent two states of one nucleon, and they interchange with each other accompanied by the virtual emission of pions which have three components π^+, π^0, π^-. The processes of such transmutations can be described by the Yukawa interactions with the symmetry of charge independence. In order to cover higher symmetry for the strong interactions among baryons, let us generalize the idea of the symmetrical meson theory and define the "interchangeability"[2] of particles in the following: If a number of fermions can be considered as different states of one particle, there

must be a number of Yukawa processes of the strong interaction with a typical symmetry. A number of bosons mediating the Yukawa processes should be introduced for establishing the invariant Hamiltonian.

In order to consider the symmetry of Hamiltonian, let us define the quantum numbers classifying fermions. First, baryons are distinguished from leptons by the baryon number (or baryon charge).[3] Next, four kinds of mass levels are divided into *three* groups (Ξ^0, Ξ^-), ($\Sigma^+\Sigma^0\Sigma^-\Lambda$) and ($p$, n) by strangeness[4),5] (-2, -1, 0) or hyper charge[6] (-1, 0, $+1$). As is well known, such classification was introduced in view of the selection rules for the interactions of baryons with mesons. However, if we lay emphasis on the symmetry properties of particles, we had better rearrange them into *two* groups; one is made of two doublets, (Ξ^0, Ξ^-) and (p, n), and the other of one triplet ($\Sigma^+\Sigma^0\Sigma^-$) and one singlet (Λ). In this respect, we follow the theory of the four-dimensional isotopic spin space,[7),8),9] and consider that the ($pn\Xi^0\Xi^-$)-group, p-group, and the ($\Sigma^+\Sigma^0\Sigma^-\Lambda$)-group, q-group, correspond to the same irreducible representation of a four dimensional rotation group. It may be conjectured here that a certain type of the interaction among the fundamental constituents of particles would be responsible for creating such division into the two groups.

Now a number of Yukawa processes are known between the p-group and the q-group accompanied by the virtual emission of kaons. According to the principle of the "interchangeability of particles" just mentioned, we shall consider that the p-group and the q-group are two states of one particle, i.e. a baryon. Just as we classify proton and neutron by the isotopic spin T, we shall introduce a new spin operator κ, and assign

$$\kappa_3 = 1/2 \quad \text{for the } p\text{-group}$$
$$= -1/2 \quad \text{for the } q\text{-group}$$

and

$$\kappa_+ = \kappa_1 - i\kappa_2, \quad \kappa_- = \kappa_1 + i\kappa_2,$$

correspond to the operators which lead to the interchange between the p-group and the q-group. We can define also "κ-charge" by the relation

$$Q_\kappa = B/2 + \kappa_3$$

(where B is the baryon number),

just as we describe the electric charge by $Q_e = B/2 + T_3$.

In this assignment, we define κ-charge independently from the electric charge. In other word, we assume that the isotopic spin space is orthogonal to the κ-space which is introduced to take place of strangeness or hyper charge. Of course, we must prove that the experimental evidences which have previously been explained by the strangeness law can also be explained by the κ-charge conservation law (§ 4).

Following the idea of the symmetrical meson theory, let us introduce a number of the Yukawa processes of the strong interactions with κ-charge independence between the p-group ($Q_\kappa=+1$) and the q-group ($Q_\kappa=0$).

$$\begin{aligned} &p_+ \to q_0 + K_+, \\ &q_0 \to p_+ + K_-, \\ &p_+ \to p_- + K_0(K_0'), \\ &q_0 \to q_0 + K_0(K_0'). \end{aligned} \tag{A}$$

Here K_+, K_-, K_0 and K_0' are mesons with $Q_\kappa=+1$, -1, 0 and 0. They should have components in the isotopic spin space.

If we assume that the conservation of the isotopic spin in the ordinary sense must hold in (A), components for $K_\pm$ should be an isospinor, and those for K_0, K_0' should be an isovector and an isoscalar:

$$\begin{aligned} K_+ &: \quad (K^+ K^0),\ (\widetilde{L^0} L^-), \\ K_- &: \quad (L^+ L^0),\ (\widetilde{K^0} K^-), \\ K_0 &: \quad (D^+ D^0 D^-)\ D^{0\prime}, \\ K_0' &: \quad (\pi^+ \pi^0 \pi^-)\ \pi^{0\prime}. \end{aligned}$$

Here the upper suffices denote electric charge, and the lower suffices κ-charge. $K_\pm$ have two isodoublets; K-kaons and L-kaons, which stand in the formal analogy with the two doublets, neucleons and the Ξ-particles, in the p-group. By introducing the four-dimensional isotopic spin space besides κ-space, we can formulate a Hamiltonian between baryons and mesons as was given in (A). It will be shown in § 2 that pions are really described by the κ-charge singlet state, while the strangeness-two mesons ($D^\pm$) are involved in the κ-charge triplet state, including the two kinds of kaons (K and L) as the other members.

According to the article of Hatsukade and one of the authors (S. N.),[10] the isotopic space was taken as the three-dimensional one. As a result, the interaction Hamiltonian contains ten coupling constants and we must require the following additional conditions,

$$\begin{aligned} ia &= b = c = d = e, \\ -ia' &= b' = c' = d' = e', \end{aligned}$$

in order to obtain the Hamiltonian of the strong interactions which is charge independent in the ordinary sense. It is remarkable, however, to note that in the present scheme of the four-dimensional isotopic formalism we need only four coupling constants g_1, g_2, g_3 and g_4, and the additional conditions are found to be $g_1=g_2$, $g_3=g_4$ (§ 2, (6a, 6b)).

In our scheme, nine kinds of new particles are introduced:

i) L-kaons (L^0, L^-, $\widetilde{L^0}$, L^+),

ii) $D^{\pm}$-mesons,
iii) D^{0}, $D^{0\prime}$,
iv) $\pi^{0\prime}$.

The problems of their observation are discussed in view of the selection rules and the interaction types (§ 3).

It must be mentioned here that the excited states of baryons and mesons (N^*, Y^*, K^*, etc.) are not originally involved in our scheme. We are in favor of the opinion that they might be deduced from consideration about the 'internal' interactions.

It is readily shown that the existing theory of the higher symmetry (charge independence, global symmetry and cosmic symmetry) and the strangeness selection rule are involved in our scheme as a special case. Since it becomes obvious that the theory of the higher symmetry cannot provide a consistent understanding for the experiments about hyperon-kaon physics, we studied other possbilities remaining in our scheme. We have broadened the frame of the charge space so that all of the strong interactions for baryons can be reviewed in terms of higher symmetry. As a result, we have found that the conservation of strangeness and the symmetry of charge independence are not well established, so far as the strong interactions involving D-mesons are concerned. We consider, therefore, a number of reactions which will provide a crucial test of the strangeness law and charge independence.

Symmetry of the weak interaction between baryons and pions is briefly discussed by formulating the Hamiltonian in the same charge spaces (§ 5).

§ 3. Interaction Hamiltonian for the strong interaction of baryon

In this section, we shall formulate the interaction Hamiltonian of Yukawa type for the strong interaction among baryons. First, let us define the properties of the wave functions for baryons in the charge space.

Next, a number of the intermediary bosons will be introduced with which the invariant Hamiltonian can be formulated in the charge space.

According to the reason mentioned in § 1, we define for baryons two vectors in the four-dimensional isotopic spin space:

$$\begin{aligned}\psi_\mu &= \left(\frac{\Xi^- + p}{\sqrt{2}}, \frac{\Xi^- - p}{\sqrt{2}\,i}, \frac{\Xi^0 - n}{\sqrt{2}}, i\frac{\Xi^0 + n}{\sqrt{2}}\right), \\ \varphi_\mu &= \left(\frac{\Sigma^- + \Sigma^+}{\sqrt{2}}, \frac{\Sigma^- - \Sigma^+}{\sqrt{2}\,i}, \Sigma^0, i\Lambda\right).\end{aligned} \tag{1}$$

In (1), ψ_μ, (μ=1, 2, 3, 0), are composed of the two doublets (p, n) and (Ξ^0, Ξ^-), while φ_μ are one triplet ($\Sigma^+\Sigma^0\Sigma^-$) and one singlet (Λ). We introduce a unitary transformation S,

$$\psi_\mu = S p_\mu, \tag{2}$$

$$\varphi_\mu = S q_\mu .$$

S will lead us to the well-known multiplets:

$$\begin{aligned} p_\mu &= (p,\ n,\ \Xi^0,\ \Xi^-), \\ q_\mu &= (\Sigma^+,\ Y^0,\ Z^0,\ \Sigma^-), \\ Y^0 &= \frac{1}{\sqrt{2}}(\Lambda - \Sigma^0), \\ Z^0 &= \frac{1}{\sqrt{2}}(\Lambda + \Sigma^0). \end{aligned} \tag{3}$$

The explicit form of S is given as follows:

$$S = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ i\frac{1}{\sqrt{2}} & 0 & 0 & -i\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & i\frac{1}{\sqrt{2}} & i\frac{1}{\sqrt{2}} & 0 \end{pmatrix}. \tag{4}$$

Now we introduce the 2×2 matrices defined by $\sigma_i = 2\kappa_i$,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5}$$

and assign

$$\begin{aligned} \kappa_3 &= 1/2 \quad \text{for } \psi, \\ &= -1/2 \quad \text{for } \varphi. \end{aligned}$$

Then we consider the Yukawa processes of the strong interactions given by (A). We require that the interaction Hamiltonian should be described by an invariant quantity in the four-dimensional isotopic spin space as well as in the three-dimensional κ-space. This automatically defines the character of bosons which mediate the Yukawa interactions among baryons: i.e. they must be dual tensor or scalar in the four-dimensional isotopic spin space, and vector or scalar in the three-dimensional κ-space. In view of the fact that only the three components π^+, π^0, π^- are known for the pion field, it may be natural to limit ourselves to the bosons of self-dual tensor or scalar in the four-dimensional isotopic spin space.

Of course, the baryon field is four-dimensional vector in the isotopic spin space and spinor in the κ-space.

Then the expression for the interaction Hamiltonian between baryons through mediation of bosons with the spin 0 becomes:

$$H_0=g_1\Psi^* T_i\sigma_0\Psi\phi_i^0+g_2\Psi^* T_0\sigma_0\Psi\phi_0^0+h.\,c., \tag{6a}$$

$$H_1=g_3\Psi^* Z_i\sigma_j\Psi\Phi_i^j+g_4\Psi^* Z_0\sigma_j\Psi\Phi_0^j+h.\,c., \tag{6b}$$

$$\Psi\to D^{1/2}_{1/2,1/2},\ \phi_0^0\to D^0_{0,0},\ \phi_i^0\to D^0_{1,0},\ \Phi_0^j\to D^1_{0,0},\ \Phi_i^j\to D^1_{0,1}. \tag{7}$$
$$(i,j=1,2,3)$$

Here $\Psi=\begin{pmatrix}\psi\\ \varphi\end{pmatrix}$, lower suffices refer to the isotopic spin space, and upper suffices to the κ-space. T_i and Z_i are given by the Schwinger representation[7] (τ_i and ζ_i in his notation),

$$T_i\phi_i^0=\begin{pmatrix} 0 & -i\phi_3^0 & i\phi_2^0 & -i\phi_1^0\\ i\phi_3^0 & 0 & -i\phi_1^0 & -i\phi_2^0\\ -i\phi_2^0 & i\phi_1^0 & 0 & -i\phi_3^0\\ i\phi_1^0 & i\phi_2^0 & i\phi_3^0 & 0\end{pmatrix}, \tag{8}$$

$$T_0\phi_0^0=\begin{pmatrix}\phi_0^0 & 0 & 0 & 0\\ 0 & \phi_0^0 & 0 & 0\\ 0 & 0 & \phi_0^0 & 0\\ 0 & 0 & 0 & \phi_0^0\end{pmatrix}, \tag{9}$$

$$Z_i\Phi_i^j=\begin{pmatrix} 0 & -i\Phi_3^j & i\Phi_2^j & i\Phi_1^j\\ i\Phi_3^j & 0 & -i\Phi_1^j & i\Phi_2^j\\ -i\Phi_2^j & i\Phi_1^j & 0 & i\Phi_3^j\\ -i\Phi_1^j & -\Phi_2^j & -i\Phi_3^j & 0\end{pmatrix}, \tag{10}$$

$$Z_0\Phi_0^j=\begin{pmatrix}\Phi_0^j & 0 & 0 & 0\\ 0 & \Phi_0^j & 0 & 0\\ 0 & 0 & \Phi_0^j & 0\\ 0 & 0 & 0 & \Phi_0^j\end{pmatrix}. \tag{11}$$

If we change the representation by S, we obtain:

$$\Psi_\mu=\begin{pmatrix}\psi_\mu\\ \varphi_\mu\end{pmatrix}\to\Psi_\mu'=\begin{pmatrix}p_\mu\\ q_\mu\end{pmatrix}, \tag{12}$$

$$S^{-1}T_iS=\begin{pmatrix}\tau_i & \\ & \tau_i\end{pmatrix}$$

(τ_i is given by the 2×2 Pauli matrices, which correspond to the ordinary isotopic spin materices.).

$$S^{-1}Z_1S=\begin{pmatrix}0&0&-1&0\\0&0&0&-1\\-1&0&0&0\\0&-1&0&0\end{pmatrix}, \tag{13a}$$

$$S^{-1}Z_2S=\begin{pmatrix}0&0&i&0\\0&0&0&i\\-i&0&0&0\\0&-i&0&0\end{pmatrix}, \tag{13b}$$

$$S^{-1}Z_3S=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}. \tag{13c}$$

If we put

$$\phi_i^0=\pi_i, \quad \phi_0^0=\pi_0, \tag{14}$$

where

$$\pi_1=\frac{\pi^-+\pi^+}{\sqrt{2}}, \ \pi_2=\frac{\pi^--\pi^+}{\sqrt{2}i}, \ \pi_3=\pi^0, \ \pi_0=\pi^{0\prime}, \tag{15}$$

we can show that H_0 is the interaction Hamiltonian between pions and baryons with the global symmetry.[11]

$$\begin{aligned}H_0\to{}&g_1(N_1^*\tau_iN_1\pi_i+N_4^*\tau_iN_4\pi_i+N_2^*\tau_iN_2\pi_i+N_3^*\tau_iN_3\pi_i)\\&+g_2(N_1^*N_1+N_4^*N_4+N_2^*N_2+N_3^*N_3)\pi_0+h.\,c.\end{aligned} \tag{16}$$

$$N_1=\begin{pmatrix}p\\n\end{pmatrix}, \ N_2=\begin{pmatrix}\Sigma^+\\Y\end{pmatrix}, \ N_3=\begin{pmatrix}Z^0\\\Sigma^-\end{pmatrix}, \ N_4=\begin{pmatrix}\Xi^0\\\Xi^-\end{pmatrix}. \tag{17}$$

From the triplet terms in κ-space, we obtain

$$Z_i'\Phi_i^j=\begin{pmatrix}\Phi_3^j&-\sqrt{2}\,\Phi_+^j\\-\sqrt{2}\,\Phi_-^j&-\Phi_3^j\end{pmatrix}. \tag{18}$$

Here Φ_3^j, etc., are given by the 2×2 diagonal matrices. We also have

$$Z_i'\sigma_j\Phi_i^j=\begin{pmatrix}\Phi_3^3&-\sqrt{2}\,\Phi_+^3&\Phi_3^1-i\Phi_3^2&-\sqrt{2}\,(\Phi_+^1-i\Phi_+^2)\\-\sqrt{2}\,\Phi_-^3&-\Phi_3^3&-\sqrt{2}\,(\Phi_-^1-i\Phi_-^2)&-(\Phi_3^1-i\Phi_3^2)\\\Phi_3^1+i\Phi_3^2&-\sqrt{2}\,(\Phi_+^1+i\Phi_+^2)&-\Phi_3^3&\sqrt{2}\,\Phi_+^3\\-\sqrt{2}\,(\Phi_-^1+i\Phi_-^2)&-(\Phi_3^1+i\Phi_3^2)&\sqrt{2}\,\Phi_-^3&\Phi_3^3\end{pmatrix}, \tag{19}$$

$$Z_0'\sigma_j\Phi_0^j=\begin{pmatrix} \Phi_0^3 & 0 & \Phi_0^1-i\Phi_0^2 & 0 \\ 0 & \Phi_0^3 & 0 & \Phi_0^1-i\Phi_0^2 \\ \Phi_0^1+i\Phi_0^2 & 0 & -\Phi_0^3 & 0 \\ 0 & \Phi_0^1+i\Phi_0^2 & 0 & -\Phi_0^3 \end{pmatrix}. \tag{20}$$

If we take $\Phi_3^3=D^0$, $\Phi_\pm^3=D^\pm$, $\Phi_0^3=D^{0\prime}$ we have from the κ-charge neutral terms in (19) and (20):

$$\begin{aligned} H_D\to{}& g_3\{(N_1^*N_1-N_4^*N_4-N_2^*N_2+N_3^*N_3)D^0 \\ & -\sqrt{2}\,N_1^*N_4D^+-\sqrt{2}\,N_4^*N_1D^-+\sqrt{2}\,N_2^*N_3D^+ +\sqrt{2}\,N_3^*N_2D^-\} \\ & +g_4(N_1^*N_1+N_4^*N_4-N_2^*N_2-N_3^*N_3)D^{0\prime}+h.\,c. \end{aligned} \tag{21}$$

In (21), it is easily seen that $D^\pm$-mesons behave as the strangeness-two mesons for the N_1-N_4 coupling, but they behave as the strangeness-zero mesons for the N_2-N_3 coupling. On the other hand, D^0 and $D^{0\prime}$ correspond to the strangeness-zero mesons. $D^\pm$, D^0 and $D^{0\prime}$ are all isoscalar in the ordinary sense, with regard to multiplets of $pn\Xi^0\Xi^-$ and $\Sigma^+Y^0Z^0\Sigma^-$.

From the κ-charge exchange terms in (19) and (20), we obtain

$$\begin{aligned} H_K\to{}& N_1^*N_2\{g_3(\Phi_3^1-i\Phi_3^2)+g_4(\Phi_0^1-i\Phi_0^2)\} \\ & -N_1^*N_3g_3\sqrt{2}\,(\Phi_+^1-i\Phi_+^2)-N_4^*N_2g_3\sqrt{2}\,(\Phi_-^1-i\Phi_-^2) \\ & -N_4^*N_3\{g_3(\Phi_3^1-i\Phi_3^2)-g_4(\Phi_0^1-i\Phi_0^2)\}+h.\,c. \end{aligned} \tag{22}$$

It can be shown that (22) will lead to the kaon-baryon couplings if we assume the following correspondence:

$$\begin{aligned} & K^+,K^0,\widetilde{K}^0,K^-\to K_1,K_2,K_3=\frac{1}{\sqrt{2}}(K^0+\widetilde{K}^0),\ K_0=\frac{1}{\sqrt{2}\,i}(\widetilde{K}^0-K^0), \\ & L^+,L^0,\widetilde{L}^0,L^-\to L_1,L_2,L_3=\frac{1}{\sqrt{2}}(L^0+\widetilde{L}^0),\ L_0=\frac{1}{\sqrt{2}\,i}(\widetilde{L}^0-L^0), \\ & \Phi_+^1=\frac{1}{\sqrt{2}}(K^++L^+),\ \Phi_-^1=\frac{1}{\sqrt{2}}(K^-+L^-),\ \Phi_3^1=-\frac{1}{\sqrt{2}}(K_3+L_3), \\ & \Phi_0^1=-\frac{1}{\sqrt{2}}(K_3-L_3),\ \Phi_+^2=\frac{1}{\sqrt{2}\,i}(L^+-K^+),\ \Phi_-^2=-\frac{1}{\sqrt{2}\,i}(L^--K^-), \\ & \Phi_3^2=-\frac{1}{\sqrt{2}}(K_0-L_0),\ \Phi_0^2=-\frac{1}{\sqrt{2}}(K_0+L_0),\ \Phi_1^1=\frac{1}{\sqrt{2}}(K_1+L_1), \\ & \Phi_2^1=\frac{1}{\sqrt{2}}(K_2+L_2),\ \Phi_1^2=\frac{1}{\sqrt{2}}(K_2-L_2),\ \Phi_2^2=-\frac{1}{\sqrt{2}}(K_1-L_1). \end{aligned} \tag{23}$$

Then, by assuming

$$g_3=g_4\equiv g_K, \tag{24}$$

we have

$$H_K \to -2g_K(N_1^* N_2 K^0 + N_1^* N_3 K^+ + N_4^* N_2 L^- - N_4^* N_3 L^0) + h.c. \quad (25)$$

It is interesting to note that in (25) we have two kinds of kaons: One is the K-kaon which couples nucleons with the Σ- and Λ-particles, while the other is the L-kaon which couples the Ξ-particles with the Σ- and Λ-particles. However, it may be regarded that (25) is a substitute of the kaon-baryon couplings with the cosmic symmetry.[12,13]

We have summarized the quantum number (charge) of baryons and mesons in Table I. In our scheme, the electric charge of particles is defined only by their quantum numbers in the four-dimensional isotopic space:

$$Q_e = \tau_3 + \zeta_3, \quad Q_\kappa = \frac{B}{2} + \kappa_3 \text{ (fermion)}, \quad Q_\kappa = \kappa_3 \text{ (meson)}. \quad (26)$$

This can be compared to the relation in the current theory of hypercharge U:

$$Q_e = I_3 + \frac{U}{2}, \quad U = \left(\frac{1}{2} + \kappa_3\right) 2\zeta_3 \quad \text{(fermion)}.$$

Table I. Quantum numbers of particles

	τ	τ_3	ζ	ζ_3	κ	κ_3
P	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
n	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
Ξ^0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
Ξ^-	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
Σ^+	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
Y^0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
Z^0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
Σ^-	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
π^+	1	1	0	0	0	0
π^0	1	0	0	0	0	0
π^-	1	−1	0	0	0	0
$\pi^{0\prime}$	0	0	0	0	0	0
K^+	0	0	1	1	1	1
K^0	0	0	1, 0	0	1	1
L^-	0	0	1	−1	1	1
$\tilde{L}^0$	0	0	1, 0	0	1	1
D^+	0	0	1	1	1	0
D^0	0	0	1	0	1	0
D^-	0	0	1	−1	1	0
$D^{0\prime}$	0	0	0	0	1	0
L^+	0	0	1	1	1	−1
$\tilde{K}^0$	0	0	1, 0	0	1	−1
K^-	0	0	1	−1	1	−1
L^0	0	0	1, 0	0	1	−1

§ 4. Selection rule

In this section we shall discuss the consequences of the selection rules derived from our scheme, by comparing them with those of the strageness rule. A) We have to make sure that we can give a reasonable explanation of the forbidden processes which have so far been accounted for by the strangeness selection rule. The Hamiltonian given by (6a)—(6b) is rotationally invariant in κ-space and also in four-dimensional isotopic space. Since the ζ-assignment for the neutral kaons, however, violates the full four-dimensional invariance, we can infer the five conserved quantities: τ, τ_3, ζ_3, κ, κ_3.

As a result, we have the selection rules

$$\Delta\tau=0, \quad \Delta\tau_3=0, \quad \Delta\zeta_3=0, \quad \Delta\kappa=0, \quad \Delta\kappa_3=0. \tag{27}$$

In view of the selection rules in our scheme, we find that the following processes are forbiddden:

i)	K,	L,	$D \not\to m\pi$			$m=1, 2, \cdots$
κ	1	0				$\Delta\kappa=1$,
ii)	$n \not\to \Sigma^+ + K^-$,					
κ_3	1/2	−1/2	−1			$\Delta\kappa_3=-2$,
ζ_3	1/2	1/2	−1			$\Delta\zeta_3=-1$,
	$n \not\to \Sigma^+ + L^-$,					
κ_3	1/2	−1/2	1			$\Delta\kappa_3=0$,
ζ_3	1/2	1/2	−1			$\Delta\zeta_3=-1$,
iii)	$\pi^- + N \not\to \begin{matrix} Y^0 \\ Z^0 \end{matrix} + K^-$,					
κ_3	0	1/2	−1/2	−1		$\Delta\kappa_3=-2$,
ζ_3	0	1/2	$\begin{matrix} 1/2 \\ -1/2 \end{matrix}$	−1		$\Delta\zeta_3=\begin{matrix} -1 \\ -2 \end{matrix}$,
	$\pi^- + n \not\to \begin{matrix} Y^0 \\ Z^0 \end{matrix} + L^-$,					
κ_3	0	1/2	−1/2	1		$\Delta\kappa_3=0$,
ζ_3	0	1/2	$\begin{matrix} 1/2 \\ -1/2 \end{matrix}$	−1		$\Delta\zeta_3=\begin{matrix} -1 \\ -2 \end{matrix}$,
	$\pi^- + p \not\to \Sigma^+ + K^-$,					
κ_3	0	1/2	−1/2	−1		$\Delta\kappa_3=-2$,
ζ_3	0	1/2	1/2	−1		$\Delta\zeta_3=-1$,

(28)

$$\begin{array}{lccccl} & \pi^- + p \to \Sigma^+ + L^-, & & & & \\ \kappa_3 & 0 & 1/2 & -1/2 & 1 & \Delta\kappa_3=0, \\ \zeta_3 & 0 & 1/2 & 1/2 & -1 & \Delta\zeta_3=-1. \end{array}$$

Thus we see that the main results of the strangeness rule are reproduced by the selection rules $\Delta\kappa_3=0$ and $\Delta\zeta_3=0$.

B) We shall discuss the processes that are allowed by the strangeness rule but are forbidden by the $\Delta\kappa_3=0$ rule.

$$\left.\begin{array}{lccccl} & \Xi^- + p \to \Lambda + \Lambda, & & & & \\ S & -2 & 0 & -1 & -1 & \Delta S=0, \\ \kappa_3 & 1/2 & 1/2 & -1/2 & -1/2 & \Delta\kappa_3=-2, \\ & D^- + p + p \to \Sigma^+ + \Lambda, & & & & \\ S & -2 & 0 & 0 & -1 & -1 \quad \Delta S=0, \\ \kappa_3 & 0 & 1/2 & 1/2 & -1/2 & -1/2 \quad \Delta\kappa_3=-2, \\ & K^+ + p \to D^+ + \Sigma^+, & & & & \\ S & 1 & 0 & 2 & -1 & \Delta S=0, \\ \kappa_3 & 1 & 1/2 & 0 & -1/2 & \Delta\kappa_3=-2. \end{array}\right\} \quad (29)$$

C) We have to consider the processes that are allowed by the $\Delta\kappa_3=0$ rule, but are forbidden by the $\Delta\zeta_3=0$ rule.

$$\begin{array}{lcccccccc} & K^- + p \not\to D^+ + \Sigma^-, & & & & K^- + n \not\to D^- + Z^0, & & & & \\ S & -1 & 0 & 2 & -1 & -1 & 0 & -2 & -1 & \Delta S=-2, \\ \kappa_3 & -1 & 1/2 & 0 & -1/2 & -1 & 1/2 & 0 & -1/2 & \Delta\kappa_3=0, \\ \zeta_3 & -1 & 1/2 & 1 & -1/2 & -1 & 1/2 & -1 & -1/2 & \Delta\zeta_3=1. \end{array}$$

It is interesting to note that this process is forbidden by the $\Delta\zeta_3=0$ rule, yielding a result identical with that of the strangeness rule.

D) We now cite the processes that are forbidden by the strangeness rule but are allowed in our scheme.

$$\begin{array}{lcccccccc} & K^- + p \to D^- + \Sigma^+, & & & & K^- + n \to D^- + Y^0, & & & & \\ S & -1 & 0 & -2 & -1 & -1 & 0 & -2 & -1 & \Delta S=-2, \\ \kappa_3 & -1 & 1/2 & 0 & -1/2 & -1 & 1/2 & 0 & -1/2 & \Delta\kappa_3=0, \\ \zeta_3 & -1 & 1/2 & -1 & +1/2 & -1 & 1/2 & -1 & 1/2. & \Delta\zeta_3=0. \end{array}$$

Our conclusion that the strangeness rule is, at best, incomplete in defining the selection rule for the strong interactions involving strange particles is of

great interest, because the experiments with the higher energy will be feasible in the near future.

§ 5. Break down of higher symmetry

Now we turn to the difficulties of higher symmetry in view of the observation on the kaon-hyperon scattering. First, we mention that our frame of (ψ, φ) or (p, q) for the isotopic spin representation is not equivalent to that of $(\Xi, \Sigma, \Lambda, N)$ which has been used in describing the ordinary charge independence (CI). Therefore, we need to investigate how CI comes about in our scheme. From the Hamiltonian for D-meson interactions given by (21), the D-mesons behave as an isoscalar to the $N\Xi D$, NND^0, $\Xi\Xi D^0$ couplings but as an isovector to the $\Sigma\Sigma D$, $\Sigma\Lambda D$ couplings. As a result, from the D-meson contributions of the higher order we have to break down charge independence for the processes in which nucleon (Ξ) and $\Sigma(\Lambda)$ appear simultaneously in the external lines. Second, we see that the observed processes cited in the following are forbidden by the $\Delta\zeta_3=0$ rule.

$$
\begin{array}{llllll}
\text{I} & p + \pi^+ \to K^+ + \Sigma^+, & & & & \\
 & \kappa_3 \quad 1/2 & 0 & 1 & -1/2 & \Delta\kappa_3=0, \\
 & \zeta_3 \quad 1/2 & 0 & 1 & 1/2 & \Delta\zeta_3=1, \\
\text{II} & K^+ + n \to K^0 + p, & & & & \\
 & \kappa_3 \quad 1 & 1/2 & 1 & 1/2 & \Delta\kappa_3=0, \\
 & \zeta_3 \quad 1 & 1/2 & 0 & 1/2 & \Delta\zeta_3=-1, \\
\text{III} & K^- + p \to \Xi^- + L^+, & & & & \\
 & \kappa_3 \quad -1 & 1/2 & 1/2 & -1 & \Delta\kappa_3=0, \\
 & \zeta_3 \quad -1 & 1/2 & -1/2 & 1 & \Delta\zeta_3=1.
\end{array}
\qquad (30)
$$

As was pointed out by Pais,[14] the current scheme of the global symmetry with that of the cosmic symmetry stands in the same situation with our scheme. Let us try simple examples which break down the symmetry in the charge spaces but solve the difficulties mentioned above.

Example 1.

It is obvious that the origin of the difficulties for the processes (I)—(III) lies in the fact that the intermediary hyperon-lines connecting to the final state are $Y^0(Z^0)$ whereas those to the initial state are $Z^0(Y^0)$. If we change the components of Ψ by changing the phase of Λ into $-\Lambda$ and denote as Ψ', we obtain the following expression instead of (6b),

$$H_1'=g_3\Psi'^*\zeta_i\sigma_j\Psi'\Phi_i^j+g_4\Psi'^*\zeta_0\sigma_j\Psi'\Phi_0^j+h.c. \qquad (6b')$$

On the other hand, we still keep the same representation for H_0 as is given by (6a). In this case, we are led to the representation

$$\begin{aligned}\psi' &\equiv (pn\Xi^0\,\Xi^-) = \psi,\\ \varphi' &\equiv (\Sigma^+ - Z^0 - Y^0\,\Sigma^-),\\ \Psi' &\equiv \begin{pmatrix}\psi'\\ \varphi'\end{pmatrix}.\end{aligned} \tag{31}$$

It is readily seen that the processes (I), (II) and (III) will now be allowed from the modified representation, but we have then to face other difficuties.

The contributions from the processes of the higher order involving the virtual emision of D-mesons will lead to the occurrence of the unwanted processes of $\pi+$ nucleon $\to L^- + (\Sigma,\ \Lambda)$ given by (6a)—(6b′). This, being a matter of the strong interactions, cannot be overlooked because of a higher order correction.

Let us utilize the varieties of parity and violate κ-independence ($\Delta\kappa=0$) in addition to change of phase of Λ in H_0 and H_1. We thus obtain a tentative solution :

1) D-meson is asssumed to be scalar.
2) Even Σ-Λ parity is assumed.
3) Odd N-Ξ parity is assumed.
4) Intrinsic parities of K-kaons and that of L-kaons are taken to be opposite, yielding a result that the relative parities of K-kaons to the nucleon-$\Sigma(\Lambda)$ pairs are same to those of L-kaons to the Ξ-$\Sigma(\Lambda)$ pairs.
5) We require that parity must be conserved in the strong interactions.

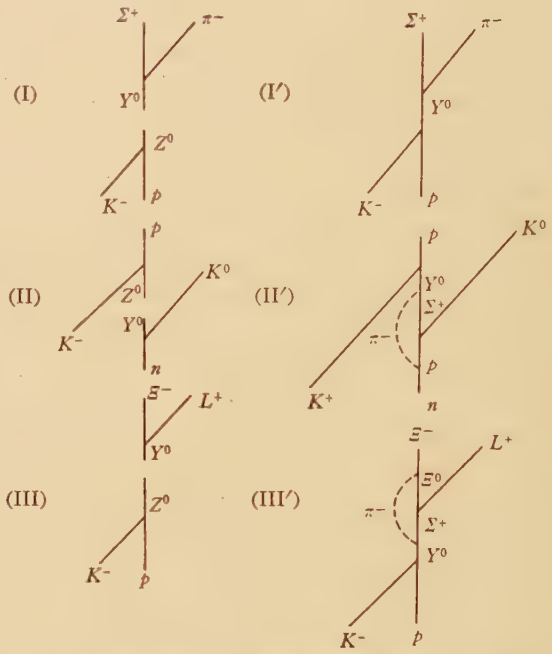

Fig. 1. The Feynman diagrams for the processes, given by (30), in which the difficulties of the theory of higher symmetry appear. (I′), (II′) and (III′) indicate the possible solution in (31). It is interesting to note that (II′) and (III′) are derived only from the higher oider terms, while (I′) is given from the lowest order term.

Under these requirements 1)—5), we are led to the elimination of some of the D-meson processes :

$$H_{N_1{}^*N_1D^0} = H_{N_4{}^*N_4D^0} = H_{N_2{}^*N_2D^0} = H_{N_3{}^*N_3D^0} = H_{N_2{}^*N_3D^+} = H_{N_3{}^*N_2D^-} = 0. \tag{32}$$

Of course, one may argue the question why we have to make a choice of the case given by 1)—5) among all others. With results of (32), however, we are

able to prove the validity of the strangeness rule and CI. The unwanted contribution from the D-meson processes of higher order will, of course, disappear. In this solution, we retain the couplings for the strangeness-two meson, D^+ and D^-, by asssuming the conditions 2)—4), otherwise all D-couplings will be lost.

Example 2.

We may obtain the other solution by changing some of the requirements given by Example 1. Instead of 2)—3), we now require as follows:

2′) Odd Σ-Λ parity is assumed.

3′) Even N-Ξ parity is assumed.

We thus obtain from 2′)—3′)

$$H_{\Xi ND}{}^{\pm}=H_{\Sigma\Sigma D}=H_{NND^0}=H_{\Xi\Xi D^0}=0,\ H_{\Sigma\Lambda D}\neq 0. \tag{33}$$

For the pion-baryon couplings, we can take the following solution by taking account of 2′)—3′).

$$H_{\Xi\Xi\pi}\neq 0,\ H_{NN\pi}\neq 0,\ H_{\Sigma\Sigma\pi}\neq 0,\ H_{\Sigma\Lambda\pi}=0. \tag{34}$$

From (33) and (34) we are again led to the revival of CI and strangeness rule. In this case, D-mesons correspond to strangeness-0 mesons of isovector.

We have in this way broken down the four-dimensional symmetry in the isotopic spin space and also κ-independence in order to obtain an agreement with experiment. It must be noted, however, that $\Delta\kappa_3=0$ rule still holds. It follows that the situation will not change for the processes allowed by the strangeness rule but forbidden by the $\Delta\kappa_3=0$ rule.

$$K^+ + p \not\to D^+ + \Sigma^+,$$
$$K^+ + n \not\to D^+ + \begin{matrix}\Sigma^0\\ \Lambda^0\end{matrix}.$$

And the production processes of $D^{\pm}$ will come out as follows:

$$\pi^- + p \to D^+ + \Xi^-,$$
$$L^+ + p \to D^+ + \Sigma^+,$$
$$L^+ + n \to D^+ + \begin{matrix}\Sigma^0\\ \Lambda\end{matrix}.$$

Example 3.

If we turn to the case:

2″) odd Σ-Λ parity is assumed,

3″) odd Ξ-N parity is assumed,

we have the following solution:

$$H_{\Xi ND}{}^{\pm}\neq 0,\ H_{\Xi\Xi D}=H_{NND}=0,\ H_{\Sigma\Lambda D}\neq 0,\ H_{\Sigma\Sigma D}=0. \tag{35}$$

Then the unobserved processes such as $\pi^-+N \to \Lambda^0+L^-$ will occur from the higher order contributions (See Fig. 2.). Also we have in this case to face the violation of CI in the higher order terms.

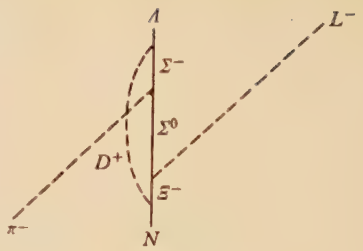

Fig. 2.

We can easily see that in this solution D-mesons correspond to strangeness-two mesons of isoscalar for the Ξ-N couplings, but to strangeness-zero mesons of isovector to the Σ-Λ couplings. Although this solution seems very unlikely in view of the available experiments, we need more precise measurements for its absolute elimination, because it concerns with the higher order approximations.

§ 6. Discussions

A) Implication of the N-κ representation to the weak interaction in the non-leptonic mode

As was already discussed in the preceding articles,[15] the four-dimensional representation for isotopic spin enables us to formulate a Hamiltonian of the weak interaction between baryons and pions as an invariant in the same representation with that of the strong interaction. In fact, we can write down an invariant Hamiltonian of the weak interaction in N-space.

$$\begin{aligned} H_1&=f_1\psi_\mu^*\pi^{\mu\nu}\varphi_\nu, \\ H_2&=f_2\psi_\mu^*\pi^{\mu\nu\prime}\varphi_\nu, \\ H_1^0&=f_1^0\psi_\mu^*\delta^{\mu\nu}\varphi_\nu\pi_0, \\ \delta^{\mu\nu}&=1, \quad \mu=\nu \\ &=0, \quad \mu\neq\nu. \end{aligned} \tag{36}$$

Here

$$\pi^{\mu\nu}=\begin{pmatrix} 0 & -i\pi_3 & i\pi_2 & -i\pi_1 \\ i\pi_3 & 0 & -i\pi_1 & -i\pi_2 \\ -i\pi_2 & i\pi_1 & 0 & -i\pi_3 \\ i\pi_1 & i\pi_2 & i\pi_3 & 0 \end{pmatrix}, \quad \pi^{\mu\nu\prime}=\begin{pmatrix} 0 & -i\pi_3 & i\pi_2 & i\pi_1 \\ i\pi_3 & 0 & -i\pi_1 & i\pi_2 \\ -i\pi_2 & i\pi_1 & 0 & i\pi_3 \\ -i\pi_1 & -i\pi_2 & -i\pi_3 & 0 \end{pmatrix},$$

$$\begin{aligned} H_1&=f_1(\boldsymbol{\psi}^*\times\boldsymbol{\pi}\cdot\boldsymbol{\varphi}-\psi_0^*\boldsymbol{\pi}\cdot\boldsymbol{\varphi}-\boldsymbol{\psi}^*\cdot\boldsymbol{\pi}\varphi_0), \\ H_1^0&=f_1^0(\boldsymbol{\psi}^*\pi_0\boldsymbol{\varphi}+\psi_0^*\pi_0\varphi_0), \\ H_2&=f_2(\boldsymbol{\psi}^*\times\boldsymbol{\pi}\cdot\boldsymbol{\varphi}+\psi_0^*\boldsymbol{\pi}\cdot\boldsymbol{\varphi}+\boldsymbol{\psi}^*\cdot\boldsymbol{\pi}\varphi_0). \end{aligned}$$

Here $\boldsymbol{\psi}=(\psi_1, \psi_2, \psi_3)$, $\boldsymbol{\varphi}=(\varphi_1, \varphi_2, \varphi_3)$ and $\boldsymbol{\pi}=(\pi_1, \pi_2, \pi_3)$.

If we transform the representation from the ψ-φ scheme to the p-q scheme, we obtain (see § 3, (4))

$$H_1=f_1(N_1^*\boldsymbol{\tau}N_2+N_4^*\boldsymbol{\tau}N_3)\boldsymbol{\pi}, \tag{37}$$

$$H_1^0=f_1^0(N_1^*N_2+N_4^*N_3)\pi_0, \tag{38}$$

$$H_2=f_2(N_1^*N_2\pi^0-\sqrt{2}\,N_1^*N_3\pi^+-\sqrt{2}\,N_4^*N_2\pi^--N_4^*N_3\pi^0), \tag{39}$$

$$f_1=f_1^0=f_2$$

$H_1\rightarrow$

$$(p^*\ n^*\ \Xi^{0*}\ \Xi^{-*})\begin{pmatrix} \pi^0 & \sqrt{2}\,\pi^+ & 0 & 0 \\ \sqrt{2}\,\pi^- & -\pi^0 & 0 & 0 \\ 0 & 0 & \pi^0 & \sqrt{2}\,\pi^+ \\ 0 & 0 & \sqrt{2}\,\pi^- & -\pi^0 \end{pmatrix}\begin{pmatrix} \Sigma^+ \\ Y^0 \\ Z^0 \\ \Sigma^- \end{pmatrix}, \tag{37a}$$

$H_2\rightarrow$

$$(p^*\ n^*\ \Xi^{0*}\ \Xi^{-*})\begin{pmatrix} \pi^0 & 0 & -\sqrt{2}\,\pi^+ & 0 \\ 0 & +\pi^0 & 0 & -\sqrt{2}\,\pi^+ \\ \sqrt{2}\,\pi^- & 0 & -\pi^0 & 0 \\ 0 & -\sqrt{2}\,\pi^- & 0 & -\pi^0 \end{pmatrix}\begin{pmatrix} \Sigma^+ \\ Y^0 \\ Z^0 \\ \Sigma^- \end{pmatrix}. \tag{39a}$$

As was done in the previous articles,[15] if we insert the Dirac operators in $\gamma_\mu(\gamma_\mu\gamma_5)$ in (37a), $\gamma_\mu\gamma_5(\gamma_\mu)$ in (39a), we are led to the parity violating interactions for $\Sigma^+\rightarrow\pi^0+p$, $\Lambda\rightarrow\pi^-+p$, $\Lambda\rightarrow\pi^0+n$, and the parity conserving interactions for $\Sigma^\pm\rightarrow\pi^\pm+n$. This result can provide a nice explanation to the experiments on the asymmetry of the Σ-decay given by Cool et al.[16]

It can be easily seen that we have the selection rules in the four-dimensional isotopic spin space $\Delta\tau_3=0$, $\Delta\zeta_3=0$ for (37) and (38), $\Delta\tau_3=\pm1$, $\Delta\zeta_3=\mp1$ for (39). On the other hand, the third component κ_3 obeys the selection rule $\Delta\kappa_3=\pm1$ in (37)—(39).

B) New particles

In the above sections, we have introduced, besides seven kinds of the known mesons (K-kaons and pions), nine kinds of new mesons. This is only because we intend to establish the interaction Hamiltonian between baryons and mesons with spin 0 based on the philosophy[2] of the interchangeability of particles. We need, therefore, to check the existence of the new mesons in view of the experiments.

i) L-kaons

L-kaons are introduced as the counter part of K-kaons in an analogy to the relationship of the Ξ-particles to nucleons. Therefore, it must not be confused with K^*, which has recently been discovered, as an excited state of K-kaons.

The production processes for L-kaons in our scheme can be cited:

$$\begin{aligned} K^- + p &\to L^+ + \Xi^-, \\ K^- + n &\to L^0 + \Xi^-. \end{aligned} \tag{40}$$

According to the experiments made in Berkeley,[17] the Ξ^--particle is produced:

$$K^- + p \to \Xi^- + K^+.$$

In our scheme, this process is forbidden because $\Delta\kappa_3 = 2$. It is tempting to us to identify this type of process as the associate production of the Ξ-particles with L-kaons given by (40). Of course, we must know the mass, M_L, of L-kaons. Simple perturbation of the self energy terms for the processes $L \to \Xi + \Sigma(\Lambda) \to L$ and $K \to N + \Sigma(\Lambda) \to K$ leads us to the same magnitudes with regard to their leading terms. We can easily infer that the decay schemes of L-kaons are the same with those of K-kaons, as long as the present discussions about symmetry are concerned. As a result, there could be no objection to interpret the Ξ^--production experiment as the one accompanied by the L-production.

ii) $D^{\pm}$-mesons

There have been several reports[18)-19)] about the possible existence of strange ness-two mesons. If we assume that

$$M_{D^{\pm}} > M_{\pi} + M_K \text{ (or } M_L), \tag{41}$$

the following decay processes will be expected in our scheme:

$$\begin{aligned} D^{\pm} &\to \pi^{\pm} + K^0(L^0), \\ &\to \pi^0 + K^{\pm}(L^{\pm}). \end{aligned} \tag{42}$$

On the other hand, the production scheme of $D^{\pm}$-mesons depends on the tentative solutions i), ii) and iii) given in § 4.

$$\begin{aligned} \pi^- + p &\to \Xi^- + D^+, \\ \pi^0 + p &\to \Xi^0 + D^+, \end{aligned} \qquad \text{(i), (iii)}$$

$$\begin{aligned} K^- + p &\to \Sigma^- + D^+, \\ K^- + n &\to \Sigma^0(\Lambda) + D^-. \end{aligned} \qquad \text{(ii), (iii)}$$

iii) $\pi^{0\prime}$-meson

There are no evidence to accept the neutral isoscalar $\pi^{0\prime}$ meson which is strongly coupled to nucleon, having the mass about equal to pion. According to Konuma and one of the present authors (S. N.),[15] the branching ratios for the pion decay of hyperons can reasonably be understood if $\pi^{0\prime}$ couples weakly with hyperons. It may, therefore, be that the strong $\pi^{0\prime}$-interaction with nucleon is described in terms of the derivative (vector) coupling and has no effect in the actual phenomena.

iv) D^0, $D^{0\prime}$-mesons

These neutral modes of D-mesons are also introduced in our scheme. To know their behavior, however, we have to define the interaction types more precisely, which is outside the present discussons.

C) Implication of higher symmetry in our scheme for the conjecture about the structure of particles

In conclusion let us add a few more words about the implication of fragile symmetry to the internal structure and fundamental fields of particles. We define higher symmetry in the three-dimensional κ space and in the four-dimensional isotopic spin space, which are orthogonal to each other. It is a typical feature of our scheme of symmetry that fermions are all described in terms of a vector in the N-space and a spinor in κ-space, while bosons are all described in terms of self-dual tensor or scalar in N-space and vector or scalar in

	isotopic spin space	κ-space
baryons	four-dimensional vector	three-dimensional spinor
mesons	four-dimensional self-dual tensor or scalar	three-dimensional vector or scalar

κ-space. It seems likely that our frame of symmetry is simpler than the strangeness frame in conjecturing the unifield model of the structure of particles.

As was derived in § 4, our scheme of symmetry gives a more stringent selection rule than that derived from the strangeness selection rule. In this respect, it is keenly interesting to investigate a number of the processes that are allowed by the strangeness rule but are forbidden by the $\Delta\kappa_3$-rule. For instance, from the $\Delta\kappa_3$-selection rule we are led to expect the existence of Ξ-hyperfragments which from the strangeness selection rule alone should decay instantaneously.

Analysis on the processes of the strong interactions tells us necessity of breaking-down the symmetry in our scheme. We have to violate at least the three conservation rules: $\Delta\kappa=0$, $\Delta\tau=0$, $\Delta\zeta=0$ and are left with the remaining two: $\Delta\kappa_3=0$, $\Delta(\tau_3+\zeta_3)=0$ (We are not sure whether conservation of the individual components τ_3 and ζ_3 holds or not. In the case of the weak interaction between baryons and pions, we have $\Delta(\tau_3+\zeta_3)=0$ but $\Delta\tau_3=0,\ \pm1$, $\Delta\zeta_3=0,\ \mp1$, and $\Delta\kappa_3=\pm1$.). This may be compared to the violation of conservation of parity, which may be interpreted to tell us something about the structure of particles. We may regard fragile symmetry as a feature of the structure of particles, the spherical symmetry of the interactions being violated by a certain type of internal interactions. Conservation of the third components, κ_3 and $\tau_3+\zeta_3$, reminds us of the axial symmetry in molecular structure.

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Propagation of Shock Waves in Inhomogeneous Medium. IV

——*Second Order Approximation*——

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Ôno, Sakashita and Yamazaki presented a first order approximation formula for the propagation of shock waves in inhomogeneous medium by extending Chisnell's method. In this paper, the corrections to their formula are obtained taking into account the effects of the second-order reflected waves. The calculation is restricted to the case of strong shock limit and it is shown that the rate of growth of the shock strength obtained by the first approximation is little influenced, so far as the strong shock limit is concerned, by taking the effects of the second-order waves into consideration.

§ 1. Introduction

Concerning the problem of the propagation of shock waves in inhomogeneous medium, Chisnell[1] presented a method in which the non-uniform medium is regarded as a stack of layers of uniform density separated by elementary contact discontinuities and then the interaction between these discontinuities and the incident shock is taken into account in order to obtain the relation between the variation of shock strength and that of density. He assumed that initially the medium has variable density and uniform pressure. Extending his method to the case in which the initial medium varies both in density and pressure under the gravitational force, Ôno, Sakashita and Yamazaki[2]* derived a more general formula for the shock propagation, and they applied it to the problems of the propagation of magnetohydrodynamic shock waves[3] and of spherical shock waves[4] in non-uniform medium. But the formula they derived is that of the first order approximation which considers only the effects of the first-order reflected waves generated when the shock front passes the discontinuities, and neglects the interaction between the higher order reflected waves and the shock wave.

In this paper, we examine the effects of the second-order reflected waves in order to obtain the correction to the first order approximation formula. Since it is difficult to treat the second order approximation in general, our calculation is restricted to the case of strong shock, where, as will be shown below, mathematical analysis is considerably simplified.

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* This paper will be called Paper I hereafter.

The relationship between the shock strength z and the density ρ, derived by Ôno et al. to the first order approximation, is expressed as

$$\frac{k}{\rho}\frac{d\rho}{dz}=\left[\frac{1}{\lambda^2+z}-\frac{2}{z-1}-\frac{2}{z-1}\sqrt{\frac{1+\lambda^2 z}{(1+\lambda^2)z}}\right]\left[\frac{k-1}{k}+2\sqrt{\frac{1+\lambda^2 z}{(1+\lambda^2)z}}\right]^{-1}, \tag{1·1}$$

where the initial medium is assumed to be a polytrope of index k, i.e. $p\propto\rho^k$, and $\lambda^2=(\gamma-1)/(\gamma+1)$ where γ is the ratio of specific heat. For $z\gg 1$ this equation is easily integrated and we have a simple relation as

$$\rho(z)/\rho(z_1)=(z_1/z)^{a(\gamma,k)}, \tag{1·2}$$

where

$$a(\gamma,k)=\left(1+2\sqrt{\frac{\lambda^2}{1+\lambda^2}}\right)\left(k-1+2k\sqrt{\frac{\lambda^2}{1+\lambda^2}}\right)^{-1}. \tag{1·3}$$

Next, we consider the state of the medium behind the shock front (see Fig. 1). As mentioned in Paper I, there exist the following conditions among the densities and the pressures in each side of the discontinuity:

$$p_5-p_1=p_4-p_3=\Delta p,$$

$$\rho_5-\rho_1=\Delta\rho,$$

$$\Delta p/p_1=k\Delta\rho/\rho_1.$$

For $z\gg 1$ we have $\rho_3\cong\rho_2\cong(1/\lambda^2)\rho_1$ and $\rho_4\cong(1/\lambda^2)(\rho_1+\Delta\rho)$, and therefore

$$(\rho_4-\rho_3)/\rho_3\cong\Delta\rho/\rho_1.$$

On the other hand, we have

$$(p_4-p_3)/p_3\cong\Delta p/zp_1=(k/z)(\Delta\rho/\rho_1).$$

Hence, the relation between the density and the pressure behind the shock is expressed approximately as

$$p\propto\rho^{k'},$$

where

$$k'\approx k/z\ll k.$$

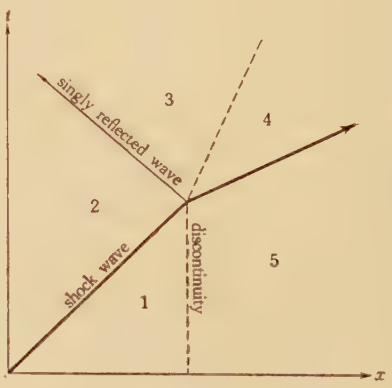

Fig. 1.

Thus it may be seen that the pressure behind the shock front is approximately constant in comparison with the variation of the density.

In the following sections, calculations will be made under the following two assumptions: (1) There is no pressure jump across the density discontinuities in the region behind the shock front, and (2) the asymptotic formula (1·2) is used for the first order approximation formula.

In § 2, the general formulation for the second order approximation will be given. In § 3, the correction to the first order approximation will be estimated assuming the appropriate spatial distributions of the density, and it will be shown that the correction is not so large as to alter the qualitative feature of the first approximation. Conclusions will be given in § 4.

§ 2. Formulation of the second order approximation

In this section, we derive the general formula for the second order approximation using the methods and the notations given in reference 1). The behavior of the singly and the doubly reflected waves is investigated first, and then the effect of the doubly reflected wave on the shock wave is calculated.

2.1 Singly reflected wave

The path of fluid particle which is originally of density $\rho(z_0)$ is called "z_0-particle path", and the crossing point of the z_0-particle path with the singly reflected wave which is generated when the incident shock moves from the z to the $z+dz$ particle path, is denoted by the point (z, z_0) where z is related to $\rho(z)$ by (1·2) (see Fig. 2). Here we adopt (z, z_0) as the coordinate system instead of the usual (x, t) coordinate system.

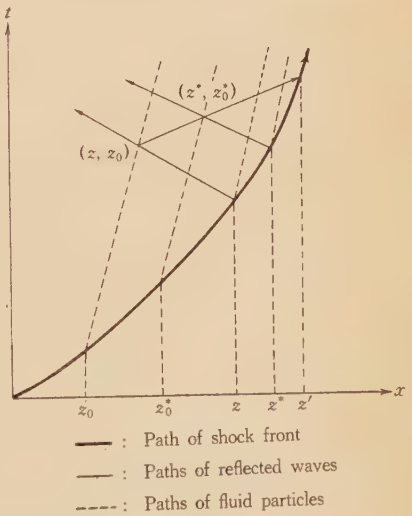

Fig. 2.

The density immediately behind the shock front $\rho(z, z)$ is given by

$$\rho(z, z)=\rho(z)\frac{\lambda^2+z}{1+\lambda^2 z}\simeq\frac{1}{\lambda^2}\rho(z), \quad (2\cdot1)$$

and substituting this into (1·2), we have

$$\rho(z, z)=(1/\lambda^2)(z_1/z)^a\rho(z_1). \quad (2\cdot2)$$

The strength of the singly reflected wave immediately behind the shock front is given, referring to Paper I, as

$$1-g\frac{dz}{z}, \quad (2\cdot3)$$

where

$$g=ka-1=\left[k-1+2k\sqrt{\frac{\lambda^2}{1+\lambda^2}}\right]^{-1}. \quad (2\cdot4)$$

It is found from (1·3), (2·3) and (2·4) that if z is a decreasing function of ρ, i.e. $a>0$, the singly reflected wave is a rarefaction wave.

Let the singly reflected wave, which is originally generated when the shock moves from the z- to the $(z+dz)$-particle path, have the strength

$$1-\alpha(z, z_0)g\frac{dz}{z}, \tag{2·5}$$

when it reaches the z_0-particle path. Since the pressure is approximately constant in the region behind the shock front, putting $k=0$ in (1·1) and expanding it in powers of $(z-1)$, as in reference 1), we have

$$\alpha(z, z_0)=\left\{\frac{\rho(z, z_0)}{\rho(z, z)}\right\}^{1/4}. \tag{2·6}$$

Since the relationship between $\alpha(z, z_0)$ and $\rho(z, z_0)$ is given by

$$\frac{\partial\rho(z, z_0)}{\partial z}dz=-\frac{1}{\gamma}\rho(z, z_0)\alpha(z, z_0)g\frac{dz}{z}, \tag{2·7}$$

substituting (2·6) into (2·7) and integrating it from z_0 to z, we obtain

$$\rho^{-1/4}(z, z_0)=\rho^{-1/4}(z_0, z_0)+\frac{g}{4\gamma}\int_{z_0}^{z}\rho^{-1/4}(x, x)\frac{dx}{x}. \tag{2·8}$$

By using (2·2), this is reduced to

$$\rho^{-1/4}(z, z_0)=(1-\zeta)\left\{\frac{1}{\lambda^2}\rho(z_1)\left(\frac{z_1}{z_0}\right)^{\alpha}\right\}^{-1/4}+\zeta\left\{\frac{1}{\lambda^2}\rho(z_1)\left(\frac{z_1}{z}\right)^{\alpha}\right\}^{-1/4}, \tag{2·9}$$

where

$$\zeta=g/\gamma a=\left\{\gamma\left(1+2\sqrt{\frac{\lambda^2}{1+\lambda^2}}\right)\right\}^{-1}, \tag{2·10}$$

and then $\alpha(z, z_0)$ is expressed as

$$\alpha^{-1}(z, z_0)=(1-\zeta)(z_0/z)^{\alpha/4}+\zeta. \tag{2·11}$$

Hence, the strength of singly reflected wave at (z, z_0) is given by

$$1-\alpha(z, z_0)g\frac{dz}{z}=1-\left\{(1-\zeta)\left(\frac{z_0}{z}\right)^{\alpha/4}+\zeta\right\}^{-1}g\frac{dz}{z}. \tag{2·12}$$

2.2 Doubly refelcted wave

Now we consider the reflection of the singly reflected wave at the discontinuity. As was done in Paper I, we put (see Fig. 3)

$$\begin{aligned}
&z_{12}=1, \quad z_{23}=1, \quad z_{76}=1-\alpha(z, z_0)g\frac{dz}{z},\\
&\rho_1=\rho(z, z_0), \quad \rho_7=\rho(z, z_0)+\frac{\partial\rho(z, z_0)}{\partial z_0}dz_0,\\
&p_1=p_7=p(z, z_0),
\end{aligned} \tag{2·13}$$

and we obtain the strength of the doubly reflected wave z_{65} at (z, z_0) as follows:

$$z_{65}=1+\left[\frac{1}{4}\left\{\frac{\rho(z, z_0)}{\rho(z, z)}\right\}^{1/4}\rho^{-1}(z, z_0)\frac{\partial\rho(z, z_0)}{\partial z_0}\right]g\frac{dz}{z}dz_0$$

$$\equiv 1-\beta(z, z_0)g\frac{dz}{z}dz_0. \tag{2·14}$$

Substitution of (2·9) and (2·11) into (2·14) gives

$$z_{65}=1-\frac{a}{4}(1-\zeta)\left(\frac{z_0}{z}\right)^{a/4}$$
$$\times\left\{(1-\zeta)\left(\frac{z_0}{z}\right)^{a/4}+\zeta\right\}^{-2}g\frac{dz}{z}\frac{dz_0}{z_0}. \tag{2·15}$$

It is found from (2·15) that the doubly reflected wave is a rarefaction or a compression wave according as the singly reflected wave is a rarefaction or a compression wave respectively.

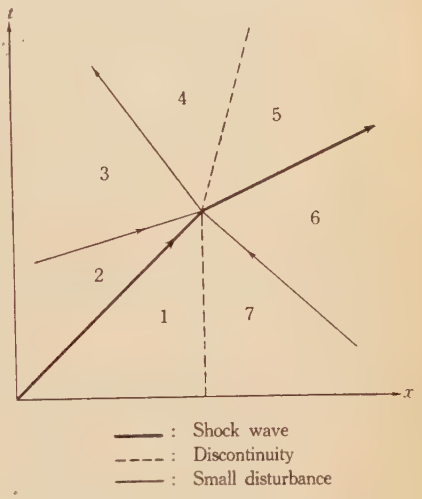

Fig. 3

2.3 *Interaction between the singly reflected wave and the doubly reflected wave*

We consider the situation that the doubly reflected wave generated at (z, z_0) collides, on the z_0^*-particle path, with the singly reflected wave generated at (z^*, z^*) (see Fig. 2).

Let the strength of the doubly reflected wave before the encounter be

$$1-\beta(z, z_0, z_0^*)g\frac{dz}{z}dz_0 \tag{2·16}$$

and as in § 2.2, put

$$z_{12}=1,\quad z_{23}=1-\beta(z, z_0, z_0^*)g\frac{dz}{z}dz_0,$$
$$z_{76}=1-\alpha(z^*, z_0^*)g\frac{dz^*}{z^*}, \tag{2·17}$$
$$\rho_1=\rho_7=\rho(z^*, z_0^*),\quad p_1=p_7=p(z^*, z_0^*),$$

then it follows that

$$z_{65}=1-g\beta(z, z_0, z_0^*)\frac{dz}{z}dz\left\{1+\frac{\lambda^2}{1+\lambda^2}\alpha(z^*, z_0^*)g\frac{dz^*}{z^*}\right\}. \tag{2·18}$$

The strength of this wave element when it reaches the $(z_0^*+dz_0^*)$-particle path is given by (2·6), and expressed as

$$1-g\beta(z, z_0, z_0^*)\frac{dz}{z}dz_0\left\{1+\frac{\lambda^2}{1+\lambda^2}\alpha(z^*, z_0^*)g\frac{dz^*}{z^*}\right\}$$

$$\times\left\{\frac{\rho(z, z_0, z_0^*+dz_0^*)}{\rho(z, z_0, z_0^*)+\dfrac{\partial\rho(z, z_0, z_0^*)}{\partial z^*}dz^*}\right\}^{1/4}, \qquad (2\cdot19)$$

where the second term of the denominator in the last bracket represents the variation in the density across the wave element generated when the shock front moves from the z^* to the (z^*+dz^*)-particle path, and it is given by uisng (2·7). From (2·16) and (2·19) we have

$$\beta^{-1}(z, z_0, z_0^*)\frac{\partial\beta(z, z_0, z_0^*)}{\partial z_0^*}dz_0^*=\frac{1}{4}\rho^{-1}(z, z_0, z_0^*)\frac{\partial\rho(z, z_0, z_0^*)}{\partial z_0^*}dz_0^*$$

$$+\frac{2\gamma-1}{4\gamma}g\alpha(z^*, z_0^*)\frac{dz^*}{z^*}. \qquad (2\cdot20)$$

Integrating this equation from z_0 to z_0^* and making use of (2·6) and (2·14), we have

$$\beta(z, z_0, z_0^*)=-\left\{\frac{\rho(z, z_0, z_0^*)}{\rho(z, z_0)}\right\}^{1/4}\frac{1}{4}\left\{\frac{\rho(z, z_0)}{\rho(z, z)}\right\}^{1/4}\rho^{-}(z, z_0)\frac{\partial\rho(z, z_0)}{\partial z_0}$$

$$\times\exp\left[\frac{2\gamma-1}{4\gamma}g\int_z^{z^*}\left\{\frac{\rho(z^*, z_0^*)}{\rho(z^*, z^*)}\right\}^{1/4}\frac{dz^*}{z^*}\right], \qquad (2\cdot21)$$

where the identity $\beta(z, z_0, z_0)=\beta(z, z_0)$ is used. In order to calculate (2·21), z_0^* must be expressed in terms of z, z_0 and z^*, i.e. the spatial distribution of the pressure and the density in the initial medium must be given explicitly.

2.4 The effect of the doubly reflected wave on the shock wave

If an element of the doubly reflected wave denoted by (2·16) catches up the shock wave of the strength z', the total strength of the doubly reflected wave which comes between z' and $z'+dz'$ is expressed, by using (2·16), as

$$1-dz'\int_{z_1}^{z'}\frac{g}{z}\beta(z, z_0, z')\frac{\partial z(z_0, z')}{\partial z'}dz_0. \qquad (2\cdot22)$$

Taking this wave into account, the conditions for the shock front which passes through the discontinuity is expressed as (see Fig. 3)

$$z_{12}=z', \quad z_{23}=1-dz'\int_{z_1}^{z'}\frac{g}{z}\beta(z, z_0, z')\frac{\partial z(z_0, z')}{\partial z'}dz_0,$$

$$z_{76}=1, \quad z_{65}=z'+dz',$$

$$\rho_7=\rho_1+d\rho, \quad p_7-p_1=p_5-p_4=dp, \quad dp/p_1=k\,d\rho/\rho. \qquad (2\cdot23)$$

Keeping only the first order terms in $d\rho$, dp and dz', as in Paper I, we obtain the second order approximation formula

$$\frac{d\rho}{\rho}=\left[\left\{\frac{1}{\lambda^2+z'}-\frac{2}{z'-1}-\frac{2}{z'-1}\sqrt{\frac{1+\lambda^2 z'}{z'(1+\lambda^2)}}\right\}dz'+\frac{4z'}{z'-1}\sqrt{\frac{1+\lambda^2}{z'(1+\lambda^2)}}\,dz''\right]$$
$$\times\left[k-1+2k\sqrt{\frac{1+\lambda^2 z'}{z'(1+\lambda^2)}}\right]^{-1}, \qquad (2\cdot24)$$

where $z_{23}=1+dz''$ for brevity. The bracketed term in the first square bracket is same as the first order approximation and another term gives the correction due to the second-order waves.

Since (2·24) is significant only for $z'\gg1$, this equation reduces to

$$\frac{\rho(z')}{\rho(z_1)}=\left(\frac{z_1}{z'}\right)^{\alpha}\exp\left[4g\sqrt{\frac{\lambda^2}{1+\lambda^2}}\int_{z_1}^{z'}dz''\right]. \qquad (2\cdot25)$$

Denote the correction factor, exp [⋯], by $\Phi(z')$, it follows finally from (2·21) and (2·22) that

$$\ln\Phi(z')=\sqrt{\frac{\lambda^2}{1+\lambda^2}}\,g^2\int_{z_1}^{z'}dz'\int_{z_1}^{z'}dz_0\,\frac{1}{z}\,\frac{\partial z(z_0,z')}{\partial z'}\left\{\frac{\rho(z,z_0,z')}{\rho(z,z)}\right\}^{1/4}\frac{\partial\ln\rho(z,z_0)}{\partial z_0}$$
$$\times\exp\left[\frac{2\gamma-1}{4\gamma}g\int_{z}^{z'}\left\{\frac{\rho(z^*,z_0^*)}{\rho(z^*,z^*)}\right\}^{1/4}\frac{dz^*}{z^*}\right]. \qquad (2\cdot26)$$

§ 3. Calculation of the correction factor for a few simple cases

3.1 The expression of z_0^* in terms of z, z_0 and z^*

As mentioned in subsection 2.3, it is necessary for the calculation of $\beta(z,z_0,z_0^*)$ to give z_0^* as a function of z, z_0 and z^*. To do this, we put forward for simplicity two assumptions about the medium behind the shock, such as (1) the sound velocity along each particle path remains constant in time, and (2) the distance between the neighboring particle paths also remains constant in time. These two assumptions may be acceptable because it is expected that among the doubly reflected wave elements only those generated at the regions not far from the shock front can give considerable effect on the shock wave.

Let $U(z)$ be the shock velocity at the strength z, and let $a(z,z)$ be the sound velocity along the z-particle path. If an infinitesimal distance between two neighboring particle paths in the initial medium, $dx_1(z)$, becomes $dx_2(z)$ after passage of the shock front, we may derive simple relations between them as follows:

$$\int_{y=z}^{z^*}U^{-1}(y)\frac{dx_1(y)}{dy}dy-\int_{y=z^*}^{z_0^*}a^{-1}(y,y)\frac{dx_2(y)}{dy}dy$$

$$= -\int_{y=z}^{z_0} a^{-1}(y, y)\frac{dx_2(y)}{dy}dy + \int_{y=z_0}^{z_0^*} a^{-1}(y, y)\frac{dx_2(y)}{dy}dy \qquad (3\cdot1)$$

and

$$\frac{dx_2(z)}{dz} = \frac{1+\lambda^2 z}{\lambda^2+z}\frac{dx_1(z)}{dz} \simeq \lambda^2\frac{dx_1(z)}{dz}. \qquad (3\cdot2)$$

Putting $p=K\rho^k$ and considering $z\gg1$, we have

$$a(z, z) = \left\{\frac{\lambda^2(1+\lambda^2)K}{1-\lambda^2}\rho^{k-1}(z)z\right\}^{1/2} = \{\lambda^2(1+\lambda^2)\}^{1/2}U(z). \qquad (3\cdot3)$$

Substitution of (3·2) and (3·3) into (3·1) gives

$$\int_z^{z^*} a^{-1}(y, y)\frac{dx_1(y)}{dy}dy + \sqrt{\frac{\lambda^2}{1+\lambda^2}}\left\{\int_{z^*}^{z_0^*} + \int_z^{z_0} - \int_{z_0}^{z_0^*}\right\} a^{-1}(y, y)\frac{dx_1(y)}{dy}dy = 0 \qquad (3\cdot4)$$

where

$$\int a^{-1}(y, y)\frac{dx_1(y)}{dy}dy = \sqrt{\frac{1-\lambda^2}{\lambda^2(1+\lambda^2)K}}\int \rho^{-(k-1)/2}y^{-1/2}\frac{dx_1(y)}{dy}dy. \qquad (3\cdot5)$$

In order to calculate (3·5) the spatial distribution of density must be given. Here we consider the following two types of the density distribution in which mathematical analysis is extremely simplified.

Case i. Let the density distribution be

$$\rho(x) = \rho_0\left(1-\frac{x}{R}\right)^n, \qquad (3\cdot6)$$

where ρ_0, R and n are constants. Then, making use of (1·2) and putting $z=y$ at $x=x'$, we have

$$x = R\left\{1+\left(\frac{\rho(z_1)}{\rho_0'}\right)^{1/n}\left(\frac{z_1}{y}\right)^{a/n}\right\}. \qquad (3\cdot7)$$

From (3·4), (3·5) and (3·7) we can express z_0^* in terms of z, z_0 and z^* as follows:

$$z_0^{*m} = (1-\xi)^{-1}(z^{*m}-z^m)+z_0^m, \qquad (3\cdot8)$$

where

$$\xi = \left(1+\sqrt{\frac{\lambda^2}{1+\lambda^2}}\right)\Big/2\sqrt{\frac{\lambda^2}{1+\lambda^2}} \qquad (3\cdot9)$$

and

$$m = \{na(k-1)-2a-n\}/2n. \qquad (3\cdot10)$$

Case ii. Let the density distribution be

$$\rho(x)=\rho_0 e^{-nx}, \tag{3·11}$$

where ρ_0 and n are constants. Then, as in Case i, we have

$$x=\ln\left[\frac{\rho_0}{\rho(z_1)}\left(\frac{y}{z_1}\right)^a\right]^{1/n}, \tag{3·12}$$

and (3·12) gives the same relation as (3·8) if we choose m, instead of the value given by (3·10), as

$$m=\{a(k-1)-1\}/2. \tag{3·13}$$

3.2 The reduction of the correction factor

The correction factor (2·26) can be reduced, by using (3·8), to a more explicit form suitable for numerical calculation.

The integral with respect to z^* in (2·26) is expressed, by using (2·11), as

$$\int_z^{z'}\left\{\frac{\rho(z^*, z_0^*)}{\rho(z^*, z^*)}\right\}^{1/4}\frac{dz^*}{z^*}=\int_z^{z'}\left\{(1-\zeta)\left(\frac{z_0^*}{z^*}\right)^{a/4}+\zeta\right\}\frac{dz^*}{z^*}. \tag{3·14}$$

Since the relationship among z, z_0 and z' is given, by putting $z_0^*=z^*=z'$ in (3·8), as

$$z^m=\xi z'^m+(1-\xi)z_0^m, \tag{3·15}$$

it follows from this relation and (3·8) that

$$\left(\frac{z_0^*}{z^*}\right)^{a/4}=\left[\frac{1}{1-\xi}\left\{1-\xi\left(\frac{z'}{z^*}\right)^m\right\}\right]^{a/4m}. \tag{3·16}$$

Hence the exponential factor in (2·26) is expressed, by using (3·14) and (3·16), as

$$\exp[\cdots]=\exp\left[\frac{2\gamma-1}{4\gamma}g\int_z^{z'}\left[(1-\zeta)(1-\xi)^{-a/4m}\left\{1-\xi\left(\frac{z'}{z^*}\right)^m\right\}^{a/4m}+\zeta\right]^{-1}\frac{dz^*}{z^*}\right] \tag{3·17}$$

where the lower end of integration is given by (3·15). The other factors in the integrand of (2·26) are expressed, by using (2·2), (2·11) and (3·15), as

$$\frac{\partial}{\partial z_0}\ln\rho(z, z_0)=-\frac{a}{z_0}\left[1+\frac{\zeta}{1-\zeta}\left\{\xi\left(\frac{z'}{z_0}\right)^m+1-\xi\right\}^{a/4m}\right]^{-1}, \tag{3·18}$$

$$\frac{\partial}{\partial z'}\ln z(z_0, z')=\frac{\xi}{z'}\left(\frac{z'}{z_0}\right)^m\left\{\xi\left(\frac{z'}{z_0}\right)^m+1-\xi\right\}^{-1} \tag{3·19}$$

and

$$\left\{\frac{\rho(z, z_0, z')}{\rho(z, z)}\right\}^{1/4}=\left\{\frac{\rho(z', z')}{\rho(z, z)}\right\}^{1/4}=\left(\frac{z_0}{z'}\right)^{a/4m}\left\{\xi\left(\frac{z'}{z_0}\right)^m+1-\xi\right\}^{a/4m}. \tag{3·20}$$

Substituting (3·17)∼(3·20) into (2·26), we finally obtain

$$\ln\Phi(z')=-\frac{\xi(1-\xi)}{1+\xi}ag^2\int_{z_1}^{z'}\frac{dz'}{z'}\int_{z_1}^{z'}\frac{dz_0}{z_0}\left[1+\frac{\zeta}{1-\zeta}\left\{\xi\left(\frac{z'}{z_0}\right)^m+1-\xi\right\}^{a/4m}\right]^{-1}$$
$$\times\left[\left(\frac{z'}{z_0}\right)^{-1}\left\{\xi\left(\frac{z'}{z_0}\right)^m+1-\xi\right\}^{1/m}\right]^{a/4-m}$$
$$\times\exp\left[\frac{2\gamma-1}{4\gamma}g\int_z^{z'}\frac{dz^*}{z^*}\left[(1-\zeta)(1-\xi)^{-a/4m}\left\{1-\xi\left(\frac{z'}{z^*}\right)^m\right\}^{a/4m}+\zeta\right]\right]. \tag{3·21}$$

3.3 Numerical estimation of the correction factor

Here we shall carry out somewhat rough estimation of the correction factor (3·21).

Performing the transformations

$$(z'/z^*)^m=u,\quad (z'/z_0)^m=v \text{ and } (z'/z_1)^m=w, \tag{3·22}$$

we can rewrite (2·21)

$$\ln\Phi(z')=-P\int_{(z'/z_1)^m}^{1}\frac{dw}{w}\int_{w}^{1}\frac{dv}{v}H(v)\{G(v)\}^{\alpha+1}\exp\left[Q\int_{G(v)}^{1}\frac{du}{u}F(u)\right], \tag{3·23}$$

where

$$F(u)=[(1-\zeta)(1-\xi)^{\alpha}(1-\xi u)^{-\alpha}+\zeta]^{-1}, \tag{3·24}$$

$$G(v)=v(\xi v+1-\xi)^{-1}, \tag{3·25}$$

$$H(v)=[1+\zeta(1-\zeta)^{-1}(\xi v+1-\xi)^{-\alpha}]^{-1}, \tag{3·26}$$

and

$$P=\frac{16\xi(1-\xi)}{1+\xi}\gamma^2\zeta^2\alpha^2 a\ (=0.576\alpha^2 a \text{ for } \gamma=5/3), \tag{3·27}$$

$$Q=(2\gamma-1)\zeta\alpha\ (=0.741\alpha \text{ for } \gamma=5/3), \tag{3·28}$$

$$\alpha=-a/4m. \tag{3·29}$$

Assuming that $a>0$ and $n>0$, we have $m<0$ both in (3·10) and (3·13), so it is found that $0<u, v, w\leq 1$.

By using m given by (3·13), α is found to be expressed as a function of γ alone, and in particular $\alpha=1.06$ for $\gamma=5/3$. On the other hand, by using m given by (3·10), it is found that $0<\alpha<1.06$ for $\gamma=5/3$, the upper and the lower limit for α corresponding to $n=\infty$ and $n=0$ respectively. Taking these facts into account, we shall examine the behavior of the functions $F(u)$, $G(v)$ and $H(v)$.

First, $F(u)$ is found to be a slowly decreasing function of u, which takes values between the minimum $F(1)=1$, and the maximum which does not exceed $F(0)=[(1-\zeta)(1-\xi)^{\alpha}+\zeta]^{-1}$. Therefore we may approximate $F(u)$ by a constant $\bar{F}$, which lies between $F(0)$ and $F(1)=1$.

Since it is also found that $H(v)$ is a slowly increasing function of v, we may substitute a constant $\bar{H}$ for $H(v)$, where $H(0)\leq\bar{H}\leq H(1)=1-\zeta$. By using these simplifications (3·23) is reduced to

$$\ln\Phi(z')=-P\bar{H}\int_{(z'/z_1)^m}^{1}\frac{dw}{w}\int_{w}^{1}\frac{dv}{v}\left(\frac{v}{\xi v+1-\xi}\right)^{\mu}, \tag{3·30}$$

where

$$\mu=\alpha+1-Q\bar{F}. \tag{3·31}$$

As we are mainly interested in the asymptotic formula for the change in the shock strength, we shall estimate the value of (3·30) in the case of $z'\gg z_1$. In this case, we may approximately put $w=0$ for the lower end of integration with respect to v. Then we have

$$\Phi(z')\cong(z'/z_1)^{P\bar{H}mC}, \tag{3·32}$$

where

$$C=\int_{0}^{1}\frac{dv}{v}\left(\frac{v}{\xi v+1-\xi}\right)^{\mu}, \tag{3·33}$$

and hence it follows that

$$\rho(z')/\rho(z_1)=(z_1/z')^{\alpha-P\bar{H}mC}. \tag{3·34}$$

For $\gamma=5/3$ and $k=5/3$, it gives

$$a=0.878,$$

and by using m in (3·10)

$$-Pm=\begin{cases}0.0224 & (n=1),\\ 0.0377 & (n=2),\\ 0.118 & (n=\infty),\end{cases}$$

where $n=\infty$ in (3·10) gives the same value of m in (3·13).

For $\gamma=5/3$ and $k=4/3$ we have similarly

$$a=1.242,$$

and

$$-Pm=\begin{cases}0.0449 & (n=1),\\ 0.0756 & (n=2),\\ 0.236 & (n=\infty).\end{cases}$$

Considering that both $\bar{H}$ and C take the values near unity, it may be seen that the magnitude of the correction term $P\bar{H}mC$ is at most 20 percent of the first order term a. Any density distribution other than those given by (3·6) or (3·11) may not give a very different result from that given above, provided it is a smoothly decreasing function of x.

§ 4. Conclusions

From the calculations made above the following conclusions may be derived.

The growth of the shock strength, which occurs as the shock propagates from the region of high density and high pressure to the region of low density and low pressure, is restrained by the effect of the second-order waves. But the correction to the first-order approximation formula is found to be relatively small as far as the case of the strong shock limit is concerned. Therefore, we may conclude that the first order approximation formula may well be used in the problems of the propagation of strong shock waves. For example, in the case of supernovae explosion in which one may expect that shock waves generated at the deep interior of a star propagate toward its surface to get very large strength, we can apply the first-order formula to the phenomena, especially to those near the stellar surface, with fairly good approximation.

Although we have not treated in this paper the case in which the shock strength is not much larger than unity it may be expected that the correction due to the second-order wave is significant in this case because the strength of the singly reflected wave element, $1-g(dz/z)$, increases as z decreases so that the strength of the second-order wave is also expected to increase. Therefore the second-order effect may become important especially when the shock strength decreases in the course of its shock propagation.

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Stability of Axially Asymmetric Nuclear Deformation

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The γ-stability of axially asymmetric nuclear deformation is investigated on the basis of the generalized collective model. It is possible in the vibrational region to make the non-axial deformation stable for the ground state, but it is necessary in this formalism to introduce a very strong potential to make the axially asymmetric shape stable in the rotational region.

§ 1. Introduction

Recently Davydov and his collaboraters[1),2),3)] have shown that it is possible to explain consistently the properties of low-lying states of even-even nuclei over the vibrational and the rotational region. Their fundamental assumption is that the nucleus has rigid axially asymmetric deformation during its rotational and β-vibrational motion.

The purpose of the present paper is to investigate the rigidity or stability of the asymmetric nuclear deformation. For this purpose we proceed from the generalized collective model in which the γ-potential is introduced to stabilize the nucleus to the axially asymmetric shape. Assuming the β-γ correlation and the rotation-vibration coupling to be small, it can be seen whether the region, in which the wave function of γ-vibrational motion has an appreciable value, is narrow or not for each γ-potential. When the mean value of γ is not zero and this region is narrow, we can assert that the nucleus has the stable axially asymmetric deformation. In this manner the condition for the γ-potential can be derived, under which the axially asymmetric nuclear deformation may be stable. These discussions may shed some light on the γ-stability of the nucleus.

§ 2. Formalism and method of analysis

The Schroedinger equation for the γ-vibration in adiabatic approximation is

$$\left[-\frac{\hbar^2}{2B_\gamma}\frac{1}{\sin 3\gamma}\frac{d}{d\gamma}\left(\sin 3\gamma\frac{d}{d\gamma}\right)+V(\gamma)\right]G(\gamma)=E_\gamma G(\gamma), \qquad (1)$$

where B_γ is the mass parameter which determines the kinetic energy of the γ-vibration. The function of $\cos 3\gamma$ is to be used for the γ-potential $V(\gamma)$ in order to preserve the invariance of the total Hamiltonian under the transfor-

mation $\gamma \to \gamma + 2\pi/3$. The variation interval of γ can be limited to $0 \leqq \gamma \leqq \pi/3$ by this invariancy. Furthermore, it is also required that $V(\gamma)$ has a minimum value at $\gamma = \gamma_0$ in order to stabilize the nucleus at the axially asymmetric deformation corresponding to the intrinsic non-axiality γ_0. Then without losing generality, we can put

$$V(\gamma) = V(\cos 3\gamma) = \frac{C_\gamma}{2} v(\cos 3\gamma\,;\ \cos 3\gamma_0) \tag{2}$$

where C_γ is the rigidity parameter.

In solving Eq. (1), we take into account the first order differential term in the kinetic energy operator exactly. To do so, it is convenient to put $\cos 3\gamma = z$ and carry out the following transformation for the wave function $G(z)$:

$$G(z) = (1-z^2)^{-1/2} g(z), \quad -1 \leqq z \leqq 1. \tag{3}$$

Then

$$[-d^2/dz^2 + \mathcal{V}(z\,;\, z_0,\, d,\, \varepsilon)]\, g(z) = 0 \tag{4}$$*

and $g(\pm 1) = 0$, where

$$\mathcal{V}(z\,;\, z_0,\, d,\, \varepsilon) = -\frac{1}{(1-z^2)^2} + d\frac{v(z,\, z_0)}{1-z^2} - \frac{2\varepsilon}{1-z^2}, \tag{5}$$

$d = C_\gamma/(9\hbar^2/B_\gamma)$ and $\varepsilon = E_\gamma/(9\hbar^2/B_\gamma)$. Hereafter, subscript γ of B_γ, C_γ and E_γ is suppressed, unless it is necessary, and we call $\mathcal{V}(z)$ the fictitious potential.

For simplicity, we adopt the following form for $v(z, z_0)$:

$$v(z,\, z_0) = (z - z_0)^2 = (\cos 3\gamma - \cos 3\gamma_0)^2. \tag{6}$$

$\mathcal{V}(z)$ is then rewritten as

$$\mathcal{V}(z\,;\, z_0,\, d,\, \varepsilon) = \frac{A^+}{1+z} + \frac{A^-}{1-z} - \frac{1}{4(1+z)^2} - \frac{1}{4(1-z)^2} - d, \tag{7}$$

where

$$A^\pm = \frac{d}{2}(1 \pm z_0)^2 - \varepsilon - \frac{1}{4}. \tag{8}$$

For the moment we restrict ourselves to the case of the maximum non-axiality, $\gamma_0 = 30°$ (i.e. $z_0 = 0$), which corresponds to $A^+ = A^- \equiv A$. For $A > 3/4$, $\mathcal{V}(z)$ has minimum and maximum values at $z = 0$ and $z = \pm a$ respectively, where

* With the use of Eq. (7), Eq. (4) is regarded as the Schroedinger equation which has an eigenvalue d, but d is actually a certain constant given by the physical quantities, B and C, of the system. The eigenvalue equation derived from Eq. (4) gives a relation between the energy eigenvalue E and the physical constants of the system. Such a procedure was used in the relativistic two-body problem.

$$a^2=\frac{4A-3}{4A+1}. \tag{9}$$

For convenience of computation $\mathcal{V}(z)$ is approximated as follows: i) The part of $\mathcal{V}(z)$ with $|z|\leqq a$ is replaced by a square well, and ii) in the part with $a\leqq|z|\leqq 1$, $\mathcal{V}(z)$ is replaced by its main term, i.e. by $-1/4(1\mp z)^2$. Fig. 1 shows this approximated fictitious potential which we denote as $U(z)$. In Fig. 1, U_0 and U_∞ are defined respectively as

$$U_0=\mathcal{V}(0)=\frac{1+a^2}{1-a^2} \tag{10}$$

and

$$U_\infty=\mathcal{V}(a)+\frac{1}{4(1-a)^2}=\frac{5+2a+a^2}{4(1-a^2)^2}. \tag{11}$$

$U(z)$ is then represented by only one parameter a. The potential of the original system is a harmonic oscillator type, $v(z)=z^2$, so that $\varepsilon>0$. Furthermore it may be convenient to take into account the condition for $\mathcal{V}(z)$, $\mathcal{V}_{max}=\mathcal{V}(a)>d$, because the state for a smaller value of d is more stable. Then, the allowed range of the parameter a is limited to $a_{min}\leqq a\leqq a_{max}$, where a^2_{min} and a^2_{max} are expressed as $1-(1/d)^{1/2}$ and $1-(2/d)$ respectively from Eqs. (7) and (8) with $z_0=0$.

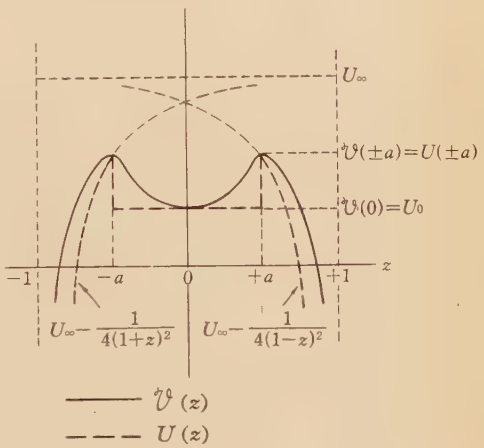

Fig. 1. Schematic figure of the potentials $\mathcal{V}(z)$ and $U(z)$ for the case of $z_0=0$ ($\gamma_0=30°$).

The solutions for the approximated potential $U(z)$ can be written as $\sin((d-U_0)^{1/2}+m\pi/2)$ in the inner region, $[(U_\infty-d)^{1/2}(1\mp z)]^{1/2}I_0((U_\infty-d)^{1/2}(1\mp z))$ in the outer region respectively, except the normalization factor; where $I_n(x)=i^{-n}J_n(ix)$. Putting $\xi=(d-U_0)^{1/2}a$ and $\eta=(U_\infty-d)^{1/2}(1-a)$, these are the function of a and d from Eqs. (10) and (11). Continuity conditions at $z=\pm a$ give

$$\frac{a}{1-a}\left[\frac{1}{2}+\eta\frac{I_1(\eta)}{I_0(\eta)}\right]=\begin{cases}\xi\tan\xi & \text{for } m=\text{odd}\\ -\xi\cot\xi & \text{for } m=\text{even.}\end{cases} \tag{12}$$

It is easily shown that both sides of Eq. (12) are monotonic functions of a, parametrized by d, in the allowed interval $a_{min}\leqq a\leqq a_{max}$. By regarding the left-hand side of Eq. (12) as a function of ξ parametrized by d, the graphical solution of eigenvalue equation (12) is shown in Fig. 2. Corresponding roots ξ, η and a are tabulated in Table I for a set of values of the parameter d.

Energy eigenvalue ε_ν of the γ-vibration for a given d-value is written in the form

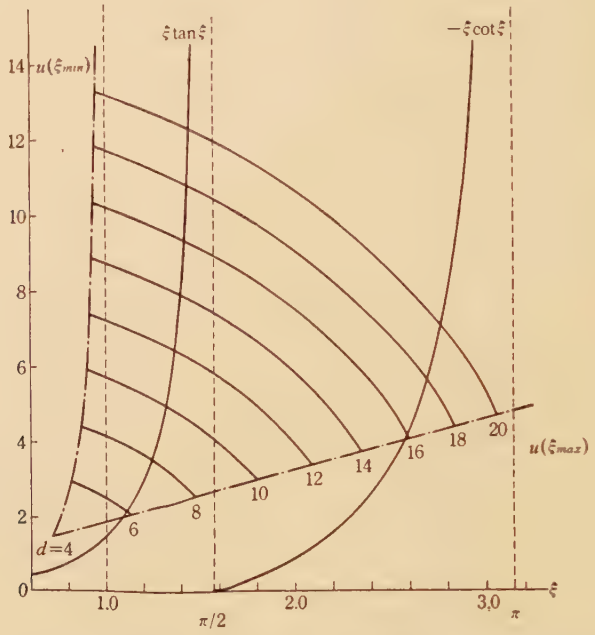

Fig. 2. Graphical solutions of eigenvalue equation (12) for each d-value. $u(\xi_{max})$ and $u(\xi_{min})$ correspond to a_{min} and a_{max} respectively. Crossing points give the ξ as roots of Eq. (12) and corresponding to them η and a are determined.

Table I.

d	ξ_0	η_0	a_0	ε_0	$\langle z^2\rangle_0$
6	1.10	.513	.775	.499	.253
8	1.24	.586	.841	.577	.230
10	1.32	.621	.879	.620	.216
12	1.36	.641	.902	.638	.210
14	1.39	.654	.918	.659	.207
16	1.41	.663	.930	.642	.207
18	1.43	.669	.938	.646	.204
20	1.44	.674	.945	.663	.201
d	ξ_1	η_1	a_1	ε_1	$\langle z^2\rangle_1$
16	2.59	.510	.869	3.93	.431
18	2.69	.571	.894	4.03	.417
20	2.76	.609	.913	4.00	.410

$$\varepsilon_\nu=\frac{d}{2}-\frac{1}{1-a_\nu^2} \tag{13}$$

by using Eqs. (8) and (9), where a_ν are roots of Eq. (12) and subscript $\nu(=0, 1, \cdots)$ specifies different eigenstates.

§ 3. Results and discussions

With the use of Eq. (13), $0.50\lesssim\varepsilon_0\lesssim0.67$ for $6\lesssim d\lesssim20$ and $\varepsilon_1\sim3.9$ for $16\lesssim d\lesssim20$. The γ-vibrating system has the ground state, which satisfies the requirement given below Eq. (11), for $d\gtrsim6$ and the first excited state may exist for $d\gtrsim16$. The γ-stability of these states can be examined by computing $\langle(z-z_0)^2\rangle$. In the case of $z_0=0$ $(\gamma_0=30°)$,

$$\langle z^2\rangle_\nu=\int_0^{\pi/3}\cos^2 3\gamma\,|G_\nu(\gamma)|^2|\sin 3\gamma|\,d\gamma. \tag{14}$$

The d-dependence of $\langle z^2\rangle_\nu$ is indicated in Table I and it is seen that $0.25\gtrsim\langle z^2\rangle_0\gtrsim0.20$ for $6\lesssim d\lesssim20$ and $\langle z^2\rangle_1\sim0.42$ for $16\lesssim d\lesssim20$. If we put $\langle z^2\rangle=\cos^2 3\bar{\gamma}$ and $\Delta\gamma=\gamma_0-\bar{\gamma}$, then $10°\gtrsim\Delta\gamma\gtrsim9°$ for the ground states. Taking account of the approximation for $\mathcal{V}(z)$ we can expect that $\Delta\gamma$ is actually smaller than the above values and γ-stability may be further increased, because the original $\mathcal{V}(z)$ with a depressed part makes the wave function decrease for $|z|\to a$ more rapidly than $U(z)$. Fig. 3 shows $|G_0(\gamma)|^2$ determined from $U(z)$ with respect to $d=6$, which decreases rapidly for $|z|\to1$, comparing with $|G_0(\gamma)|^2$ $(=\text{const.})$ for $\mathcal{V}=0$. For the first excited states $\Delta\gamma\cong13°$, therefore the stability of these states is somewhat poorer than the ground states.

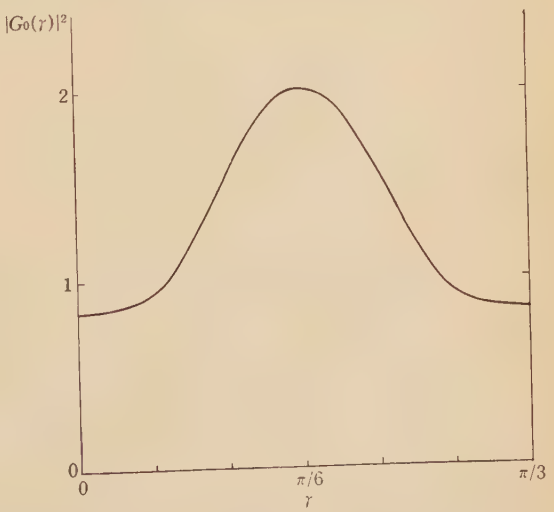

Fig. 3. $|G_0(\gamma)|^2$ calculated from $U(z)$ and $d=6$. $G_0(\gamma)$ is normalized to $\int_0^{\pi/3}|G_0(\gamma)|^2|\sin 3\gamma|\,d\gamma=1$. $d=6$ corresponds to $a_0=0.775$ as is seen from Table I.

We have so far investigated the behaviour of γ-vibration and the condition of γ-stability in the case of $\gamma_0=30°$ in detail. In order to proceed to the case of $\gamma_0<30°$, we must repeat the same self-consistent procedure as the case of $\gamma_0=30°$. This procedure, however, is troublesome, so we make the following qualitative consideration. When the wave function $G(\gamma)$, which is finite at

boundaries, gives the γ-stable state, the function $g(z)=G(\gamma)\sin 3\gamma$ is more localized in the depressed part of $\mathcal{V}(z)$. This means that the depth of the depressed part of the fictitious potential is larger than the kinetic energy of the state. Therefore we can estimate minimum d-values giving the γ-stable state for each γ_0 by making use of the relation $\overline{\mathcal{V}} > 1/\bar{z}^2$, where $\overline{\mathcal{V}}$ and $\bar{z}$ are respectively the depth and the width of the depressed part of $\mathcal{V}(z)$ which is located around $z \approx z_0$. In the case of $\gamma_0=30°$, this relation leads to the condition, $d-2\varepsilon \gtrsim 4.65$. This is consistent with the results obtained by the previous method, that is, $d_{min}\sim 6$ and $\varepsilon_0 \sim 0.5$. Fig. 4 shows the γ_0-dependence for the minimum d-value satisfying the above relations.

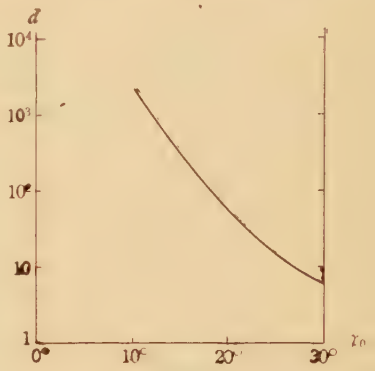

Fig. 4. The qualitative behaviour for γ_0-dependence of minimum d-value which corresponds to the γ-stable ground state. The curve corresponds to the case of $v(z, z_0)=(z-z_0)^2$ and $\varepsilon \sim 1$.

In the previous note[4] we gave the systematics of γ_0 for medium and heavy even-even nuclei and it was pointed out that the vibrational and the rotational regions correspond to $\gamma_0 \sim 26°$ and $\gamma_0 \sim 10°$ respectively. From Fig. 4, $d \sim 10$ for $\gamma_0 \sim 26°$ and $d \sim 2000$ for $\gamma_0 \sim 10°$. Putting $B_\gamma = B_l \beta_0^2$ and $\hbar^2/B_l\beta_0^2 = p(p+1)^{-1} \times E_1(2^+)$,[1] where $p=E_2(2^+)/E_1(2^+)$, the value of C_γ for these d-values are about 30 Mev for $\gamma_0 \sim 26°$ and 850 Mev for $\gamma_0 \sim 10°$, when $p \sim 2.2$, 16 and $E_1(2^+) \sim 0.5$, 0.05 Mev respectively. It seems that the value of C_γ is reliable in the vibrational region. Its value for the rotational region is much larger than Bohr-Mottelson's one. It is necessary to note that our treatment is based on the periodic boundary condition and our purpose is to investigate the condition by which the wave function is well localized in the vicinity of γ_0, while the approximation of small vibration is assumed from the beginning in the Bohr-Mottelson model.[5]

In conclusion, it is possible in the vibrational region to make the nuclear shape γ-stable by introducing the reliable collective γ-potential. On the other hand, it is required to introduce a γ-potential with very large C_γ in order to make the nuclear shape γ-stable in the rotational region. This does not always mean, however, that the nucleus is fairly γ-unstable for small γ_0, when C_γ is not so large. There may be another origin which makes the nucleus γ-stable in the rotational region.

The present work was carried out as a part of the "Annual Research Project for the Nuclear Force and Nuclear Structure" organized by the Research Institute for Fundamental Physics in 1960. The authors would like to express their sincere thanks to Prof. S. Takagi, Dr. R. Tamagaki and Dr. Y. Shono and other members of our project group.

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Letters to the Editor

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Nambu Model for Elementary Particles and the Composite Model for Pion

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April 12, 1961

Recently Nambu[1] has proposed a dynamical model of elementary particles based on an analogy with the theory of superconductivity.[2),3)] The essential points of the Nambu model are as follows: 1) The nucleon mass is created due to some attractive interaction between massless bare fermions. 2) The pion is caused as secondary product from requirement of the γ_5-gauge invariance of the theory. However, the mass of such a pion must necessarily be zero. Then we show here one example of the Nambu model with a finite pion mass in which γ_5-invariance of the theory is violated. That is, the nucleon mass is created on the basis of the Nambu model and the pion mass is treated on the composite model for pion.[4)]

At first we choose the following Hamiltonian in order to create the nucleon mass and make the bound state of nucleon pairs,

$$H=\bar{\psi}\gamma_i\partial_i\psi+g_0[(\bar{\psi}\gamma_5\psi)^2+(\bar{\psi}\gamma_5\tau_i\psi)^2] \tag{1}$$

where ψ represents the massless bare fermion field operator and g_0 is the bare coupling constant which has dimensions $(\text{mass})^{-2}$. Next we use the method of the so-called "dangerous graphs"[3),5)] to derive the Hamiltonian of nucleon system from (1). We describe (1) by means of second quantized operators and carry out the following transformation analogous to Bogoliubov's one in the superconductivity theory on these bare fermion operators,

$$\begin{aligned}
&\alpha_p^\rho=u_p a_p^\rho+v_p b_{-p}^{\rho *},\\
&\alpha_p^{\rho *}=u_p a_p^{\rho *}+v_p b_{-p}^\rho,\\
&\beta_{-p}^\rho=u_p b_{-p}^\rho-v_p a_p^{\rho *},\\
&\beta_{-p}^{\rho *}=u_p b_{-p}^{\rho *}-v_p a_p^\rho,
\end{aligned} \tag{2}$$

$$u_p^2+v_p^2=1 \tag{3}$$

where α_p^* and β_p^* (α_p and β_p) are creation (annihilation) operators for massless fermion and antifermion respectively and a_p^* and b_p^* (a_p and b_p) creation (annihilation) operators for nucleon and antinucleon respectively. u_p and v_p are the coefficients of transformation and ρ is helicity. This transformation conserves the ordinary gauge invariance but violates γ_5-gauge invariance. The "dangerous terms"

corresponding to the creation and annihilation of nucleon pair is contained in the transformed Hamiltonian and they violate the stability of the vacuum. To remove this difficulty, we choose the coefficient of all terms leading from the vacuum to the nucleon pair state and *vice versa* to compensate one another. So we get

$$2|\boldsymbol{p}|u_p v_p=(u_p^2-v_p^2)\frac{4g_0}{(2\pi)^3}\int d\boldsymbol{p}_1 u_{p_1}v_{p_1}. \tag{4}$$

It is clear that (4) contains a trivial solution $u_p v_p=0$ corresponding to the massless fermion. In order to find the non-trivial solution, we put

$$m=\frac{4g_0}{(2\pi)^3}\int d\boldsymbol{p}_1 u_{p_1}v_{p_1}. \tag{5}$$

Then, making use of (3), we obtain

$$u_p^2=\frac{1}{2}\left(1+\frac{|\boldsymbol{p}|}{\sqrt{\boldsymbol{p}^2+m^2}}\right),$$
$$v_p^2=\frac{1}{2}\left(1-\frac{|\boldsymbol{p}|}{\sqrt{\boldsymbol{p}^2+m^2}}\right), \tag{6}$$

$$u_p v_p=\frac{1}{2}\frac{m}{\sqrt{\boldsymbol{p}^2+m^2}} \tag{7}$$

from (4) and (5). Substituting (7) in (5), we have an equation for constant m:

$$1=\frac{2g_0}{(2\pi)^3}\int d\boldsymbol{p}\frac{1}{\sqrt{\boldsymbol{p}^2+m^2}}. \tag{8}$$

Thus by using (4), (5), (6) and (7), the total Hamiltonian becomes

$$H=H_0+H_I \tag{9}$$

where

$$H_0=\sum_\rho\int d\boldsymbol{p}\sqrt{\boldsymbol{p}^2+m^2}(a_p^{\rho*}a_p^\rho+b_{-p}^{\rho*}b_{-p}^\rho)$$
$$-4\int d\boldsymbol{p}(\sqrt{\boldsymbol{p}^2+m^2}-|\boldsymbol{p}|) \tag{10}$$

and H_I is the effective interaction Hamiltonian to make the bound state of the nucleon pair. m is the nucleon mass which is determined by (8) and arises as the self-energy of the massless fermion in the lowest order. Further, the difference of vacuum energy level of bare fermion system and nucleon system is $-4\int d\boldsymbol{p}(\sqrt{\boldsymbol{p}^2+m^2}-|\boldsymbol{p}|)$.

Now we may consider the eigenvalue problem for the nucleon pair starting from (9). The eigenvalue equation for the one nucleon pair excited state is obtained as derived in reference 4).

$$1=\frac{8g_0}{(2\pi)^3}\int d\boldsymbol{p}(u_{p+k/2}u_{p-k/2}$$
$$+v_{p+k/2}v_{p-k/2})^2\left(\frac{1}{E_{p+k/2}+E_{p-k/2}-\omega_k}\right.$$
$$\left.+\frac{1}{E_{p+k/2}+E_{p-k/2}+\omega_k}\right). \tag{11}$$

This equation has two types of solutions. One of them is

$$\omega_k=E_{p+k/2}+E_{p-k/2} \tag{12}$$

which corresponds to a free nucleon pair and the other is

$$\omega_k\sim\sqrt{\boldsymbol{k}^2+\mu^2} \tag{13}$$

which corresponds to a bound state. If we take $g_0M^2\sim 6.6$ and cut the integral in (8) and in (11) at $2M$ and M (M is the nucleon mass) respectively, we get $m=M$ and adjust μ to the real pion mass. Furthermore, it is needless to say that we easily have an effective Hamiltonian for the pion-nucleon system if we use the treatment of § 3 in reference 4).

Thus it seems to us that the Nambu model can be related to the actual pion-nucleon phenomena through the previous treatment of the composite model for pion, while the latter is given the firm foundation by the former.

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Another Possible Model of Quantum Field Theory with Indefinite Metric

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April 28, 1961

In the previous paper* we proposed a kind of convergent model of the unitary S-matrix to the case where a physical neutral scalar field $A(x)$ (mass μ) and a physical spinor field $\psi(x)$ (mass κ) interact with each other indirectly by way of four kinds of unphysical spinor fields, $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_1'(x)$ and $\varphi_2'(x)$. By using the total Lagrangian given in I, it was shown that the contents of the theory is such that every vertex in the usual theory is exactly replaced with some kind of extended vertex without any alteration of the other results.

On the other hand, however, it was also shown that there exists another possible model which is equivalent to the case of the extended vertex (38) in the last section in I, starting from the following total Lagrangian:

$$\begin{aligned}
&\mathcal{L}=\mathcal{L}_0+\mathcal{L}', \quad \mathcal{L}'=\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3,\\
&\mathcal{L}_1=f(\bar{\varphi}_1+\bar{\varphi}_1'+\bar{\varphi}_2+\bar{\varphi}_2')(\varphi_1+\varphi_1'\\
&\qquad +\varphi_2+\varphi_2')A,\\
&\mathcal{L}_2=g\bar{\psi}(\varphi_1-\varphi_1')+g\bar{\psi}(\varphi_2-\varphi_2')\\
&\qquad +\text{adjoint conj.},\\
&\mathcal{L}_3=g'(\bar{\varphi}_1\varphi_2-\bar{\varphi}_1'\varphi_2)+\text{adjoint conj.},
\end{aligned} \tag{1}$$

where $\mathcal{L}_0$ is taken as that used for the case of $\Gamma(k)$ in (38) in I, that is, it is taken so as to get the negative anticommutators for φ_1 and φ_2'.

Although the total Lagrangian $\mathcal{L}$ in (1) gives an equivalent extended vertex to that of (38) in I for the processes described by the actual S-matrix ($\widetilde{S}$) and also gives a guarantee for the unitarity of the $\widetilde{S}$ and the convergency of all results, the internal expressions of the theory naturally differ from those given in I.

The aim of this letter is to report these results briefly. For convenience, we follow the same notations as those of I.

First, for the case of the interaction $\mathcal{L}_3$ only, solving the same equations as (37) in I, we get

$$F_{11}(k)=-\frac{g'^2(i\gamma k+M)}{(i\gamma k+M)(i\gamma k+m)+g'^2},$$

* K. Yokoyama, Prog. Theor. Phys. **26** (1961), 131, quoted as I in this letter.

$$F_{22}(k)=+\frac{g'^2(i\gamma k+m)}{(i\gamma k+M)(i\gamma k+m)+g'^2}, \tag{2}$$

$$F_{12}(k)=F_{21}(k)$$
$$=-\frac{g'(i\gamma k+M)(i\gamma k+m)}{(i\gamma k+M)(i\gamma k+m)+g'^2}.$$

Similarly to I, for the case of the interactions $\mathcal{L}_2$ and $\mathcal{L}_3$, we at once obtain

$$G(k)=0,$$
$$G_1(k)=-g-gS_m(k)F_{21}(k)$$
$$+gS_M(k)F_{11}(k)=-g(i\gamma k+M)$$
$$\times\frac{(i\gamma k+m+g')}{(i\gamma k+M)(i\gamma k+m)+g'^2}, \tag{3}*$$
$$G_2(k)=-g+gS_M(k)F_{12}(k)$$
$$-gS_m(k)F_{22}(k)=-g(i\gamma k+m)$$
$$\times\frac{(i\gamma k+M-g')}{(i\gamma k+M)(i\gamma k+m)+g'^2}.$$

Thus, for the case of the total interaction Lagrangian $\mathcal{L}'$, we obtain the following results through the same procedures as used in I:

$$F_1(k)=1-S_M(k)F_{11}(k)$$
$$+S_m(k)F_{21}(k)=(i\gamma k+M)$$
$$\times\frac{(i\gamma k+m+g')}{(i\gamma k+M)(i\gamma k+m)+g'^2}, \tag{4}$$
$$F_2(k)=1+S_m(k)F_{22}(k)$$
$$-S_M(k)F_{12}(k)=(i\gamma k+m)$$
$$\times\frac{(i\gamma k+M-g')}{(i\gamma k+M)(i\gamma k+m)+g'^2},$$

and for $\Gamma(k)$,

$$\Gamma(k)=2gS_M(k)F_1(k)-2gS_m(k)F_2(k)$$
$$=-2S_M(k)G_1(k)+2gS_m(k)G_2(k)$$
$$=2g\frac{2g'-M+m}{(i\gamma k+M)(i\gamma k+m)+g'^2}. \tag{5}$$

The form of the above $\Gamma(k)$ is equivalent to that of (38) in I apart from their numerical factors. The local limit is achieved by taking

$$M \text{ and } m\to\infty,$$

keeping the relation

$$\left|2g\ \frac{2g'-M+m}{Mm+g'^2}\right|=1,$$

which has always real solutions for g and g'.

As mentioned in detail in I, however, the singular points of $\Gamma(k)$ in (5), which are the roots of the equation

$$(\lambda-M)(\lambda-m)+g'^2=0,$$

do not always become real in this case. Thus, if we want to obtain the equivalent results to the case of the extended vertex given by (22) in I, it is evident that the further condition

$$|M-m|>|g'|$$

is required.

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* $S_m(k)\equiv-\frac{1}{i\gamma k+m}$ and $S_M(k)\equiv-\frac{1}{i\gamma k+M}$.

Test of the Intermediary Boson Theory

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May 4, 1961

Tanikawa[1] once proposed a theory of universal Fermi interactions in terms of the intermediary action of bosons with baryon number. Recently, Kinoshita[2] suggested a test experiment of the Tanikawa theory by observing the possible resonance effect which should be produced in the scattering of the high energy neutrinos. We now point out that there still remain simple effects which are not yet measured, which should as well be expected from the Tanikawa theory. These are the energy spectra of the electrons emitted in the β-decay, Λ-decay and Σ-decay.

1. *The β-decay*

We begin with calculating the spectrum, $W(E)$, of the electron in β-decay. Tanikawa and Saeki[3] determined the lower limit, $m_B \approx 2300\, m_e$, for the mass of the B-meson in order to prove the effect to be negligible. Since the precise measurements of the spectrum shape will, however, be feasible in the near future, it may be worthwhile to estimate the nonlocal effect in a greater detail. In order to derive the V-A interaction, we are left with one possibility[4] for the B-meson with spin 0:

$$H=f_\beta\{\bar{p}(1-\gamma_5)e^c + \bar{n}(1-\gamma_5)\nu^c\}B_\beta^{(0)}+h.c. \qquad (1)$$

From the Hamiltonian (1) we can easily obtain the correction factor $C_\mp(E)$, which is to be multiplied to the spectrum of electron $(-)$ or positron $(+)$.

$$C_-(E) \sim \frac{1}{(1+\alpha_\beta(E_0-E))^2}, \qquad (2)$$

$$C_+(E) \sim \frac{1}{(1+\alpha_\beta E)^2}, \qquad (3)$$

$$\alpha_\beta=\frac{2m_N}{m_{B_\beta}^2-m_N^2}.$$

Here E denotes the energy of electron or positron, and E_0 is its maximum value. The amount of deviation, α_β, depends on the mass, m_B, of B-meson. For the β-decay of $N^{12}\to C^{12}$ ($E_0=17.7$ Mev) and $B^{12}\to C^{12}$ ($E_0=13.4$ Mev),

$$\left.\begin{aligned}\alpha_\beta E_0 &\approx 0.045 \quad m_{B_\beta}=1150 \text{ Mev}\\ &\approx 0.021 \quad m_{B_\beta}=1500 \text{ Mev}\\ &\approx 0.009 \quad m_{B_\beta}=2000 \text{ Mev.}\end{aligned}\right\} \qquad (4)$$

It must be noted that the same order of correction factors could be deduced also from the theory of the vector current[5] or from the contributions of the second forbidden transition.[6]

2. *The β-decay of Λ-hyperon*

We now turn to the effect on the

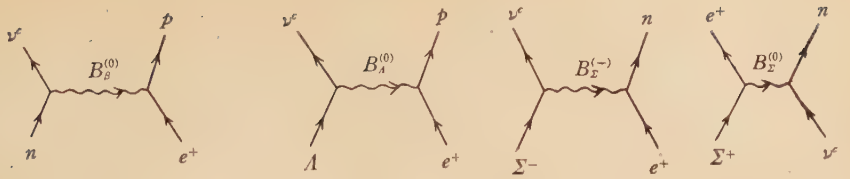

Fig. 1.

Λ-decay in the leptonic mode. By taking an analogy to the β-decay, we assume the following Hamiltonian:

$$H=f_\Lambda\{\bar{p}(1-\gamma_5)e^e + \bar{\Lambda}(1-\gamma_5)\nu^e\}B_\Lambda^{(0)}+h.c. \quad (5)$$

The energy spectrum, $W_\Lambda(E)$, of the electron from the Λ-decay will be given as follows:

$$W_\Lambda(E)\propto\frac{2+\alpha_\Lambda E_0}{\alpha_\Lambda}\ln\left[1+\frac{2\alpha_\Lambda E(E_0-E)}{(m_\Lambda-2E)\{1+\alpha_\Lambda(E_0-E)\}}\right] - \frac{2E(E_0-E)}{m_\Lambda-2E}\left[1+\frac{1+\alpha_\Lambda W}{\left\{1+\alpha_\Lambda\frac{m_\Lambda(E_0-E)}{m_\Lambda-2E}\right\}\{1+\alpha_\Lambda(E_0-E)\}}\right], \quad (6)$$

$$\alpha_\Lambda=\frac{2m_\Lambda}{m_{B_\Lambda}^2-m_\Lambda^2}.$$

It is interesting to note that the expected shape of the spectrum is appreciably weighted on the high energy side as compared with that of the local limit $m_B\to\infty$, i. e. the direct four-fermion interaction. As is seen in Fig. 2, the amount of the asymmetry depends upon the mass of the B-meson.

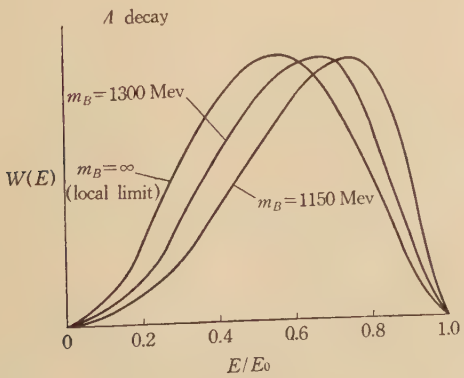

Fig. 2.

3. *The β-decay of Σ-hyperon*

Oneda[7] pointed out that the Σ^+-mode is not allowed in the Sakata model, since here the β-decay of hyperon accompanied by the emission of e^+ is not included as a fundamental interaction. In the weak current theory, we cannot allow the Σ^+-decay without violating the rule $\Delta s/\Delta Q\neq -1$, which is required in order to understand the observed mass-difference, $M_{K_1^0}-M_{K_2^0}\approx 10^{-6}$ ev, between K_1^0 and K_2^0.

We can derive the $\Sigma^\pm$-decay in terms of the B-meson processes, without such troubles.

Σ^--decay

$$H_{\Sigma^-}=f_\Sigma\{\bar{n}(1-\gamma_5)e^c+\bar{\Sigma}^-(1-\gamma_5)\nu^c\}B_\Sigma^{(-)}+h.c. \tag{7}$$

Σ^+-decay

$$H_{\Sigma^+}=f_\Sigma\{\bar{\Sigma}^+(1-\gamma_5)e^c+\bar{n}(1-\gamma_5)\nu^c\}B_\Sigma^{(0)}+h.c. \tag{8}$$

$$W_{\Sigma^-}(E)\sim\left\{\frac{2+\alpha_\Sigma E_0}{\alpha_\Sigma}\ln\left[1+\frac{2\alpha_\Sigma E(E_0-E)}{(m_\Sigma-2E)\{1+\alpha_\Sigma(E_0-E)\}}\right]\right.$$
$$\left.-\frac{2E(E_0-E)}{m_\Sigma-2E}\left[1+\frac{1+\alpha_\Sigma W}{\left\{1+\alpha_\Sigma\frac{m_\Sigma(E_0-E)}{m_\Sigma-2E}\right\}\{1+\alpha_\Sigma(E_0-E)\}}\right]\right. \tag{9}$$

$$W_{\Sigma^+}(E)\sim\left\{\frac{2E^2(E_0-E)^2(m_\Sigma-E_0)}{(m_\Sigma-2E)^2}-\frac{8}{3}\frac{E^3(E_0-E)^3}{(m_\Sigma-2E)^3}\right\}(1+\alpha_\Sigma E)^{-2}. \tag{10}$$

$$\alpha_\Sigma=\frac{2m_\Sigma}{m_{B_\Sigma}^2-m_\Sigma^2}.$$

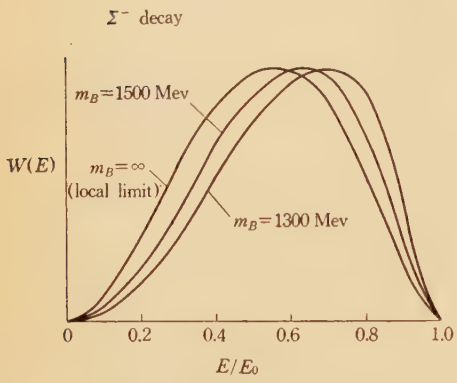

Fig. 3.

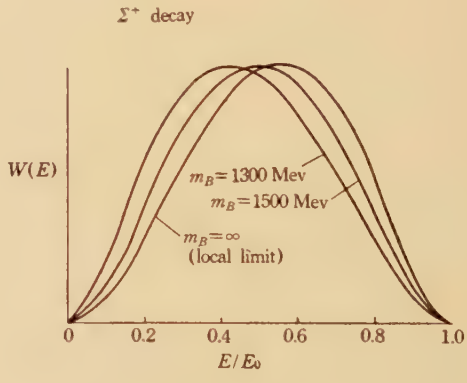

Fig. 4.

Figs. 3 and 4 show the energy spectra of the electron emitted from $\Sigma^\pm$-decay in the case of m_{B_Σ}=1300 Mev, 1500 Mev and $m_{B_\Sigma}=\infty$ (local limit). In comparing to the case of Λ-decay, the feature of asymmetry is similar for Σ^--decay but opposite for Σ^+-decay.

Lastly, let us add a few more words about the lifetime of hyperon decay relative to that of β-decay. Now that the amount of the coupling constants, f's, and the mass of the B-meson could in general be taken as arbitray parameters in each decay process, it is not surprising to find the diversities of the effective coupling constants in the case of β-decay, Λ-decay and $\Sigma^\pm$-decay in terms of the universal Fermi interaction.

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Hypothesis of the Isotopic Spin Independence in Extremely High Energy Region

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Pomeranchuk has proved that the total cross section of a particle A on a target B and the total cross section of the antiparticle $\bar{A}$ on the same target B have the same asymptotic value, provided that these cross sections are constant in the extremely high energy limit.[1] Recent experiment shows that p-p and $\bar{p}$-p cross sections seem to be approaching to a constant, though about 30% difference is observed even for $E_{p,\bar{p}} \sim 10$ Bev.[2]

The pion-nucleon collisions, $\pi^+ + p$ and $\pi^- + p$ are taken as example of our discussion. The total isotopic spin of the former is 3/2, and those of the latter are 3/2 and 1/2. Actually, the total cross section is written down in terms of the total cross sections with $I=3/2$ and 1/2 as

$$\sigma(\pi^+ + p) = \sigma(I=3/2), \tag{1}$$

$$\sigma(\pi^- + p) = \frac{1}{3}\sigma\left(I=\frac{3}{2}\right) + \frac{2}{3}\sigma\left(I=\frac{1}{2}\right). \tag{2}$$

According to Pomeranchuk's theorem, we have

$$\sigma(\pi^+ + p) = \sigma(\pi^- + p) \tag{3}$$

at extreme high energy,in other words,

$$\sigma\left(I=\frac{3}{2}\right) = \sigma\left(I=\frac{1}{2}\right). \tag{4}$$

In this situation we would like to make the following speculation: If a particle A with isotopic spin I_A collides on a target B with isotopic spin I_B, *the total cross sections corresponding to the total isotopic spin* $I=I_A+I_B,\ I_A+I_B-1,\ \cdots,\ |I_A-I_B|$ *tend to a constant independent of* I,

$$\sigma(I=I_A+I_B) = \sigma(I=I_A+I_B-1) = \cdots = \sigma(I=|I_A-I_B|) = \text{const.} \tag{5}$$

The present speculation is a generalization of the relation (4).

The following relations are immediately obtained, which are, at the same time, criterions of the hypothesis:

$$\sigma(p+p) = \sigma(p+n) = \sigma(\bar{p}+n) = \sigma(\bar{n}+p) \tag{6}$$

for the energy where $\sigma(p+p) = \sigma(\bar{p}+p)$ holds. Recent experiment supports $\sigma(p+p) = \sigma(p+n)$.[3] Further, one obtains

$$\sigma(K^- + p) = \sigma(K^- + n) = \sigma(K^+ + p) = \sigma(K^+ + n),$$
$$\sigma(\Sigma^- + p) = \sigma(\Sigma^- + n) = \sigma(\Sigma^+ + p) = \sigma(\Sigma^+ + n) \tag{7}$$

for the energy where $\sigma(\Sigma^+ + p) =$

$\sigma(\bar{\Sigma}^+ + p)$ holds. It may be suggested to have

$$\sigma(N+p) \approx \sigma(\Lambda+p) \approx \sigma(\Sigma+p) \approx \sigma(\Xi+p) \qquad (8)$$

in the high energy region where mass difference among baryons is not important.

If the present speculation is justified by experiment, part of the nuclear interaction may be clarified.

Further, highly speculative assumption is made that *the S-matrices are independent of the total isotopic spin.* For example, the $\boldsymbol{\tau}^{(1)}\boldsymbol{\tau}^{(2)}$ terms in the nucleon-nucleon interaction and the $\boldsymbol{\tau\omega}$ terms in the pion-nucleon interaction would be less important than other terms in the extremely high energy region, where $\boldsymbol{\omega}$ is the isotopic spin matrix for pion.

This assumption is understood in the following way for the elastic scattering, for example, *n-p* scattering. The main part of the elastic scattering is interpretable in terms of the diffraction scattering, which gives a sharp forward peak. This part is considered to be independent of the total isotopic spin, and this does not contribute to the charge exchange scattering. On the other hand, the charge exchange term, the $\boldsymbol{\tau}^{(1)}\boldsymbol{\tau}^{(2)}$ term, gives a backward peak. The rate of violation of the present assumption could, therefore, be seen by comparing the forward and backward peaks of the *n-p* elastic scattering. As is known, the peripheral collision is important in such energy region, so that the $\boldsymbol{\tau}^{(1)}\boldsymbol{\tau}^{(2)}$ term due to the one-pion exchange would be of second importance.

This idea is examined by the elastic scattering. For instance, we have

$$\sigma(\pi^+ + p \to \pi^+ + p) = \sigma(\pi^- + p \to \pi^- + p),$$

$$\sigma(\pi^- + p \to \pi^0 + n) = 0, \qquad (9)$$

$$\sigma(p + p \to p + p) = \sigma(n + p \to n + p). \qquad (10)$$

It is possible to find counter examples against this speculation. For example, $\pi^- + p \to \Lambda^0 + K^0$ contributes only to $\sigma(I=1/2)$ in (2), so that it might be possible to satisfy (4) even if the $\boldsymbol{\tau\omega}$ term remains finite. In any case, this assumption on the *S*-matrix is sufficient to satisfy (5), but not necessary. For inelastic scattering this assumption is less plausible.

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K^--p Interactions at 1.15 Bev/c

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In addition to the discovery of an isobaric state Y_1^* in the Λ-π system, the existence of a new isobaric state Y_0^* in the Σ-π system seems to be confirmed by the recent experiment

for $K^-+p\to\Sigma+\pi+\pi+\pi$ reaction at 1.15 Bev/c.[1] According to the experimental results the mass of Y_0^* is nearly equal to 1405 Mev. And the assumption of $I_{Y_0^*}=0$ ($I_{Y_0^*}$ indicates the isotopic spin of Y_0^*) seems to be consistent with the experimental results for the branching ratio of the decay of Y_0^*. Since the discovery of Y_1^*, the Y_1^* has often been regarded by many authors as a bound state in the $\bar{K}$-N system. Such an interpretation as this will be supported by the existence of this Y_0^*, because Y_1^* and Y_0^* may correspond respectively to the bound states with isotopic spin 1 and 0 in the $\bar{K}$-N system.*

In the previous paper[2] we have predicted the following property of $\bar{K}$-N interaction at high energy. In the neighborhood of the energy 675 Mev (lab.) which corresponds to the threshold energy of K^* production, we may expect the large cross section for K^--p scattering** and these resonance phenomena may be due to the strong interaction in the $I=0$ state, where I indicates the isotopic spin of the $\bar{K}$-N system. Making use of these results we should like to study the K^--p reactions at 1.15 Bev/c which may be regarded as the phenomena in the neighborhood of the energy 675 Mev (lab.). In this paper we examine which states of total angular momentum and parity are responsible for this resonance.

As was mentioned in our previous paper,[2] the resonance phenomena in K^--p scattering may be explained in terms of the effects due to the contributions from the intermediate K^*-N state (cf. Fig. 1). When the K^* in the intermediate state decays into a real $\bar{K}$ and a real π,

$$K^-+p\to\bar{K}+\pi+N. \qquad \text{(I)}$$

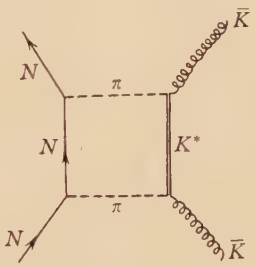

Fig. 1.

When the π in the intermediate $(\pi\bar{K})$-N state is re-absorbed by the nucleon,

$$K^-+p\to\bar{K}+N. \qquad \text{(II)}$$

When the $\bar{K}$ in the intermediate $(\pi\bar{K})$-N state is bound by the nucleon,

$$K^-+p\to Y_1^*+\pi \qquad \text{(III)}$$

or

$$K^-+p\to Y_0^*+\pi. \qquad \text{(III)}'$$

In our model, the resonance phenomena in the neighborhood of 675 Mev are due to the strong interaction in the $I=0$ state. Then, as was pointed out by Alston et al.,[1] the (III)′ process turns out to be forbidden owing to the conservation law of isotopic spin.

Now let us pay attention to the reaction

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* This model is remarkably different from the Goldhaber model in which Σ and Λ were regarded respectively as the bound states with isotopic spin 1 and 0 in the $\bar{K}$-N system.

** These resonance phenomena seem to be confirmed by the experiment (Berkeley Conference (1960)).

$$K^- + p \rightarrow K^* + N. \qquad \text{(IV)}$$

It seems that the experimental results[3] for this reaction are consistent with the assumption that K^* is produced predominantly in the s-state and with $S=0$ or $S=1$, where S indicates the spin of K^*. From these results it follows that the resonance phenomena are due to the strong interaction in the state of total angular momentum $J=1/2$ or $J=3/2$. If $J=1/2$ state is responsible for the resonance, the angular distribution of Y_1^* in the reaction (III) ought to be isotropic under the assumption that the spin of Y_1^* is equal to $1/2$. If the $J=3/2$ state is responsible for the resonance, the angular distribution of Y_1^* ought to have a form of $\sim(1+3\cos^2\theta)$. The experimental results[4] seem to be consistent with the assumption of $J=3/2$. Then we may conclude that $S=1$ is more promising than $S=0$. The same conclusion as this has been obtained in the study for $K^- + p \rightarrow \bar{K}^0 + \pi^- + p$ reaction.[5),6)]

Thus K^* may be regarded as a p-wave resonant state in the $\bar{K}$-π system. From this it follows that the parity of K^* is the same with that of $\bar{K}$ meson. In order to satisfy the parity conservation law, the orbital angular momentum l in the initial K^--p state must be even ($l=2n$) because of the assumption that K^* is produced in the s-state. The initial $d_{3/2}$-state is only the state which satisfies the conditions both of $J=3/2$ and of $l=$even. Thus we may conclude as follows. *The resonance phenomena in K^--p scattering which are expected in the neighborhood of 675 Mev are due to the strong interaction in $I=0$, $d_{3/2}$-state.*

Finally we want to make some discussion about the reason why Y_1^* has not been observed in the final state ($\Lambda+\pi+\pi+\pi$). The $K^- + p \rightarrow Y_1^* + \pi + \pi$ reaction cannot be forbidden by the same reason as in the case of (III)′ reaction. Since the $I=0$ state plays the most important role, the isotopic spin of 2π-system in the ($Y_0^*+\pi+\pi$) state must be equal to 0, while that of 2π-system in the ($Y_1^*+\pi+\pi$) state must be equal to 1. We denote by l_0 and l_1 the orbital angular momenta of Y_0^* and Y_1^* respectively with respect to the center of mass for two final pions. Since the available energy in these final states is considerably low in K^--p collision at 1.15 Bev/c, we may expect that $l_0=0$ and $l_1=0$ states respectively play the most important role in these reactions. If both Y_1^* and Y_0^* correspond to the s-states in the $\bar{K}$-N system, the parity of these bound states is the same with that of $\bar{K}$ meson. Then l_0 and l_1 must be even and odd respectively, because the resonance phenomena in K^--p collision are due to the strong interaction in the d-state. This may be one of the reasons why Y_1^* has not been observed in the final state ($\Lambda+\pi+\pi+\pi$).

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Field Theory of Nucleon-Nucleon Scattering

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We calculate the one-pion-exchange and two-pion-exchange parts of the S-matrix for nucleon-nucleon scattering using field theory, decompose them into partial wave amplitudes, and calculate the phase shifts. In order to secure the unitarity we regard the antihermitian part of the S-matrix as the K-matrix. The two-pion-exchange part of the S-matrix is written as

$$\begin{aligned}
&\langle q_1, q_2|S_2|p_1, p_2\rangle \\
&\quad = -M^2(16p_{10}p_{20}q_{10}q_{20})^{-1/2} \\
&\quad \times \delta^4(p_1+p_2-q_1-q_2) \\
&\quad \times \Big\{ \int d^4k_1 d^4k_2 \delta^4(p_1+k_1-q_1-k_2) \\
&\quad \times \bar{u}(q_1) T(q_1 k_2; p_1 k_1) u(p_1) (k_1^2+\mu^2)^{-1} \\
&\quad \times (k_2^2+\mu^2)^{-1} \bar{u}(q_2) T(q_2 k_1; p_2 k_2) u(p_2) \\
&\quad - 4i(2\pi)^{-4} \int d^4k_1 d^4k_2 d^4k_1' d^4k_2' \\
&\quad \times \delta^4(p_1+k_1-q_1-k_2) \\
&\quad \times \delta^4(p_2+k_2'-q_2-k_1') \\
&\quad \times \bar{u}(q_1) T(q_1 k_2; p_1 k_1) u(p_1) (k_1^2+\mu^2)^{-1} \\
&\quad \times (k_2^2+\mu^2)^{-1} G(k_1, -k_2; k_1', -k_2') \\
&\quad \times (k_1'^2+\mu^2)^{-1} (k_2'^2+\mu^2)^{-1} \\
&\quad \times \bar{u}(q_2) T(q_2 k_1'; p_2 k_2') u(p_2) \Big\}, \qquad (1)
\end{aligned}$$

where G is the two-pion Green's function obeying the integral equation

$$\begin{aligned}
&G(k_1, k_2; k_3, k_4) = K(k_1, k_2; k_3, k_4) \\
&\quad - 4i(2\pi)^{-4} \int d^4k_1' d^4k_2' \\
&\quad \times \delta^4(k_1+k_2-k_1'-k_2') \\
&\quad \times K(k_1, k_2; k_1', k_2') (k_1'^2+\mu^2)^{-1} \\
&\quad \times (k_2'^2+\mu^2)^{-1} G(k_1', k_2'; k_3, k_4). \qquad (2)
\end{aligned}$$

$T(qk_2; pk_1)$ is the sum of contributions of all those Feynman graphs each of which has two external nucleon lines with momenta p and q and two external pion lines with momenta k_1 and k_2, but cannot be divided into the part having the external nucleon lines p and q and the one having the external pion lines k_1 and k_2 by opening two internal pion lines. $K(k_1, k_2; k_3, k_4)$ is the sum of all those Feynman graphs each of which has four external pion lines k_1, k_2, k_3 and k_4, but cannot be divided into the part having the pion lines k_1 and k_2 and the one having the pion lines k_3 and k_4 by opening two internal pion lines. We can write T in the following form (explicitly writing the charge indices of pions) :

$$T_{\beta\alpha}(qk_2\,;\,pk_1)=-\delta_{\beta\alpha}A^{(+)}$$
$$-\frac{1}{2}[\tau_\beta,\,\tau_\alpha]A^{(-)}+\frac{i}{2}\gamma(k_1+k_2)$$
$$\times\left\{\delta_{\beta\alpha}B^{(+)}+\frac{1}{2}[\tau_\beta,\,\tau_\alpha]B^{(-)}\right\},\quad(3)$$

where $A^{(\pm)}$ and $B^{(\pm)}$ are Lorentz-invariant functions of p, q, k_1 and k_2. If $T_{\beta\alpha}$ were the sum of contributions of all the Feynman graphs which have two external nucleon lines and two external pion lines, and do not have any more restrictions, and if momenta of the external pion lines were on the mass shell, $T_{\beta\alpha}$ would be the pion-nucleon scattering amplitude, and $A^{(\pm)}$ and $B^{(\pm)}$ would satisfy the following dispersion relations:[1]

$$A^{(\pm)}=\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}dW^2\,\mathrm{Im}\,A^{(\pm)}(W^2,\,\kappa^2)$$
$$\times\left\{[(p+k_1)^2+W^2]^{-1}\right.$$
$$\left.\pm[(p-k_2)^2+W^2]^{-1}\right\},\quad(4)$$

$$B^{(\pm)}=g^2\left\{[(p+k_1)^2+M^2]^{-1}\right.$$
$$\left.\mp[(p-k_2)^2+M^2]^{-1}\right\}$$
$$+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}dW^2\,\mathrm{Im}\,B^{(\pm)}(W^2,\,\kappa^2)$$
$$\times\left\{[(p+k_1)^2+W^2]^{-1}\right.$$
$$\left.\mp[(p-k_2)^2+W^2]^{-1}\right\},\quad(5)$$

where $\kappa^2=(p-q)^2/4$. We assume that, if we replace $\mathrm{Im}\,A^{(\pm)}$ and $\mathrm{Im}\,B^{(\pm)}$ by those parts of them which correspond to the elastic pion-nucleon scattering, (3) with (4) and (5) becomes a good approximation to the correct $T_{\beta\alpha}$. We further neglect all the partial wave amplitudes other than (3-3) amplitude in $\mathrm{Im}\,A^{(\pm)}$ and $\mathrm{Im}\,B^{(\pm)}$, and make the sharp-resonance approximation[1] to the (3-3) amplitude.

We have calculated the phase shifts for the proton-proton scattering at

Table I. Nuclear bar phase shifts in degrees for proton-proton scattering at 310 Mev.

	Theory			Experiment			
	I	II	III	set 1	set 2	set 3	set 6
$^1S_0$	−50.18	86.82	84.19	−10.9	−22.1	−11.2	0.4
$^1D_2$	2.45	30.47	52.30	12.1	4.4	13.4	12.1
$^1G_4$	0.92	3.26	4.31	1.2	1.1	1.2	−1.3
$^3P_1$	−32.51	64.47	6.41	−26.2	−11.1	−19.8	−14.1
$^3F_3$	−4.72	2.98	15.96	−4.6	−0.5	−3.0	2.5
$^3H_5$	−1.31	−0.57	−0.31	−0.4	−1.5	0.5	−2.0
$^3P_0$	50.18	73.88	−33.71	−14.0	−34.4	−4.4	−65.7
$^3P_2$	5.52	69.99	59.18	16.6	19.2	22.4	8.2
$^3F_2$	2.82	6.25	22.83	1.3	0.1	−1.8	−1.3
ϵ_2	−6.49	−1.70	−1.66	−1.4	−8.6	1.4	0.9
$^3F_4$	0.92	8.88	13.11	3.2	2.9	0.8	3.4
$^3H_4$	0.55	1.03	1.24	1.7	2.4	−0.4	2.3
ϵ_4	−1.64	−1.56	−1.50	−1.4	−1.7	−1.4	0.8
$^3H_6$	0.24	1.02	1.27	1.2	1.4	−0.3	0.5

310 Mev, neglecting the second term between the curly brackets in (1). The results are listed in Table I together with the experimental values given by Cziffra et al.[2] Column I corresponds to the one-pion-exchange part alone. Column II has been obtained by neglecting the terms other than the Born term in (4) and (5), and inclusion of these terms gives Column III. Since we have neglected the exchange of three or more pions, it is meaningless to compare the S and P phase shifts with the experiment. This is the case also for ${}^3F_2$ phase shift because of the coupling between ${}^3F_2$ and ${}^2P_2$ states. Even if these facts are taken into consideration, the discrepancy between theory and experiment is serious, and this suggests the importance of the second term between the curly brackets in (1), i.e. the term due to the pion-pion interaction.

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Effect of the Bond Polarization on the I·I-coupling in Hydrocarbons

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Recently, interesting experiments have been performed showing that the coupling constant of the **I**·**I**-coupling between protons H and H' in a hydrocarbon, $J_{HH'}$, decreases with increasing electro-negativity of the atom situated at the end of the molecule.[1)~5)] These experimental results may be explained with the aid of the previously derived formula.[6)]

The previous calculation[6)] was based on the second-order perturbation theory, in which the wave functions of the excited states are eliminated by the use of the approximation replacing the energy differences between the ground state and the excited states by their suitable average ΔE.[7)] Following the valence bond method, the wave function of the ground electronic state Ψ was expressed as a linear combination of independent canonical structures Ψ_ξ's, i.e. $\Psi = \sum_\xi C_\xi \Psi_\xi$. An approximate formula for $J_{HH'}$ which is convenient for qualitative discussion was derived in addition to a general one.

Now we shall assume that, when the atom situated at the end of the molecule is substituted by another atom, ΔE and C_ξ do not change appreciably. By using this assumption and the above mentioned approximate formula, it can be shown that $J_{HH'}$ depends almost only on the characters of the chemical bonds H-X and H'-X' where $X(X')$ denotes an atom forming a chemical bond with $H(H')$. The formula for $J_{HH'}$ is shown to be of the form

$$J_{HH'} \propto N(hx)\{1+I(h)S(hx)\} \times N(h'x')\{1+I(h')S(h'x')\}, \quad (1)$$

where $N(hx)$ is the normalization constant of ϕ_{hx}, the orbital part of the wave function representing the electronic state of the chemical bond H-X (see Eq. (6) in reference 6)). The explicit formula for $N(hx)$ is as follows,

$$N(hx)=\Big[2+4S(hx)\{I(h)+I(x)\}+2S^2(hx)\{1+I(h)I(x)\}+I^2(h)+I^2(x)\Big]^{-1/2}. \quad (2)$$

In these equations, $I(h)$ and $I(x)$ are the coefficients of the ionic terms indicating the H^-X^+ and H^+X^- characters in the H-X bond, and $S(hx)$ is the overlap integral between atomic orbitals of atoms H and X. $N(h'x')$, $I(h')$, $I(x')$ and $S(h'x')$ are the respective quantities for the atoms H' and X'.

Since the electro-negativity of the atom X is larger than that of hydrogen atom H, $I(h)$ and $I(h')$ are fairly smaller than unity. Therefore they may be neglected in Eqs. (1) and (2). Then we obtain

$$J_{HH'}\propto\Big\{1+S^2(hx)+2S(hx)I(x)+\frac{1}{2}I^2(x)\Big\}^{-1/2}\Big\{1+S^2(h'x')+2S(h'x')I(x')+\frac{1}{2}I^2(x')\Big\}^{-1/2}. \quad (3)$$

From Eq. (3), the following result is obtained. As the atom at the end of the molecule is substituted by a more electro-negative one, the electrons in the molecule are drawn to the substituted atom. Then, the electrons in the bonds H-X and H'-X' are polarized toward the atoms X and X' respectively, that is, $I(x)$ and $I(x')$ increase. This results in a decrease of $J_{HH'}$, which is in accordance with the results of the experiments. The observed variations of $J_{HH'}$ in vinyl compounds[1)~5)] can be explained by Eq. (3), if we take the changes in $I(x)$ and $I(x')$ about $\pm 1/2$.

Details of this calculation and the explicit comparisons with the experiments will be published in the near future.

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Single Particle versus Collective Excitation in the Direct Interactions

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A detailed but phenomenological analysis is made in estimating the contribution of the single-particle excitations to the integrated cross sections $\sigma_{pp'}$ and σ_{pn} of the (p, p') and (p, n) reactions. It is found that, if the experimentally known collective-excitation cross section is added (which is in fact added only to the former), the resulting integrated cross sections agree very well with experiments and reasonably explain why $\sigma_{pp'} \gtrsim \sigma_{pn}$. It should be noted, however, that the contribution of the collective excitation to $\sigma_{pp'}$ turned out, in our analysis, to be only a small fraction of $\sigma_{pp'}$, contrary to the supposition of Cohen that it is the main part of $\sigma_{pp'}$. To complement the above phenomenological analysis, somewhat more basic considerations of the mechanism of the direct interactions are made, using the T-matrix formalism. Discussions are given concerning the way how to describe the collective excitation in the inelastic scattering process as a superposition of single-particle excitations, in accordance with other authors, and also concerning the reason why the excitation of the collective states is expected to be small in the transmutation reactions.

§ 1. Introduction

The bumps observed in the energy spectra of the protons,[1-3] deuterons[4),5)] and α-particles[6),7)] inelastically scattered by various nuclei, are now known to be associated with the excitation of the collective vibrational states. Some time ago several theoretical works[8)] appeared which showed, by treating the collective motions phenomenologically and working in the framework of the direct interaction theory, that the cross section for the inelastic scattering, leaving the target nucleus in the collective states, could become quite large, and this result was taken to support the above interpretation of the bumps.[9),10)]

In recent years, however, a new approach[11-13)] has been developed, which made it possible to describe the above excitations of the collective states as due to a coherent superposition of the scattering of the incident particle with the individual nucleons in the target nucleus, and thus to predict large cross sections for the excitation of the collective states, without introducing any collective parameters explicitly.

This approach has a further advantage that it can treat the excitations of the collective and the single-particle states on the same basis. If one takes this fact for granted then he can argue that every direct interaction, including

the excitations of the collective states, could be caused primarily by the collision of the incident particle with the individual nucleons in the target nucleus, and that, in spite of the large difference between the magnitudes of the cross sections of the excitation of the collective and the single-particle states, these two are the same kind of process.

It would then be expected that the integrated cross sections of the (p, n) and the (p, p') reactions are of the same order of magnitudes, because they are caused, respectively, by the p-n and the p-p collisions.* Experimentally, however, as was pointed out by Cohen,[10] it is known that

$$\int \sigma(p, n)dE \ll \int \sigma(p, p')dE \tag{1}$$

where the left-hand side means the cross section of the (p, n) reaction, integrated over an appropriate energy range, while the right-hand side is the corresponding one of the (p, p') reaction. Thus Cohen considered that the excitations of the collective and the single-particle states are phenomena completely different from each other. This is a view which is just opposite to that given in the preceding paragraph.

Several people considered the way to remove this apparent contradiction between these two views, by finding out the reason why the cross section for the collision of the incident proton with neutron in the target nucleus is so much smaller than that for its collision with protons. Thus Hintz[14] considers that the Coulomb interaction helps in increasing the cross section for the $(p$-$p')$ process, while Brown[15] suggests that the $(p$-$p')$ process can be increased by a factor of four over that of the $(p$-$n)$ process, because of the antisymmetrization. Finally, Weisskopf[16] suggests the possibility of the weakening of the effective p-n interaction inside the nuclear matter.

Based on these speculations casting doubt on the above-mentioned Cohen's interpretation, Eisberg[17] argued that it is not necessary to believe that the single-particle and the collective excitations are two completely different phenomena involved in direct interactions.

It is our opinion that Eisberg's view is more natural to be accepted than Cohen's. We consider, however, that although the above speculations of Hintz, Brown and Weisskopf taken by Eisberg as the basis of his argument might be correct to some extent, they are not the whole reason to explain the experimental fact given by (1). The relation (1) could rather be explained in a more natural way by looking at the experimental materials somewhat more carefully, and to show the possibility is the main purpose of the present paper. For this purpose we evaluate in § 2 the absolute magnitude of the cross section of the processes

* In fact $(p$-$n)$ collision also contributes to (p, p') reaction. Thus, as was noticed by Cohen,[10] the ratio of the integrated cross sections of (p, p') and (p, n) reactions would be about twice as large.

which can be considered to proceed through an elementary nucleon-nucleon collison. The integrated cross sections are evaluated in § 3 and the relation (1) is discussed. § 4 is devoted to somewhat more basic considerations of the contents of §§ 2 and 3, and finally concluding remarks are given in § 5.

§ 2. Evaluation of the cross sections of the elementary nucleon-nucleon interactions in nuclei

In accomplishing our aim to give support to the view that every direct interaction is primarily caused by the particle-particle interaction, the first thing we have to know is the absolute magnitude of the cross sections of the processes which proceed through the collision of the incident nucleon with only one nucleon in the target nucleus. It is very difficult to know this magnitude solely from theoretical considerations, even when we have knowledge of the free nucleon-nucleon interaction. Thus the best way would perhaps be to look at the experimental data concerning the process in which the initial state of the target nucleus and the final state of the residual nucleus can both be considered as pure single-particle states with high accuracy.

For simplicity let us denote these cross sections as σ^{sp}_{p-p} and σ^{sp}_{p-n}, the superfix sp meaning that they are cross sections of the single-particle process. Unfortunately, to our knowledge, there exist no data on the (p, n) reaction which supply us with information on σ^{sp}_{p-n}, perhaps because of experimental difficulties. It is possible, however, to get information on σ^{sp}_{p-n}, if one looks at the data for the inelastic scattering of protons by a nucleus, in which both the initial and the final states of the latter are pure single-neutron states. Similarly, information on σ^{sp}_{p-p} could be obtained from the corresponding data which involve pure single-proton states.

It is unfortunate that even the amount of this kind of data is quite limited. The data which we are going to use are those obtained by Cohen and Rubin,[3]

Table I-a. Experimental data of Pb^{207} (p, p') Pb^{207*}.

ΔE	configuration	$d\sigma(p, p')_{90°}$	σ^{sp}_{p-n}	$Av\,\sigma^{sp}_{p-n}$
0.5 Mev	$(2f_{5/2})^{-1}$	0.03 mb/ster	1.2 mb	
0.9 〃	$(3p_{3/2})^{-1}$	0.015 〃	0.6 〃	0.8 mb
1.6 〃	$(1i_{13/2})^{-1}$	0.015 〃	0.6 〃	

Table I-b. Experimental data of Bi^{209} (p, p') Bi^{209*}.

ΔE	configuration	$d\sigma(p, p')_{90°}$	σ^{sp}_{p-p}	$Av\,\sigma^{sp}_{p-p}$
1.0 Mev	$(1i_{13/2})$	0.10 mb/ster	4.0 mb	2.3 mb
1.5 〃	$(2f_{7/2})$	0.015 〃	0.6 〃	

from the inelastic scattering of 23 Mev protons by Pb^{207} and Bi^{209}, and they are summarized in Tables I–a and I–b.

In both of these tables the first column gives the excitation energy and the second column gives the possible assignment of the relevant single-particle states. The third column gives the differential cross sections of the excitation of these states at 90°, which are obtained in the following way. We first subtract, from the experimental excitation curve given at 90°, the back ground due to the tail of the elastic scattering, this back ground being assumed as known from the experimental data of the Pb^{208} (p, p') Pb^{208*} reaction.[18] Then, by graphical integration over the spread of the energy, we get the magnitude of the excitation of each state in arbitrary units. Similar quantities can be obtained at 90° and 60° for the excitation of the 3^- collective state at 2.6 Mev, and, particularly at 60°, its absolute magnitude is known. We can thus infer the absolute value of the excitation cross section of the 3^- state at 90°, and in turn those of the other excited states, by assuming that the angular distributions of the excitation cross sections are the same for all the states. The results are the contents of the column three.

The differential cross section of the excitation of 3^- state is given for a wider range of angle, and by numerical integration the total cross section for the excitation of this state is estimated. Then again assuming the similarity of the differential cross section for all the states, the total cross section of the excitation of the other states can be estimated and are given in the fourth column of Table I. These are nothing but the required σ^{sp}_{p-p} and σ^{sp}_{p-n}. The fifth column simply gives the arithmetic mean of the values given in the fourth column.

The assignment of the configuration of the relevant states is fairly certain in Pb^{207}, but there is no direct experimental evidence concerning the assignment of the states Bi^{209}, and the content of the second column of Table I-b is quite arbitrary.

Summarizing, we get the following results from the experimental data in the Pb region:

$$\sigma^{sp}_{p-n}=0.8 \text{ mb}, \quad \sigma^{sp}_{p-p}=2.3 \text{ mb}, \quad \sigma^{sp}_{Av}=1.4 \text{ mb}, \tag{2}$$

where σ^{sp}_{Av} is the mean value of the σ^{sp}_{p-n} and σ^{sp}_{p-p}.

There exists an alternative way of estimating the magnitude of σ^{sp}_{Av}. Cohen[3] gives a diagram in which the magnitudes of the differential cross sections at 90° of the excitation of the collective states by the (p, p') reaction are compared with the enhancement factor of the electromagnetic transition probabilities. This diagram is reproduced as Fig. 1 of the present paper and, as is seen, these two quantities are proportional to each other, which means that it is possible to express their relation as

$$d\sigma^{90°}_{collect}=C[B(E2)_{coul}/B(E2)_{sp}]. \tag{3}$$

It is clear that this proportionality constant C gives just the magnitude of the differential cross section at 90° of the single-particle process, and thus has similar meaning as the contents of the column three (and four) of Table I.

If one neglects in Fig. 1 the data for $Z=28\sim30$, which give an abnormally large value to C, we get $C=0.03$ mb, and thus get $\sigma_{Av}^{sp}=1.2$ mb which is consistent with the value given in (2). On the other hand, if one employs only the data for $Z=28\sim30$, it gives $C=0.06$ mb and in turn $\sigma_{Av}^{sp}=2.4$ mb, which is twice as large as* the values of (2). Summarizing the results obtained in this alternative way, one gets

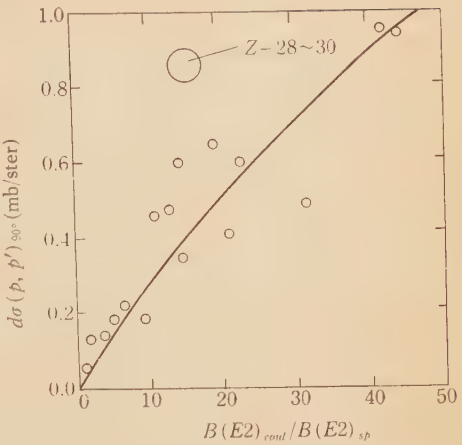

Fig. 1. Correlation between the cross sections of excitation of collective states by the inelastic scattering and the Coulomb excitation. This figure is reproduced from ref. 3).

$$\begin{aligned} &\text{Case I} \quad \sigma_{Av}^{sp}=1.2 \text{ mb}, \\ &\text{Case II} \quad \sigma_{Av}^{sp}=2.4 \text{ mb}. \end{aligned} \tag{4}$$

§ 3. Evaluation of the integrated cross sections of the (p, n) and (p, p') reactions

We now have an idea of the magnitude of the cross sections of the elementary processes and thus can proceed to the evaluation of the integrated cross sections. The relevant experimental data are summarized by Cohen,[10] and a part of it is reproduced in Table II of the present paper. Cohen gave similar data for other targets too, but the nuclei for which the cross sections of both the (p, n) and (p, p') reactions are given are only Th, Pb–Bi, Cu and Al. Among them Al seems to be too light to represent the average behaviour

Table II. The integrated cross sections (in mb).

Target	σ_{pn} by activation		$\sigma_{pp'}$ ($\Delta E \leq 8$ Mev)	
	23 Mev	31 Mev	23 Mev	31 Mev
Cu	40	20	110	—
Pb–Bi	~17	~21	120	120

* It is noticed recently by Cohen and Price[5] that if $d\sigma_{collect}$ is plotted against $d\sigma_{coul}$ the distinction between the $Z=28\sim30$ group and the rest is largely removed.

of the other nuclei, and thus we limit our consideration to Pb–Bi and Cu. As for Th a discussion is given in § 5.

To save the writing, we denote $\int_0^{\Delta E_{max}} \sigma(p, n)dE$ and $\int_0^{\Delta E_{max}} \sigma(p, p')dE$ simply as σ_{pn} and $\sigma_{pp'}$ in the following. Then from Table II we see that

$$\begin{aligned} &\text{For Cu:} \quad \sigma_{pp'}/\sigma_{pn} \fallingdotseq 2.7 \sim 5.5. \\ &\text{For Pb–Bi:} \quad \sigma_{pp'}/\sigma_{pn} \fallingdotseq 6.0. \end{aligned} \tag{5}$$

Here it is assumed that $\sigma_{pp'}=110$ mb for Cu at $E_p=31$ Mev, i.e. same as at $E_p=23$ Mev.

In calculating the theoretical ratio of $\sigma_{pp'}/\sigma_{pn}$, it is necessary to recollect first how the contents of Table II are obtained. As is explained by Cohen,[10] σ_{pn} of Table II is observed as an activation cross section, and is interpreted to represent the cross sections of the direct (p, n) reactions, in which the energy of the emitted neutron is within 8 Mev of the incident proton energy (i.e. $\Delta E \leq 8$ Mev), 8 Mev being the binding energy of protons or neutrons secondarily emitted so as to lead to (p, np) or $(p, 2n)$ reactions. Correspondingly $\sigma_{pp'}$ is obtained by integrating the observed cross sections of the (p, p') reaction for the energy loss of the proton up to 8 Mev.

To evaluate σ_{pn}, we shall assume that the process occurs as a knock-out of a neutron from the target by the incident proton and that the struck neutron must have occupied a definite single-particle orbit, while the proton must jump into a definite unoccupied orbit. As will be shown in § 4, the collective mode gives no essential contributions to the (p, n) reaction. Therefore σ_{pn} is considered simply as the number of the combinations of taking these possible orbitals, which we denote as $N_{n\to p}$, times the average cross section for such a process, i.e. σ^{sp}_{p-n} obtained in § 2. Namely we get

$$\sigma^{th}_{pn}=N_{n\to p}\times\sigma^{sp}_{p-n}. \tag{6}$$

On the other hand $\sigma_{pp'}$ consists of three parts: (i) excitation of a neutron from an orbit to a higher one, (ii) excitation of a proton in the target in the same way and (iii) excitation of the collective states.

Thus denoting as $N_{n\to n}$ and $N_{p\to p}$ the possible number of combinations of orbitals which, respectively, correspond to the above processes (i) and (ii), we have*

$$\sigma^{th}_{pp'}=N_{n\to n}\times\sigma^{sp}_{p-n}+N_{p\to p}\times\sigma^{sp}_{p-p}+\sigma_{collect}. \tag{7}$$

The evaluation of $N_{n\to p}$, etc., is done as follows.

We first note that the above limitation of $\Delta E \leq 8$ Mev for the (p, n) process is inappropriate, because, although it is true that the binding energy of the neutrons, which could be emitted as the second member in the $(p, 2n)$ re-

* For the validity of the expressions (6) and (7), see the discussion in the Appendix.

actions, is close to 8 Mev, that of the protons in the Pb–Bi isotopes is of the order of 4 Mev.[19] This means that if $\Delta E \geq 4$ Mev the final nucleus which is observed by activation cannot be the one that is given as the initial nucleus plus a proton minus a neutron. Therefore in evaluating $N_{n\to p}$ it is enough to consider only the combination of the orbitals for neutron and proton for which $\Delta E \leq 4$ Mev. The corresponding relation in Cu is $\Delta E \leq 6$ Mev.[19]

The second piece of information we must have is on the spacing of the orbitals, and we assume that the relative positions of the relevant levels are given by the Nilsson model[20] (with $\delta = 0$). In obtaining the absolute spacing, however, we utilize more direct experimental evidence and take

$$\begin{aligned} &\text{For Pb–Bi:} \quad \hbar\omega_{exp} = 5.5 \text{ Mev } (= \hbar\omega_{Nilsson}/1.3). \\ &\text{For Cu:} \quad \hbar\omega_{exp} = 3.2 \text{ Mev } (= \hbar\omega_{Nilsson}/3.2). \end{aligned} \tag{8}$$

$\hbar\omega_{exp}$ for Pb-Bi is determined from the spacing[21] of the hole states in Pb^{207}, while that for Cu is obtained from the spacing of the 3s, 2p and other states as inferred for this mass number region from the (d, p) reaction studies of Schiffer et al.[22]

The last thing we have to know is the pairing energy. This is needed because in some transition a pair has to be broken up and this requires an extra energy in addition to that due to the orbital jump. For nucleons which occupy the topmost orbitals, the pairing energy can be inferred directly from experiments, while that of deep lying nucleons has to be estimated theoretically. We follow the argument of Horie and Sasaki[23] and assume that the pairing energies are given for protons and neutrons respectively by $40(2j+1)/A$ Mev and 30 $(2j+1)/A$ Mev.

Using these knowledges, we finally get the following results.

$$\text{For Pb–Bi:} \quad N_{n\to n} = 54, \; N_{p\to p} = 29, \; N_{n\to p} = 24. \tag{9·1}$$

$$\text{For Cu:} \quad N_{n\to n} = 39, \; N_{p\to p} = 44, \; N_{n\to p} = 44. \tag{9·2}$$

Thus if one inserts (9), (2) and the experimental value[3] of $\sigma_{collect}$ into (6) and (7), it is found that for Pb

$$\sigma_{pp'}^{th} = 54\times 0.8 + 29\times 2.3 + 36 = 146 \text{ mb}, \tag{10·1}$$

$$\sigma_{pn}^{th} = 24\times 0.8 = 19 \text{ mb}. \tag{10·2}$$

Here (10·2) is quite close to the corresponding value of Table II, i.e. 17–21 mb, while (10·1) is somewhat too large.

In a similar manner, one gets for Cu

$$\sigma_{pp'}^{th} = 39\times 0.8 + 44\times 2.3 + 34 = 166 \text{ mb}, \tag{11·1}$$

$$\sigma_{pn}^{th} = 44\times 0.8 = 35 \text{ mb}. \tag{11·2}$$

In this case again the theoretical values are somewhat too large.

In obtaining these results (10) and (11), it is assumed that $\sigma^{sp}_{p-n}=0.8$ mb and $\sigma^{sp}_{p-p}=2.3$ mb. It is not certain, however, whether these values of σ^{sp} should be taken literally or not because of the poor statistics. If it were taken so, σ^{sp}_{p-p} is three times as large as σ^{sp}_{p-n} and is in accordance with the previous arguments.[14)-16)] It should, however, be noted that theoretically the result $\sigma_{pp'}/\sigma_{pn}\gg 1$ mainly comes from the fact that there contributed three terms to $\sigma_{pp'}$, while only one term contributes to σ_{pn} as is seen from (6) and (7), and further that a wider range of energy is required for the integration in obtaining $\sigma_{pp'}$ than for σ_{pn}. This situation is more clearly seen by estimating $\sigma_{pp'}$ and σ_{pn} under the assumption that $\sigma^{sp}_{p-p}=\sigma^{sp}_{p-n}=\sigma^{sp}_{Ao}=1.2$ mb, as is the case I of (4). The results are

$$\begin{aligned} &\text{Pb–Bi}: \quad \sigma^{th}_{pp'}=136 \text{ mb}, \ \sigma^{th}_{pn}=29 \text{ mb}, \\ &\text{Cu}: \quad \sigma^{th}_{pp'}=152 \text{ mb}, \ \sigma^{th}_{pn}=53 \text{ mb}. \end{aligned} \tag{12}$$

These values are again somewhat too large compared with experiment, but the ratio $\sigma_{pp'}/\sigma_{pn}$, which is equal to 4.7 and 3.0 respectively for Pb–Bi and Cu, agrees well with the corresponding experimental values which are approximately equal to 6.0 and 3.0.

We have thus obtained the results which explain experiments at least qualitatively, but the magnitudes of the cross sections $\sigma_{pp'}$ and σ_{pn} are in all cases too large. To obtain smaller results, we look at the experimental data[2] more carefully and note that in the energy spectrum of the (p, p') reaction, the contribution to the integrated cross section increases with increased ΔE up to, say, $\Delta E=4\sim5$ Mev, but it ceases to increase afterwards and in some cases even begins to decrease. On the other hand the contribution to $N_{n\to p}$, etc., defined in § 3, are certainly rapidly increasing functions of ΔE and thus, if σ^{sp}_{p-p} and σ^{sp}_{p-n} are assumed to be independent of ΔE as is assumed so far, it is clear that the above energy spectrum should keep increasing. Therefore the fact that the latter does not in fact increase means that σ^{sp} are decreasing functions of ΔE, and so, if this fact is taken into account properly, this will result in samller integrated cross sections than those given in (10), (11) and (12).

To see how strong the dependence of σ^{sp} on ΔE is, we reproduce in Fig. 2a the energy spectrum of the Pb^{207}

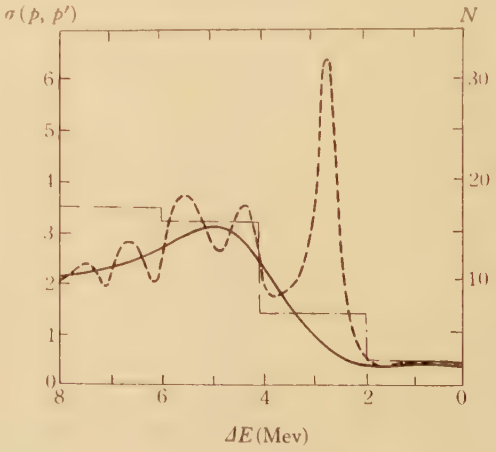

Fig. 2a. ----- Experimental energy spectrum[2] of 23 Mev protons inelastically scattered by Pb^{207}; ——— mean of the experimental spectrum; —·—·— N, the possible number of excitation modes per 1 Mev.

(p, p') Pb^{207*} reaction, obtained by Cohen and Mosko.[2] Here the dotted line is the experimental energy spectrum, while the full line gives its mean energy spectrum (except the excitation of the collective state). In this figure is plotted also the number of the possible combinations of the orbital energies per unit energy, as considered in § 3, at each excitation energy, and the points are connected by a dot-dash line. By comparing this with the above mean curve, the experimental knowledge on the ΔE dependence of σ^{sp} can be obtained and the result is shown in Fig. 2b.

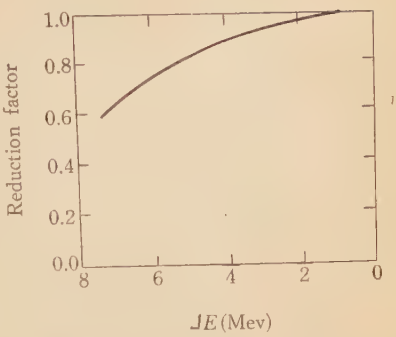

Fig. 2b. Reduction factor of the effective cross section for the single-particle mode of excitation.

By taking this ΔE dependence of σ^{sp} into account, the integrated cross sections are reevaluated and the results are given as

(i) (2) for σ^{sp} at $\Delta E \approx 0$;

Pb–Bi: $\sigma_{pp'}^{th}=117$ mb, $\sigma_{pn}^{th}=13$ mb, (13·1)

Cu: $\sigma_{pp'}^{th}=130$ mb, $\sigma_{pn}^{th}=26$ mb. (13·2)

(ii) Case I of (4) for σ^{sp} at $\Delta E \approx 0$;

Pb–Bi: $\sigma_{pp'}^{th}=110$ mb, $\sigma_{pn}^{th}=19$ mb, (14·1)

Cu: $\sigma_{pp'}^{th}=106$ mb, $\sigma_{pn}^{th}=38$ mb. (14·2)*

It is thus seen that pretty good agreement with experiment can be obtained in either of the above two cases, although the agreement seems somewhat better in (14) than in (13).

It should be remarked that what we have thus shown is that we can get correct order of magnitude for the integrated cross section $\sigma_{pp'}$ of the (p, p') reaction, if the contribution from the known collective state and the one from the single-particle states obtained in the way described above are added together, and that the contribution of the latter exceeds that of the former, contrary to Cohen's idea that the main part of the contribution comes from various collective states.

* Instead of using the data of Schiffer et al.[22] for obtaining the spacings of the single-particle states, if one uses the spacing of the $3s_{1/2}$ and $1g_{9/2}$ states at $A \approx 65$ from Fig. 9 of Cohen and Price,[22-a] one gets $\hbar\omega_{exp}=5$ Mev, instead of $\hbar\omega_{exp}=3.2$ Mev of (8) for Cu. By using this new value of $\hbar\omega_{exp}$ and Case II of (4), which is more appropriate to use than the Case I of (4) for Cu, we get $\sigma_{pp'}=108$ mb, $\sigma_{pn}=34$ mb, which agrees with experiment to the same extent as does (14·2)

§ 4. Discussion on the magnitudes of various cross sections

In this section we shall give somewhat more basic considerations which will supplement the contents of the preceding sections. One of them concerns the relative magnitudes of the cross sections of the single-particle processes which proceeds, respectively, via elementary p-p and p-n collisions. The other concerns the description of the excitation of the collective states based on the single-particle model.

4·1 *Relative magnitude of* σ_{p-p}^{sp} and σ_{p-n}^{sp}

As was mentioned in the Introduction, there are several arguments which imply that σ_{p-p}^{sp} could be much larger than σ_{p-n}^{sp}. The considerations given in § 2 and resulted in (2) might seem to support these arguments. It might, nevertheless, be dangerous to take these results too seriously, because the statistics of the experimental values which were employed there are quite poor. We shall thus give a brief consideration in this section, based on the T-matrix formalism, on the relative magnitudes of the above two kinds of elementary cross sections.

The scattering matrices of the nucleon-nucleon collisions which lead to the (p, p') and (p, n) reaction may be written as follows.

$$S_{if}^{(p-p) \text{ or } (p-n)}(p, p') = \langle \varphi_+^f(0)\phi_{\alpha'\pm}(1)|T|\varphi_+^i(0)\phi_{\alpha\pm}(1)\rangle - \langle \varphi_+^f(0)\phi_{\alpha'\pm}(1)|T|\varphi_+^i(1)\phi_{\alpha\pm}(0)\rangle, \tag{15·1}$$

$$S_{if}(p, n) = \langle \varphi_-^f(0)\phi_{\alpha'+}(1)|T|\varphi_+^i(0)\phi_{\alpha-}(1)\rangle - \langle \varphi_-^f(0)\phi_{\alpha'+}(1)|T|\varphi_+^i(1)\phi_{\alpha-}(0)\rangle. \tag{15·2}$$

Here the superfix $(p$-$p)$ or $(p$-$n)$ in (15·1), respectively, means that the relevant S-matrix is that for the collision of the incident proton with a proton or a neutron in the target nucleus, both contributing in the (p, p') reaction. $\varphi_\pm^i(0)$ and $\varphi_\pm^f(0)$ are the wave functions of the incident and the outgoing particle, while $\phi_{\alpha\pm}(1)$ means the wave function of the single particle which occupies initial and final orbitals specified by the quantum numbers α. The suffices $+$ and $-$ mean that they describe protons and neutrons, respectively.

If one employes the impulse approximation, the T-matrix is given by

$$\begin{aligned}(\boldsymbol{k}_f|T|\boldsymbol{k}_i) &\equiv T_0(\boldsymbol{k}_f, \boldsymbol{k}_i; \boldsymbol{\sigma}_0, \boldsymbol{\sigma}_1) + T_1(\boldsymbol{k}_f, \boldsymbol{k}_i; \boldsymbol{\sigma}_0, \boldsymbol{\sigma}_1)\cdot(\boldsymbol{\tau}_0\cdot\boldsymbol{\tau}_1)\\ &= [T_0 + T_1\cdot(\tau_{0z}\cdot\tau_{1z})] + 2T_1[(\tau_{0+}\cdot\tau_{1-}) + (\tau_{0-}\cdot\tau_{1+})],\end{aligned} \tag{16}$$

in terms of the free two-nucleon scattering amplitudes. In the final version of (16) the first and the second terms describe, respectively, the charge non-exchange and the charge exchange scattering. $\boldsymbol{k}_i$ and $\boldsymbol{k}_f$ are the momentum vectors of the incident and the outgoing particles. The meaning of the rest of the notation will be clear.

Inserting (16) into (15), the S-matrices are expressed as

$$S_{if}^{(p-p)}(p, p') = \{\langle T_0(\boldsymbol{k}_f, \boldsymbol{k}_i)\rangle + \langle T_1(\boldsymbol{k}_f, \boldsymbol{k}_i)\rangle\}\int \psi_{\alpha'}(\boldsymbol{k}-\boldsymbol{q})\psi_\alpha(\boldsymbol{k})\,d\boldsymbol{k}$$
$$-\int \{\langle T_0(\boldsymbol{k}_f, \boldsymbol{k})\rangle + \langle T_1(\boldsymbol{k}_f, \boldsymbol{k})\rangle\}\psi_{\alpha'}(\boldsymbol{k}-\boldsymbol{q})\psi_\alpha(\boldsymbol{k})\,d\boldsymbol{k},$$
$$S_{if}^{(p-n)}(p, p') = \{\langle T_0(\boldsymbol{k}_f, \boldsymbol{k}_i)\rangle - \langle T_1(\boldsymbol{k}_f, \boldsymbol{k}_i)\rangle\}\int \psi_{\alpha'}(\boldsymbol{k}-\boldsymbol{q})\psi_\alpha(\boldsymbol{k})\,d\boldsymbol{k}$$
$$-\int \{2\langle T_1(\boldsymbol{k}_f, \boldsymbol{k})\rangle\}\psi_{\alpha'}(\boldsymbol{k}-\boldsymbol{q})\psi_\alpha(\boldsymbol{k})\,d\boldsymbol{k},$$
$$S_{if}(p, n) = \{2\langle T_1(\boldsymbol{k}_f, \boldsymbol{k}_i)\rangle\}\int \psi_{\alpha'}(\boldsymbol{k}-\boldsymbol{q})\psi_\alpha(\boldsymbol{k})\,d\boldsymbol{k}$$
$$-\int \{\langle T_0(\boldsymbol{k}_f, \boldsymbol{k})\rangle - \langle T_1(\boldsymbol{k}_f, \boldsymbol{k})\rangle\}\psi_{\alpha'}(\boldsymbol{k}-\boldsymbol{q})\psi_\alpha(\boldsymbol{k})\,d\boldsymbol{k}. \qquad (17)$$

Here $\psi_\alpha(\boldsymbol{k})$, for example, is the Fourier transform of $\phi_\alpha(\boldsymbol{r})$, while $\boldsymbol{q}=\boldsymbol{k}_f-\boldsymbol{k}_i$ is the momentum transfer.

To get an idea about the relative magnitudes of various elementary cross sections, we make here the assumption that ϕ_α and $\phi_{\alpha'}$ are the same for all the processes considered here. Then it is clearly seen from (17) that

$$S_{if}^{(p-p)}(p, p') = S_{if}^{(p-n)}(p, p') + S_{if}(p, n). \qquad (18)$$

The magnitudes of the T-matrices appearing in (17) may be inferred from the table prepared by Kerman et al.[24] Unfortunately, the lowest energy considered in this paper is 40 Mev, which is somewhat too high compared with the energy considered in the present paper. By assuming that there might not be a drastic difference in the behaviour of various quantities at 40 Mev and 20–30 Mev, we use the result of Kerman et al. for 40 Mev. Neglecting further the last terms, i.e. the exchange terms in (17), one gets the following result:

$$\sigma^{(p-p)}(p, p') : \sigma^{(p-n)}(p, p') : \sigma(p, n) = 0.7 : 1.1 : 1.0. \qquad (19)$$

It is thus seen that all the elementary cross sections are more or less of the same order of magnitudes.* One might expect that $\sigma^{(p-p)}(p, p')$ is large compared with the others, because, as is seen in (18), $S_{if}^{(p-p)}(p, p')$ is the sum of two terms $S_{if}^{(p-n)}(p, p')$ and $S_{if}(p, n)$. The reason that $\sigma^{(p-p)}(p, p')$ is small, on the contrary, is that the contributions of $S_{if}^{(p-n)}(p, p')$ and $S_{if}(p, n)$ to $S_{if}^{(p-p)}(p, p')$ are out of phase, i.e. they interfere destructively.

Finally, we would like to mention that what is meant by (18) is that, as may be seen easily by looking at (16), the amplitude for the scattering of two protons is a sum of amplitudes with and without charge exchange. This consideration may have some connection with the suggestion by Brown,[15] although

* We also made an estimation for the case in which the exchange terms are included. The latter, i. e. the second terms of (17), are evaluated by replacing the factor $\langle T\rangle$ by its most probable value, and then taking out of the integral sign. It is found that the result (19) is essentially unchanged.

the exchange term considered by him is not the same as that considered here.*

The fact that all the three cross sections which appear in (19) have almost the same magnitude may justify the use of (4) over that of (2), and this result may be in accordance with the fact that we got somewhat better result in (14) than in (13).

4·2 *Excitation of the collective state in the* (p, p') *and the* (p, n) *reactions*

Generalizing the argument of the preceding subsection, by taking into consideration the spin dependence of the T-matrix explicitly, the transition matrix element $\langle f|\sum_n T(q)e^{-iqr_n}|i\rangle$ may be expressed as[24] (forward scattering)

$$\begin{aligned}
&\sum_l a_l q^l \langle f|\sum_n r_n^l Y_{l0}(\Omega_n)|i\rangle T_0^{(0)} \\
&\quad+\sum_{l,\lambda}(-)^\lambda a_l q^l\langle f|\sum_n r_n^l Y_{l0}(\Omega_n)\tau_{n,\lambda}|i\rangle\tau_{-\lambda}T_1^{(0)} \\
&\quad+\sum_{l,J,\mu}(-)^\mu b_{lJ\mu}q^l\langle f|\sum_n r_n^l \mathcal{Y}_{lJ,\mu}(\Omega_n,\sigma_n)|i\rangle T_{0,\mu}^{(1)} \\
&\quad+\sum_{l,J,\lambda,\mu}(-)^{\lambda+\mu}b_{lJ\mu}q^l\langle f|\sum_n r_n^l \mathcal{Y}_{lJ,\mu}(\Omega_n,\sigma_n)\tau_{n,\lambda}|i\rangle\tau_{-\lambda}T_{1,\mu}^{(1)},
\end{aligned}\tag{20}$$

with

$$\mathcal{Y}_{lJ,\mu}=\sum_{\mu',\mu''}(l1\mu'\mu''|J\mu)\,Y_{l\mu'}\sigma_{\mu''}.$$

In (20) a_l and $b_{lJ\mu}$ are appropriate c-number coefficients. $T_0^{(0)}$, $T_1^{(0)}$, $T_{0,\mu}^{(1)}$ and $T_{1,\mu}^{(1)}$ are functions of q, the momentum transfer, and in addition the latter two are vectors in the spin space of the incident particle.

In (20) the first term and a part of the second term would contribute to exciting the collective states with $T=0$ and $J=2^+$, 3^-, etc., while the second term alone will excite states with $T=1$ and $J=1^-$, 2^+, etc. The third and the fourth terms of (20) will excite states with $T=0$ and 1, and $J=1^+$, etc., with excitation being accompanied with a spin flip.[25),26)]

(20) is quite general and in making a little more detailed discussion on the mechanism of the excitation of the collective states, we restrict our argument to the 2^+ state with $T=0$ for simplicity.

Now the wave function of the collective state in an even-even nucleus, which is described as a superposition of the excitation of the two quasi-protons and two quasi-neutrons, is obtained[27)] by slightly extending the argument of Baranger[12)] as

$$|\Psi_{collect}\rangle=\frac{1}{N}\{\sum_{a,b}f_{ab}A_{ab}^+ + \sum_{a,b}g_{ab}A_{ab} + \sum_{\alpha,\beta}f_{\alpha\beta}A_{\alpha\beta}^+ + \sum_{\alpha,\beta}g_{\alpha\beta}A_{\alpha\beta}\}|\Psi_0\rangle \quad Q_2^+|\Psi_0\rangle,\tag{21}$$

$|\Psi_0\rangle$ being the wave function of the true vacuum.

In (21) A_{ab}^+ is the creation operator of the quasi-proton pair coupled to an angular momentum of two units, while A_{ab} is the corresponding annihilation

* See, however, the preceding footnote.

operator. $A^{+}_{\alpha\beta}$ and $A_{\alpha\beta}$ are corresponding operators for the quasi-neutron pair. f_{ab} and g_{ab} are defined by

$$\begin{Bmatrix} f_{ab} \\ g_{ab} \end{Bmatrix} = C_p q_{ab} / \{\omega \mp (E_a + E_b)\}\,; \quad \begin{Bmatrix} f_{\alpha\beta} \\ g_{\alpha\beta} \end{Bmatrix} = C_n q_{\alpha\beta} / \{\omega \mp (E_\alpha + E_\beta)\}, \tag{22}$$

where E_a is the energy of the quasi-proton. $q_{ab} = \langle a \| r^2\ Y_{2m} \| b \rangle (u_a v_b + u_b v_a)$, u_a and v_a, etc., describing the fractional occupation of the quasi-proton state a. C_p and C_n of (22) are constants.

After the Bogoliubov transformation[28] the operator $\mathcal{M}_\gamma$, i.e. a part of (20) which has non-vanishing contribution to the matrix element between the ground state and the collective state (21), is given as

$$\mathcal{M}_\gamma = (T_0 + T_1) \sum_{a,b} q_{ab} (A^{+}_{ab} + A_{ab}) + (T_0 - T_1) \sum_{\alpha,\beta} q_{\alpha\beta} (A^{+}_{\alpha\beta} + A_{\alpha\beta}). \tag{23}$$

Then the matrix element which we are seeking is obtained in the following very simplified form:

$$\begin{aligned} &\langle \Psi_{collect} | \mathcal{M}_\gamma | \Psi_0 \rangle \\ &= \frac{1}{N} \left\{ (T_0 + T_1) C_p \sum_{a,b} \frac{2(E_a + E_b) q^2_{ab}}{\omega^2 - (E_a + E_b)^2} + (T_0 - T_1) C_n \sum_{\alpha,\beta} \frac{2(E_\alpha + E_\beta) q^2_{\alpha\beta}}{\omega^2 - (E_\alpha + E_\beta)^2} \right\} \\ &\equiv \frac{1}{N} \{ (T_0 + T_1) C_p X_p + (T_0 - T_1) C_n X_n \}, \end{aligned} \tag{24}$$

X_p and X_n being defined by the final version of (24).

The matrix element for the $E2$ transition is given similarly as

$$\langle \Psi_{collect} | \mathcal{M}(E2) | \Psi_0 \rangle = \frac{1}{N} (e_p C_p X_p + e_n C_n X_n), \tag{25}$$

e_p and e_n meaning effective charge.[27] Comparison (24) with (25) shows that the (p, p') reaction is enhanced just as the $E2$ transition is done, and it is easily seen from the first version of (24) that this enhancement is caused because there appears a double summation over quasi-particle states and the terms which appear under the summation symbol all have the same sign.*

Turning to the discussion on the excitation cross section of the collective state due to the (p, n) reaction, we have to note first that if the relevant collective state is to be a state in an even-even nucleus the initial nucleus had to be an odd-odd nucleus which is usually not realized. To make the discussion more realistic and at the same time definite, we shall here consider an odd nucleus as target and thus the final nucleus is also an odd-A nucleus. Then to make the comparison with the (p, p') reaction possible we further mention that an essentially similar discussion can be made for the cross section for the

* This feature has been discussed sometime ago in the usual shell model language by Tamura and Choudhury.[11]

excitation of the core part of an odd-A nucleus as for the excitation of an even-even nucleus and thus some of the states in an odd-A nucleus can have large cross sections for the (p, p') reaction.

If we assume that the initial nucleus is an odd-neutron nucleus, then using the similar notation as for the even nucleus, and denoting the mixing coefficients as I_γ and F_c, the initial and the final state may be written as

$$|\Psi_i\rangle=\{I_{\gamma_0}\alpha_{\gamma_0}^{+}+\sum_{\gamma} I_\gamma \alpha_\gamma{}^{+} Q_2{}^{+}\}|\Psi_0\rangle$$

and

$$|\Psi_f\rangle=\{F_{c_0}\alpha_{c_0}^{+}+\sum_{c} F_c \alpha_c{}^{+} Q_2{}^{+}\}|\Psi_0\rangle, \tag{26}$$

where $\alpha_c{}^{+}$ and $\alpha_\gamma{}^{+}$ are the creation operators of the quasi-proton and the quasi-neutron, respectively. On the other hand the operator which replaces (23) is given by

$$\mathcal{M}_\beta=2T_1\sum_{a,\alpha}\{q_{a\alpha}^{(1)}\alpha_a{}^{+}\alpha_\alpha+q_{a\alpha}^{(2)}\alpha_a\alpha_\alpha{}^{+}+q_{a\alpha}^{(3)}\alpha_a\alpha_\alpha+q_{a\alpha}^{(4)}\alpha_a{}^{+}\alpha_\alpha{}^{+}\}, \tag{27}$$

where

$$q_{a\alpha}^{(1),(2),(3),(4)}=\langle a\|r^2\,Y_{2m}\|b\rangle(u_a u_\alpha,\ v_a v_\alpha,\ v_a u_\alpha,\ u_a v_\alpha),$$

respectively. Thus the transition matrix element becomes

$$\begin{aligned}
&\langle\Psi_f|\mathcal{M}_\beta|\Psi_i\rangle\\
&=2T_1\Big\{F_{c_0}q_{c_0\gamma_0}^{(1)}I_{\gamma_0}+\frac{1}{N}\sum_{\gamma}F_{c_0}\Big(C_p\frac{q_{ac_0}q_{a\gamma}^{(3)}}{\omega-(E_{c_0}+E_a)}-C_n\frac{q_{c_0\alpha}^{(4)}q_{a\gamma}}{\omega+(E_\alpha+E_\gamma)}\Big)I_\gamma\\
&\quad+\frac{1}{N}\sum_{c}F_c\Big(-C_p\frac{q_{ac}q_{a\gamma_0}^{(3)}}{\omega+(E_a+E_c)}+C_n\frac{q_{c\alpha}^{(4)}q_{a\gamma_0}}{\omega-(E_a+E_{\gamma_0})}\Big)I_{\gamma_0}\\
&\quad+\sum_{c,\gamma}F_c q_{c\gamma}^{(1)}I_\gamma+\frac{2C_p{}^2}{N^2}\sum_{c,\gamma}F_c\Big(\sum_{a,b}\frac{q_{ba}q_{ac}q_{b\gamma}^{(1)}}{\{\omega-(E_a+E_b)\}\{\omega-(E_a+E_c)\}}\Big)I_\gamma\\
&\quad+\frac{2C_n{}^2}{N^2}\sum_{c,\gamma}F_c\Big(\sum_{\alpha,\beta}\frac{q_{c\alpha}^{(1)}q_{\gamma\beta}q_{\beta\alpha}}{\{\omega-(E_\beta+E_\gamma)\}\{\omega-(E_\alpha+E_\beta)\}}\Big)I_\gamma\\
&\quad+\frac{2C_pC_n}{N^2}\sum_{c,\gamma}F_c\Big(\sum_{a,\alpha}\frac{q_{ac}q_{a\alpha}^{(2)}q_{a\gamma}}{\{\omega-(E_a+E_c)\}\{\omega-(E_\alpha+E_\gamma)\}}\Big)I_\gamma\Big\}.
\end{aligned}\tag{28}$$

In (28) the first term originates from the first terms of Ψ_i and Ψ_f of (26) and its magnitude is just the single-particle value. The second and the third terms of (28) are the cross terms of the first and the second terms of Ψ_i and Ψ_f, and if the (p, p') reaction were considered rather than the (p, n) reaction these terms should essentially be the same as (24), i.e. these are just the terms that explain the collective enhancement in the (p, p') reaction. In the (p, n) case the terms that appear under the summation symbol have no definite sign, and in addition the summation is only one-fold rather than two-fold as in (24)

and thus these terms would in general be quite small compared with (24), which means that no collective enhancement is expected.

The rest of the terms in (28) are somewhat complicated, but because of the randomness of the signs of various terms in summation and because of the normalization condition of Ψ_i and Ψ_f of (26), it is easy to see that their magnitudes are at most of the order of the single-particle value.

It is thus seen that the collective excitation in the (p, n) reaction is very hard to realize.

The excitation of the collective states other than the 2^+ with $T=0$ state of even-even nucleus in (p, p') reaction may be obtained as in (24). Particularly, the excitation of $T=1$ collective states would be caused as a coherent superposition of the collisions of protons with neutrons, and the cross section is expected to be large. Cohen and Price[5] suggested that these states may also be excited with large cross sections in the (p, n) reactions, but it seems that a discussion similar to that following (28) can be applied here too and the cross section could not be very large.

So far the discussion has been limited to the reactions caused by the inpinging protons. It is experimentally known, however, that the collective states are strongly enhanced in the (d, d') reaction, for instance, and this fact was one of the main reasons that led Cohen to consider that the collective excitation is quite different from the single-particle one. It is, however, seen from experiments that the magnitude of the cross section for exciting a single-particle state due to the (d, d') reaction is of the same order of magnitude as that due to the (p, p') reaction, and as the argument on the enhancement discussed in this subsection can equally well be applied to the (d, d') reaction, the strong excitation of the collective state in the (d, d') reaction can as well be explained as a superposition of the single-particle processes. In other words, what we should question is why the elementary cross sections are the same for the (d, d') and the (p, p') reactions, and not why the collective state is so much excited in the (d, d') reaction.

§ 5. Concluding remarks

It has been shown that the idea that all the direct interactions are primarily caused by the nucleon-nucleon interaction is consistent with all the data known at present, i.e. this idea explains the collective and the single-particle excitations in the process of the inelastic scattering, and also the (p, n) reaction.

In proving this fact, we had to use the magnitude of the cross sections of the elementary processes, which are gathered from rather scarce experimental data, and thus we have been suffering from very poor statistics. This cross section is an intrinsically fluctuating quantity, but if more experimental data were supplied, a more reliable average value of the relevant quantities could be

obtained, and the whole argument given in the present paper could be based on a firmer foundation. It would thus be of interest to measure the absolute magnitude of the cross sections of the inelastic scattering which excites the known single-particle levels of nuclei like Ca^{41}, K^{39} and others.

In spite of the crudeness of our estimation, and the various difficulties mentioned above, it is gratifying that we have already got an idea on the order of magnitude of σ^{sp}. This quantity was estimated in § 2 in several different ways and it is found that we always get a value which is quite close to 1.0 mb. This magnitude might thus be taken as a single-particle unit, and using this one may define the enhancement or forbidden factor, just as is done for the electromagnetic transition probabilities.

Unfortunately, we do not have sufficient experimental data to get a corresponding knowledge for nuclei in the deformed region, and this is the reason why we excluded Th from the arguments of § 3. Experimental data on the integrated cross sections of Th for the (p, p') and (p, n) reactions show, however, that they are of the same order of magnitude as in other nuclei (Pb-Bi and Cu), and this fact means that, if we consider that the amplitude to excite a Mayer-Jensen state is distributed among several Nilsson states which originated from the former, essentially the same argument as in § 3 could be applied to Th too.

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Appendix

The purpose of this appendix is to show the validity of (6) and (7) of the text for the integrated cross sections.

We assume that the main body of the process in which we are interested proceeds through the direct interaction, and so the distorted wave Born approximation, for instance, could be used to formulate the description. Then the cross section which leaves the residual nucleus in a state n may be described as

$$\sigma_{0n}=c|\langle\Phi_n\varphi_{k',n}|V|\Phi_0\varphi_k\rangle|^2, \qquad (A\cdot1)$$

where Φ_0 and Φ_n, respectively, describe the wave functions of the initial state of the target, and the final state of the residual nucleus. φ_k and $\varphi_{k',n}$ describe the motion of the incident and the outgoing nucleons. The integrated cross section is then given by

$$\sigma=\int_0^{\Delta Emax}\sigma(E)\,dE=\sum_n^{\Delta Emax}\sigma_{0n}=c\sum_n^{\Delta Emax}|\langle\Phi_n\varphi_{k',n}|V|\Phi_0\varphi_k\rangle|^2. \qquad (A\cdot2)$$

In general Φ_n can be expanded in terms of ψ_i^{sp} and ϕ_λ, where ψ_i^{sp} is the wave function of a state in which one particle is excited, while ϕ_λ describes a state in which two or more particles are excited. Thus

$$\Phi_n=\sum_i a_{ni}\psi_i^{sp}+\sum_\lambda \alpha_{n\lambda}\phi_\lambda, \qquad (A\cdot3)$$

and so

$$\sum_n|\Phi_n\rangle\langle\Phi_n|=\sum_{i,j}\Big(\sum_n^{\Delta Emax}a_{ni}^*a_{nj}\Big)|\psi_i^{sp}\rangle\langle\psi_j^{sp}|+\sum_{i,\lambda}\Big(\sum_n^{\Delta Emax}a_{ni}^*\alpha_{n\lambda}\Big)|\psi_i^{sp}\rangle\langle\phi_\lambda|$$
$$+\sum_{i,\lambda}\Big(\sum_n^{\Delta Emax}\alpha_{n\lambda}^*a_{ni}\Big)|\phi_\lambda\rangle\langle\psi_i^{sp}|+\sum_{\lambda,\mu}\Big(\sum_n^{\Delta Emax}\alpha_{n\lambda}^*\alpha_{n\mu}\Big)|\phi_\lambda\rangle\langle\phi_\mu|. \qquad (A\cdot4)$$

Taking into account the fact that $\langle\phi_\lambda\varphi_{k'}|V|\Phi_0\varphi_k\rangle$ vanishes, as V is a two-body operator, σ of (A·2) is reduced to

$$\sigma=c\sum_{i,j}\Big(\sum_n^{\Delta Emax}a_{ni}^*a_{nj}\Big)\langle\psi_i^{sp}\varphi_{k'}|V|\Phi_0\varphi_k\rangle^*\langle\psi_j^{sp}\varphi_{k'}|V|\Phi_0\varphi_k\rangle. \qquad (A\cdot5)$$

Here it is assumed that $\varphi_{k',n}$ is independent of the final state n, as the energy of the outgoing particle is assumed to be very large compared with the energy of excitation.

The amplitude of a single-particle excited state is distributed, as is seen from (A·3), over a number of actual excited states n, but we make here an assumption that the range of n over which this distribution occurs is not too wide. Then one gets

$$\sum_n^{\Delta Emax}a_{ni}^*a_{nj}\fallingdotseq\delta_{ij}, \qquad (A\cdot6)$$

which, if inserted into (A·5), gives

$$\sigma=c\sum_i^{\Delta Emax}|\langle\psi_i^{sp}\varphi_{k'}|V|\Phi_0\varphi_k\rangle|^2=N\times\sigma_{Av}^{sp}. \qquad (A\cdot7)$$

This is the result we have aimed to obtain, i.e. the justification of (6).

Of couse, there will be contributions from some single-particle states ψ_i^{sp}, with the excitation energy E_i higher than ΔE_{max} to the above summation, but at the same time, part of the contributions from the lower states with $E_i<\Delta E_{max}$ will be missed in the summation, and on the average these contributions will cancel out.

(A·7) may be correct for the (p, n) reaction, but may not be so for reactions where the collective motion is expected to play an important role. If ΔE_{max} is not very large, many higher single-particle states with $E_i>\Delta E_{max}$ would

contribute coherently to the collective state, and the negative contribution, from states with $E_i \lesssim \Delta E_{max}$ in the sense of last paragraph, would not be large enough to cancel this out. In other words, for the case of (p, p') reaction, there would exist a net contribution from higher single-particle states to the collective state, and this contribution has to be added to (A·7) in obtaining the integrated cross section for the (p, p') reaction. We thus replace (A·7) by

$$\sigma_{pp'} = c \sum_{i}^{\Delta E_{max}} \varepsilon_i |\langle \varphi_i'^{sp} \varphi_{K'} | V | \Phi_0 \varphi_K \rangle|^2 + \sigma_{collect} \doteqdot \sum^{\Delta E_{max}} N(E) \times \sigma_{An}^{sp}(E) + \sigma_{collect}. \quad (A \cdot 8)$$

This expression justifies the use of (7) of the text. In (A·8) $\varepsilon_i \leq 1$ is considered to take into account the fact that the single-particle states with $E_i \leq \Delta E_{max}$ also partially contribute to the collective state, and so their contributions to the first term have to be reduced correspondingly. In this way the danger of counting the same thing twice is avoided.

In case when ΔE_{max} is so large that the main contribution to the collective state comes from the states with $E_i \leq \Delta E_{max}$, there remains no reason to treat the collective state separately. In other wordes (A·8) is reduced to (A·7) and the integrated cross sections for the (p, p') and the (p, n) reaction may become of the same order of magnitude.

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The Singularity of Propagators in Field Theory and the Structure of Space-Time

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The singularity of propagators in field theory can be removed by introducing the distribution, or the fluctuation, of space-time. The fluctuation of space-time is assumed to result from the existence of elementary particles.

§ 1. Introduction

As is well known, the difficulty of infinities found in the present field theory is originated in the singularity of propagators on the light cone. Even if the self-effects are taken into account, the propagators behave at least as singularly as the free propagators.[1] We shall show that the singularity of propagators is closely connected with the structure of space-time.

Our fundamental assumption is that the existence of the elementary particle causes the distribution, or the fluctuation, of space-time. The fluctuation of space-time has influence on the motion of the elementary particle and removes the singularity of the propagator on the light cone. The existence of matter and the structure of space-time, therefore, are never independent of each other. In addition, it may be that each elementary particle will be characterized by its own distribution function (§ 2). An example of field theory with indefinite metric given by Markov[2] is included in our theory as a special case. The correspondence to the usual electrodynamics is also examined (§ 3). Bopp's generalized electrodynamics[3] is discussed from the standpoint of the distribution of space-time. The removal of the singularity of propagators by the fluctuation of space-time is not anything but the introduction of the indefinite metric in Hilbert space or of the negative energy, either of which should be dealt with so as to present itself as the result of some other kind of physical cause. The same can be said not only of the electromagnetic field, but also of the meson field (§ 4). In Appendix are shown the formulae on the Fourier transformation of Lorentz invariant functions.

§ 2. The fundamental assumption

In this paper, we attempt to establish a theory which is founded on the

idea that the existence of matter and the structure of space-time can never be independent of each other. The existence of the elementary particle is assumed to produce the distribution, or the fluctuation, of space-time and, in turn, the motion of the elementary particle itself must be influenced by the fluctuation of space-time.

Below is given the fundamental assumption: If the elementary particle exists, space-time is distributed, or fluctuated, the metric of which is given in the form of

$$\tau'^2=\tau^2-a^2, \qquad \tau^2=x_\mu{}^2=x_0{}^2-x_1{}^2-x_2{}^2-x_3{}^2, \tag{1}$$

a having the probability distribution $p(a)$. The modification of the metric fundamental form may be understood geometrically as the introduction of the fifth dimension, which is conditioned by the probability distribution. The fifth dimension is a probability space as well as a metric space. Namely, our fluctuating four-dimensional space is structurally a five-dimensional probability distribution of four-dimensional subspaces corresponding to various values of a. The distribution, or the fluctuation, of space-time is regarded as a physical concept characterized by a new constant, the so-called universal length. Consequently, the physical quantity in our theory can be defined as a weighted average of the corresponding quantity in each four-dimensional subspace. Considering that $\tau'^2=x_\mu{}'^2$ plays as a substitute for τ^2, each four-dimensional subspace can be treated analogically with the usual four-dimensional space-time. It is assumed, therefore, that τ' must be substituted for τ in the usual expressions of propagators and that the weighted averages of these substituted expressions with respect to a must be found, before the results of our theory can be compared with those of observation or other theories.

As for the true vacuum, the distribution function $p(a)$ must be chosen as $\delta(a)$ and the special theory of relativity does hold there strictly. It might be permissible to have such an image that the elementary particle gives life to the fifth degree of freedom, which is dead in the true vacuum.

This assumption means that the elementary particle propagates in the probability $p(a)$ between any two points in the five-dimensional space standing at interval of a in the direction of the fifth axis. In other words, $p(a)$ is nothing but the probability in which the elementary particle propagates between two points in the four-dimensional space-time, covering the distance modified by a. At a limit of vanishing mass or of extremely high energy, the free motion of the elementary particle is entirely determined by the distributed metric, because of the condition $\tau^2-a^2=0$. Accordingly, the path of the elementary particle deviates from the usual light cone in the four-dimensional space-time as well as in the direction of the fifth axis. Carrying out the weighted average with respect to a, we find that the path of the elementary particle is distributed in the four-dimensional space-time. Thus finally, does the fluctuation of the path

of elementary particles in the direction of the fifth axis reflect, as a kind of non-locality, in the usual four-dimensional space-time.

The distribution function $p(a)$ is normalized as

$$\int_{-\infty}^{\infty} p(a)\,da=1, \tag{2}$$

as it should be. It may also be assumed that the distribution function is an even function of a. If so, the weighted averages of any odd functions of a, especially the weighted average of a itself, will vanish. But the weighted average of metric fundamental form τ'^2 will be different from the Minkowski metric τ^2. That is, the distance between two points in the four-dimensional space-time must be measured in view of the fluctuation of space-time.

In passing, we have interpreted the fifth degree of freedom a as the fifth dimension,[4] however, it might also be possible to interpret it as a variable in the four-dimensional space-time.

Besides the difficulty of infinities, the main problem awaiting solution in field theory is to explain a variety of elementary particles. As for the latter problem, we expect that each elementary particle will be characterized by its own distribution function. Then, space-time will reveal itself as that defined by the distributed metric which varies according to the kind of the elementary particle.

§ 3. The discrete distribution of space-time

Before considering the distribution of metric, let us carry on some preliminary calculations. Following the assumption, we substitute τ' for τ in the usual invariant D-functions, which connect the field quantities at two points with each other. The substituted expressions are as follows:

$$\bar{D}(\tau')=\frac{1}{4\pi}\,\delta(\tau'^2)$$

$$=\int \bar{\Delta}(\tau,\kappa)\left[\delta(\kappa^2)-\frac{1}{2}|a|^2\frac{J_1(|a|\kappa)}{|a|\kappa}\right]d\kappa^2, \tag{3}$$

$$D^{(1)}(\tau')=-\frac{1}{2\pi}\,\frac{1}{\tau'^2}$$

$$=\int \Delta^{(1)}(\tau,\kappa)\left[\delta(\kappa^2)-\frac{1}{2}|a|^2\frac{J_1(|a|\kappa)}{|a|\kappa}\right]d\kappa^2, \tag{4}$$

$$D(\tau')=\int \Delta(\tau,\kappa)\left[\delta(\kappa^2)-\frac{1}{2}|a|^2\frac{J_1(|a|\kappa)}{|a|\kappa}\right]d\kappa^2, \tag{5}$$

where

$$\kappa^2=k_\mu{}^2=k_0{}^2-k_1{}^2-k_2{}^2-k_3{}^2. \tag{6}$$

These expressions can be derived by means of the same technique as that used by Schwinger for the derivation of the analytical forms of Δ-functions[2] and by means of the formulae on the Fourier transformation of Lorentz invariant functions which are shown in Appendix.

In each subspace the above functions (3), (4) and (5) can be considered as Green's functions of the wave equation with the differential opeator

$$\frac{d^2}{d\tau'^2}+\frac{3}{\tau'}\frac{d}{d\tau'} \tag{7}$$

concluded by analogy with the differential operator of the usual four-dimensional wave equation

$$\Box=\frac{\partial^2}{\partial x_0^2}-\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}-\frac{\partial^2}{\partial x_3^2}=\frac{d^2}{d\tau^2}+\frac{3}{\tau}\frac{d}{d\tau}. \tag{8}$$

It may be added that these functions are not Green's functions of the five-dimensional wave equation, which, as is well known, does not satisfy Huygens' principle.

As the simplest example of the distribution of metric, let us consider a sharp distribution

$$p(a)=\frac{1}{2}\delta(|a|-l), \qquad l: \text{ positive constant.} \tag{9}$$

The above distribution function is normalized and even with respect to a, as is required by the assumption. It is easily seen that the weighted average of the metric fundamental form gives τ^2-l^2, where l may be taken as the so-called universal length.

To apply the distribution (9) to the functions (3), (4) and (5) is to replace $|a|$ by l in these functions. The resulting functions are D-functions in this kind of distributed space, and these D-functions satisfy the wave equation having the differential operator (7) with the replacement of $|a|$ by l. The sharp distribution, therefore, leads to the spectral function

$$\rho(\kappa^2)=\delta(\kappa^2)-\frac{1}{2}l^2\frac{J_1(l\kappa)}{l\kappa}, \tag{10}$$

which is not positive-definite. Such a modification of spectral function as written above implies the introduction of indefinite metric in Hilbert space.

The above is nothing but the result given by Markov.[2] Markov's theory is thus included in our theory as a special case. It may here be emphasized incidentally that his physical interpretation is different from ours, i.e. in Markov's theory the propagator of photon is modified by the existence of electron, but in our theory the propagator of photon is modified by the existence of photon itself.

It must be added that, in the case of sharp distribution, Green's function D satisfies the differential equation of infinitely high order, which does not have a propagation character.[2,5] This is expected from the fact that the spectral function (10) comprises the infinite numbers of masses.

To examine the correspondence to the usual electrodynamics, let us apply the distribution

$$p(a)=\delta(a) \tag{11}$$

to the functions (3), (4) and (5). Then, as we anticipated, we obtain Green's functions in the usual four-dimensional spac-time:

$$\int \bar{D}(\tau')\delta(a)da=\bar{D}(\tau), \tag{12}$$

$$\int D^{(1)}(\tau')\delta(a)da=D^{(1)}(\tau), \tag{13}$$

$$\int D(\tau')\delta(a)da=D(\tau). \tag{14}$$

By the way, the general choice of metric fundamental form $f(\tau^2)$ satisfying the conditions that the algebraic equation

$$f(\tau^2)=0$$

has its roots α_i^2 and

$$df(\alpha_i^2)/d(\tau^2)\neq 0,$$

would not produce much difference from the previous case in the way of calculation, because $\delta[f(\tau^2)]$ can be transformed as follows:

$$\delta[f(\tau^2)]=\sum_i \frac{1}{|f'(\alpha_i^2)|}\delta(\tau^2-\alpha_i^2) \tag{15}$$

and the right-hand side of Eq. (15) is a combination of terms appearing in the previous case. It means that the general choice of metric fundamental form $f(\tau^2)$ corresponds to attaching a discrete distribution to the metric fundamental form τ^2-a^2.

§ 4. The continuous distribution of space-time

As an example of indefinite metric which cannot be caused by the discrete distribution of metric fundamental form in space-time, let us consider the generalized electrodynamics proposed by Bopp,[3] which has been interpreted as a theory of mixed field with negative energy or a theory with indefinite metric. Bopp's theory succeeded in making the self-energy of electron finite, but the indefinite metric or the negative energy should be dealt with so as to present itself as the result of some other kind of physical cause. In Bopp's theory,

Green's function has a shape of Bessel function, and by means of selecting any kind of $f(\tau^2)$ whatever, we could not transform Green's function into $\delta[f(\tau^2)]$. It is impossible, therefore, to attribute such an indefinite metric to the mere modification of metric fundamental form, or the discrete distribution of metric, in space-time.

To discuss Bopp's theory from the standpoint of the distribution of metric, we introduce the continuous distribution

$$p(a)=\frac{1}{2}\kappa_0 J_1(\kappa_0|a|), \qquad \kappa_0: \text{ constant.} \tag{16}$$

This distribution function is even with respect to a and normalized as below:

$$\int_{-\infty}^{\infty}\frac{1}{2}\kappa_0 J_1(\kappa_0|a|)\,da=\int_0^{\infty}\kappa_0 J_1(\kappa_0 a)\,da=1. \tag{17}$$

Let us apply the above distribution to the functions (3), (4) and (5) in § 3, to clarify the reason why we should have selected the distribution (16).

$$\int_{-\infty}^{\infty}\bar{D}(\tau')\frac{1}{2}\kappa_0 J_1(\kappa_0|a|)\,da=\bar{D}(\tau)-\bar{J}(\tau,\kappa_0), \tag{18}$$

$$\int_{-\infty}^{\infty}D^{(1)}(\tau')\frac{1}{2}\kappa_0 J_1(\kappa_0|a|)\,da=D^{(1)}(\tau)-J^{(1)}(\tau,\kappa_0), \tag{19}$$

$$\int_{-\infty}^{\infty}D(\tau')\frac{1}{2}\kappa_0 J_1(\kappa_0|a|)\,da=D(\tau)-J(\tau,\kappa_0). \tag{20}$$

These are nothing but the D-functions in Bopp's theory. Our assumption thus brings out the modification of propagator, or of quantization rule.

Needless to say, these D-functions make the self-energy of electron finite and satisfy the wave equation with such a differential operator of the fourth order as

$$\Box(\Box+\kappa_0^2)=\left(\frac{d^2}{d\tau^2}+\frac{3}{\tau}\frac{d}{d\tau}\right)\left(\frac{d^2}{d\tau^2}+\frac{3}{\tau}\frac{d}{d\tau}+\kappa_0^2\right), \tag{21}$$

which certainly has a propagation character.[5]

In the reduction of Eqs. (18), (19) and (20), the following formula was used:

$$\int_0^{\infty}aJ_1(\kappa a)\kappa_0 J_1(\kappa_0 a)\,da=\delta(\kappa-\kappa_0). \tag{22}$$

By this integration formula is established the relation between the Hankel transformation and its inverse transformation.[6]

Knowing the above D-functions, we can calculate the electromagnetic potential by the usual procedure. As regards the static potential, the weighted average of Coulomb potential in each subspace also leads to the same result:

$$\int_{-\infty}^{\infty} \frac{e}{\sqrt{r^2+a^2}} \frac{1}{2}\kappa_0 J_1(\kappa_0|a|)\,da = \frac{e}{r}[1-\exp(-\kappa_0 r)]. \tag{23}$$

This potential deviates from the Coulomb's law in the neighborhood of electron, namely, as for the static potential, the fluctuation of space-time is effective only in the neighborhood of electron. In fact, the right-hand side of the above approaches a finite value $e/(1/\kappa_0)$ as r approaches zero.[3] This is an example to show that our assumption has effect not only on the quantum level, but also on the classical level. We would like to remark again that the concept of the fluctuation of space-time is deeper than the classical or quantum concept.

Leaving off the discussion about Bopp's theory, let us proceed to the general case. For the general choice of distribution function $p(a)$, the spectral function is given as below:

$$\rho(\kappa^2) = \int_0^{\infty} [\delta(\kappa^2) - \delta(\kappa^2-\lambda^2)] G(\lambda)\,d\lambda, \tag{24}$$

$$G(\lambda) = \int_{-\infty}^{\infty} |a| J_1(|a|\lambda) p(a)\,da, \tag{25}$$

provided the integrations with respect to λ and a are commutable. This shows that our assumption necessarily produces the indefinite metric in Hilbert space. And, as $G(\lambda)$ satisfies the condition

$$\int_0^{\infty} G(\lambda)\,d\lambda = 1, \tag{26}$$

the spectral function (24) is nothing but the well-known Feynman's relativistic cut-off,[8] which guarantees the finiteness of the self-energy of electron.

In the expression (24), $G(\lambda) = \delta(\lambda-\kappa_0)$ or $p(a) = 1/2 \cdot \kappa_0 J_1(\kappa_0|a|)$ corresponds to Bopp's theory and $G(\lambda) = lJ_1(l\lambda)$ or $p(a) = 1/2 \cdot \delta(|a|-l)$ corresponds to Markov's theory.

For another example, Feynman[8] gave $G(\lambda) = 3\lambda/\kappa_0^2 \cdot [1+(\lambda/\kappa_0)^2]^{-5/2}$, which leads to Green's function like $1/8\pi \cdot \kappa_0^2 \exp(-\kappa_0|\tau|)$. This kind of indefinite metric can be derived by introducing the distribution: $1/2 \cdot \kappa_0^2|a| \exp(-\kappa_0|a|)$, κ_0: constant. In fact, the following integral is proved:

$$\int_{-\infty}^{\infty}|a|J_1(|a|\lambda)\frac{1}{2}\kappa_0^2|a|\exp(-\kappa_0|a|)da=3\frac{\lambda}{\kappa_0^2}[1+\left(\frac{\lambda}{\kappa_0}\right)^2]^{-5/2}. \qquad (27)^{7)}$$

Different from the previous oscillating distribution function $1/2\cdot\kappa_0 J_1(\kappa_0|a|)$, the present distribution function $1/2\cdot\kappa_0^2|a|\exp(-\kappa_0|a|)$ is a non-negative function. The characteristic width of this distribution is in the order of $1/\kappa_0$, which may correspond to the so-called universal length.

The spectral function (24) is not limited in the cases of the mixture of two fields or infinite numbers of fields. It is easily seen that, to obtain the mixture of finite numbers of fields, we may apply the following distribution function

$$p(a)=\sum_i C_i\frac{1}{2}\kappa_i J_1(\kappa_i|a|), \qquad (28)$$

$$\sum_i C_i=1. \qquad (29)$$

The condition (29) comes from the normalization of distribution function $p(a)$. If the absolute value of coefficients C_i is restricted to one, because of this condition, the fields of positive and negative energies have to appear alternately.[5)]

Comparing the various spectral functions, we can realize that a sharp distribution of a leads to a broad distribution of κ and a sharp distribution of κ is led from a broad distribution of a. The conjugate property between a and κ is also seen in the comparison between quadratic forms $x_\mu^2-a^2$ and $k_\mu^2-\kappa^2$.

We may select the distribution functions in such a way that the resulting propagation function is properly regularized[9)] and, at the same time, the differential equation satisfied by the resulting propagation function exhibits a propagation character.[5)] Until the deeper principle will be found, our guiding principle of finding a proper distribution function is not more than inductive.

It must be noted that the distribution of space-time concerning one elementary particle is independent of the existence of another elementary particles. The distribution should be taken, therefore, with respect to each path of each elementary particle. As the result, the fluctuation of space-time leads to the modification of propagator and of quantization rule, and the product of propagators can be expressed as the product of regularized propagators.

In such a manner, we leave the advantages of the indefinite metric or the negative energy as they are, and, on the other hand, we attribute their disadvantages to the fluctuation of space-time. Because of the close connection with the structure of space-time, our theory is expected to be more comprehensive and more profound than that of mixed field which is led from the mere modification of quantization rule.

The essential point of our result to be explained is that the motion of photon varies with the geometry of the space-time involved, which, in turn, is conditioned by photon. The non-singularity of propagator is ascribed to the

fluctuation of light cone, in other words, the smearing of light cone due to the appearence, on equal footing with Minkowski space, of all other suitably modified spaces. This situation is analogous to the theory proposed by Deser,[10] who considered the correction of propagators by gravitation.

The above theory can be said not only of the electromagnetic field, but also of the meson field. The most general expression of spectral function for the meson field is given as follows:

$$\rho(\kappa^2)=\int[\delta(\kappa^2-\kappa^2{}_m)-\delta(\kappa^2-\kappa^2{}_m-\lambda^2)]\,[\int|a|J_1(|a|\lambda)p(a)da]d\lambda, \tag{30}$$

κ_m : meson mass.

We expect, moreover, that the distribution function might characterize the kind of the elementary particle. As previously stated, a is the degree of freedom connecting the structure of space-time with the existence of elementary particle. From the standpoint of space-time the fifth variable a plays a role of fifth dimension and from the standpoint of elementary particle it behaves as if it were an internal coordinate. From the standpoint of space-time the probability distribution $p(a)$ corresponds to the fluctuation of space-time and from the standpoint of elementary particle it corresponds to a kind of extension of elementary particle. Speculatively speaking, $p(a)$ might serve for finding the mass spectrum and a might play a part of radial variable of the internal space.

§ 5. Conclusions

We have tried to set up a theory which will lead to the conclusion that the existence of matter and the structure of space-time can never be independent of each other. In our theory the singularity of propagators on the light cone is removed by introducing the distribution, or the fluctuation, of space-time. The fluctuation of space-time is assumed to be caused by the existence of elementary particles. This fundamental assumption means that the motion of elementary particles varies with geometry of space-time, which, in turn, is conditioned by the elementary particles. The fluctuation of space-time is regarded as a concept characterized by the so-called universal length. The general theory of relativity exists on the basis of the dependence of both the existence of matter and the structure of space-time on each other in the macroscopic world, and the present theory is on the basis of the same dependence in the microscopic world.

The removal of the singularity of propagators by means of the fluctuation of space-time is not anything but the introduction of the indefinite metric in Hilbert space or of the negative energy. This meets the requirement that the indefinite metric or the negative energy should be dealt with so as to present itself as the result of some other kind of physical cause. It also seems interesting that

the distribution of metric in space-time clarifies relations between Markov's theory and Bopp's theory. The present theory goes over to the usual field theory when the fluctuation of space-time does not rise.

To change the distribution function according to the kind of the elementary particle, it may be that each elementary particle will be characterized by its own distribution function. In its appearence, therefore, space-time will be defined by the metric varying according to the kind of the elementary particle.

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Appendix

The Fourier transformation in four-dimensional Minkowski space is written as follows:

$$F(x_0, x_1, x_2, x_3)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k_0, k_1, k_2, k_3)e^{ikx}\, dk_0dk_1dk_2dk_3, \qquad (A\cdot 1)$$

$$kx=k_0x_0-k_1x_1-k_2x_2-k_3x_3.$$

If $f(k_0, k_1, k_2, k_3)$ is a function of the variable $\kappa=\sqrt{k_0^2-k_1^2-k_2^2-k_3^2}$, that is, a Lorentz invariant function, the above transformation can be reduced to the transformations by Bessel functions.

$f(\kappa)$ is even with respect to k_0:

$$F = \begin{cases} \displaystyle\int_0^{\infty} f(\kappa)\kappa^3\frac{N_1(\kappa\tau)}{\kappa\tau}\, d\kappa, & \tau^2>0, \\ \displaystyle\frac{2}{\pi}\int_0^{\infty} f(\kappa)\kappa^3\frac{K_1(\kappa\sqrt{-\tau^2})}{\kappa\sqrt{-\tau^2}}\, d\kappa, & \tau^2<0, \end{cases} \qquad (A\cdot 2)$$

$$=2\pi\int_0^{\infty}\Delta^{(1)}(\tau, \kappa)f(\kappa)\, d\kappa^2. \qquad (A\cdot 3)$$

$f(\kappa)$ is odd with respect to k_0:

$$F=i\varepsilon(x_0)\left[2\delta(\tau^2)\int_0^{\infty} f(\kappa)\kappa d\kappa-\theta(\tau^2)\int_0^{\infty} f(\kappa)\kappa^3\frac{J_1(\kappa\tau)}{\kappa\tau}d\kappa\right], \qquad (A\cdot 4)$$

$$=2\pi i\int_0^\infty \Delta(\tau, \kappa)f(\kappa)d\kappa^2. \qquad (A\cdot 5)$$

We will reserve the proof of these formulae for another occasion.

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The Polarization of Deuteron

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The polarization in the elastic scattering of deuteron from spin zero target is treated on the basis of diffraction theory by using the polarization formalism of Wolfenstein, Lakin and Stapp. The *D*-state contribution of deuteron is small, especially for small angle, in the result of cross section and polarization at 94 Mev deuteron elastic scattering.

§ 1. Introduction

Recently, the polarization effects in the elastic scattering of deuteron by carbon and various other nuclei were measured,[1),2),3)] and were interpreted on the basis of the impulse approximation. Especially at 400 Mev deuteron scattering,[2)] the tensor polarization, in addition to the vector polarization, was measured, and was found to be appreciable. The general theory of polarization of the particle with any spin was discussed by Wolfenstein.[4)] The general polarization formalism of deuteron has been presented by Lakin.[5)] He constructs a complete set of nine 3×3 matrixes as follows:

$$
\begin{aligned}
T_{11} &= -\frac{1}{2}\sqrt{3}\,(s_x+is_y),\\
T_{10} &= \left(\frac{3}{2}\right)^{1/2} s_z,\\
T_{20} &= \left(\frac{1}{2}\right)^{1/2}(3s_z^2-2),\\
T_{21} &= -\frac{1}{2}\sqrt{3}\left[(s_x+is_y)s_z+s_z(s_x+is_y)\right],\\
T_{22} &= \frac{1}{2}\sqrt{3}\,(s_x+is_y)^2,\\
T_{J,-M} &= (-1)^M T_{J,M}^{\dagger}.
\end{aligned}
\tag{1}
$$

The secondly scattered intensity is given by

$$
\begin{aligned}
I(\theta,\varphi) = I_0(\theta,\varphi)[1&+\langle T_{20}\rangle_1\langle T_{20}\rangle_2+2\{\langle iT_{11}\rangle_1\langle iT_{11}\rangle_2-\langle T_{21}\rangle_1\langle T_{21}\rangle_2\}\cos\varphi\\
&+2\langle T_{22}\rangle_1\langle T_{22}\rangle_2\cos 2\varphi]
\end{aligned}
\tag{2}
$$

where $\langle T_{ij}\rangle_1$ and $\langle T_{ij}\rangle_2$ are the quantum mechanical expectation value of T_{ij} averaged over the spin directions of beam, at the first and the second scattering. The other general formalism of deuteron polarization has been given by Stapp,[6] taking the D-state of deuteron into consideration. He pointed out the important effect of simultaneous scattering of both particles of deuteron, especially for large scattering angles.

On the other hand, the diffraction theory of neutron inelastic scattering has been given by Inopin,[7] and this method has been extended for the inelastic scattering of α particle, deuteron and proton,[8] in which the surface vibration of nucleus is induced. The experimental results are fairly well explained by this model. An experimental verification of this model, which has been pointed out by Blair, is that the oscillation of inelastic scattering angular distribution curve, where the excitation involves no change of parity, is just out of phase with those of elastic angular distribution curve. In the stripping process of deuteron, it has been shown that a significant role was played by the dissociation process arising from diffractive effect.[9] For elastic scattering, the dissociation of deuteron and the polarization of nucleon stripped from deuteron at high energy has been treated by Akhieser and Sitenko[10),11] on the assumption of diffraction theory, taking the spin orbit interaction into consideration. In the present paper, the polarization of deuteron elastic scattering is treated by the diffraction theory on the assumption of completely black nucleus, in which account is taken of spin orbit interaction. Although the D-state contribution of deuteron to both the cross section and polarization is small for small angle and low energy. The component of tensor polarization $\langle T_{21}\rangle$ is exactly equal to zero in the case that deuteron constitutes only of S-state.

The calculations of elastic and inelastic scattering of deuteron have been done by many authors.[12),13),14)] It is shown that the electric break-up of deuteron[14] plays an important role in the elastic scattering of deuteron, but we neglect this process in the present paper.

§ 2. Calculation

The free motion of deuteron in a plane perpendicular to the direction of incident deuteron (the Z axis) is descrived by the wave function

$$\psi_{\Delta K}(\rho, r) = \frac{1}{L}\exp[i\Delta\boldsymbol{K}\cdot\boldsymbol{\rho}]\phi_d(r), \tag{3}$$

where L is the normalization length, $|\Delta\boldsymbol{K}| = K\sin\theta \approx K\theta$ and $\boldsymbol{\rho}$ are the projection of the wave vector $\boldsymbol{K}$ and radius vector of the centre of mass of deuteron, and $\boldsymbol{r}$ is the relative coordinate between neutron and proton. $\phi_d(r)$ is the wave function of deuteron

$$\phi_d(r) = [s(r) + 8^{-1/2}\ S_{12}d(r)]. \tag{4}$$

$s(r)$ is the S-state wave function and $d(r)$ is the D-state wave function. For triplet state, the tensor operator[15] S_{12} is given by

$$S_{12}=6\left(\boldsymbol{s}\cdot\frac{\boldsymbol{r}}{r}\right)^2-4 \tag{5}$$

where $\boldsymbol{s}$ is the total spin of deuteron $\boldsymbol{s}=1/2(\boldsymbol{\sigma}_n+\boldsymbol{\sigma}_p)$. For the incident wave function, $\Delta\boldsymbol{K}$ of (3) is zero, and so ψ_0 is simply $(1/L)\phi_d(r)$. Owing to the presence of an absorbing nucleus, incident wave of deuteron is diffracted and the wave function can be written in the form

$$\psi(\rho, r)=\sum_{\Delta\boldsymbol{K}}\hat{a}_{\Delta\boldsymbol{K}}\psi_{\Delta\boldsymbol{K}}(\rho, r). \tag{6}$$

The scattered amplitude $M(\theta, \varphi)$ is given by

$$M(\theta, \varphi)=-i(L^2K/2\pi)\hat{a}_{\Delta\boldsymbol{K}}. \tag{7}$$

We shall consider the effect of spin orbit coupling in the case of diffractional scattering, so $\hat{a}_{\Delta\boldsymbol{K}}$ is calculated in the following way.[11]

$$\hat{a}_{\Delta\boldsymbol{K}}=\iint\psi^*(\rho, r)[\hat{\omega}(r_n)+\hat{\omega}(r_p)-\hat{\omega}(r_n)\hat{\omega}(r_p)]\psi_0(\rho, r)\,d\boldsymbol{\rho}d\boldsymbol{r}$$

where

$$\hat{\omega}(r_i)=\omega(r_i)\left[1+\gamma\left(\frac{\hbar}{Mc}\right)^2 e^{i(\delta-\delta')}\cdot\frac{\sigma_i}{2}\cdot\left[\boldsymbol{k}_i\times\frac{\partial}{\partial r_i}\right]\right].$$

δ and δ' are phase shifts specifying the complex parts of the central and spin orbit coupling interaction, γ is the spin orbit coupling constant. $\boldsymbol{\sigma}$ is the spin matrix of nucleon.

$$\hat{a}_{\Delta\boldsymbol{K}}=a_{\Delta\boldsymbol{K}}+\frac{i\gamma}{2}\left(\frac{\hbar}{Mc}\right)^2 e^{i(\delta-\delta')}\int e^{i\Delta\boldsymbol{K}\cdot\boldsymbol{\rho}}\omega(\rho)\,d\boldsymbol{\rho}\int e^{-(i/2)\Delta\boldsymbol{K}\cdot\boldsymbol{r}}\phi_d^*(r)[\boldsymbol{s}\cdot\boldsymbol{K}\times\Delta\boldsymbol{K}]\phi_d(r)\,d\boldsymbol{r}$$

$$+\frac{i\gamma}{2}\left(\frac{\hbar}{Mc}\right)^2 e^{i(\delta-\delta')}\int e^{i\Delta\boldsymbol{K}\cdot\boldsymbol{\rho}}\omega(\rho)\,d\boldsymbol{\rho}\int e^{-(i/2)\Delta\boldsymbol{K}\cdot\boldsymbol{r}}\phi_d^*(r)[(\sigma_n-\sigma_p)\cdot\boldsymbol{k}\times\Delta\boldsymbol{K}]\phi_d(r)\,d\boldsymbol{r} \tag{8}$$

where $\boldsymbol{K}=\boldsymbol{k}_n+\boldsymbol{k}_p$ and $\boldsymbol{k}=(1/2)(\boldsymbol{k}_n-\boldsymbol{k}_p)$,

$$\omega(\rho)=\begin{cases}1 & \rho<R\\ 0 & \rho>R.\end{cases} \tag{9}$$

R is radius of nucleus, and

$$a_{\Delta\boldsymbol{K}}=\iint\psi^*(\rho, r)[\omega(r_n)+\omega(r_p)-\omega(r_n)\omega(r_p)]\psi_0(\rho, r)\,d\boldsymbol{\rho}d\boldsymbol{r}. \tag{10}$$

The last term in Eq. (8) is proportional to $(\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_p)$, whose expectation value vanishes as the deuteron is in triplet state and thus can drop out from Eq. (8). The scattered amplitude is given by

$$M(\theta, \varphi) = a(\theta, \varphi) + b(\theta, \varphi)(\boldsymbol{s}_i \boldsymbol{n}_i) + c(\theta, \varphi)(s_{ij} e_i e_j) \tag{11}$$

where

$$\boldsymbol{n} = \frac{\boldsymbol{K} \times \Delta \boldsymbol{K}}{|\boldsymbol{K} \times \Delta \boldsymbol{K}|}$$

and e_i is the component of unit vector e with direction of $\Delta \boldsymbol{K}$. The unit vector e is approximately equal to the unit vector $\boldsymbol{E} = \frac{\boldsymbol{K}_{out} - \boldsymbol{K}_{in}}{|\boldsymbol{K}_{out} - \boldsymbol{K}_{in}|}$ defined by Stapp. We neglected the effect of simultaneous scattering $\omega_n \omega_p$ in the calculation of the scattering amplitude. We shall discuss the term with $\omega_n \omega_p$ in § 3. s_i is the component of the total spin s of deuteron and s_{ij} is the symmetrical tensor,

$$s_{ij} = \frac{1}{2}(s_i s_j + s_j s_i) - \frac{2}{3}\delta_{ij} I.$$

The quantum mechanical expectation values of s_i and s_{ij} are given as follows.

$$\begin{aligned} I_0 \langle s_i \rangle &= \mathrm{Tr}\left[\frac{1}{3}\bar{M}(\theta, \varphi) M(\theta, \varphi) s_i\right] = \frac{2}{3}\mathrm{Re}\left[b\left(2a - \frac{c}{3}\right)^*\right], \\ I_0 \langle s_{ij} \rangle &= \mathrm{Tr}\left[\frac{1}{3}\bar{M}(\theta, \varphi) M(\theta, \varphi) s_{ij}\right] = \frac{1}{3}\left[bb^*\left(n_i n_j - \frac{1}{3}\delta_{ij}\right)\right. \\ &\qquad \left. + \left(2\mathrm{Re}ac^* - \frac{1}{3}cc^*\right)\left(e_i e_j - \frac{1}{3}\delta_{ij}\right)\right], \end{aligned} \tag{12}$$

and

$$I_0 = a^2 + \frac{2}{3}b^2 + \frac{2}{9}c^2. \tag{13}$$

The differential cross section after the second scattering, under the assumption that the vector e is approximately equal to the unit vector $\boldsymbol{E}$, is given by

$$I(\theta, \varphi) = I_0(\theta, \varphi)\left[1 + \frac{3}{2}\langle \boldsymbol{s}_i \rangle_1 \langle \boldsymbol{s}_i \rangle_2 + 3\langle s_{ij} \rangle_1 \langle s_{ij} \rangle_2\right]. \tag{14}$$

We can calculate the vector polarization $\langle iT_{11} \rangle$ and tensor polarization $\langle T_{20} \rangle$, etc., from Eq. (14) in comparison with Eq. (2).

§ 3. Results and discussion

We calculate the elastic scattering cross section and polarization of deuteron by diffraction theory, and the comparison with experimental results at 94 Mev deuteron energy is shown in Figs. 1 and 2. We employ the wave function of deuteron for the potential without core, introduced by Sugawara and Hulthen,[17] which has 4% of D-state probability. Spin orbit coupling constant $\gamma = 1/2$, if spin orbit interaction is caused by Thomas precession.[18] However, in the shell

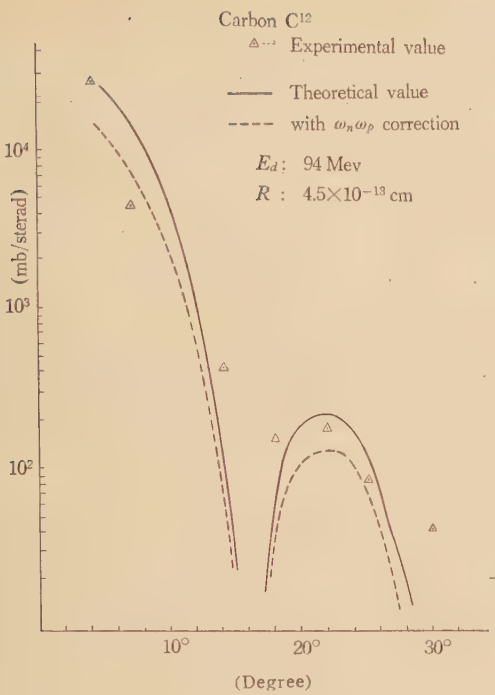

Fig. 1.

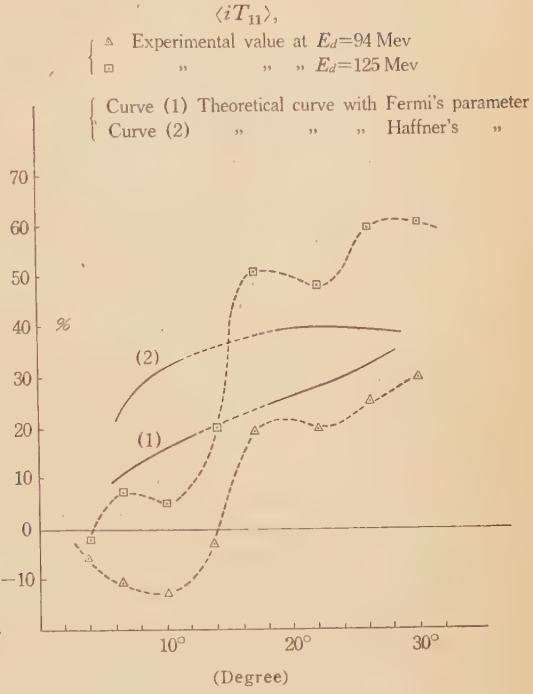

Fig. 2.

model and recent study of polarization of nucleon, γ is very large compared with 1/2. We employ $\gamma=15$; and further $\delta=34.3°$ and $\delta'=0$ given by Fermi.[19] Haffner analyzed the data of polarization of proton employing the Wood Saxon potential[20] and chose $\gamma=22$, $\delta=68.5°$ $\delta'=0$. The vector polarization $\langle iT_{11}\rangle$ is very large compared with experimental result if we employ the parameter given by Haffner. The D-state contribution is very small for both cross section and polarization in small angle scattering. The D-state contribution for $\langle iT_{11}\rangle$ is about 5% of the value of $\langle iT_{11}\rangle$. The component of tensor polarization $\langle T_{21}\rangle$ is exactly equal to zero without D-state. It is less than 1% even if we take account of D-state and even at energy as high as 94 Mev. This is much smaller at lower energies and the statement of Hird, etc., that the negative coefficient of $\cos\varphi'$ in (28) is caused by the effect of $\langle T_{21}\rangle$, seems doubtful. $\langle T_{22}\rangle$ is less than 2% and $\langle T_{20}\rangle$ is less than 4% at 94 Mev.

In § 2 and § 3, we neglected the term with $\omega_n\omega_p$. This approximation corresponds to the impulse approximation which neglects the simultaneous scattering of the particles composing the deuteron by nucleus.[6] Stapp pointed out that the simultaneous scattering may play an important role for large angle scattering. For small angles, which we have treated in the present paper the effect of simultaneous scattering may be small. The term with $\omega_n\omega_p$ has been evaluated by Akhieser and Sitenko, assuming that the wave function of deuteron

is $rs(r)=Ne^{-\alpha r}$ and that the nuclear radius R is greater than the radius of deuteron $R_d=1/2\alpha$.

The results with the correction of $\omega_n\omega_p$ are shown in Fig. 1, with broken lines. We take the radius of carbon $R=4.5\times10^{-13}$ cm, which Haffner employ in the analysis of deuteron inelastic scattering.[16] We cannot explain the fact that the negative sign of vector polarization in smaller angles less than 10° appears at 94 Mev and does not appear at 125 Mev.

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Nuclear Matter at Low Density with Attractive and Singular Interactions*

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The theory of nuclear superfluidity is presented by making use of the independent-pair-approximation developed in the Brueckner theory. The relation between the both theories will be clarified in the course of this derivation. A stable configuration of the system is determined by the correlation function between the pairs which satisfies the Cooper-Mills-Sessler (CMS) equation. This equation is simplified in such a way that the correct energy gap can be obtained even after the simplification. The variational principle used by CMS is found to give a lower bound to the energy gap. The simplified equation is solved in the low density limit without making any particular assumption about the two-body potential. When the attraction is so weak as to give a negative scattering length, a diffused Fermi distribution becomes stable. The exact energy gap in this limit is shown to be smaller by the factor $\exp(-2)$ than that of Emery and Sessler. If the attraction is strong enough to create a bound state the Fermi distribution is completely diffused. The energy gap is twice the binding energy. Finally, the insufficiency of the independent-pair-approximation is indicated.

§ 1. Introduction

Nuclear matter shows a characteristic phenomenon of the saturation. It is a bound state of many nucleons but not a collapsed state. Such a bound state may result from correlations among the nucleons. When the effects of many-body forces are negligibly small, the two-body potential must have a repulsive part within a short distance and an attractive part around it. The former is necessary in order to keep the system from collapsing and the latter to form the bound state. If the repulsive part has a hard core, nucleon collisions in the nuclear matter must be carefully treated. Brueckner and many other authors[3] have developed the method for treating such a problem. One of the important assumptions made in those theories is the validity of the independent-pair-approximation in which only such processes are considered that arise from the couplings of particle-particle and hole-hole pairs with the same total momentum, definite total spin and isotopic spin. This approximation has enabled one to treat the hard core interactions between the nucleons. One may expect that the two-nucleon clusterings due to the attractive interaction could in principle be included in this approximation, but this possibility has

* A preliminary account of this work was published in references 1) and 2).

been excluded by the second assumption that the configuration of the system is not drastically altered from that of a degenerate Fermi gas. The effect of the attractive interaction has been considered after the elimination of the hard core interaction.[4] Meanwhile, Cooper, Mills and Sessler[5] have applied the method used in the theory of superconductivity[6] to the problem of the nuclear matter. They have been able to treat the attractive- and the hard core interactions simultaneously. The relation between these two methods has not been clear.* It will be clarified in this paper that the latter method will be derived by making use of the independent-pair-approximation if one does not assume the stability of the degenerate gas state.

The structure of the dynamical system will be studied here in the independent-pair-approximation by making use of a familiar method treating the small oscillation about a stable point.[8] The following example may be instructive to explain the treatment. Considering a motion of a particle in a complicated potential by quantum mechanics, one will find a minimum point of the potential and expand it around this point. The potential is approximated by the first two terms of the expansion, so that the system is simulated by the harmonic oscillator. Then one will be able to apply quantum mechanics to this system to find the motion of the particle. This approximation is satisfactory if the zero point energy of the small oscillation is sufficiently small compared with the depth of the potential. The same principle can be applied to our Fermion system since the Hamiltonian of the system can be rewritten in terms of spin variables assigned to each pair state whose stable configuration may by found in a classical way.[9] It corresponds to the minimum point of the potential. The configuration must be more stable than that of the former example since the kinetic energy is also taken into account at the same time. It turns out that the stable configuration will be determined by the correlation function identical with that of a superfluid nuclear matter.[5] This means that the independent-pair-approximation is able to describe the two-particle clustering if the statistics of Fermion pair is taken into account.

The equation for the correlation function has been investigated by many authors. Cooper, Mills and Sessler have derived a variational principle which gives the criterion for superfluidity,[10] but the relation between the variational principle and the energy gap has not been noticed. Emery and Sessler[11] have calculated the energy gap transforming the equation for the correlation function into a form suitable for numerical solution. They have obtained too small energy gap in the case of the actual density. In this paper the equation for the correlation function is solved by another method. It is simplified so as to

* Recently, Marumori et al. have discussed on this point and obtained the same result with this paper.[7] The stability of the Bogoliubov ground state is assumed *a priori* in their paper, but it will not be assumed here.

give the accurate form of the function in the region of small distances in the coordinate space. Since the criterion for the existence of a superfluid state is given by the continuity condition at the surface where the short range two-body potential vanishes, this simplification is expected to give a satisfactory result. The simplified equation is found to be the Euler equation resulting from the variational principle for the energy gap, which is nothing but the variational principle used in CMS. In the case of a low density system the simplified equation will be solved in two cases without making any particular assumption about the two-body potential: One is the case where the interaction is attractive and weak leading to a negative scattering length and the other is the case where the interaction is strong enough to give a bound state. The energy gap will be obtained from the eigenvalue.

In § 2 the Hamiltonian of the system is simplified under the assumption of the independent-pair-approximation and the minimum of the energy is found classically. The equation for the correlation function is derived and it is shown to be the Euler equation of the variational principle used in CMS. The equation is solved in § 3 when the eigenvalue is not larger than the energy of the Fermi level. § 4 deals with the case where the eigenvalue is independent of the Fermi momentum. Finally in § 5 discussion is given on our results and on the problems to be considered.

§ 2. Independent-pair-approximation

In order to treat two-particle collisions rigorously, let us first introduce explicitly an operator of a pair of particles. If the pair has the total momentum $2\boldsymbol{p}$, relative momentum $2\boldsymbol{k}$, total spin I with the z component m, total isotopic spin T with the third component n, the annihilation operator of this pair is defined by

$$C_{\boldsymbol{k}}(\boldsymbol{p}\,;\,I,\,m\,;\,T,\,n)=\sum_{\substack{ss'\\tt'}}\left(\frac{1}{2}\,s\,\frac{1}{2}\,s'|Im\right)\left(\frac{1}{2}\,t\,\frac{1}{2}\,t'|Tn\right)c^{st}_{\boldsymbol{p}+\boldsymbol{k}}\,c^{s't'}_{\boldsymbol{p}-\boldsymbol{k}},$$

where $c^{st}_{\boldsymbol{p}+\boldsymbol{k}}$ means the annihilation operator of a nucleon with momentum $\boldsymbol{p}+\boldsymbol{k}$, the z component of spin s and the third component of isotopic spin t. For simplicity, the two-body potential is assumed to be of static and central type. The Hamiltonian of the system is then given by

$$\begin{aligned}
&H=H_0+V,\\
&H_0=\sum_{\boldsymbol{k}st}(\boldsymbol{k}^2/2m)\,c^{st*}_{\boldsymbol{k}}\,c^{st}_{\boldsymbol{k}},\\
&V=\sum_{\boldsymbol{p}ImTn}V(\boldsymbol{p}\,;\,I,\,m\,;\,T,\,n),\\
&V(\boldsymbol{p}\,;\,I,\,m\,;\,T,\,n)\\
&\quad=\frac{1}{\Omega}\sum_{\boldsymbol{k}\boldsymbol{k}'}{}'C_{\boldsymbol{k}}{}^{*}(\boldsymbol{p}\,;\,I,\,m\,;\,T,\,n)\,(\boldsymbol{k}|V_{IT}|\boldsymbol{k}')\,C_{\boldsymbol{k}'}(\boldsymbol{p}\,;\,I,\,m\,;\,T,\,n),\qquad(2\cdot1)
\end{aligned}$$

where m is the nucleon mass, Ω the normalization volume and the prime on the summation symbol means that the summation is taken only over half the $\boldsymbol{k}$ space; $k_z > 0$. The quantity $(\boldsymbol{k}|V_{IT}|\boldsymbol{k}')$ is the sum of the Fourier transforms of various exchange potentials and is given in the Appendix.

The equation of motion of a pair is found to be

$$[C_k(\boldsymbol{p}; I, m; T, n), H] = \{(\boldsymbol{p}+\boldsymbol{k})^2 + (\boldsymbol{p}-\boldsymbol{k})^2\}/2m \cdot C_k(\boldsymbol{p}; I, m; T, n)$$
$$+[C_k(\boldsymbol{p}; I, m; T, n), C_k^*(\boldsymbol{p}; I, m; T, n)] \frac{1}{\Omega} \sum_{\boldsymbol{k}'}{}' (\boldsymbol{k}|V_{IT}|\boldsymbol{k}') C_{k'}(\boldsymbol{p}; I, m; T, n)$$
$$+\text{coupling terms with different } (\boldsymbol{p}, I, m, T, n).$$

The last terms on the right-hand side are to be neglected under the assumption of the independent-pair-approximation. The average effect of those terms may be taken into account by determining a single-particle potential selfconsistently. The single-particle energy $\boldsymbol{k}^2/2m$ will be replaced by ω_k which is the energy in the selfconsistent potential but we here shall not discuss the determination of ω_k.*

Now one will take the reduced Hamiltonian in such a way that it will give the same equation of motion for the pair. The operators $C_k(\boldsymbol{p}; I, m; T, n)$ and $C_{k'}^*(\boldsymbol{p}'; I', m'; T', n')$ may be regarded as commutative if any one of $\boldsymbol{p}$, I, m, T or n is different from the corresponding value with prime. The system is then equivalent to an assembly of independent dynamical systems each one of which is specified by a distinct set $(\boldsymbol{p}, I, m, T, n)$. The shift of the ground state energy due to the interaction is called the energy shift. The energy shift of the total system is obtained as the sum of those of each system.

When we consider a particular system with a definite set $(\boldsymbol{p}, I, m, T, n)$, the interaction energy is given by (2·1) and the notations of paired- and unpaired states can be used as follows. Every state with $\boldsymbol{p}+\boldsymbol{k}$, s and t has a partner which is characterized by the quantum numbers $\boldsymbol{p}-\boldsymbol{k}$, $m-s$ and $n-t$. These two states may be regarded to form a pair of states. If one of the states in a pair is occupied by a nucleon and the other is unoccupied at the beginning (these will be called unpaired states), the nucleon is not influenced by the interaction and the energy of the nucleon can be easily found. If both states are occupied or unoccupied at the beginning (paired states) those are always occupied or unoccupied simultaneously. The operator for such a pair of the nucleons is given by

$$C_k(s, t) = c_{p+k}^{st} c_{p-k}^{m-s\ n-t}, \qquad (2\cdot2)$$

where $(\boldsymbol{p}, I, m, T, n)$ is suppressed on the left-hand side. They satisfy the relation

* The determination of ω_k has been discussed in reference 7).

$$c_{p+k}^{mt*}c_{p+k}^{st}\Psi=c_{p-k}^{m-s\ n-t*}c_{p-k}^{m-s\ n-t}\Psi=C_k^*(s,t)C_k(s,t)\Psi, \tag{2·3}$$

where Ψ is any state which is to be considered in treating the system with a definite set $(\boldsymbol{p}, I, m, T, n)$. By means of this the kinetic energy of the system can be written

$$\sum_{pst}\omega_p c_p^{st*}c_p^{st}\to H_0(\boldsymbol{p}\,;\,I,\,m\,;\,T,\,n)=\sum_{kst}{}'\,(\omega_{p+k}+\omega_{p-k})C_k^*(s,t)C_k(s,t)$$
$$+\sum_{kst}{}''\,\{\omega_{p+k}c_{p+k}^{st*}c_{p+k}^{st}+\omega_{p-k}c_{p-k}^{st*}c_{p-k}^{st}\},$$

where the first sum is taken over the paired states and the second over the unpaired states. The reduced Hamiltonian is now found to be

$$H_{red}=\sum_{pImTn}H(\boldsymbol{p}\,;\,I,\,m\,;\,T,\,n), \tag{2·4}$$

where

$$\begin{aligned}
&H(\boldsymbol{p}\,;\,I,\,m\,;\,T,\,n)=H_0(\boldsymbol{p}\,;\,I,\,m\,;\,T,\,n)+V(\boldsymbol{p}\,;\,I,\,m\,;\,T,\,n),\\
&V(\boldsymbol{p}\,;\,I,\,m\,;\,T,\,n)=\frac{1}{\Omega}\sum_{\substack{kst\\k's't'}}{}'\,C_k^*(s,t)\,(kst\ V_{IT}^{mn}|k's't')C_{k'}(s',t'),\\
&(kst|V_{IT}^{mn}|k's't')=\left(\frac{1}{2}s\frac{1}{2}m-s|Im\right)\left(\frac{1}{2}t\frac{1}{2}n-t|Tn\right)(k|V_{IT}|k')\\
&\qquad\times\left(\frac{1}{2}s'\frac{1}{2}m-s'|Im\right)\left(\frac{1}{2}t'\frac{1}{2}n-t'|Tn\right).
\end{aligned} \tag{2·5}$$

In (2·4) the operators $C_k(s,t)$ and $C_{k'}^*(s',t')$ which belong to different $H(\boldsymbol{p}\,;\,I,\,m\,;\,T,\,n)$'s must be regarded as commutative. Instead of this we have added the kinetic energy to the respective Hamiltonian $H(\boldsymbol{p}\,;\,I,\,m\,;\,T,\,n)$. One can obtain the desired equation of motion for the pair by making use of (2·4).

Now let us consider a particular Hamiltonian $H(\boldsymbol{p}\,;\,I,\,m\,;\,T,\,n)$ hereafter. $C_k(s,t)$ and $C_k^*(s,t)$ satisfy the commutation relations

$$\begin{aligned}
&[C_k(s,t),\,C_{k'}(s',t')]=0,\\
&[C_k(s,t),\,C_{k'}^*(s',t')]=0,\quad \textit{if}\ (k,s,t)\neq(k',s',t'),\\
&[C_k(s,t),\,C_k^*(s,t)]_+=1,\\
&[C_k(s,t),\,C_k(s,t)]_+=0,
\end{aligned}$$

which are derived from (2·2) and (2·3). These relations can be satisfied by the following representation,

$$\begin{aligned}
&C_k(s,t)=\frac{1}{2}\{\sigma_x(kst)+i\sigma_y(kst)\}=m_x(kst)+im_y(kst),\\
&C_k^*(s,t)C_k(s,t)=\frac{1}{2}-m_z(kst),
\end{aligned} \tag{2·6}$$

where σ's are the Pauli spin matrices and m's the angular momentum operators of magnitude one half. Thus one assigns a spin to each paired state in such a way that if it is occupied the spin has a direction downwards, and if unoccupied upwards. This assignment has been used by Takano, Fukuda and the present author in the discussion on the theory of superconductivity.[9] The Hamiltonian (2·5) can be written in terms of them as

$$H(\boldsymbol{p}; I, m; T, n)=\frac{1}{2}\sum_{kst}'\{\omega_{p+k}+\omega_{p-k}\}-\sum_{kst}'\{\omega_{p+k}+\omega_{p-k}\}m_z(kst)$$
$$+\frac{1}{\Omega}\sum_{\substack{kst\\k's't'}}'(kst|V_{IT}^{mn}|k's't')\{m_x(kst)m_x(k's't')+m_y(kst)m_y(k's't')\}$$
$$+\sum_{kst}''\{\omega_{p+k}c_{p+k}^{st*}c_{p+k}^{st}+\omega_{p-k}c_{p-k}^{st*}c_{p-k}^{st}\}.$$

For simplicity we will consider the Hamiltonian $H(0; I, m; T, n)$ hereafter and assume that there are no unpaired states. This is the correlation which Bardeen, Cooper and Schrieffer[6] have taken into account in their theory of superconductivity. Now one will apply the method of small oscillations around the stable point. As the nucleon number is not definite in this method, one imposes a subsidiary condition

$$2\sum_{kst}'C_k^*(s,t)C_k(s,t)=\sum_{kst}'(1-2m_z(kst))=A,$$

which must be satisfied as an average, A being the total number of nucleons. Then one makes the expression

$$\mathfrak{H}=H(0; I, m; T, n)-\lambda[\sum_{kst}'(1-2m_z(kst))-A] \tag{2·7}$$

minimum by treating the spins as classical quantities and putting

$$m_z(kst)=\frac{1}{2}\cos\theta(kst),$$
$$m_x(kst)=\frac{1}{2}\sin\theta(kst)\cos\varphi(kst), \tag{2·8}$$
$$m_y(kst)=\frac{1}{2}\sin\theta(kst)\sin\varphi(kst),$$

where λ is the chemical potential. The Fermi momentum k_F is defined by

$$\lambda=\omega_{k_F}.$$

We will assume the same Fermi momenta for the neutron- and the proton-system. The minimum point is given by the solution of the following equations:

$$0=\varepsilon(k)\sin\theta(kst)+\frac{1}{\Omega}\cos\theta(kst)\sum_{k's't'}'(kst|V_{IT}^{mn}|k's't')\sin\theta(k's't')$$
$$\times\cos(\varphi(kst)-\varphi(k's't')),$$

$$0=\frac{1}{\Omega}\sum_{k's't'}{}' (\boldsymbol{k}st|V_{IT}^{mn}|\boldsymbol{k}'s't') \sin\theta(\boldsymbol{k}st) \sin\theta(\boldsymbol{k}'s't') \sin(\varphi(\boldsymbol{k}st)-\varphi(\boldsymbol{k}'s't')),$$

where

$$\varepsilon(\boldsymbol{k})=2(\omega_k-\lambda).$$

From these follow the relations

$$\varphi(\boldsymbol{k}st) \text{ is independent of } (\boldsymbol{k}, s, t),$$

$$\cos\theta(\boldsymbol{k}st)=\frac{\varepsilon(\boldsymbol{k})}{\sqrt{\varepsilon^2(\boldsymbol{k})+4f^2(\boldsymbol{k})}}, \quad \sin\theta(\boldsymbol{k}st)=-\frac{2f(\boldsymbol{k}st)}{\sqrt{\varepsilon^2(\boldsymbol{k})+4f^2(\boldsymbol{k})}} \tag{2·9}$$

in which $f(\boldsymbol{k}st)$ and $f^2(\boldsymbol{k})$ are defined by

$$f(\boldsymbol{k}st)=\frac{1}{2\Omega}\sum_{k's't'}(\boldsymbol{k}st|V_{IT}^{mn}|\boldsymbol{k}'s't') \sin\theta(\boldsymbol{k}'s't'),$$

$$f(\boldsymbol{k})^2=f(\boldsymbol{k}st)^2,$$

and satisfy the equation

$$f(\boldsymbol{k}st)=-\frac{1}{\Omega}\sum_{k's't'}{}' (\boldsymbol{k}st|V_{IT}^{mn}|\boldsymbol{k}'s't')\frac{f(\boldsymbol{k}'s't')}{\sqrt{\varepsilon(\boldsymbol{k}')^2+4f(\boldsymbol{k}')^2}}. \tag{2·10}$$

This is nothing but the equation found in the theory of the superfluid nuclear matter. Introducing the Fourier transform of the correlation function $\chi(\boldsymbol{x})$ by

$$\chi(\boldsymbol{k})=\frac{1}{2}\sum_{st}\left(\frac{1}{2}s\frac{1}{2}m-s|Im\right)\left(\frac{1}{2}t\frac{1}{2}n-t|Tn\right)\frac{f(\boldsymbol{k}st)}{\sqrt{\varepsilon(\boldsymbol{k})^2+4f(\boldsymbol{k})^2}},$$

one can rewrite (2·10) as

$$\chi(\boldsymbol{k})=-\frac{1}{\sqrt{\varepsilon(\boldsymbol{k})^2+4f(\boldsymbol{k})^2}}\cdot\frac{1}{\Omega}\sum_{k'}V(\boldsymbol{k}-\boldsymbol{k}', I, T)\chi(\boldsymbol{k}'), \tag{2·11}$$

where $\chi(-\boldsymbol{k})$ is defined by

$$\chi(-\boldsymbol{k})=-(-1)^{I+T}\chi(\boldsymbol{k}), \tag{2·12}$$

and $V(\boldsymbol{k}-\boldsymbol{k}', I, T)$ is given in (A·1). It is evident that Eq. (2·11) has only a trivial solution $\chi(\boldsymbol{k})=0$ if the interaction is repulsive, i.e. $V(\boldsymbol{k}-\boldsymbol{k}', I, T)>0$.

An approximation will be made here. If $f(\boldsymbol{k})^2$ in the denominator of (2·11) is slowly varying around the point $k=k_F$, it may be replaced by a constant $f(\boldsymbol{k}_F)^2=\varepsilon_0^2$, since it is appreciable only at that point. At k far from k_F, ε_0^2 may be neglected in comparison with $\varepsilon(\boldsymbol{k})^2$. These properties can be retained if one replaces $\sqrt{\varepsilon(\boldsymbol{k})^2+4\varepsilon_0^2}$ by $|\varepsilon(\boldsymbol{k})|+\mu\varepsilon_0$.* μ will be determined so as to satisfy the relation

* The relations used by Emery and Sessler[11] can be obtained even with this approximation if Eq. (2·13) is satisfied. This means that the above approximation gives ε_0 with a sufficient accuracy.

$$\int_{k_F-\kappa}^{k_F+\kappa} \frac{dk}{|\varepsilon(k)|+\mu\varepsilon_0} = \int_{k_F-\kappa}^{k_F+\kappa} \frac{dk}{\sqrt{\varepsilon(k)^2+4\varepsilon_0^2}}, \tag{2·13}$$

where κ is a quantity such that

$$\varepsilon(k_F \pm \kappa)^2 \gg \varepsilon_0^2,$$

$$|\kappa - k_F| \ll 1.$$

The quantity (2·13) must be kept invariant as the function of ε_0 under any approximation, since ε_0 is determined by an equation which involves only the quantity (2·13).[12] Eq. (2·13) gives

$$\mu = 1.$$

This approximation must be satisfactory as far as ε_0 is sufficiently small. Physically speaking, it changes the exponential decrement of $\chi(x)$ to the inverse power decrement for large $|x|$. However, the correlation function can be well reproduced for small $|x|$ since the equation for $\chi(k)$ is accurate for large $|k|$. Therefore the present application must give the correct criterion for the existence of a superfluid state. Eq. (2·11) takes the form[13]

$$|\omega_k - \omega_{k_F}|\chi(k) = -\frac{1}{2\Omega}\sum_{k'} V(k-k', I, T)\chi(k') - \frac{\varepsilon_0}{2}\chi(k). \tag{2·14}$$

This is the Euler equation resulting from the variational principle

$$-\frac{\varepsilon_0}{2} = \text{Min.}\left\{\sum_k \chi(k)^* |\omega_k - \omega_{k_F}|\chi(k) + \frac{1}{2\Omega}\sum_{kk'}\chi(k)^* V(k-k', I, T)\chi(k')\right\} \Big/ \sum_k \chi(k)^*\chi(k). \tag{2·15}$$

This has been used in CMS as the criterion for a superfluid phase. It is now known that this variational principle gives a lower bound for ε_0. From (2·6), (2·8) and (2·9) ε_0 is found to give the magnitude of diffuseness of the Fermi surface.

Substitution of (2·8) and (2·9) in (2·7) gives the classical minimum energy. The energy shift is obtained as the difference from the Fermi energy as follows:

$$\Delta E = \sum_k \varepsilon(k) + \omega_{k_F} A - 4\sum_{k \geq k_F}\omega_k - \sum_k \frac{\varepsilon(k)^2}{\sqrt{\varepsilon(k)^2+4f(k)^2}} + 2\Omega\int dx\,\chi(x)^* V(x, I, T)\chi(x), \tag{2·16}$$

where

$$V(x, I, T) = \frac{1}{\Omega}\sum_k e^{-ikx} V(k, I, T)$$

and

$$\chi(\boldsymbol{x})=\frac{1}{\Omega}\sum_{k}e^{-i\boldsymbol{k}\boldsymbol{x}}\chi(\boldsymbol{k}).$$

This is the same result as that given in CMS in the case of neutral Fermions.

§ 3. S-wave solution in the case of weak attraction

As the potential is spherically symmetric Eq. (2·14) can be reduced to a set of eigenvalue equations for each partial wave. Let us consider the S-wave solution. It requires $I+T$ to be unity from (2·12). The correlation function now takes the form

$$\chi(\boldsymbol{x})=u(r)/r,$$

where we have used the non-dimensional variable

$$\boldsymbol{r}=k_F\boldsymbol{x}. \tag{3·1}$$

Eq. (2·14) can be rewritten as

$$\frac{d^2u}{dr^2}+u=\frac{m^*}{k_F{}^2}\left[v(r,\,I,\,T)u(r)-2\int_0^\infty ds\,v(s,\,I,\,T)u(s)\chi(r,\,s)\right]$$
$$+\beta^2\left[u(r)-2\int_0^\infty ds\,u(s)\chi(r,\,s)\right], \tag{3·2}$$

where the effective mass approximation

$$\omega_k=\boldsymbol{k}^2/2m^*$$

is adopted and we put

$$v(r,\,I,\,T)=V(\boldsymbol{r}/k_F,\,I,\,T),$$
$$\chi(r,\,s)=\frac{1}{\pi}\left\{\frac{\sin(r-s)}{r-s}-\frac{\sin(r+s)}{r+s}\right\},$$
$$\beta^2=m^*\varepsilon_0/k_F{}^2.$$

The two-body potential $v(r,\,I,\,T)$ is assumed to be

$$v(r,\,I,\,T)=\begin{cases}\infty, & \text{at } r<k_Fr_c=c,\\ \text{arbitrary but non-singular} & \text{at } c<r<a=k_Fr_a,\\ 0, & \text{at } a<r.\end{cases}$$

The method used by Bethe and Goldstone[14] to study their equation applies to (3·2). The function

$$w(r)=\frac{m^*}{k_F{}^2}v(r,\,I,\,T)\,u(r)$$

takes the form

$$w(r)=A\delta(r-c)+w_1(r) \quad \text{at} \quad 0<r\leq c,$$

where

$$A=u_1'(c+0).$$

For convenience, the following notations will be used:

$$u(r)=\begin{cases} u_1(r), & \text{at} \quad c<r<a, \\ u_2(r), & \text{at} \quad a<r. \end{cases} \tag{3·3}$$

From (3·2), $w_1(r)$ satisfies

$$w_1(r)=2A\chi(r, c)+2\int_0^c ds w_1(s)\chi(r, s)$$

$$+\frac{2m^*}{k_F^2}\int_c^a ds v(s, I, T)u_1(s)\chi(r, s)+2\beta^2\int_c^\infty ds u(s)\chi(r, s). \tag{3·4}$$

At $r>c$, Eq. (3·2) can be rewritten as

$$\frac{d^2u}{dr^2}+\left(1-\frac{m^*}{k_F^2}v(r, I, T)-\beta^2\right)u(r)=-\frac{2m^*}{k_F^2}\int_c^a ds v(s, I, T)u_1(s)\chi(r, s)$$

$$-2\beta^2\int_c^\infty ds u(s)\chi(r, s)-2A\chi(r, c)-2\int_0^c ds w_1(s)\chi(r, s). \tag{3·5}$$

Now we shall consider a low density system. The Fermi momentum k_F and the range parameters, a and c, are small. The quantity ε_0 is now assumed not to be larger than k_F^2 in this limit. It will be seen later that this assumption will correspond to the case of the weak attraction. Let us first consider the solution $u_1(r)$ at $c<r<a$. From (3·3) and (3·4) the k_F dependence of A and w_1 is found to be

$$A\sim u(r)/k_F,$$

$$w_1\sim k_F u(r).$$

When only the terms proportional to $1/k_F$ are retained in (3·5) it is simplified to

$$\frac{d^2u_1}{dr^2}=\frac{m^*}{k_F^2}v(r, I, T)u_1,$$

which is the same equation as for zero energy scattering of the two particles with reduced mass $m^*/2$. If the scattering length of such particles is denoted by l (in the non-dimensional unit (3·1)) the logarithmic derivative of the correlation function at $r=a$ is given by

$$\frac{u_1'(a)}{u_1(a)}=\frac{1}{a-l}. \tag{3·6}$$

The equation for $u_2(r)$ at $r>a$ is reduced to

$$\frac{d^2u_2}{dr^2}+(1-\beta^2)u_2=-\frac{2m^*}{k_F^2}\int_c^a ds\,v(s,\,I,\,T)u_1(s)\chi(r,\,s)$$

$$-2\beta^2\int_0^\infty ds\,u_2(s)\chi(r,\,s)-2A\chi(r,\,c), \tag{3·7}$$

where in the last integral the asymptotic form of $u_2(s)$ is used at $0<s<a$. This modification can be made within the low density approximation. The first term on the right-hand side can be rewritten

$$-\frac{2m^*}{k_F^2}\int_c^a ds\,v(s,\,I,\,T)u_1(s)\chi(r,\,s)=-2\int_c^a ds\,\frac{d^2u_1}{ds^2}\chi(r,\,s)$$

$$=-2\,\frac{du_1(a)}{da}\chi(r,\,a)+2u_1(a)\,\frac{d\chi(r,\,a)}{da}+2A\chi(r,\,c)-2\int_c^a ds\,u_1(s)\,\frac{d^2\chi(r,\,s)}{ds^2},$$

in which the last term may be neglected as it is proportional to k_F^2. When the unknown $u_2(r)$ is represented in the form

$$u_2(r)=\int_0^\infty d\kappa\,\eta(\kappa)\sin\kappa r,$$

$\eta(\kappa)$ satisfies the relation

$$\int_0^\infty d\kappa(1-\beta^2-\kappa^2)\eta(\kappa)\sin\kappa r=\int_0^1 d\kappa\left[-2\beta^2\eta(\kappa)+\frac{4}{\pi}u_1(a)\,\frac{l\kappa}{l-a}\right]\sin\kappa r, \tag{3·8}$$

where $\chi(r,\,a)$ is written in the form

$$\chi(r,\,a)=\frac{2}{\pi}\int_0^1 d\kappa\sin\kappa a\sin\kappa r,$$

and Eq. (3·6) is used. As the function represented by the sine transform is necessarily an odd function, it must have a discontinuity at the origin $r=0$ if it does not vanish there. The second derivative of such a function has the form

$$\frac{d^2u_2(r)}{dr^2}=2u_2(0)\,\delta'(r)+\text{non-singular term.}$$

After adding the first term on the right-hand side to (3·8) one equates the coefficients of $\sin\kappa r$ and finds

$$\eta(\kappa)=\begin{cases}\dfrac{2}{\pi}u_2(0)\dfrac{\kappa}{\kappa^2-1+\beta^2}, & \kappa>1,\\ \dfrac{2}{\pi}F\dfrac{\kappa}{\kappa^2-1-\beta^2}, & \kappa<1,\ F=u_2(0)-\dfrac{2u_1(a)l}{l-a}.\end{cases} \tag{3·9}$$

The function $u_2(r)$ now takes the form

$$u_2(r)=\frac{2}{\pi}F\int_0^1 d\kappa\frac{\kappa}{\kappa^2-1-\beta^2}\sin\kappa r+\frac{2}{\pi}u_2(0)\int_1^\infty d\kappa\frac{\kappa}{\kappa^2-1+\beta^2}\sin\kappa r.$$

In order to derive the boundary conditions at $r=a$, $u_2(r)$ is developed in power series in r since it is the expansion in powers of k_F,

$$u_2(r)=u_2(0)+\frac{r}{\pi}(F\omega_1-u_2(0)\omega_0)\log\beta^2$$
$$+\frac{2}{\pi}r\left\{F(1-\omega_1\log(1+\omega_1))-u_2(0)(1-\omega_0\log(1+\omega_0))\right\},$$

where

$$\omega_0=\sqrt{1-\beta^2}\quad\text{and}\quad\omega_1=\sqrt{1+\beta^2}.$$

As $\log\beta^2$ will be found to be proportional to $1/k_F$, the first two terms are independent of k_F and the last is of the order of k_F. The boundary conditions

$$u_1(a)=u_2(a),\quad u_2'(a)/u_2(a)=1/(a-l)$$

and the definition of F, (3·9), lead to

$$\pi=l(\omega_0+\omega_1)\log\beta^2+2l\{2-\omega_0\log(1+\omega_0)-\omega_1\log(1+\omega_1)\}. \tag{3·10}$$

The left-hand side is independent of k_F but the right-hand side is proportional to it except for $\log\beta^2$. Therefore $|\log\beta^2|$ must be large, that is, β^2 must be small. As $\log\beta^2$ is negative, its coefficient must be negative. Thus it is concluded that we have a non-trivial solution for (2·14) in the low density limit if the scattering length l is negative. Since the effective mass m^* should be reduced to m in the low density limit, the nuclear matter would have a superfluid state in this limit as the scattering length in the singlet state is actually negative. This possibility was first pointed out by Moszkowski[15] in the case of singular interactions. ε_0 is obtained from (3·10) as

$$\varepsilon_0=\frac{4k_F^2}{m^*}\exp\left(\frac{\pi}{2l}-2\right), \tag{3·11}$$

which is not larger than k_F^2. This verifies our former assumption made below (3·5). The quantity (3·11) is smaller than that obtained by Emery and Sessler.[11] As already remarked, ε_0 gives the magnitude of diffuseness of the Fermi surface, and Eq. (3·11) shows that in the case of weak attraction only the surface layer is diffused as indicated in the figure in reference 2).

§ 4. S-wave solution in the case of strong attraction

When the attractive interaction is strong enough to create a bound state, the scattering length takes a positive value. According to the results in § 3, ε_0 seems to vanish. This strange result should be due to the improper assumption that ε_0 is not larger than k_F^2. A low density system with such an interaction would be an assembly of the composite particles. Therefore the distribution of particles in the momentum space should be determined by the internal structure of the composite particle. The quantity ε_0 is now assumed to be independent of k_F which will be verified later.

Let us consider the solution $u_1(r)$ of (3·4) and (3·5) at $c<r<a$. The k_F dependence of A and w_1 is now given by

$$A \sim u(r)/k_F,$$

$$w_1 \sim u(r)/k_F.$$

If one retains in (3·5) only the terms proportional to $1/k_F^2$, it is simplified to

$$\frac{d^2u_1(r)}{dr^2} - \left(\beta^2 + \frac{m^*}{k_F^2} v(r, I, T)\right) u_1(r) = -2\beta^2 \int_0^\infty ds u_2(s)\, \chi(r, s), \qquad (4\cdot1)$$

where the same modification applies as (3·7). In the same way Eq. (3·5) for $u_2(r)$ is reduced to

$$\frac{d^2u_2(r)}{dr^2} - \beta^2 u_2(r) = -2\beta^2 \int_0^\infty ds u_2(s)\, \chi(r, s). \qquad (4\cdot2)$$

Putting

$$u_2(r) = \int_0^\infty d\kappa \eta(\kappa) \sin \kappa r, \qquad (4\cdot3)$$

in (4·2) one finds

$$\eta(\kappa) = \begin{cases} \dfrac{2}{\pi} u_2(0) \dfrac{\kappa}{\kappa^2+\beta^2}, & \kappa > 1, \\ \dfrac{2}{\pi} u_2(0) \dfrac{\kappa}{\kappa^2-\beta^2}, & \kappa < 1, \end{cases}$$

where use has been made of the same method applied to (3·7). The function $u_2(r)$ now takes the form

$$u_2(r) = \int_1^\infty \frac{2u_2(0)}{\pi} d\kappa \frac{\kappa}{\kappa^2+\beta^2} \sin \kappa r + \frac{2}{\pi} u_2(0) \int_0^1 d\kappa \frac{\kappa}{\kappa^2-\beta^2} \sin \kappa r.$$

One may extend the first integral to the origin, since the integral along $0<\kappa<1$

gives a small contribution, and may neglect the second integral which leads to

$$u_2(r) = u_2(0) \exp(-\beta r).$$

This is the wave function of the two-particle relative motion in a bound state. The two-particle correlation in a medium is determined by themselves alone when the attractive interaction is strong, that is, the correlation function decreases even in a distance smaller than the mean distance. The right-hand side of (4·1) may be neglected as one can see by substituting (4·3). The function $u_1(r)$ satisfies the Schrödinger equation of a two-particle system. Thus we find that the criterion of existence of a non-trivial solution for the eigenvalue problem is nothing but the criterion for the existence of bound state in the two-particle system. This will be satisfied in a low density system since the two-particle system has a bound state in the triplet state. The magnitude of ε_0 is equal to the binding energy which is independent of the density. The previous assumption is verified in this way. Since the quantity ε_0 is larger than $k_F^2/2m^*$ in a low density system, the Fermi distribution is completely diffused as shown in a figure of reference 2). This verifies the expectation that the system is an assembly of composite particles.

§ 5. Discussions

It is a common feature of the two solutions in the preceding sections that the correlation functions, in the region of small distance, take the form of the wave function of a two-particle system. In the large distances outside the mean distance of the particles it deviates gradually from that of the two-particle system on account of the effect of other particles, if the interaction is weak. When the interaction is strong enough to create a bound state the correlation function is the same with the wave function even in the large distance. The characteristics of those solutions can be summarized as follows. The quantity ε_0 vanishes and the Fermi distribution is stable, if the two-body potential is repulsive, or in other words, if the scattering length of the two-particle system is positive and there are no bound states. On the other hand, the quantity ε_0 is not larger than the energy of a one-particle level at the Fermi surface, if the interaction is attractive and weak leading to a negative scattering length, so that a diffused Fermi distribution becomes stable. The energy gap is shown to be smaller by the factor $\exp(-2)$ than that of Emery and Sessler (Eq. (19) in reference 11)). This suggests the fact that the estimation of the energy gap by making use of the scattering phase shift might not be reliable except in the low density limit. Finally, if the interaction is strong enough to give a bound state, the quantity has the same magnitude with the binding energy of the two-particle system which is independent of the medium density, so that the Fermi distribution is diffused completely.

There are some remaining problems to be considered. The results of various experiments suggest the existence of an energy gap with the order of magnitude 1 Mev. However, any theoretical calculation ever made could not derive such a large energy gap for the infinite matter. It is desirable to obtain a satisfactory solution of (2·14) in the case of actual density.

We have so far considered only the correlation between the nucleons with zero total momentum which is given by the Hamiltonian $H(0\,;\,I,\,m\,;\,T,\,n)$. It has given the energy shift (2·16) which is proportional to Ω. If the other correlations described by $H(\boldsymbol{p}\,;\,I,\,m\,;\,T,\,n)$ would give such large energy shifts independently, the total energy shift becomes of the order of Ω^2. This fact indicates the insufficiency of the independent-pair-approximation. This approximation may be applied when the system has a configuration similar to that of the degenerate Fermi gas, where pairs with different total momenta, or with different total spins or isotopic spins, may be regarded as independent, since the commutators of the operators referring to such pairs have always the vanishing expectation value with respect to the ground state. This may not be true when the configuration is drastically altered. We may be able to overcome this difficulty if one uses the description in terms of quasi-particles introduced by Bogoliubov.[16] Since the ground state may be well approximated by the vacuum with respect to the quasi-particles the independent-pair-approximation will hold for them.

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Appendix

The most general two-body potential of the exchange type (static and central) has form

$$V=V_W(\boldsymbol{x})+V_M(\boldsymbol{x})\,P^M+V_B(\boldsymbol{x})\,P^B+V_H(\boldsymbol{x})\,P^H,$$

where P's are the exchange operators defined by

$$P^B=\frac{1}{2}(1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2),$$

$$P^H=-\frac{1}{2}(1+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2),$$

$$P^M=-\frac{1}{4}(1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(1+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2).$$

Suffices 1 and 2 refer to the two interacting particles, respectively. Introducing the Fourier transforms

$$V_i(\boldsymbol{k}) = \int V_i(\boldsymbol{x})\, e^{i\boldsymbol{k}\boldsymbol{x}}\, d\boldsymbol{x},$$

one makes use of the notations

$$V(\boldsymbol{k}-\boldsymbol{k}',\, I,T) = V_W(\boldsymbol{k}-\boldsymbol{k}') - (-1)^{I+T} V_M(\boldsymbol{k}-\boldsymbol{k}') - (-1)^I V_B(\boldsymbol{k}-\boldsymbol{k}') + (-1)^T V_H(\boldsymbol{k}-\boldsymbol{k}'). \tag{A·1}$$

The quantity $(\boldsymbol{k}|V_{IT}|\boldsymbol{k}')$ in (2·1) is defined by

$$(\boldsymbol{k}|V_{IT}|\boldsymbol{k}') = V(\boldsymbol{k}-\boldsymbol{k}',\, I,\, T) - (-1)^{I+T} V(\boldsymbol{k}+\boldsymbol{k}',\, I,\, T). \tag{A·2}$$

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Integral Representations for Scattering Amplitudes in Perturbation Theory

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Some integral representations for scattering amplitudes are proposed, which are more general than the Mandelstam representation. Their support properties are investigated for some practical cases and for the general case in every order of perturbation theory. In the Appendices Nambu-Symanzik's formula is proved in terms of the Feynman parametric integral, and an example (for the two-particle scattering) is given in which stability conditions are satisfied but no dispersion relation holds.

§ 1. Introduction

Mandelstam[1] proposed a conjecture on the possibility of a double dispersion representation for the two-particle scattering amplitude. Its validity has been investigated by many authors,[2] and it seems that optimistic standpoints are generally accepted. But until now no complete proof is given even in perturbation theory[3] in spite of many efforts. So it will be also possible to take a more pessimistic standpoint. Even if the Mandelstam representation is indeed correct, its proof will be too difficult to be given within a few years ahead and moreover this representation cannot be generalized to production processes as is easily seen.

In these circumstances, it will probably be worthwhile to investigate some other integral representations for scattering amplitudes. The present author already proved integral representations for the vertex function with two or three variables.[4] The purpose of this paper is to propose such representations for scattering amplitudes which are valid in every order of perturbation theory. This is not a trivial generalization.

In § 2, as preliminaries Nambu-Symanzik's formula[5] for the general term of perturbation theory is given in terms of the Feynman parametric integral, and corresponding integral representations are proposed. In § 3, the support properties of weight functions are investigated in the cases of the nucleon-nucleon, pion-nucleon and kaon-nucleon scatterings. The subsequent two sections are devoted to the consideration of support properties in the general case.

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In this connection we introduce a new concept which is convenient for indicating mass-spectral conditions in *each Feynman graph*. This concept seems to be very essential in graphical considerations.

In the final section the relations between the proposed integral representations and the Mandelstam representation and the dispersion relation are discussed.

Throughout this paper we use the terminology defined in § 2 of N.*

§ 2. Integral representations

The general term of perturbation theory (corresponding to a Feynman graph G) is given by

$$\text{const.} \int_0^1 \cdots \int_0^1 \frac{\delta(1-\sum_{i=1}^{N} x_i) \prod_{i=1}^{N} dx_i}{U^2(V-i\varepsilon)^{N-2n}}, \tag{2·1}$$

where we have assumed for simplicity that all particles are scalar and all couplings are direct. The notations used in (2·1) are as follows:

N: number of internal lines belonging to G,

n: number of independent circuits of G,

ε: infinitesimal positive quantity,

x_i: Feynman parameter,

$U = \sum x_{\nu_1} x_{\nu_2} \cdots x_{\nu_n}$ where the summation goes over all possible sets such that any circuit in G contains at least one line among $\{\nu_1, \nu_2, \cdots, \nu_n\}$;

and

$$V \equiv \sum_{i=1}^{N} x_i m_i^2 + \frac{1}{U} \sum_{S} W_S k_S^2, \tag{2·2}$$

where

m_i: internal mass,

S: intermediate state (the summation $\sum_S$ goes over all possible intermediate states),

k_S: effective external momentum in the reduced graph which is obtained by shrinking each line of $G-S$ to one point (metric: $k^2 = \boldsymbol{k}^2 - k_0^2$),

W_S: $U^{(H_1)} U^{(H_2)} \prod_{i \in S} x_i$ where H_1 and H_2 are two subgraphs obtained by the Schnitt S and where $U^{(H)}$ means the U-function for a graph H ($U^{(H)} = 1$ if H includes no circuit).

A proof of (2·2) will be presented in Appendix A. The formula (2·1) with (2·2) is just the Feynman-parametric form of Nambu-Symanzik's formula,[5] in which integration parameters are inversely proportional to the Feynman parameters.

* N stands for reference 4).

We denote the set of (effective) external lines of G by g and a subset of g by h. The complement of h is denoted by $h^*(=g-h)$. Then it is obvious that

$$\begin{aligned}&(h^*)^*=h,\\&(h_1\cap h_2)^*=h_1^*\cup h_2^*,\\&(h_1\cup h_2)^*=h_1^*\cap h_2^*.\end{aligned}\tag{2·3}$$

A Schnitt S divides g into two parts h and h^*. Then the momentum k_S is expressed as

$$k_S=\sum_{A_j\in h} k_j=-\sum_{A_j\in h^*} k_j.\tag{2·4}$$

So we may denote it by k_h. Then (2·2) is rewritten as

$$V=\sum_{i=1}^{N} x_i m_i^2-\sum_h \zeta_h s_h.\tag{2·5}$$

with

$$s_h\equiv -k_h^2=-k_{h^*}^2,\tag{2·6}$$

$$\zeta_h\equiv\frac{1}{U}\sum_{S\in\mathfrak{S}(h)} W_S\geqq 0.\tag{2·7}$$

The summation $\sum_h'$ in (2·5) means to take the sum over all possible divisions $(h|h^*)$ of g. The summation in (2·7) means to take the sum over all intermediate states corresponding to the division $(h|h^*)$.

We denote the number of the elements belonging to a set $\mathfrak{N}$ by $\nu(\mathfrak{N})$. Then the number of the squares s_h is equal to $2^{\nu(g)-1}-1$. So when $\nu(g)\geqq 4$, all squares are not linearly independent.*

Now, we consider integral repesentations. We multiply (2·1) by

$$1\equiv\int_0^\infty d\kappa\,\delta(\kappa-\sum_{i=1}^{N} x_i m_i^2/\xi)\prod_h\int_0^1\delta(z_h-\zeta_h/\xi)\,dz_h\tag{2·8}$$

with

$$\xi=\sum_h \zeta_h\geqq 0.\tag{2·9}$$

After integrations by parts, we obtain an integral representation

$$\int_0^\infty d\kappa\int_0^1\cdots\int_0^1\delta(1-\sum_h z_h)\prod_h dz_h\frac{\varphi(\kappa,z_h)}{\kappa-\sum_h z_h s_h-i\varepsilon}.\tag{2·10}$$

The most important problem is to find the support of the weight function $\varphi(\kappa, z_h)$.

* The number of independent squares is $4\nu(g)-10$ for $\nu(g)\geqq 4$

In particular, when $\nu(g)=2$, (2·10) reduces to

$$\int_{M^2}^{\infty} d\kappa \frac{\varphi(\kappa)}{\kappa - s - i\varepsilon} \tag{2·11}$$

where M stands for the lowest intermediate-state mass. This is the well-known Umezawa-Kamefuchi-Källén-Lehmann representation for the one-body propagator. When $\nu(g)=3$, (2·10) becomes

$$\int_0^{\infty} d\kappa \int_0^1 dz_A \int_0^1 dz_B \int_0^1 dz_C \delta(1-z_A-z_B-z_C) \frac{\varphi(\kappa, z_A, z_B, z_C)}{\kappa - z_A s_A - z_B s_B - z_C s_C - i\varepsilon} \tag{2·12}$$

where A, B and C are the three external lines of the vertex part. The support property of $\varphi(\kappa, z_A, z_B, z_C)$ was investigated in § 16 of N, that is, φ vanishes unless

$$\begin{aligned} \kappa \geqq \mathrm{Max}[&M_A^2 z_A + M_B^2 z_B + (M_A - M_B)^2 z_C, \\ &M_A^2 z_A + (M_A - M_C)^2 z_B + M_C^2 z_C, \\ &(M_B - M_C)^2 z_A + M_B^2 z_B + M_C^2 z_C], \end{aligned} \tag{2·13}$$

where M_A, M_B and M_C are the respective lowest intermediate-state masses, provided that they satisfy triangular inequalities.*

The case $\nu(g)=4$ is the two-particle scattering. We denote the four external lines by A, B, C, D. The possible divisions of g are expressed by

$$h = \{A\},\ \{B\},\ \{C\},\ \{D\},\ \{AB\},\ \{AC\},\ \{AD\}. \tag{2·14}$$

The corresponding squares s_h are denoted by M_a^2, M_b^2, M_c^2, M_d^2, s, t, u, respectively. Between them the well-known identity

$$M_a^2 + M_b^2 + M_c^2 + M_d^2 = s + t + u \tag{2·15}$$

holds. The first four squares are usually fixed on mass shells. Then putting

$$\lambda = \frac{\sum_{i=1}^{N} x_i m_i^2 - (\zeta_A + \zeta_0) M_a^2 - (\zeta_B + \zeta_0) M_b^2 - (\zeta_C + \zeta_0) M_c^2 - (\zeta_D + \zeta_0) M_d^2}{\zeta_{AB} + \zeta_{AC} + \zeta_{AD} - 3\zeta_0} \tag{2·16}$$

and

$$\begin{aligned} z_1 &= \frac{\zeta_{AB} - \zeta_0}{\zeta_{AB} + \zeta_{AC} + \zeta_{AD} - 3\zeta_0}, \\ z_2 &= \frac{\zeta_{AC} - \zeta_0}{\zeta_{AB} + \zeta_{AC} + \zeta_{AD} - 3\zeta_0}, \end{aligned} \tag{2·17}$$

* If triangular inequalities are not satisfied, instead of (2·13) we get

$$\kappa \geqq \mathrm{Max}\,[M_A^2 z_A + M_B^2 z_B,\ M_A^2 z_A + M_C^2 z_C,\ M_B^2 z_B + M_C^2 z_C]$$

by the method which will be stated in § 4 and § 5.

$$z_3=\frac{\zeta_{AD}-\zeta_0}{\zeta_{AB}+\zeta_{AC}+\zeta_{AD}-3\zeta_0}$$

instead of (2·8), where ζ_0 is an arbitrary function satisfying

$$\zeta_0 \leqq \mathrm{Min}(\zeta_{AB}, \zeta_{AC}, \zeta_{AD}), \tag{2·18}$$

we get

$$\int_{-\infty}^{\infty} d\lambda \int_0^1 dz_1 \int_0^1 dz_2 \int_0^1 dz_3 \delta(1-\sum_{j=1}^{3} z_j) \frac{\psi(\lambda, z_1, z_2, z_3)}{\lambda - z_1 s - z_2 t - z_3 u - i\varepsilon}. \tag{2·19}$$

The support in this integral representation is ambiguous on account of the identity (2·15). For example, if ζ_0 tends to minus infinity, the support shrinks to a point

$$\begin{aligned} &\lambda=(M_a^2+M_b^2+M_c^2+M_d^2)/3, \\ &z_j=1/3 \quad (j=1, 2, 3). \end{aligned} \tag{2·20}$$

Thus (2·19) is not interesting.

In order to avoid such an ambiguity, we put

$$\zeta_0=\mathrm{Min}(\zeta_{AB}, \zeta_{AC}, \zeta_{AD}). \tag{2·21}$$

Then (2·19) reduces to

$$\begin{aligned} &\int_{-\infty}^{\infty} d\alpha \int_0^1 dx \frac{\rho_{12}(\alpha, x)}{\alpha - xs - (1-x)t - i\varepsilon} + \int_{-\infty}^{\infty} d\beta \int_0^1 dy \frac{\rho_{23}(\beta, y)}{\beta - yt - (1-y)u - i\varepsilon} \\ &+ \int_{-\infty}^{\infty} d\gamma \int_0^1 dz \frac{\rho_{31}(\gamma, z)}{\gamma - zu - (1-z)s - i\varepsilon} \end{aligned} \tag{2·22}$$

plus three possible single dispersion terms (and a constant term). Here the first, second and third terms in (2·22) correspond to the cases $\zeta_0=\zeta_{AD}$, $\zeta_0=\zeta_{AB}$, and $\zeta_0=\zeta_{AC}$, respectively. In the next section the support properties of (2·22) will be investigated for some practical cases.

When M_a^2, M_b^2, M_c^2, M_d^2 are considered to be variables, we can derive an integral representation, which consists of seven terms such as

$$\begin{aligned} &\int_0^{\infty} d\kappa \int_0^1 dz_A \cdots \int_0^1 dz_{AC} \delta(1-z_A-\cdots-z_{AC}) \\ &\times \frac{\varphi(\kappa, z_A, z_B, z_C, z_D, z_{AB}, z_{AC})}{\kappa - z_A M_a^2 - z_B M_b^2 - z_C M_c^2 - z_D M_d^2 - z_{AB} s - z_{AC} t - i\varepsilon} + \cdots \end{aligned} \tag{2·23}$$

with some possible subtraction terms, by a procedure similar to the above ($\zeta_0=$ Min $(\zeta_A, \zeta_B, \zeta_C, \zeta_D, \zeta_{AB}, \zeta_{AC}, \zeta_{AD})$ in the present case). The support properties will be investigated in § 5.

We can likewise deal with the cases $\nu(g) \geqq 5$. But the number of terms will become enormously large.

§ 3. Support properties in some practical cases

In this section the support properties of the weight functions in (2·22) are investigated for the nucleon-nucleon, pion-nucleon and kaon-nucleon scatterings. For this purpose, it is convenient to make use of the method used in the proof of dispersion relations (see § 14 and § 15 of N).

We denote the four external momenta by k_A, k_B, k_C, k_D. The V-function for the two-particle scattering is

$$\begin{aligned} V &= \sum_{i=1}^{N} x_i m_i^2 + \sum_{J=A,B,C,D} \zeta_J k_J^2 + \zeta_{AB}(k_A+k_B)^2 + \zeta_{AC}(k_A+k_C)^2 + \zeta_{AD}(k_A+k_D)^2 \\ &= \sum_{i=1}^{N} x_i m_i^2 + \sum_{J} (\zeta_J+\zeta_{AD}) k_J^2 + (\zeta_{AB}-\zeta_{AD})(k_A+k_B)^2 + (\zeta_{AC}-\zeta_{AD})(k_A+k_C)^2. \end{aligned} \tag{3·1}$$

We assume that the graph G contains no one-particle line, otherwise it reduces to simpler graphs.

We first consider the equal-mass case. Let k_1 and k_2 be two vectors having the following properties:

$$-k_1^2 = -k_2^2 = \mu^2, \quad k_1 k_2 = 0. \tag{3·2}$$

We abbreviate $a_1 k_1 + a_2 k_2$ as (a_1, a_2). We then have proved in § 15 of N that if the four external momenta are given by (1, 1), (1, −1), (−1, 1) and (−1, −1) then the V-function is non-negative definite. For example, when we put

$$\begin{aligned} k_A &= (1, 1), \qquad k_B = (1, -1), \\ k_C &= (-1, 1), \quad k_D = (-1, -1), \end{aligned} \tag{3·3}$$

we get

$$\sum_i x_i \mu^2 - \sum_J (\zeta_J+\zeta_{AD}) \cdot 2\mu^2 - (\zeta_{AB}-\zeta_{AD})(2\mu)^2 - (\zeta_{AC}-\zeta_{AD})(2\mu)^2 \geqq 0. \tag{3·4}$$

Since

$$\alpha = \frac{\sum_i x_i \mu^2 - \sum_J (\zeta_J+\zeta_{AD})\mu^2}{\zeta_{AB}+\zeta_{AC}-2\zeta_{AD}}$$

and

$$x = \frac{\zeta_{AB}-\zeta_{AD}}{\zeta_{AB}+\zeta_{AC}-2\zeta_{AD}}, \tag{3·5}$$

from (3·4) we obtain

$$\alpha \geqq (2\mu)^2 \tag{3·6}$$

as a restriction for the support of $\rho_{12}(\alpha, x)$. Those of ρ_{23} and ρ_{31} are the same.

In practical cases we put $\mu = m_\pi$. The nucleon number conservation law means that there are some nucleon paths (i.e. nucleon open-polygons) in G.

Then we can add *either* $(f, 0)$ and $(-f, 0)$ *or* $(0, f)$ and $(0, -f)$ to both ends of each nucleon path, where

$$f=\frac{m_N-m_\pi}{m_\pi}. \tag{3·7}$$

In the nucleon-nucleon scattering there are two nucleon paths (either $P(AC)$ and $P(BD)$ or $P(AD)$ and $P(BC)$, where $P(AC)$ stands for a path connecting A and C). We take external momenta as follows:

$$\begin{aligned} &k_A=(f+1, 1), \quad k_B=(f+1, -1), \\ &k_C=(-f-1, 1), \quad k_D=(-f-1, -1). \end{aligned} \tag{3·8}$$

We then obtain

$$-k_A^2=-k_B^2=-k_C^2=-k_D^2=m_N^2+m_\pi^2>m_N^2 \tag{3·9}$$

and

$$\begin{aligned} &k_A+k_B=(2m_N/m_\pi, 0), \quad -(k_A+k_B)^2=(2m_N)^2, \\ &k_A+k_C=(0, 2), \quad\quad\quad -(k_A+k_C)^2=(2m_\pi)^2. \end{aligned} \tag{3·10}$$

Therefore the support of ρ_{12} is included in the domain

$$\alpha \geqq (2m_N)^2 x+(2m_\pi)^2(1-x). \tag{3·11a}$$

Likewise we get

$$\beta \geqq \mathrm{Min}[(2m_\pi)^2 y+(2m_N)^2(1-y), \ (2m_N)^2 y+(2m_\pi)^2(1-y)], \tag{3·11b}$$

$$\gamma \geqq (2m_\pi)^2 z+(2m_N)^2(1-z). \tag{3·11c}$$

Next we consider the pion-nucleon scattering. Putting

$$\begin{aligned} &k_A=(f+1, 1), \quad k_B=(1, -1), \\ &k_C=(-f-1, 1), \quad k_D=(-1, -1), \end{aligned} \tag{3·12}$$

so that

$$\begin{aligned} &-k_A^2=-k_C^2=m_N^2+m_\pi^2>m_N^2, \\ &-k_B^2=-k_D^2=2m_\pi^2>m_\pi^2, \end{aligned} \tag{3·13}$$

and

$$\begin{aligned} &-(k_A+k_B)^2=(m_N+m_\pi)^2, \\ &-(k_A+k_C)^2=(2m_\pi)^2, \end{aligned} \tag{3·14}$$

we obtain

$$\alpha \geqq (m_N+m_\pi)^2 x+(2m_\pi)^2(1-x). \tag{3·15a}$$

Likewise we have

$$\beta \geqq (2m_\pi)^2 y + (m_N+m_\pi)^2(1-y), \tag{3·15b}$$

$$\gamma \geqq \mathrm{Max}[(m_N+m_\pi)^2 z + \{(m_N-m_\pi)^2+(2m_\pi)^2\}(1-z),$$
$$\{(m_N-m_\pi)^2+(2m_\pi)^2\} z + (m_N+m_\pi)^2(1-z)], \tag{3·15c}$$

where the latter has been obtained by putting

$$k_A = -k_{A'} = (f+1, 1),\ k_B = -k_{B'} = \pm(-1, 1). \tag{3·16}$$

Finally, we consider the kaon-nucleon scattering. In this case, we have not succeeded in proving the non-forward scattering dispersion relation for experimental mass values. So we must content ourselves with making use of the proof of the forward scattering dispersion relation. We obtain

$$\alpha \geqq (m_N+m_K)^2 x, \tag{3·17a}$$

$$\beta \geqq (m_A+m_\pi)^2(1-y), \tag{3·17b}$$

$$\gamma \geqq \mathrm{Max}[(m_A+m_\pi)^2 z + \{2m_N{}^2+2m_K{}^2-(m_A+m_\pi)^2\}(1-z),$$
$$(m_N-m_K)^2 z + (m_N+m_K)^2(1-z)]. \tag{3·17c}$$

§ 4. Generalized intermediate state

Before entering into the consideration on the support properties in the general case, we will introduce some new concepts and investigate their properties.

Let Z be a set of internal lines such that any external line belonging to h does not connect with any one of h^* without passing through Z. We denote the set of Z by $\boldsymbol{Z}(h)$. Then it is evident that

$$\boldsymbol{Z}(h^*) = \boldsymbol{Z}(h), \tag{4·1}$$

$$\boldsymbol{Z}(h) \neq \phi, \tag{4·2}$$

where ϕ means "empty set". The latter is because $\boldsymbol{Z}(h)$ always contains the set of all internal lines of G, provided that all elements of g are effective external lines. If $Z' \supset Z$ for $Z \in \boldsymbol{Z}(h)$, then one has $Z' \in \boldsymbol{Z}(h)$.

Let $\boldsymbol{Z}$ be the union of $\boldsymbol{Z}(h)$ where h moves over all subsets of g except for ϕ and g. We call an element of $\boldsymbol{Z}$ a "generalized intermediate state".

When $Z_1 \in \boldsymbol{Z}(h_1)$ and $Z_2 \in \boldsymbol{Z}(h_2)$, $Z_1 \cup Z_2$ separates g into four parts $h_1 \cap h_2$, $h_1 \cap h_2{}^*$, $h_1{}^* \cap h_2$ and $h_1{}^* \cap h_2{}^*$. We therefore get

$$Z_1 \cup Z_2 \in \boldsymbol{Z}(h_1 \cap h_2), \tag{4·3a}$$

$$Z_1 \cup Z_2 \in \boldsymbol{Z}(h_1 \cap h_2{}^*), \tag{4·3b}$$

$$Z_1 \cup Z_2 \in \boldsymbol{Z}(h_1 \cup h_2{}^*), \tag{4·3c}$$

$$Z_1 \cup Z_2 \in \boldsymbol{Z}(h_1 \cup h_2) \tag{4·3d}$$

by using (2·3) and (4·1).

An intermediate state S is defined by

$$S \in \boldsymbol{Z} \tag{4·4}^*$$

and

$$\forall R < S \quad R \notin \boldsymbol{Z}.$$

Let $\boldsymbol{S}$ be the set of intermediate states. We write

$$\boldsymbol{S}(h) = \boldsymbol{Z}(h) \cap \boldsymbol{S}. \tag{4·5}$$

Then $\boldsymbol{S}(h^*) = \boldsymbol{S}(h)$ holds, but $\boldsymbol{S}(h) \neq \phi$ is not always assured. If $h_1 \neq h_2$ and $h_2{}^*$, $\boldsymbol{S}(h_1) \cap \boldsymbol{S}(h_2) = \phi$.

It is an important property that if $Z \in \boldsymbol{Z}$, then $\exists S \subset Z \; S \in \boldsymbol{S}$, because if $Z \notin \boldsymbol{S}$ then $\exists Z_1 < Z \; Z_1 \in \boldsymbol{Z}$ on account of (4·4), and if $Z_1 \notin \boldsymbol{S}$ then $Z_2 < Z_1 \; Z_2 \in \boldsymbol{Z}$; since the series $Z > Z_1 > Z_2 > \cdots \in \boldsymbol{Z}$ is of course a finite one, there exists the final element $Z_r : \forall R < Z_r \; R \notin \boldsymbol{Z}$, namely, $Z_r \in \boldsymbol{S}$. But it should be noticed that $\exists S \subset Z$ $S \in \boldsymbol{S}(h)$ does not follow from $Z \in \boldsymbol{Z}(h)$.

When H is a set of internal lines in G, we define the "mass measure" of H by

$$m(H) \equiv \sum_{i \in H} m_i. \tag{4·6}$$

This evidently has the following properties:

i) $m(\phi) = 0,$ (4·7a)

ii) $m(H) \geqq 0,$ (4·7b)

iii) $m(H_1 \cup H_2) + m(H_1 \cap H_2) = m(H_1) + m(H_2),$ (4·7c)

especially, $m(H_1 \pm H_2) = m(H_1) \pm m(H_2).$ (4·7d)

We define the "lowest intermediate-state mass of G" and the "lowest generalized-intermediate-state mass of G" as follows:

$$M_G(h) = \operatorname*{Min}_{S \in \boldsymbol{S}(h)} m(S) \quad \text{if } \boldsymbol{S}(h) \neq \phi, \tag{4·8}$$

$$= \infty \quad \text{if } \boldsymbol{S}(h) = \phi,$$

$$\boldsymbol{M}_G(h) = \operatorname*{Min}_{Z \in \boldsymbol{Z}(h)} m(Z). \tag{4·9}$$

Then it is obvious that

$$M_G(h^*) = M_G(h), \quad \boldsymbol{M}_G(h^*) = \boldsymbol{M}_G(h) \tag{4·10}$$

and

$$M_G(h) \geqq \boldsymbol{M}_G(h) \geqq 0. \tag{4·11}$$

$\boldsymbol{M}_G(h)$ has some remarkable properties. When $h_1 \cap h_2 = \phi$, the triangular inequalities

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* $\mathfrak{A} < \mathfrak{B}$ means $\mathfrak{A} \subset \mathfrak{B}$ and $\mathfrak{A} \neq \mathfrak{B}$.

$$|\boldsymbol{M}_G(h_1)-\boldsymbol{M}_G(h_2)|\leqq \boldsymbol{M}_G(h_1+h_2)\leqq \boldsymbol{M}_G(h_1)+\boldsymbol{M}_G(h_2) \tag{4·12}$$

always holds.

Proof. From (4·7c) and (4·7b) we have

$$\boldsymbol{M}_G(h_1)+\boldsymbol{M}_G(h_2)=\operatorname*{Min}_{\substack{Z_1\in \boldsymbol{Z}(h_1)\\ Z_2\in \boldsymbol{Z}(h_2)}}[m(Z_1)+m(Z_2)]\geqq \operatorname{Min} m(Z_1\cup Z_2). \tag{4·13}$$

Since for any $Z_1\in \boldsymbol{Z}(h_1)$ and $Z_2\in \boldsymbol{Z}(h_2)$

$$Z_1\cup Z_2\in \boldsymbol{Z}(h_1+h_2) \tag{4·14}$$

on account of (4·3d), we obtain

$$\operatorname*{Min}_{\substack{Z_1\in \boldsymbol{Z}(h_1)\\ Z_2\in \boldsymbol{Z}(h_2)}} m(Z_1\cup Z_2)\geqq \operatorname*{Min}_{Z\in \boldsymbol{Z}(h_1+h_2)} m(Z)=\boldsymbol{M}_G(h_1+h_2). \tag{4·15}$$

The second inequality of (4·11) follows from (4·13) and (4·15). In order to prove the first one, instead of (4·14) we have only to use that for any $Z\in \boldsymbol{Z}(h_1+h_2)$ and $Z'\in \boldsymbol{Z}(h_1)$ (or $\in \boldsymbol{Z}(h_2)$)

$$Z\cup Z'\in \boldsymbol{Z}(h_2)\ \ (\text{or}\ \in \boldsymbol{Z}(h_1)) \tag{4·16}$$

on account of (4·3b) and $h_1\cap h_2=\phi$. (Q.E.D.)

Next we consider the relation between $M_G(h)$ and $\boldsymbol{M}_G(h)$. Let $G[h]$ be the self-energy graph which is obtained from G by joining the external lines of h and those of h^* respectively. If we consider $Z\in \boldsymbol{Z}(h)$ in $G[h]$, we get

$$\exists S\subset Z\quad S\in \boldsymbol{S}(h)\ \text{in}\ G[h] \tag{4·17}$$

because $\boldsymbol{S}(h)=\boldsymbol{S}$ in $G[h]$. In G, therefore, we have

$$S\in \boldsymbol{Z}(h) \tag{4·18}$$

and

$$S=\sum_{i=1}^{r} S_i\quad S_i\in \boldsymbol{S}(h_i)$$

with

$$1\leqq r\leqq \nu(g)-1. \tag{4·19}$$

We call S defined by (4·18) a "primitive generalized intermediate state with respect to h", and denote the set of primitive generalized intermediate states with respect to h by $\boldsymbol{P}(h)$. We call $\{\boldsymbol{S}(h_i)\}$ the "type of S".

$\boldsymbol{P}(h)$ is of course a subset of $\boldsymbol{Z}(h)$. Since $m(S)\leqq m(Z)$ on account of $S\subset Z$, one has

$$\boldsymbol{M}_G(h)=\operatorname*{Min}_{S\in \boldsymbol{P}(h)} m(S)=\operatorname*{Min}_{S\in \boldsymbol{P}(h)} \sum_i m(S_i). \tag{4·20}$$

Calculating the minimum of $m(S_i)$ within $\boldsymbol{S}(h_i)$ first, we obtain

$$\boldsymbol{M}_G(h)=\operatorname*{Min}_{h\text{-}prim}[\sum_i M_G(h_i)], \tag{4·21}$$

where $\underset{h\ prim}{\mathrm{Min}}$ means to take the minimum for all types, $\{\boldsymbol{S}(h_i)\}$, of primitive generalized intermediate states with respect to h. In particular, if h consists of only one external line, (4·21) becomes

$$\boldsymbol{M}_G(h)=\underset{\sum_i h_i=h^*}{\mathrm{Min}}\Big[\sum_i M_G(h_i)\Big]. \tag{4·22}$$

The mass spectral condition is usually indicated by

$$M_h=\underset{G}{\mathrm{Min}}\, M_G(h), \tag{4·23}$$

where $\underset{G}{\mathrm{Min}}$ means to take the minimum for all graphs having the same g which are permissive under given interactions. Then

$$M_h=\underset{G}{\mathrm{Min}}\,\boldsymbol{M}_G(h). \tag{4·24}$$

Proof. Because of (4·11), it suffices to show that

$$\forall G\ \ \exists G'\quad \boldsymbol{M}_G(h)\geqq M_{G'}(h). \tag{4·25}$$

Such a graph G' is constructed by joining three G's: The first G is joined to the second in h^* and the second is joined to the third in h. (Q.E.D.)

Finally, we notice that the following interesting inequality always holds in case of the two-particle scattering ($g=\{A, B, C, D\}$):

$$\boldsymbol{M}_G(A)+\boldsymbol{M}_G(B)+\boldsymbol{M}_G(C)+\boldsymbol{M}_G(D)\leqq \boldsymbol{M}_G(AB)+\boldsymbol{M}_G(AC)+\boldsymbol{M}_G(AD). \tag{4·26}$$

It should be remarked that a corresponding inequality for M_h does not hold in general.

Proof. It suffices to prove that for any $Z_1\in\boldsymbol{Z}(AB)$, $Z_2\in\boldsymbol{Z}(AC)$ and $Z_3\in\boldsymbol{Z}(AD)$

$$\exists Z_J\in\boldsymbol{Z}(J)\quad (J=A, B, C, D)$$
$$Z_A+Z_B+Z_C+Z_D\subset Z_1+Z_2+Z_3, \tag{4·27}$$

where we have assumed $Z_1\cap Z_2=Z_2\cap Z_3=Z_3\cap Z_1=\phi$. If not so, we have only to introduce some 2-vertices appropriately (then G is topologically unchanged). We denote a path connecting A and B by $P(AB)$. By definition, Z_1 intersects any one of $P(AC)$, $P(AD)$, $P(BC)$ and $P(BD)$, Z_2 does any one of $P(AB)$, $P(AD)$, $P(BC)$ and $P(CD)$, and Z_3 does any one of $P(AB)$, $P(AC)$, $P(BD)$ and $P(CD)$. Therefore $Z_1+Z_2+Z_3$ intersects any path at least at *two* lines. Now, we consider a reduced graph G_R that is obtained from G by shrinking all the lines which do not belong to $Z_1+Z_2+Z_3$. Then the four external stars (i.e. the intermediate states nearest to external lines) of G_R (they are denoted by S^A, S^B, S^C, S^D) are mutually non-overlapping, and of course

$$S^A+S^B+S^C+S^D\subset Z_1+Z_2+Z_3. \tag{4·28}$$

We put $Z_J=S^J(J=A, B, C, D)$, then $Z_J\in\boldsymbol{Z}(J)$ in G is obvious. Thus Z_A, Z_B, Z_C, Z_D satisfy (4·27). (Q.E.D.)

§ 5. Support properties in the general case

For the two-particle scattering the triangular inequalities (4·12) are

$$\begin{aligned}
&|\boldsymbol{M}_G(A)-\boldsymbol{M}_G(B)|\leqq\boldsymbol{M}_G(AB)\leqq\boldsymbol{M}_G(A)+\boldsymbol{M}_G(B),\\
&|\boldsymbol{M}_G(C)-\boldsymbol{M}_G(D)|\leqq\boldsymbol{M}_G(AB)\leqq\boldsymbol{M}_G(C)+\boldsymbol{M}_G(D),\\
&|\boldsymbol{M}_G(A)-\boldsymbol{M}_G(C)|\leqq\boldsymbol{M}_G(AC)\leqq\boldsymbol{M}_G(A)+\boldsymbol{M}_G(C),\\
&|\boldsymbol{M}_G(B)-\boldsymbol{M}_G(D)|\leqq\boldsymbol{M}_G(AC)\leqq\boldsymbol{M}_G(B)+\boldsymbol{M}_G(D),\\
&|\boldsymbol{M}_G(A)-\boldsymbol{M}_G(D)|\leqq\boldsymbol{M}_G(AD)\leqq\boldsymbol{M}_G(A)+\boldsymbol{M}_G(D),\\
&|\boldsymbol{M}_G(B)-\boldsymbol{M}_G(C)|\leqq\boldsymbol{M}_G(AD)\leqq\boldsymbol{M}_G(B)+\boldsymbol{M}_G(C).
\end{aligned}\tag{5·1}$$

Combining them and (4·26), we get

$$\begin{aligned}
&\boldsymbol{M}_G(C)+\boldsymbol{M}_G(D)\leqq\boldsymbol{M}_G(AC)+\boldsymbol{M}_G(AD),\\
&\boldsymbol{M}_G(A)+\boldsymbol{M}_G(B)\leqq\boldsymbol{M}_G(AC)+\boldsymbol{M}_G(AD),\\
&\boldsymbol{M}_G(B)+\boldsymbol{M}_G(D)\leqq\boldsymbol{M}_G(AB)+\boldsymbol{M}_G(AD),\\
&\boldsymbol{M}_G(A)+\boldsymbol{M}_G(C)\leqq\boldsymbol{M}_G(AB)+\boldsymbol{M}_G(AD),\\
&\boldsymbol{M}_G(B)+\boldsymbol{M}_G(C)\leqq\boldsymbol{M}_G(AB)+\boldsymbol{M}_G(AC),\\
&\boldsymbol{M}_G(A)+\boldsymbol{M}_G(D)\leqq\boldsymbol{M}_G(AB)+\boldsymbol{M}_G(AC).
\end{aligned}\tag{5·2}$$

Now, the following property follows from Theorems 13–1 and 13–3 in N and (4·11):

If

$$\begin{aligned}
&k_J=a_J k\quad(J=A,\ B,\ C,\ D),\\
&\sum_J a_J=0,\\
&-k^2=1,
\end{aligned}\tag{5·3}$$

and if

$$\begin{aligned}
&|a_J|\leqq\boldsymbol{M}_G(J)\qquad(J=A,\ B,\ C,\ D),\\
&|a_A+a_B|\leqq\boldsymbol{M}_G(AB),\\
&|a_A+a_C|\leqq\boldsymbol{M}_G(AC),\\
&|a_A+a_D|\leqq\boldsymbol{M}_G(AD),
\end{aligned}\tag{5·4}$$

then

$$\sum_{i=1}^{N}x_i m_i^2-\sum_J\zeta_J a_J^2-\zeta_{AB}(a_A+a_B)^2-\zeta_{AC}(a_A+a_C)^2-\zeta_{AD}(a_A+a_D)^2\geqq 0.\tag{5·5}$$

The coefficients a_J are chosen in the following way.

1) $a_A=\boldsymbol{M}_G(A)$, $a_B=-\boldsymbol{M}_G(B)$, $a_C=\boldsymbol{M}_G(AC)-\boldsymbol{M}_G(A)$,

$$a_D=\boldsymbol{M}_G(B)-\boldsymbol{M}_G(AC),\quad a_A+a_B=\boldsymbol{M}_G(A)-\boldsymbol{M}_G(B),$$
$$a_A+a_C=\boldsymbol{M}_G(AC),\quad a_A+a_D=\boldsymbol{M}_G(A)+\boldsymbol{M}_G(B)-\boldsymbol{M}_G(AC).$$

We can easily see that they satisfy the inequalities (5·4) on account of (5·1) and (5·2).

2) $a_A=\boldsymbol{M}_G(A),\quad a_B=-\boldsymbol{M}_G(B),\quad a_A+a_D=\boldsymbol{M}_G(AD),$

3) $a_A=\boldsymbol{M}_G(A),\quad a_C=-\boldsymbol{M}_G(C),\quad a_A+a_B=\boldsymbol{M}_G(AB),$

4) $a_A=\boldsymbol{M}_G(A),\quad a_C=-\boldsymbol{M}_G(C),\quad a_A+a_D=\boldsymbol{M}_G(AD),$

5) $a_A=\boldsymbol{M}_G(A),\quad a_D=-\boldsymbol{M}_G(D),\quad a_A+a_B=\boldsymbol{M}_G(AB),$

6) $a_A=\boldsymbol{M}_G(A),\quad a_D=-\boldsymbol{M}_G(D),\quad a_A+a_C=\boldsymbol{M}_G(AC),$

7) $a_B=\boldsymbol{M}_G(B),\quad a_C=-\boldsymbol{M}_G(C),\quad a_A+a_B=\boldsymbol{M}_G(AB),$

8) $a_B=\boldsymbol{M}_G(B),\quad a_C=-\boldsymbol{M}_G(C),\quad a_A+a_C=-\boldsymbol{M}_G(AC),$

9) $a_B=\boldsymbol{M}_G(B),\quad a_D=-\boldsymbol{M}_G(D),\quad a_A+a_B=\boldsymbol{M}_G(AB),$

10) $a_B=\boldsymbol{M}_G(B),\quad a_D=-\boldsymbol{M}_G(D),\quad a_A+a_D=-\boldsymbol{M}_G(AD),$

11) $a_C=\boldsymbol{M}_G(C),\quad a_D=-\boldsymbol{M}_G(D),\quad a_A+a_C=\boldsymbol{M}_G(AC),$

12) $a_C=\boldsymbol{M}_G(C),\quad a_D=-\boldsymbol{M}_G(D),\quad a_A+a_D=-\boldsymbol{M}_G(AD).$

From (5·5) with 1) we get

$$\begin{aligned}\sum x_i m_i^2 \geqq\ & \boldsymbol{M}_G(A)^2\zeta_A+\boldsymbol{M}_G(B)^2\zeta_B+\{\boldsymbol{M}_G(AC)-\boldsymbol{M}_G(A)\}^2\zeta_C\\ &+\{\boldsymbol{M}_G(AC)-\boldsymbol{M}_G(B)\}^2\zeta_D+\{\boldsymbol{M}_G(A)-\boldsymbol{M}_G(B)\}^2\zeta_{AB}\\ &+\boldsymbol{M}_G(AC)^2\zeta_{AC}+\{\boldsymbol{M}_G(A)+\boldsymbol{M}_G(B)-\boldsymbol{M}_G(AC)\}^2\zeta_{AD}.\end{aligned}\tag{5·6}$$

Hence one has

$$\sum x_i m_i^2 \geqq M_A^2\zeta_A+M_B^2\zeta_B+M_{AC}^2\zeta_{AC}\tag{5·7}$$

for all graphs. So, using the choices 2)∼12), we obtain

$$\begin{aligned}\sum x_i m_i^2 \geqq \mathrm{Max}[&M_A^2\zeta_A+M_B^2\zeta_B+M_{AC}^2\zeta_{AC},\quad M_A^2\zeta_A+M_B^2\zeta_B+M_{AD}^2\zeta_{AD},\\ &M_A^2\zeta_A+M_C^2\zeta_C+M_{AB}^2\zeta_{AB},\quad M_A^2\zeta_A+M_C^2\zeta_C+M_{AD}^2\zeta_{AD},\\ &M_A^2\zeta_A+M_D^2\zeta_D+M_{AB}^2\zeta_{AB},\quad M_A^2\zeta_A+M_D^2\zeta_D+M_{AC}^2\zeta_{AC},\\ &M_B^2\zeta_B+M_C^2\zeta_C+M_{AB}^2\zeta_{AB},\quad M_B^2\zeta_B+M_C^2\zeta_C+M_{AC}^2\zeta_{AC},\\ &M_B^2\zeta_B+M_D^2\zeta_D+M_{AB}^2\zeta_{AB},\quad M_B^2\zeta_B+M_D^2\zeta_D+M_{AD}^2\zeta_{AD},\\ &M_C^2\zeta_C+M_D^2\zeta_D+M_{AC}^2\zeta_{AC},\quad M_C^2\zeta_C+M_D^2\zeta_D+M_{AD}^2\zeta_{AD}].\end{aligned}\tag{5·8}$$

Restriction for the support of each term in the integral representation (2·23) is easily derived from (5·8). For instance, that of the first term is obtained

from (5·8) by replacing $\sum x_i m_i^2$ by κ, ζ_{AD} by 0 and ζ_h by z_h for $h \neq \{AD\}$. The restriction for support (5·8) yields, for example,

$$\sum x_i m_i^2 \geqq \frac{1}{2} \sum_J M_J^2 \zeta_J + \frac{1}{3} \{M_{AB}^2 \zeta_{AB} + M_{AC}^2 \zeta_{AC} + M_{AD}^2 \zeta_{AD}\}, \tag{5·9}$$

but one cannot prove $\sum x_i m_i^2 \geqq \sum_J M_J^2 \zeta_J$ in general (see Appendix B).

The integration parameters in (2·22) are given by

$$\alpha = \frac{\sum x_i m_i^2 - M_a^2 \zeta_A - M_b^2 \zeta_B - M_c^2 \zeta_C - M_d^2 \zeta_D - (M_a^2 + M_b^2 + M_c^2 + M_d^2)\zeta_{AD}}{\zeta_{AB} + \zeta_{AC} - 2\zeta_{AD}},$$

$$x = \frac{\zeta_{AB} - \zeta_{AD}}{\zeta_{AB} + \zeta_{AC} - 2\zeta_{AD}} \tag{5·10}$$

for $\zeta_{AB} > \zeta_{AD}$ and $\zeta_{AC} > \zeta_{AD}$, and so on. In order to assure $\alpha \geqq 0$, $\beta \geqq 0$ and $\gamma \geqq 0$ from (5·8), it is necessary and sufficient that

$$\begin{aligned} &\exists\, b_i (i = 1, 2, \cdots, 12) \quad b_i \geqq 0, \quad \sum_{i=1}^{12} b_i = 1, \\ &M_a^2 + M_b^2 + M_c^2 + M_d^2 \leqq (b_3 + b_5 + b_7 + b_9) M_{AB}^2 \\ &\quad + (b_1 + b_6 + b_8 + b_{11}) M_{AC}^2 + (b_2 + b_4 + b_{10} + b_{12}) M_{AD}^2 \end{aligned} \tag{5·11a}$$

and

$$\begin{aligned} &(b_1 + b_2 + b_3 + b_4 + b_5 + b_6) M_A^2 \geqq M_a^2, \\ &(b_1 + b_2 + b_7 + b_8 + b_9 + b_{10}) M_B^2 \geqq M_b^2, \\ &(b_3 + b_4 + b_7 + b_8 + b_{11} + b_{12}) M_C^2 \geqq M_c^2, \\ &(b_5 + b_6 + b_9 + b_{10} + b_{11} + b_{12}) M_D^2 \geqq M_d^2. \end{aligned} \tag{5·11b}$$

As is easily seen, (5·11a) is satisfied only when

$$M \equiv M_a = M_b = M_c = M_d = \frac{1}{2} \operatorname{Max}(M_{AB}, M_{AC}, M_{AD}) \tag{5·12}$$

because of inequalities $M_a + M_b > M_{AB}$, etc. If at least two among M_{AB}, M_{AC} and M_{AD} are equal to $2M$, (5·11b) is equivalent to

$$\operatorname{Max} M_J^{-2} \leqq \frac{1}{2} \sum_J M_J^{-2} \leqq M^{-2}. \tag{5·13}$$

If $M_{AB} > \operatorname{Max}(M_{AC}, M_{AD})$, (5·11b) becomes

$$M_A^{-2} + M_B^{-2} \leqq M^{-2}, \quad M_C^{-2} + M_D^{-2} \leqq M^{-2} \tag{5·14}$$

instead of (5·13).

§ 6. Discussions

We have investigated support properties for the integral representations

(2·22) and (2·23). Our main results are (3·11), (3·15) and (5·8). Of course, these restrictions for supports are not necessarily the best ones.

If the Mandelstam representation

$$\int_{M_{AB}^2}^{\infty} ds' \int_{M_{AC}^2}^{\infty} dt' \frac{\sigma_{12}(s', t')}{(s'-s-i\varepsilon)(t'-t-i\varepsilon)} + \int_{M_{AC}^2}^{\infty} dt' \int_{M_{AD}^2}^{\infty} du' \frac{\sigma_{23}(t', u')}{(t'-t-i\varepsilon)(u'-u-i\varepsilon)}$$

$$+ \int_{M_{AD}^2}^{\infty} du' \int_{M_{AB}^2}^{\infty} ds' \frac{\sigma_{31}(u', s')}{(u'-u-i\varepsilon)(s'-s-i\varepsilon)} \tag{6·1}$$

holds, the integral representation (2·22) immediately follows from it by using the Feynman identity and integrating each term by parts. Then the restrictions for supports become

$$\begin{aligned} \alpha &\geqq M_{AB}^2 x + M_{AC}^2(1-x), \\ \beta &\geqq M_{AC}^2 y + M_{AD}^2(1-y), \\ \gamma &\geqq M_{AD}^2 z + M_{AB}^2(1-z). \end{aligned} \tag{6·2}$$

So the restrictions for supports, (3·11), for the nucleon-nucleon scattering are good, while the restriction, (3·15c), of ρ_{31} for the pion-nucleon scattering should be improved to $(m_N+m_\pi)^2$, otherwise the Mandelstam representation must break down.

When t is fixed to $t=-4\Delta^2\leqq 0$, the integral representation (2·22) is reduced to a single dispersion relation if

$$\begin{aligned} \alpha &\geqq 0, \\ \beta &\geqq 0, \\ \gamma &\geqq M^2 \cdot \mathrm{Min}(z, 1-z) \end{aligned} \tag{6·3}$$

with

$$M^2 \geqq M_a^2 + M_b^2 + M_c^2 + M_d^2 + 4\Delta^2. \tag{6·4}$$

The spectral variables are defined as follows:

First term: $$s' = \frac{\alpha + 4\Delta^2(1-x)}{x}, \tag{6·5a}$$

Second term: $$u' = \frac{\beta + 4\Delta^2 y}{1-y}, \tag{6·5b}$$

Third term:

$$\begin{aligned} s' &= \frac{\gamma - z(M_a^2+M_b^2+M_c^2+M_d^2+4\Delta^2)}{1-2z} \quad \text{for } z \leqq \frac{1}{2}, \\ u' &= \frac{\gamma - (1-z)(M_a^2+M_b^2+M_c^2+M_d^2+4\Delta^2)}{-1+2z} \quad \text{for } z \geqq \frac{1}{2}. \end{aligned} \tag{6·5c}$$

If (6·2) is satisfied, we naturally have the normal dispersion relation for

$$-M_{AC}^2 \leqq 4\Delta^2 \leqq M_{AB}^2+M_{AD}^2-(M_a^2+M_b^2+M_c^2+M_d^2), \tag{6·6}$$

where Δ^2 may become negative.

A similar consideration also applies to the case in which s or u is fixed.

In the general case, $\alpha \geqq 0$, $\beta \geqq 0$ and $\gamma \geqq 0$ are not always assured. Such an example will be presented in Appendix B. We have obtained (5·12) with (5·13) or (5·14) as the condition for $\alpha \geqq 0$, $\beta \geqq 0$ and $\gamma \geqq 0$. As this condition is too stringent, our results in the general case will be far from practical uses, but it will be of theoretical interest especially when it is compared with results obtained from the axiomatic theory. Namely, the integral representation (2·10) with (5·8) reveals notable characteristics of perturbation theoretical results irrelevant to particle-number conservation laws.

Appendix A

Proof of (2·2)

1°) We have proved in § 6 of N that

$$V=\sum_{i=1}^{N} x_i m_i^2-\frac{1}{U}\sum_{i>j} W^{(ij)} k_i k_j \tag{A·1}$$

where $W^{(ij)}$ is the U-function in the graph in which external lines A_i and A_j are joined. Hence if $x_{\nu_1} x_{\nu_2} \cdots x_{\nu_{n+1}} \in W^{(ij)}$

$$\forall P(A_i A_j) \quad \exists \nu_k \in F \quad \nu_k \in P(A_i A_j), \tag{A·2}$$

where $F \equiv \{\nu_1, \nu_2, \cdots, \nu_{n+1}\}$. Therefore when all the lines of F are opened, A_i and A_j are separated (i.e. $F \in \boldsymbol{Z}$). Hence

$$\exists S \in \boldsymbol{S} \quad F \supset S. \tag{A·3}$$

2°) **Lemma.** In order that $x_{\nu_1} x_{\nu_2} \cdots x_{\nu_{n+1}}/U$ is non-vanishing when $x_i=0$ $(\forall i \in G-S)$ where S is an intermediate state, it is necessary and sufficient that

$$F = \{\nu_1, \nu_2, \cdots, \nu_{n+1}\} \supset S. \tag{A·4}$$

Proof. Since S intersects $\nu(S)-1$ independent circuits, the number of independent circuits included in $G-S$ is $n-\{\nu(S)-1\}=n+1-\nu(S)$. So U has $n+1-\nu(S)$-th order zero point at $x_i=0$ $(\forall i \in G-S)$. On the other hand, the order of zero point of the numerator is

$$\begin{aligned} &\nu(F \cap (G-S))=n+1-\nu(S) \quad \text{if } F \supset S, \\ &\nu(F \cap (G-S)) > n+1-\nu(S) \quad \text{if } F \not\supset S \end{aligned} \tag{A·5}$$

on account of $n+1=\nu(F)=\nu(F \cap S)+\nu(F \cap (G-S))$. Thus the lemma is obtained. (Q.E.D.)

3°) From Theorem 5–1 of N we have

$$U=U^{(H_1)}U^{(H_2)}U^{(R)}+O(\varepsilon_S^{\nu(S)}), \tag{A·6}$$

where $\varepsilon_S=\underset{i\in G-S}{\mathrm{Max}}\, x_i$ and where R is the reduced graph corresponding to an intermediate state S. So for $\varepsilon_S=0$ we get

$$W_S/U=\prod_{i\in S} x_i/U^{(R)}=\Big(\sum_{i\in S} 1/x_i\Big)^{-1}. \tag{A·7}$$

On the other hand, for $\varepsilon_S=0$

$$\forall S'\neq S \quad W_{S'}/U=0. \tag{A·8}$$

Proof. Let H_1' and H_2' be the two subgraphs obtained by the Schnitt S'. Since $S'-(S'\cap S)\neq\phi$, we denote one of its elements by I. Then we have

$$\begin{gathered}\exists A \quad \forall P(AI) \text{ in } H_1' \text{ or } H_2' \\ P(AI)\cap S\neq\phi\end{gathered} \tag{A·9}$$

because of the definition of S. For instance, we assume A is an external line of H_1'. Then we see that $H_1'\cap S$ is a generalized intermediate state of H_1'.

Now, from the lemma if $W_{S'}/U\neq 0$ for $\varepsilon_S=0$ then one has

$$\begin{gathered}\exists\, x_{\nu_1}x_{\nu_2}\cdots x_{\nu_m}\in U^{(H_1')}U^{(H_2')} \quad (m=n+1-\nu(S')) \\ F'\equiv S'+\{\nu_1,\nu_2,\cdots,\nu_m\}\supset S.\end{gathered} \tag{A·10}$$

Therefore $H_1'\cap F'$ includes an intermediate state (i.e. a subset of $H_1'\cap S$) of H_1'. This is inconsistent with the definition of $U^{(H_1')}$ (because each term of the U-function corresponds to a set of *independent* integration momenta (see § 3 of N)). (Q.E.D.)

4°) We write

$$V'\equiv\sum_{i=1}^{N} x_i m_i^2+\sum_S \frac{W_S}{U}k_S^2. \tag{A·11}$$

From 3°) we get

$$\forall S\in\boldsymbol{S} \quad V'=V^{(R)} \quad \text{for } x_i=0 \ (\forall i\in G-S), \tag{A·12}$$

where $V^{(R)}$ is the V-function for the reduced graph R:

$$V^{(R)}\equiv\sum_{i\in S} x_i m_i^2+\frac{\prod_{i\in S} x_i}{U^{(R)}}k_S^2. \tag{A·13}$$

So, (A·12) is rewritten as

$$\forall S\in\boldsymbol{S} \quad V-V'=0 \quad \text{for } x_i=0 \ (\forall i\in G-S) \tag{A·14}$$

on account of Theorem 5–1 of N. If $V-V'$ were not identically zero, it would be a linear combination of such terms as $x_{\nu_1}x_{\nu_2}\cdots x_{\nu_{n+1-\nu}}/U$. Since no cancellation cannot happen by putting $x_i=0$, one must have

$$\forall S\in\boldsymbol{S} \quad x_{\nu_1}x_{\nu_1}\cdots x_{\nu_{n+1}}/U=0 \quad \text{for } x_i=0 \ (\forall i\in G-S). \tag{A·15}$$

Hence from the lemma

$$\forall S\in\boldsymbol{S} \quad F\equiv\{\nu_1, \nu_2, \cdots, \nu_{n+1}\} \not\ni S. \tag{A·16}$$

This is inconsistent with (A·3). Therefore $V-V'\equiv 0$.

Appendix B

Counterexample against $\alpha\geqq 0$, $\beta\geqq 0$ and $\gamma\geqq 0$

One might suppose that the weight functions in (2·22) would vanish in general unless $\alpha\geqq 0$, $\beta\geqq 0$, $\gamma\geqq 0$ if stability conditions are satisfied. This is indeed the case in the non-trivial lowest order. For Fig. 1 we obtain

$$\alpha\geqq x(m_1-m_3)^2+(1-x)(m_2-m_4)^2\geqq 0, \tag{B·1}$$

where

$$\alpha=\frac{\sum_{i=1}^{4}x_i m_i^2-x_1x_2M_a^2-x_2x_3M_b^2-x_1x_4M_c^2-x_3x_4M_d^2}{x_1x_3+x_2x_4},$$

$$x=\frac{x_1x_3}{x_1x_3+x_2x_4}. \tag{B·2}$$

But we will show that this expectation is not correct in general.

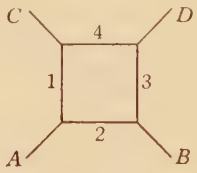

Fig. 1.

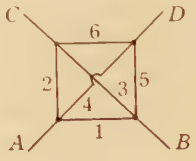

Fig. 2.

Consider Fig. 2, in which the expressions for U and W_S are as follows:

$$\begin{aligned}
U=&x_1x_2x_3+x_1x_2x_5+x_1x_2x_6+x_1x_3x_4\\
&+x_1x_3x_6+x_1x_4x_5+x_1x_4x_6+x_1x_5x_6\\
&+x_2x_3x_4+x_2x_3x_5+x_2x_4x_5+x_2x_4x_6\\
&+x_2x_5x_6+x_3x_4x_5+x_3x_4x_6+x_3x_5x_6,\\
W_A=&x_1x_2x_4(x_3+x_5+x_6),\\
W_B=&x_1x_3x_5(x_2+x_4+x_6),\\
W_C=&x_2x_3x_6(x_1+x_4+x_5),\\
W_D=&x_4x_5x_6(x_1+x_2+x_3),
\end{aligned} \tag{B·3}$$

$$W_{AB}=x_2x_3x_4x_5,$$

$$W_{AC}=x_1x_3x_4x_6,$$

$$W_{AD}=x_1x_2x_5x_6.$$

Hence, when

$$\begin{aligned}&\forall i \quad m_i=m,\\ &M_a=M_b=M_c=M_d=M,\\ &\forall i \quad x_i=1/6+\varepsilon y_i \quad (\varepsilon:\ \text{infinitesimal})\end{aligned} \tag{B·4}$$

with $\sum_{i=1}^{6} y_i=0,$

we have

$$V=m^2-\frac{M^2}{6}-\frac{\varepsilon}{16}[(y_2+y_3+y_4+y_5)s+(y_1+y_3+y_4+y_6)t+(y_1+y_2+y_5+y_6)u]. \tag{B·5}$$

On the other hand, stability conditions are satisfied if $M<3m$. Therefore if

$$6m^2<M^2<9m^2, \tag{B·6}$$

the lower limits of integrations over α, β and γ in (2·22) are indeed minus infinity in spite of the presence of stability conditions. This means that no dispersion relation holds for this example.

It might be worthwhile to notice that this example is realizable as a reduced graph if there exists the pseudoscalar meson (say $\pi^{0\prime}$) whose mass is larger than 340 Mev.

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Note added in proof: From our integral representation (2·22) we can derive the partial wave dispersion relation which has so far been derived under the assumption of the Mandelstam representation. Detailed account will be discussed elsewhere.

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K Meson-Pion Interaction and $K^-+p\to\bar{K}^0+\pi^-+p$ Reaction

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$\bar{K}$ meson-pion interaction is studied by taking into account the effects of K^*. Both the value of coupling constant for $\bar{K}\pi K^*$-interaction and the magnitude of cross section of $\bar{K}$-π scattering are estimated under the assumption that the isotopic spin of K^* is equal to 1/2. Making use of the value of coupling constant we try to investigate the reaction $K^-+p\to\bar{K}^0+\pi^-+p$ at 1.15 Bev/c and 2 Bev/c. The total cross section of this reaction, the angular distributions and momentum spectra of protons and pions are examined. It is shown that the assumption of $S=1$ (S indicates the spin of K^*) rather than that of $S=0$ is consistent not only with the experimental results for the angular distribution of $K^-+p\to K^*+p$ process but also with those for $K^-+p\to\bar{K}^0+\pi^-+p$ reaction at 1.15 Bev/c, that is, the magnitude of cross section and the energy spectrum of pion.

§ 1. Introduction

The recent experimental results for $K^-+p\to\bar{K}^0+\pi^-+p$ reaction[1] (1.15 Bev/c K^- meson) have shown the following remarkable facts.

(i) The observed distribution for momentum of proton in the final state can be best explained by the reaction through the following process,

$$K^-+p\to K^*+p\to\bar{K}^0+\pi^-+p.$$

(ii) The angular distribution for the reaction $K^-+p\to K^*+p$ is consistent with isotropy.

(iii) The total cross section for $K^-+p\to\bar{K}^0+\pi^-+p$ is about 2 mb.

(iv) The mass of K^* is nearly equal to 884 Mev.

(v) The half width of this resonant state is about 10 Mev.

(vi) The decay angular distribution seems to be consistent with $S=0$ or $S=1$, where S is the spin of K^*.

(vii) The assumption of $I_{K^*}=1/2$ (I_{K^*} indicates the isotopic spin of K^*) rather than that of $I_{K^*}=3/2$ seems to be consistent with the experimental results for the branching ratio of the decay of K^*.

In advance of the study for $K^-+p\to\bar{K}^0+\pi^-+p$ reaction, we investigate the problems of $\bar{K}$-π scattering.* In order to obtain the matrix elements (or cross

* In this paper, only the scattering process that gives rise to the resonance is taken into account. In order to satisfy the condition of crossing symmetry it is of course necessary to estimate the contribution from other processes. But these processes have not large effects on the scattering.

sections) for $\bar{K}$-π scattering, the propagation function of K^* is modified by taking into account the damping effects. But it must be noted that, as will be shown later, the magnitude of the cross section $\sigma(\bar{K}\text{-}\pi)$ at the resonance energy does not depend on the value of coupling constant for $\bar{K}\pi K^*$-interaction. Since its magnitude can also be estimated from kinematical point of view, it is possible to see whether the modified propagation function of K^* is correct or not. Based both on the correct expression for the modified propagation function and on the experimental result for the width of K^*-resonance level, the value of coupling constant for $\bar{K}\pi K^*$-interaction is estimated.

From the experimental result (vii), it is impossible to expect the strong $\bar{K}^0$-π^+ or K^--π^- interaction and there are of course the following relations between the cross sections,

$$\sigma(K^-+\pi^+\to K^-+\pi^+)/4=\sigma(K^-+\pi^-\to\bar{K}^0+\pi^0)/2=\sigma(\bar{K}^0+\pi^0\to\bar{K}^0+\pi^0)/1$$
$$=\sigma(\bar{K}^0+\pi^-\to\bar{K}^0+\pi^-)/4=\sigma(\bar{K}^0+\pi^-\to K^-+\pi^0)/2=\sigma(K^-+\pi^0\to K^-+\pi^0)/1. \tag{1}$$

Therefore only the scattering process $K^-+\pi^+\to K^-+\pi^+$ is studied in this paper.* The problems of s-wave and p-wave $\bar{K}$-π scattering are treated in § 2 and § 3 respectively.

Making use of the values of coupling constants obtained in § 2 and § 3, we should like to study the reaction $K^-+p\to\bar{K}^0+\pi^-+p$ from field theoretical point of view. First of all we pay attention to the reaction $K^-+p\to K^*+p$. In § 4 the angular distribution for this reaction is examined and the magnitude of cross section is estimated. Based on the results, it is shown that the assumption of $S=1$ rather than that of $S=0$ is consistent with the experimental results (ii) for the interaction of 1.15 Bev/c K^- meson in hydrogen. The angular distribution of proton for the reaction at 2 Bev/c has a remarkable peak in the backward direction (cf. Fig. 4 and Fig. 5). This tendency is more conspicuous in the case $S=0$ than in the case $S=1$.

In § 5 the magnitude of cross section, the angular distributions and momentum spectra of pions and protons for the reaction $K^-+p\to\bar{K}^0+\pi^-+p$ at 1.15 Bev/c and 2 Bev/c are investigated in order to see the energy dependence of them. It is shown that the angular distribution and momentum spectrum of pion depend strongly on the spin of K^*. Comparing our theoretical results for the cross section and energy spectrum of pion with the experimental ones, we may conclude that $S=1$ is more promising than $S=0$.

§ 2. S-wave $\bar{K}$-π scattering

In the case of $S=0$, the interaction Hamiltonian is given as follows:

* With respect to the isotopic spin of I_{K^*}, some discussion will be made in the last section.

$$H_1 = m_K G_S \Psi(x) \tau_\alpha \varphi(x) \phi_\alpha(x), \tag{2}$$

where Ψ, φ and ϕ_α are the wave functions of the K^*, $\bar{K}$ and π, and M, m_K and μ are the masses of K^*, $\bar{K}$ and π respectively. For the purpose of estimating the value of coupling constant $G_S^2/4\pi$ from the experimental result for the width of K^*, the propagation function of K^* is modified by means of damping theory. Of course it is necessary to get the correct expression for the modified propagation function in order to study the $\bar{K}$ meson reactions in which K^* plays the important role. In our description of $\bar{K}$-π scattering, the damping effects may be introduced if we take into account the sum of contributions from a series of graphs (a^1), (a^2), (a^3), $\cdots$ in Fig. 1.*

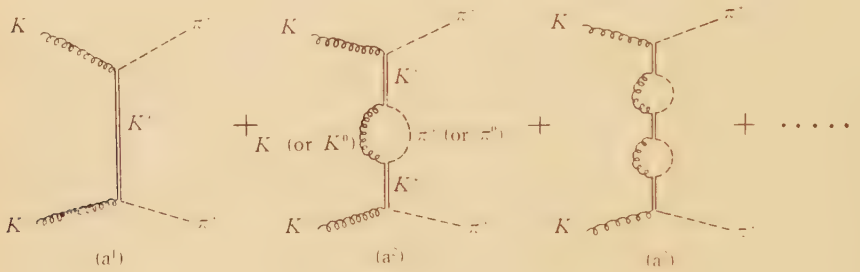

Fig. 1.

The propagation function $D_F(k)$ written in the momentum space is modified as follows,

$$\bar{D}_F(k) = D_F + D_F N D_F + D_F N D_F N D_F + \cdots \tag{3}$$

$$= \frac{D_F}{1 - N D_F},$$

where

$$D_F(k) = \frac{1}{2\pi i} \cdot \frac{1}{k^2 + M^2} \tag{4}$$

and

$$N(k) = -\frac{(2+1)}{2} m_K^2 \left(\frac{G_S^2}{4\pi}\right) \int (dt)\, \delta[(k-t)^2 + m_K^2]\, \delta(t^2 + \mu^2) \tag{5}$$

which vanishes for $k^2 + (m_K + \mu)^2 \quad 0$. Here it should be noted that the damping effects due to the occurrence both of $(K^- + \pi^+)$ and of $(\bar{K}^0 + \pi^0)$ in the intermediate state must be involved in the expression for $N(k)$. The coefficients of 2 and 1 in the factor $(2+1)$ in Eq. (5) show the effects of the former and those of the latter respectively.

* The same way as this was employed previously[2] in order to obtain the modified propagation function of the particle with spin 3/2.

The integration in Eq. (5) yields

$$N(k)=-\frac{3}{2}m_K^2\left(\frac{G_S^2}{4\pi}\right)\frac{\pi}{\sqrt{-k^2}}A, \tag{6}$$

$$A=\sqrt{\frac{[k^2+(m_K+\mu)^2][k^2+(m_K-\mu)^2]}{-4k^2}}. \tag{7}$$

Then

$$\begin{aligned}\bar{D}_F(k)&=\frac{1}{2\pi i}\ \frac{1}{k^2+M^2}\ \frac{1}{1-(2\pi i)^{-1}N(k)\,(k^2+M^2)^{-1}}\\&=\frac{1}{2\pi i}\ \frac{1}{k^2+M^2-(3/4)\,im_K^2(G_S^2/4\pi)\,A/\sqrt{-k^2}}.\end{aligned} \tag{8}$$

Such a method as this has been adopted usually in order to estimate the width of resonant state. However, we now should like to point out that the imaginary part of denominator of this modified propagation function $2\pi i\,\bar{D}_F(k)$ should be multiplied by factor 2 as will be seen from the following consideration.

By means of the relation (8), the scattering matrix R for this process can be written as

$$\begin{aligned}&R(K^-+\pi^+\to K^-+\pi^+)=2R(\bar{K}^0+\pi^0\to\bar{K}^0+\pi^0)\\&=\frac{1}{2\pi i}\ \frac{-m_K^2(G_S^2/4\pi)}{[(\hat{p}+\hat{q})^2+M^2-(3/4)\,im_K^2(G_S^2/4\pi)\,A/\sqrt{-(\hat{p}+\hat{q})^2}]},\end{aligned} \tag{9}$$

where $\hat{p}$ and $\hat{q}$ are the four momenta of the incident π and $\bar{K}$ respectively. In the center of mass system, the scattering amplitude for the process $K^-+\pi^+\to K^-+\pi^+$ is given by

$$\begin{aligned}f(K^-+\pi^+\to K^-+\pi^+)&=\frac{1}{(\sqrt{p^{*2}+m_K^2}+\sqrt{p^{*2}+\mu^2})}\\&\times\frac{m_K^2(G_S^2/4\pi)}{[(\hat{p}+\hat{q})^2+M^2-(3/4)\,im_K^2(G_S^2/4\pi)\,A/\sqrt{-(\hat{p}+\hat{q})^2}]}\end{aligned} \tag{10}$$

because

$$\frac{d\sigma(K^-+\pi^+\to K^-+\pi^+)}{d\Omega}=\frac{4\pi^2}{(\sqrt{p^{*2}+m_K^2}+\sqrt{p^{*2}+\mu^2})^2}|R(K^-+\pi^+\to K^-+\pi^+)|^2. \tag{11}$$

p^* is the momentum of the incident pion in the center of mass system.

In the scattering at the resonance energy

$$\begin{aligned}&\sqrt{p^{*2}+m_K^2}+\sqrt{p^{*2}+\mu^2}\to M,\\&(\hat{p}+\hat{q})^2+M^2\to 0,\\&A\to\sqrt{[(M^2-m_K^2-\mu^2)^2-4m_K^2\mu^2]}/4M^2=p^*.\end{aligned} \tag{12}$$

Thus the expression for scattering amplitude at the resonance energy is reduced to the following form (cf. Eq. (10)) :

$$f(K^-+\pi^+\to K^-+\pi^+)_{resonance}=\frac{4}{3}i\frac{1}{p^*}. \tag{13}$$

On the other hand, the scattering amplitude $f(\theta)$ can generally be written down as follows :

$$f(\theta)=\frac{\sqrt{\pi}}{ip^*}\sum_{I,l}\sqrt{2l+1}\,R_{2I,l}\,Y_l^0(\theta,\varphi), \tag{14}$$

$$R_{2I,l}=\exp(2i\delta_{2I,l})-1, \tag{15}$$

where $R_{2I,l}$ means the R-matrix for the state of isotopic spin I and angular momentum l. As we concentrate our attention to the resonance scattering in $I=1/2$, $l=0$-state, the scattering amplitude for the process $K^-+\pi^+\to K^-+\pi^+$ at the resonance energy is reduced to

$$f(K^-+\pi^+\to K^-+\pi^+)_{resonance}=\frac{2}{3}i\frac{1}{p^*}. \tag{16}$$

The factor (2/3) in Eq. (16) comes from the consideration about the isotopic spin space. If the effects of the other reactions (for instance, $\bar{K}+\pi\to\bar{K}+\pi+\pi$) are taken into consideration, Eq. (16) should be modified as

$$|f(K^-+\pi^+\to K^-+\pi^+)_{resonance}|\leq|\frac{2}{3}i\frac{1}{p^*}|. \tag{16}'$$

At any rate there is the difference of factor 2 between Eq. (16) and Eq. (13). Of course such a situation as this is reproduced also in the approach to pion-pion interaction[3] although any remark on this point has not been made.* In the estimation of the damping effects it is necessary to pay attention to this fact.

The correct expression for the modified propagation function can be written down as follows :

$$\begin{aligned}\bar{D}_F(k)&=\frac{1}{2\pi i}\,\frac{1}{k^2+M^2-2(2\pi i)^{-1}N(k)}\\&=\frac{1}{2\pi i}\,\frac{1}{k^2+M^2-(3/2)im_K^2(G_S^2/4\pi)A/\sqrt{-k^2}}.\end{aligned} \tag{17}$$

It is needless to say that the correct value of scattering amplitude at the resonance energy is given by the $\bar{D}_F(k)$-function. Since $M=884$ Mev, the resonance scattering takes place at the pion energy of 382 Mev (lab.), and the cross section turns out to be

* We are grateful to Dr. Y. Miyamoto and Dr. Y. Fujii for their helpful discussions on this point.

$$\sigma(K^-+\pi^+\to K^-+\pi^+)_{resonance}\cong 27 \text{ mb}. \tag{18}$$

In the center of mass system the denominator of $\bar{\bar{D}}_F(\hat{p}+\hat{q})$ can be rewritten as

$$(\hat{p}+\hat{q})^2+M^2-(3/2)im_K^2(G_S^2/4\pi)A/\sqrt{-(\hat{p}+\hat{q})^2}$$
$$=(E_r+E^*)[(E_r-E^*)-(3/2)im_K^2(G_S^2/4\pi)A/\{E^*(E_r+E^*)\}], \tag{19}$$

where E_r and E^* stand for the resonance energy and the total energy of the colliding system respectively. Paying attention to the scattering in the neighborhood of resonance and comparing the expression in Eq. (19) with the denominator in one level formula, we may get approximately the following relation,

$$\frac{\Gamma}{2}\cong\frac{3}{2}m_K^2\left(\frac{G_S^2}{4\pi}\right)\frac{A}{2E_r^2}. \tag{20}$$

This approximation may be fairly well because the width Γ of the resonance is very narrow. Putting the experimental value $\Gamma/2=(10\sim13)$ Mev into Eq. (20), we obtain

$$(G_S^2/4\pi)=(0.15\sim0.2). \tag{21}$$

The energy dependence of $\sigma(K^-+\pi^+\to K^-+\pi^+)$ in case of $(G_s^2/4\pi)=0.15$ is illustrated in Fig. 2.

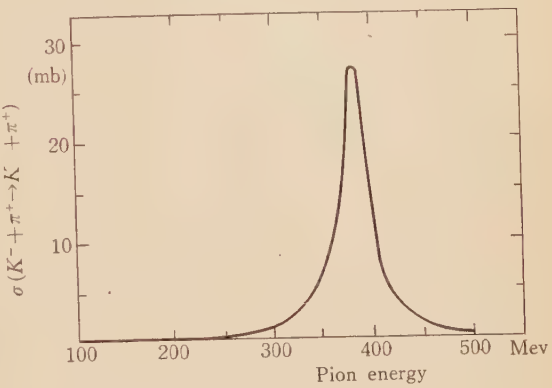

Fig. 2.

§ 3. *P*-wave $\bar{K}$-π scattering

Next let us deal with the problem of $\bar{K}$-π scattering in the case where K^* is a vector particle. As the interaction Hamiltonian, the following type may be adopted,

$$H_2=G_V\Phi_\mu\tau_\alpha\left(\varphi\frac{\partial\phi_\alpha}{\partial x_\mu}-\frac{\partial\varphi}{\partial x_\mu}\phi_\alpha\right) \tag{22}$$

or

$$H_2'=G_V'\Phi_\mu\tau_\alpha\varphi\frac{\partial\phi_\alpha}{\partial x_\mu}, \tag{22}'$$

where Φ_μ is the wave function of the K^*. When $\bar{K}$-π scattering is described in terms of such an interaction as this, another partial wave besides p-wave contributes to the scattering.* If the spin of K^* is equal to one, the $\bar{K}$-π resonance

* If the mass of $\bar{K}$ meson were equal to that of pion, the interaction Hamiltonian mentioned in Eq. (22) would mean the p-wave $\bar{K}\pi K^*$-interaction.

scattering is of course attributed to the strong p-wave interaction. We hereafter try to select the p-wave component out of the matrix elements obtained by employing the interaction Hamiltonian (22) or (22)′.

Our approach to K^--π scattering is made along the same line with the study in § 2. The $N_{\mu\nu}(k)$ which corresponds to the $N(k)$ in Eq. (5) can be expressed as follows: For the H_2-interaction

$$N_{\mu\nu}(k) = -\frac{3}{2}\left(\frac{G_V^2}{4\pi}\right)\int (dt)\,(2t_\mu - k_\mu)(2t_\nu - k_\nu)\,\delta[(k-t)^2+m_K^2]\,\delta(t^2+t'^2). \quad (23)$$

For the H_2'-interaction

$$N_{\mu\nu}'(k) = -\frac{3}{2}\left(\frac{G_V'^2}{4\pi}\right)\int (dt)\,t_\mu t_\nu\,\delta[(k-t)^2+m_K^2]\,\delta(t^2+t'^2). \quad (23)'$$

By virtue of the transformation property, $N_{\mu\nu}(k)$ or $N_{\mu\nu}'(k)$ can be expressed by the following form,

$$N_{\mu\nu}(k) = -\frac{3}{2}\left(\frac{G_V^2}{4\pi}\right)\left[B(k)\,\delta_{\mu\nu}+C(k)\,\frac{k_\mu k_\nu}{-k^2}\right], \quad (24)$$

or

$$N_{\mu\nu}'(k) = -\frac{3}{2}\left(\frac{G_V'^2}{4\pi}\right)\left[B'(k)\,\delta_{\mu\nu}+C'(k)\,\frac{k_\mu k_\nu}{-k^2}\right]. \quad (24)'$$

We can obtain the expression for $\bar{D}_F(k)$ putting the $N_{\mu\nu}(k)$ or $N_{\mu\nu}'(k)$ into the equation corresponding to Eq. (3). Thus the scattering matrix elements can be written down by using the $\bar{D}_F(\hat{p}+\hat{q})$ although they consist of infinitely many terms. In the expression for the scattering matrix elements, all the terms which contain $C(\hat{p}+\hat{q})$ are expressed in terms of $(\hat{p}'-\hat{q}')(\hat{p}'+\hat{q}')$, $(\hat{p}+\hat{q})^2$ and $(\hat{p}-\hat{q})(\hat{p}+\hat{q})$, and all the terms which contain $C'(\hat{p}+\hat{q})$ are expressed in terms of $\hat{p}'(\hat{p}+\hat{q})$, $(\hat{p}+\hat{q})^2$ and $\hat{p}(\hat{p}+\hat{q})$, where $\hat{p}'$ and $\hat{q}'$ are the four momenta of the final π and K respectively. Therefore all the matrix elements which contain $C(\hat{p}+\hat{q})$ or $C'(\hat{p}+\hat{q})$ are responsible for the s-wave scattering because $\vec{p}+\vec{q}=\vec{p}'+\vec{q}'=0$ in the center of mass system. Since p-wave scatterings are examined in our study, we have only to take into account the contributions from $B(\hat{p}+\hat{q})$ or $B'(\hat{p}+\hat{q})$-terms. Then $\bar{D}_F(\hat{p}+\hat{q})$ can be expressed by a compact form.

Tracing the similar consideration with that in § 2, we obtain the correctly modified propagation function $D_F(\hat{p}+\hat{q})$. The p-wave scattering matrix R(or R') for the process $K^-+\pi^+\to K^-+\pi^+$ can be expressed by the following compact form,

$$R(K^-+\pi^+\to K^-+\pi^+) = \frac{1}{2\pi i}\,P_{l=1}\cdot\frac{-(G_V^2/4\pi)(\hat{p}'-\hat{q}')(\hat{p}-\hat{q})}{(\hat{p}+\hat{q})^2+M^2-(3/2\pi)i(G_V^2/4\pi)B(\hat{p}+\hat{q})} \quad (25)$$

$$=\frac{1}{2\pi i}\frac{-(G_V^2/4\pi)4p^{*2}\cos\theta}{(\hat{p}+\hat{q})^2+M^2-(3/2\pi)i(G_V^2/4\pi)B(\hat{p}+\hat{q})}, \tag{26}$$

or

$$R'(K^-+\pi^+\to K^-+\pi^+)=\frac{1}{2\pi i}P_{l=1}\frac{-(G_V'^2/4\pi)(\hat{p}\hat{p}')}{(\hat{p}+\hat{q})^2+M^2-(3/2\pi)i(G_V'^2/4\pi)B'(\hat{p}+\hat{q})} \tag{25}'$$

$$=\frac{1}{2\pi i}\frac{-(G_V'^2/4\pi)p^{*2}\cos\theta}{(\hat{p}+\hat{q})^2+M^2-(3/2\pi)i(G_V'^2/4\pi)B'(\hat{p}+\hat{q})}, \tag{26}'$$

where

$$B(k)=4B'(k)=\frac{4\pi}{3\sqrt{-k^2}}A^3. \tag{27}$$

The A in Eq. (27) is given by Eq. (7), and $P_{l=1}$ means the projection operator on p-state. If we replace $4(G_V^2/4\pi)$ in Eq. (26) by $(G_V'^2/4\pi)$, the expression for $R(K^-+\pi^+\to K^-+\pi^+)$ turns out to be the same with that for $R'(K^-+\pi^+\to K^-+\pi^+)$. Thus there is no need to discuss the matrix elements R and R' separately.

The scattering cross section for this process at the resonance energy is

$$\sigma(K^-+\pi^+\to K^-+\pi^+)_{resonance}$$
$$=\frac{4}{9}\frac{3\times4\pi}{p^{*2}}\simeq 82\text{ mb}. \tag{28}$$

In the same way as in § 2, we can estimate the value of coupling constant for $\bar{K}\pi K^*$-interaction. The result is as follows,

$$(G_V^2/4\pi)=(G_V'^2/4\pi)/4$$
$$=(0.35\sim0.45). \tag{29}$$

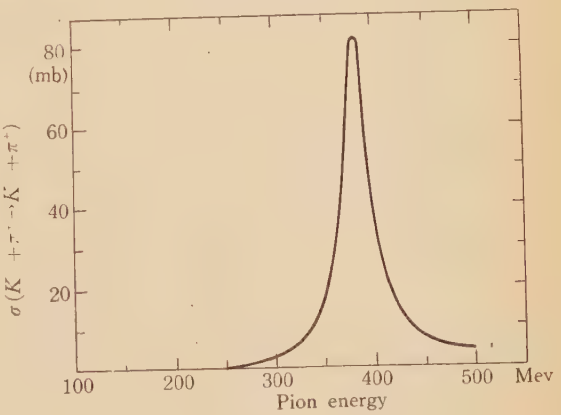

Fig. 3.

The energy dependence of $\sigma(K^-+\pi^+\to K^-+\pi^+)$ in case of $(G_V^2/4\pi)=0.35$ is illustrated in Fig. 3.

§ 4. $K^-+p\to K^*+p$ process

In this section the angular distribution and the magnitude of cross section for $K^-+p\to K^*+p$ process are examined by means of perturbation theory. Since the so-called equivalence theorem for pion-nucleon interaction can be applied to our case, there is no need to examine the results in the case of pseudoscalar coupling and those in the case of pseudovector coupling separately. Hereafter we make our discussion about the former case only.

I) *The case of* $S=0$

The differential cross sections for this reaction can be expressed as follows,

$$\frac{d\sigma(K^-+p\to K^*+p)}{d\Omega}=\frac{k\left(\frac{m_K}{m}\right)^2\left(\frac{g^2}{4\pi}\right)\left(\frac{G_S^2}{4\pi}\right)}{q(\sqrt{q^2+m^2}+\sqrt{q^2+m_K^2})^2}\cdot\frac{1}{4}\cdot\frac{\left[-2\frac{(\hat{I}\hat{F})}{m^2}-2\right]}{\left[-2\frac{(\hat{I}\hat{F})}{m^2}-2+\frac{\mu^2}{m^2}\right]^2}, \tag{30}$$

where $\hat{I}$ and $\hat{F}$ are the four momenta of proton in the initial and final state respectively, q and k are the momenta of K^- meson and K^* respectively, and m is the mass of nucleon. The angular distributions of protons are expressed by the last term in Eq. (30) and have the character shown in Fig. 4. The tendency of backward peak for angular distributions of protons becomes remarkable with the increase of energy of incident K meson. This may be due to the following situation: i) In high energy region, a number of partial waves contribute to this reaction, ii) K^* will be emitted by the collision in periphery of the nucleon.

Putting the values $g^2/4\pi=14.5$ and $G_S^2/4\pi=0.15$, we obtain the following results for total cross sections,

$$\begin{aligned}&\sigma(K^-+p\to K^*+p)\cong 0.18\text{ mb}\quad\text{at }1.15\text{ Bev}/c,\\&\sigma(K^-+p\to K^*+p)\cong 0.16\text{ mb}\quad\text{at }2\text{ Bev}/c.\end{aligned} \tag{31}$$

II) *The case of* $S=1$*

The differential cross sections for this process are given by

$$\frac{d\sigma(K^-+p\to K^*+p)}{d\Omega}\cong\frac{k\left(\frac{g^2}{4\pi}\right)\left(\frac{G_V^2}{4\pi}\right)}{q(\sqrt{q^2+m^2}+\sqrt{q^2+m_K^2})^2}\cdot\frac{1}{4}\cdot\frac{\left[-2\frac{(\hat{I}\hat{F})}{m^2}-2\right]}{\left[-2\frac{(\hat{I}\hat{F})}{m^2}-2+\frac{\mu^2}{m^2}\right]^2}\times\frac{(4q^2+k^2+4kq\cos\theta)}{m^2}, \tag{32}$$

where θ is the angle between the direction of incident K^--meson and that of proton in the final state. The angular distributions of protons for this process are shown in Fig. 5. Putting the values $g^2/4\pi=14.5$ and $G_V^2/4\pi=0.35$, we obtain the following results,

$$\begin{aligned}&\sigma(K^-+p\to K^*+p)\cong 2.3\text{ mb}\quad\text{at }1.15\text{ Bev}/c,\\&\sigma(K^-+p\to K^*+p)\cong 3.4\text{ mb}\quad\text{at }2\text{ Bev}/c.\end{aligned} \tag{33}$$

* Hereafter we adopt the interaction Hamiltonian H_2 with the value of coupling constant $G_V^2/4\pi=0.35$ (cf. Eq. (22)).

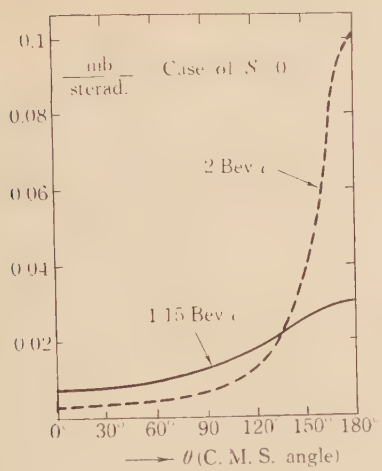

Fig. 4. Angular distribution of proton for $K^-+p\to K^*+p$.

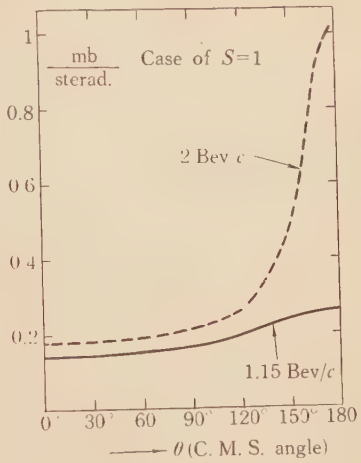

Fig. 5. Angular distribution of proton for $K^-+p\to K^*+p$.

As is seen from (31) and (33), the excitation function for this reaction depends on the spin of K^*. In the case of $S=0$ the magnitude of $\sigma(K^-+p\to K^*+p)$ at 2 Bev/c is nearly equal to that at 1.15 Bev/c, while in the case of $S=1$ its magnitude increases considerably with the energy of incident K^- meson. By comparing our results illustrated in Fig. 4 and Fig. 5 with the experimental results (ii) for the angular distribution at 1.15 Bev/c, it may be said that the assumption of $S=1$ is more promising than that of $S=0$.

§ 5. $K^-+p\to\bar{K}^0+\pi^-+p$ reaction

According to the experimental result (i) this reaction mainly takes place through the process $K^-+p\to K^*+p\to\bar{K}^0+\pi^-+p$. Now we study this reaction in the case where K^* is not observed directly in the course of collision although K^* of course plays the most important role in the virtual state for this reaction. Fig. 6 shows the diagram corresponding to the lowest order perturbation for this process. When we denote the momenta of proton and K meson in the initial state as I and q respectively and denote the momenta of proton, K meson and pion in the final state as F, l and p respectively, the cross section for this process can be expressed by the following form,

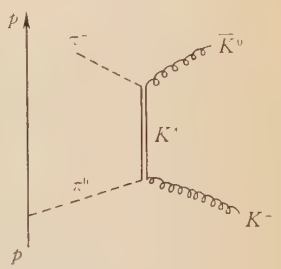

Fig. 6.

$$\sigma(K^-+p\to\bar{K}^0+\pi^-+p)=\frac{4\pi^2}{B}\int|\langle f|I|i\rangle|^2\,\frac{d\vec{F}}{F_0}\,\frac{d\vec{p}}{p_0}\,\frac{d\vec{l}}{l_0}$$

$$\times\delta(\vec{I}+\vec{q}-\vec{F}-\vec{p}-\vec{l})\,\delta(I_0+q_0-F_0-p_0-l_0)$$

$$= \frac{4\pi^2}{B}\int |\langle f|I|i\rangle|^2 \frac{p\,d\Omega_1}{(W-F_0)+F(p_0/p)\cos\theta}\frac{d\vec{F}}{F_0}, \qquad (34)^*$$

$$B=q(\sqrt{q^2+m^2}+\sqrt{q^2+m_K^2}).$$

W is the total energy in the center of mass system, θ is the angle between the direction of $\vec{F}$ and that of $\vec{p}$, $\langle f|I|i\rangle$ is the invariant matrix element for this process, and $d\Omega_1$ is the element of solid angle for the direction of $\vec{p}$. There is the following relation between p and θ owing to the energy conservation law,

$$\sqrt{p^2+\mu^2}=\frac{W-F_0}{2}-\frac{(F^2+m_K^2-\mu^2)}{2(W-F_0)}-\frac{Fp}{W-F_0}\cos\theta. \qquad (35)$$

Making use of the modified propagation function of K^* which has been obtained in § 2 and § 3, we can write down the expression for $|\langle f|I|i\rangle|^2$ as follows:

I) In the case of $S=0$,

$$|\langle f|I|i\rangle|^2=\frac{2\left(\frac{g^2}{4\pi}\right)\left(\frac{m_K}{m}\right)^4\left(\frac{G_S^2}{4\pi}\right)^2}{4^3\pi^4}\frac{1}{m^2}\frac{\left[-2-\frac{(\hat{I}\hat{F})}{m^2}-2\right]}{\left[-2-\frac{(\hat{I}\hat{F})}{m^2}-2+\frac{\mu^2}{m^2}\right]^2}$$

$$\times\left[\frac{(\hat{I}-\hat{F}+\hat{q})^2+M^2}{m^2}\right]^2+\frac{9}{4}\left(\frac{m_K}{m}\right)^4\left(\frac{G_S^2}{4\pi}\right)^2\frac{A^2}{-(\hat{I}-\hat{F}+\hat{q})^2}. \qquad (36)^{**}$$

II) In the case of $S=1$,

$$|\langle f|I|i\rangle|^2\cong\frac{2\left(\frac{g^2}{4\pi}\right)\left(\frac{G_V^2}{4\pi}\right)^2}{4^3\pi^4}\frac{1}{m^2}\frac{\left[-2-\frac{(\hat{I}\hat{F})}{m^2}-2\right]}{\left[-2-\frac{(\hat{I}\hat{F})}{m^2}-2+\frac{\mu^2}{m^2}\right]^2}$$

$$\times\frac{\left[-2\frac{F}{m}\frac{p}{m}\cos\theta-\left(\frac{F}{m}\right)^2-4\frac{p}{m}\frac{q}{m}\cos\theta_1-2\frac{F}{m}\frac{q}{m}\cos\theta\right]^2}{\left[\frac{(\hat{I}-\hat{F}+\hat{q})^2+M^2}{m^2}\right]^2+4\left(\frac{G_V^2}{4\pi}\right)^2\frac{A^2}{-(\hat{I}-\hat{F}+\hat{q})^2}\left(\frac{A^2}{m^2}\right)^2}, \qquad (37)^{**}$$

where

$$A^2=\frac{[(\hat{I}-\hat{F}+\hat{q})^2+(m_K+\mu)^2][(\hat{I}-\hat{F}+\hat{q})^2+(m_K-\mu)^2]}{-4(\hat{I}-\hat{F}+\hat{q})^2}, \qquad (38)$$

$$-(\hat{I}\hat{F})=qF\cos\theta+\sqrt{q^2+m^2}\sqrt{F^2+m^2}, \qquad (39)$$

* The integral on $d\vec{F}$ is performed at the last step of our calculation.

** The factor 2 in the first term on the right-hand side of (36) or (37) is derived from the consideration about isotopic spin space.

$$-(\hat{I}-\hat{F}+\hat{q})^2=2m^2+m_K^2+2[q^2+\sqrt{q^2+m^2}\sqrt{q^2+m_K^2}$$
$$-\sqrt{F^2+m^2}(\sqrt{q^2+m^2}+\sqrt{q^2+m_K^2})]. \tag{40}$$

θ_1 is the angle between $\vec{q}$ and $\vec{p}$, and θ the angle between $\vec{q}$ and $\vec{F}$. The terms which contain A are derived from the damping effects. The magnitude of cross section for this process, the angular distributions and momentum spectra of pions and protons can be obtained by means of the relations (34), (36) or (37). But this work is considerably troublesome. We now try to examine them under the crude assumptions which will be mentioned below.

Let us examine the following function in Eq. (34),

$$|\langle f|I|i\rangle|^2\frac{p}{(W-F_0)+F(p_0/p)\cos\theta}. \tag{41}$$

Although the term* $\frac{9}{4}\left(\frac{m_K}{m}\right)^4\left(\frac{G_S^2}{4\pi}\right)^2\frac{A^2}{-(\hat{I}-\hat{F}+\hat{q})^2}$ or $4\left(\frac{G_V^2}{4\pi}\right)^2\frac{A^2}{-(\hat{I}-\hat{F}+\hat{q})^2}\times\left(\frac{A^2}{m^2}\right)^2$ in $|\langle f|I|i\rangle|^2$ depends on F, its magnitude is very small in our case. When F/m has the value $(F/m)_r$ corresponding to the resonance of K^*, $(\hat{I}-\hat{F}+\hat{q})^2+M^2=0$. Therefore

$$\mathcal{J}_S(F/m)=\frac{1}{\left[\frac{(\hat{I}-\hat{F}+\hat{q})^2+M^2}{m^2}\right]^2+\frac{9}{4}\left(\frac{m_K}{m}\right)^4\left(\frac{G_S^2}{4\pi}\right)^2\frac{A^2}{-(\hat{I}-\hat{F}+\hat{q})^2}}$$

and

$$\mathcal{J}_V(F/m)=\frac{1}{\left[\frac{(\hat{I}-\hat{F}+\hat{q})^2+M^2}{m^2}\right]^2+4\left(\frac{G_V^2}{4\pi}\right)^2\frac{A^2}{-(\hat{I}-\hat{F}+\hat{q})^2}\left(\frac{A^2}{m^2}\right)^2}$$

have large values only in the very narrow region, that is, only in the neighborhood of $(F/m)_r$. In addition, the other terms except $\mathcal{J}_S(F/m)$ (or $\mathcal{J}_V(F/m)$) in Eq. (41) have not such a property as this with respect to the variable F/m. In other words the resonance phenomena of K^* with the narrow width play the most important role in the reaction $K^-+p\to\bar{K}^0+\pi^-+p$. This may make it possible to examine the characteristic property of this reaction by replacing approximately the variable F/m in all the other terms except $\mathcal{J}_S(F/m)$ or $\mathcal{J}_V(F/m)$ by the value of $(F/m)_r$.

In the case where this reaction takes place through the process $K^-+p\to K^*+p\to\bar{K}^0+\pi^-+p$, the main feature of angular distribution of the final proton may be represented by the form of that of proton for the reaction $K^-+p\to K^*+p$ mentioned in § 4. So far as the phenomena at 2 Bev/c (or at the energy higher than 2 Bev/c) are concerned, it may be said that the final protons in $K^-+p\to K^*+p$ process are emitted mainly into the backward direc-

* Its value at the resonance energy may be expressed in terms of $(\Gamma/2)^2$.

tion. In our study for the reaction $K^-+p\to\bar{K}^0+\pi^-+p$ at 2 Bev/c we introduce another assumption of $\theta\cong\pi-\theta_1$.

§ 5·1. $K^-+p\to\bar{K}^0+\pi^-+p$ reaction at 2 Bev/c

Under the assumption $\theta\cong\pi-\theta_1$, Eq. (35) can be written as follows:

$$\sqrt{\left(\frac{p}{m}\right)^2+\left(\frac{\mu}{m}\right)^2}=\frac{\left(\frac{W}{m}\right)-\left(\frac{F_0}{m}\right)_r}{2}-\frac{\left(\frac{F}{m}\right)_r^2+\left(\frac{m_K}{m}\right)^2-\left(\frac{\mu}{m}\right)^2}{2\left[\left(\frac{W}{m}\right)-\left(\frac{F_0}{m}\right)_r\right]}+\frac{\left(\frac{F}{m}\right)_r\left(\frac{p}{m}\right)}{\left(\frac{W}{m}\right)-\left(\frac{F_0}{m}\right)_r}\cos\theta_1. \qquad (35)'$$

From this relation it follows that (p/m) can be expressed in terms of $\cos\theta_1$. We can see the angular distributions of pions by examining the θ_1-dependence of $|\langle f|I|i\rangle|^2_{(W-F_0)-F(p_0/p)\cos\theta_1}$. They are expressed by the following forms:

I) In the case of $S=0$,

$$\frac{(p/m)}{(W/m)-(F_0/m)_r-(F/m)_r(p/p_0)\cos\theta_1}\equiv\mathcal{G}_S(p,\theta_1)\equiv\mathcal{G}_S'(\theta_1) \qquad (42)$$

because $|\langle f|I|i\rangle|^2$ does not depend on θ_1.

II) In the case of $S=1$,

$$\frac{\left[2\left(\frac{F}{m}\right)_r\left(\frac{p}{m}\right)\cos\theta_1-\left(\frac{F}{m}\right)_r^2-4\left(\frac{p}{m}\right)\left(\frac{q}{m}\right)\cos\theta_1+2\left(\frac{F}{m}\right)_r\left(\frac{q}{m}\right)\right]^2}{(W/m)-(F_0/m)_r-(F/m)_r(p/p_0)\cos\theta_1}(p/m)\cdot$$
$$\equiv\mathcal{G}_V(p,\theta_1)\equiv\mathcal{G}_V'(\theta_1). \qquad (43)^*$$

The curves of $\mathcal{G}_S'(\theta_1)$ and $\mathcal{G}_V'(\theta_1)$ are illustrated in Fig. 7. If we refer to the rest system of K^*, the angular distributions of pions should be isotropic and of a form of $\sim\cos^2\theta_1$ corresponding to the case of $S=0$ and $S=1$ respectively. Under the assumption that the proton does not change its direction considerably during a collision, K^* ought to be emitted in the forward direction. Therefore the angular distributions of pions in the center of mass system are enhanced remarkably in the forward direction compared with those in the rest system of K^* (cf. Fig. 7).

Next let us examine the momentum spectrum of pion. Differentiating Eq. (35)', we get

* The assumption of $\theta\cong\pi$ is introduced in the expression of (43).

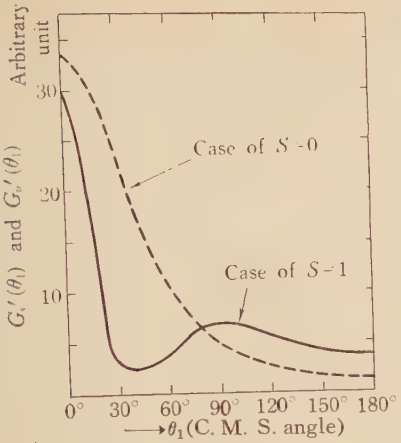

Fig. 7. Angular distribution of pion for $K^-+p\to\bar{K}^0+\pi^-+p$ at 2 Bev/c (where $F/m=(F/m)_r=0.6886$).

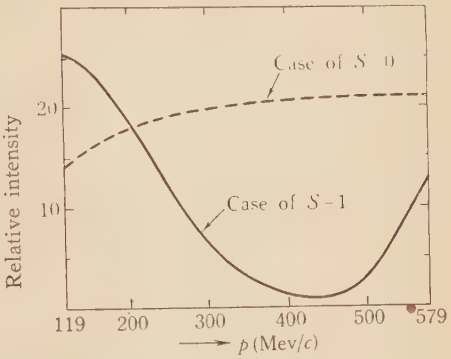

Fig. 8. Momentum spectrum of pion for $K^-+p\to\bar{K}^0+\pi^-+p$ at 2 Bev/c, where proton energy $T_p\cong 200$ Mev (corresponding to $(F/m)_r=0.6886$).

$$\sin\theta_1\, d\theta_1=\left[\cos\theta_1-\frac{p}{p_0}\,\frac{\left(\frac{W}{m}\right)-\left(\frac{F_0}{m}\right)_r}{(F/m)_r}\right]\frac{dp}{p}. \tag{44}$$

$\mathcal{G}_S(p,\theta_1)$ and $\mathcal{G}_V(p,\theta_1)$ can be re-expressed as the functions of p by means of relation (35)′.

$$\mathcal{G}_S(p,\theta_1)\equiv\mathcal{G}_S'(\theta_1)=\mathcal{G}_S''(p),$$
$$\mathcal{G}_V(p,\theta_1)\equiv\mathcal{G}_V'(\theta_1)\equiv\mathcal{G}_V''(p).$$

Therefore

$$\int\mathcal{G}_S'(\theta_1)\,d\Omega_1=2\pi\int_{p_{min}}^{p_{max}}\mathcal{G}_S''(p)\left[\frac{p}{p_0}\,\frac{\left(\frac{W}{m}\right)-\left(\frac{F_0}{m}\right)_r}{(F/m)_r}-\cos\theta_1\right]\frac{dp}{p},$$
$$\int\mathcal{G}_V'(\theta_1)\,d\Omega_1=2\pi\int_{p_{min}}^{p_{max}}\mathcal{G}_V''(p)\left[\frac{p}{p_0}\,\frac{\left(\frac{W}{m}\right)-\left(\frac{F_0}{m}\right)_r}{(F/m)_r}-\cos\theta_1\right]\frac{dp}{p}. \tag{45}$$

Through Eq. (45) we can see the momentum spectrum (or energy spectrum) of pion in the case where $F/m=(F/m)_r=0.6886$ (cf. Fig. 8).

Now we try to estimate the magnitude of cross section for this reaction and to examine the angular distribution and momentum spectrum of proton. As was mentioned above, in the expression of (34) we may approximately replace the variable F/m in all the other terms except $\mathcal{F}_S(F/m)$ or $\mathcal{F}_V(F/m)$ by the value of $(F/m)_r$ and approximately replace θ by $(\pi-\theta_1)$. Then the

integral on $d\Omega_1$ can be carried out easily. Since $\mathcal{F}_S(F/m)$ and $\mathcal{F}_V(F/m)$ do not depend on the angle θ, in our approximation the character of angular distribution of proton can be expressed by that of angular distribution in the case of $F/m=(F/m)_r$ (cf. Fig. 9).

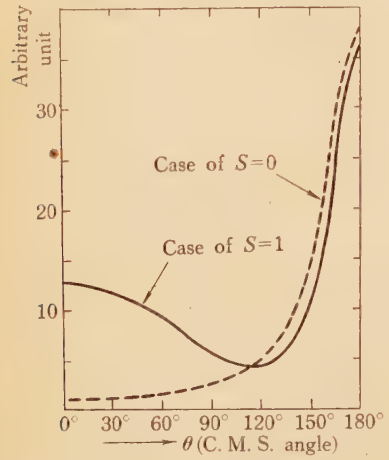

Fig. 9. Angular distribution of proton for $K^-+p\to\bar{K}^0+\pi^-+p$ at 2 Bev/c.

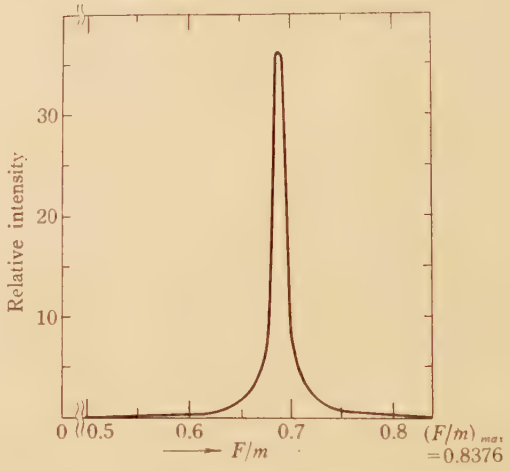

Fig. 10. Momentum spectrum of proton for $K^-+p\to\bar{K}^0+\pi^-+p$ at 2 Bev/c.

So long as K^* is not observed in the course of reaction $K^-+p\to\bar{K}^0+\pi^-+p$, the angular distribution of proton for this reaction may be more or less different from that for $K^-+p\to K^*+p$ (compare Fig. 9 with Fig. 5). The reason why the dotted line in Fig. 9 has the similar form with that in Fig. 4 is due to the following situation. Under the assumption $\theta\sim\pi-\theta_1$, the angular distribution of proton for $K^-+p\to\bar{K}^0+\pi^-+p$ depends on $|\langle f|I|i\rangle|^2$ only. In the case of $S=0$ the θ-dependence of $|\langle f|I|i\rangle|^2$ is the same with that of $d\sigma(K^-+p\to K^*+p)/d\Omega$ mentioned in Eq. (30) apart from a trivial factor.

The momentum spectrum of proton in the final state is shown in Fig. 10. Performing the integral on $d\vec{F}$ and adopting the values of coupling constants $g^2/4\pi=14.5$ and $G_S^2/4\pi=0.15$ (or $G_V^2/4\pi=0.35$), we obtain the following results:

I) In the case of $S=0$,

$$\sigma(K^-+p\to\bar{K}^0+\pi^-+p)_{at\ 2\ Bev/c}\cong 0.11\ \text{mb}. \tag{46}$$

II) In the case of $S=1$,

$$\sigma(K^-+p\to\bar{K}^0+\pi^-+p)_{at\ 2\ Bev/c}\cong 2.2\ \text{mb}. \tag{47}$$

§ 5·2. *$K^-+p\to K^0+\pi^-+p$ reaction at 1.15 Bev/c*

Although the approximation of $\theta\cong\pi-\theta_1$ may not be so useful in this case, we try to examine the angular distribution and energy spectrum (or momen-

tum spectrum) of pion by the same way as in § 5·1. The results are illustrated in Fig. 11 and Fig. 12.

First of all let us make some discussion about the angular distribution of pion. In the collision at 1.15 Bev/c, K^* cannot be emitted with high momentum. Therefore the angular distributions of pions in the center of mass system ought to have nearly the similar forms with those in the rest system of K^*, that is, they are nearly isotropic and of the forms of $\cos^2\theta_1$ corresponding to the case of $S=0$ and $S=1$ respectively. It may be said that the curves illustrated in Fig. 11 have these characteristic properties, although the angular distributions are slightly larger in the forward than in the backward direction because we assume that the proton does not change its direction remarkably during a collision.

Next let us make some discussion about the energy spectrum (or momentum spectrum) of pion. We show in Fig. 12 the energy spectrum of pion in the case where proton momentum in the final state is equal to $(F/m)_r=0.2091$ (proton energy $T_p\simeq 20$ Mev). From these curves we can see the following characteristic properties :

i) In the case of $S=0$, the emitted pions have a uniform energy spectrum.

ii) In the case of $S=1$, the energy spectrum of pion has a minimum value in the neighborhood of $T_{\pi^-}=190$ Mev.

Comparing these results with the experimental ones for the final proton energy $T_p\simeq 20$ Mev, we may conclude that the assumption of $S=1$ is more promising than that of $S=0$.

Finally we try to estimate the magnitude of cross section for this reaction.

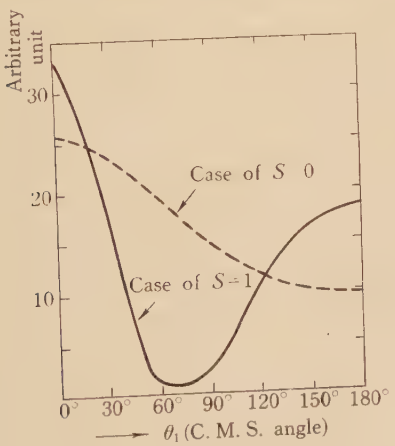

Fig. 11. Angular distribution of pion for $K^-+p\to\bar{K}^0+\pi^-+p$ at 1.15 Bev/c.

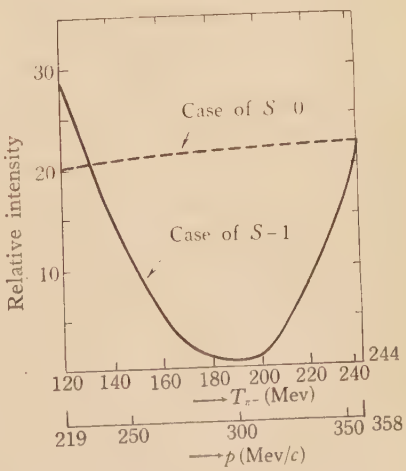

Fig. 12. Energy spectrum of pion for $K^-+p\to\bar{K}^0+\pi^-+p$ at 1.15 Bev/c, where kinetic energy of proton $T_p\cong 20$ Mev (corresponding to $(F/m)_r=0.2091$).

Since the approximation of $\theta \cong \pi - \theta_1$ may not be so useful in this case, it is necessary to examine the value of

$$\frac{(p/m)}{(W/m)-(F_0/m)_r+(F/m)_r(p_0/p)\cos\theta} \equiv \mathcal{H}_0(p,\theta). \tag{48}$$

Using the relation of (35), in the case of $F/m=(F/m)_r=0.2091$ we get

$$\begin{aligned} &\mathcal{H}_0(p,\theta)_{max} \cong 0.515, \quad \mathcal{H}_0(p,\theta)_{min} \cong 0.192, \\ &\langle \mathcal{H}_0(p,\theta) \rangle_{mean} \cong 0.331. \end{aligned} \tag{49}$$

Therefore

$$\begin{aligned} &(0.192)4\pi \leq I_0 = \int \mathcal{H}_0(p,\theta)\, d\Omega_1 \leq (0.515)4\pi, \\ &\int \langle \mathcal{H}_0(p,\theta) \rangle_{mean}\, d\Omega_1 \cong (0.331)4\pi. \end{aligned} \tag{50}$$

On the other hand, if we assume $\theta=\pi-\theta_1$,

$$I_0 = \int \mathcal{H}_0(p, \pi-\theta_1)\, d\Omega_1 \cong (0.317)4\pi. \tag{51}$$

Thus it may be said that the assumption $\theta \cong \pi-\theta_1$ is applicable to the case of $S=0$ in estimating the order of $\sigma(K^-+p \to \bar{K}^0+\pi^-+p)$. Performing the integral on $d\vec{F}$ and adopting the values of coupling constants $g^2/4\pi=14.5$ and $G_S^2/4\pi=0.15$, we obtain the following result. In the case of $S=0$,

$$\sigma(K^-+p \to \bar{K}^0+\pi^-+p)_{at\ 1.15\ Bev/c} \cong 0.12 \text{ mb}. \tag{52}$$

Comparing this with the experimental value (2.0 ± 0.3) mb, we may draw the following conclusion. The assumption of $S=0$ is not consistent with the experimental results for cross section of $K^-+p \to \bar{K}^0+\pi^-+p$ reaction.

In the case of $S=1$ it is necessary to estimate the contributions from the following $\mathcal{H}_1(p,\theta)$ and $\mathcal{H}_2(p,\theta)$-terms in addition to $\mathcal{H}_0(p,\theta)$,

$$\begin{aligned} \mathcal{H}_1(p,\theta) &= (p/m)\mathcal{H}_0(p,\theta), \\ \mathcal{H}_2(p,\theta) &= (p/m)^2\mathcal{H}_0(p,\theta). \end{aligned}$$

Using the relation (35), in the case of $F/m=(F/m)_r=0.2091$ we get

$$\begin{aligned} &\mathcal{H}_1(p,\theta)_{max} \cong 0.197, \quad \mathcal{H}_1(p,\theta)_{min} \cong 0.045, \\ &\langle \mathcal{H}_1(p,\theta) \rangle_{mean} \cong 0.106, \end{aligned} \tag{53}$$

and

$$\begin{aligned} &\mathcal{H}_2(p,\theta)_{max} \cong 0.075, \quad \mathcal{H}_2(p,\theta)_{min} \cong 0.01, \\ &\langle \mathcal{H}_2(p,\theta) \rangle_{mean} \cong 0.035. \end{aligned} \tag{54}$$

Therefore

$$\langle I_0\rangle=\int\langle\mathcal{H}_0(p,\theta)\rangle_{mean}d\Omega_1=(0.331)4\pi,$$

$$\langle I_1\rangle=\int\langle\mathcal{H}_1(p,\theta)\rangle_{mean}\cos\theta_1 d\Omega_1=0, \tag{55}$$

$$\langle I_2\rangle=\int\langle\mathcal{H}_2(p,\theta)\rangle_{mean}\cos^2\theta_1 d\Omega_1=(0.0117)4\pi.$$

On the other hand, if we assume $\theta\cong\pi-\theta_1$,

$$I_0=\int\mathcal{H}_0(p,\theta)d\Omega_1=(0.317)4\pi,$$

$$I_1=\int\mathcal{H}_1(p,\theta)\cos\theta_1 d\Omega_1=(0.025)4\pi, \tag{56}$$

$$I_2=\int\mathcal{H}_2(p,\theta)\cos^2\theta_1 d\Omega_1=(0.0118)4\pi.$$

Let us tentatively estimate the order of $\sigma(K^-+p\to\bar{K}^0+\pi^-+p)$ at 1.15 Bev/c by using the values of (56). Although there is a considerable difference between $\langle I_1\rangle$ in (55) and I_1 in (56), this difference has not large effect on the final result. Adopting the values $g^2/4\pi=14.5$ and $G_V^2/4\pi=0.35$, we obtain

$$\sigma(K^-+p\to\bar{K}^0+\pi^-+p)_{at\ 1.15\ Bev/c}\cong 1.5\text{ mb}. \tag{57}*$$

This value agrees fairly well with the experimental one (2.0 ± 0.3) mb. Thus it may be said that the assumption of $S=1$ is consistent with the experimental result.

§ 6. On the isotopic spin of K^*

We have studied the $\bar{K}$ meson reaction with the assumption of $I_{K^*}=1/2$. The description of the reaction in the case of $I_{K^*}=3/2$ can be made along the same line with the above one. But it may be difficult to expect the case of $I_{K^*}=3/2$ on the basis of the reason mentioned in § 1. We should like to make some additional disscusion about this point.

Previously we made the following prediction about the $\bar{K}$-N scattering at high energy.[4] In the neighborhood of the energy 675 Mev (lab.) which corresponds to the threshold energy of $\bar{K}+N\to K^*+N$, we may expect the large cross sections of $\bar{K}$-N scattering. And this resonance is due to the strong interaction in the $I=0$ state if the isotopic spin of K^* is equal to 1/2. On the other hand, if the isotopic spin of K^* is equal to 3/2, this resonance is due to the strong interaction in the $I=1$ state.

According to the recent experiment,[5] there seems to be some evidence of

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* The numerical value mentioned in (57) should not be interpreted so seriously because of our rough estimation.

resonance in K^--p scattering in the neighborhood of the threshold energy of $K+N \rightarrow K^*+N$, but such resonance phenomena are not observed in K^--n scattering. Thus it may be said that not only the experimental results for the branching ratio of the decay of K^* but also these experimental results for $\bar{K}$-N scattering are consistent with the assumption of $I_{K^*}=1/2$.

After completing this work the author saw the papers by Chia-Hwa Chan[6] and M. A. B. Bég et al.[7] in which the similar conclusion to ours is mentioned, that is, a vector K^* fits the experimental data better than a scalar K^*.

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Elastic π^--p Scattering at 1.4 Bev

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The analysis of experimental results for π^--p collision at 1.4 Bev is made again by taking into account the effects of real parts of scattering amplitude. It is shown that the approximation in which the real parts of phase shifts are neglected is not necessarily good.

§ 1. Introduction

Previously we tried to analyze the experimental data for π^--p collision at 1.4 Bev under the crude assumptions of $R_l^+ = R_l^- = R_l (= \eta_l e^{2i\delta_l} - 1)$ and of δ_l's $= 0$,[1] where R_l^+ and R_l^- stand for the R-matrices for the l-wave belonging to total angular momenta $J = l + 1/2$ and $J = l - 1/2$ states respectively. As an empirical formula for differential cross sections of elastic scattering we adopted

$$d\sigma/d\Omega = 0.2/(1.13 - \cos\theta)^2 \text{ mb/sterad.} \tag{1}$$

and were able to explain the experimental results for cross section of pion production. But there was an essential contradiction in our previous analysis owing to roughness of our assumption. The value of η_l for $l = 0$ state turned out to be negative, that is, $\eta_0 = -0.4655$. It is needless to say that all the η_l must satisfy the following relation,

$$0 \leq \eta_l \leq 1. \tag{2}$$

Taking into account the real parts of scattering amplitudes* and getting rid of the defect of $\eta_l < 0$, we should like to try the analysis again and to examine the possibility of reproducing the small bump in the backward hemisphere which seemed to be confirmed in the experiment of angular distribution for elastic scattering.

§ 2. Some consideration about real phase shifts

First of all we show in Table I the results obtained in our previous analysis.[1]

* Koba et al.[2] examined the value of $|\mathrm{Re}\, f(0^\circ)_l|^2/|\mathrm{Im}\, f(0^\circ)_l|^2$, spin dependence of the nuclear interaction and the presence of a quasi-resonance state in order to get rid of the contradiction in simple theory which has been revealed in proton-proton collision at Bev region.

Table I.

l	η_l	σ_l^{scatt}(mb)	σ_l^{prod}(mb)
0	−0.4655	4.91	1.79
1	0.3920	2.54	5.80
2	0.7023	1.01	5.79
3	0.8446	0.39	4.59
4	0.9161	0.15	3.31
5	0.9536	0.05	2.28
6	0.9737	0.02	1.54
total		9.1 mb	25.1 mb

The magnitude of both elastic and inelastic cross sections for s-wave is by no means larger than $\pi/k^2 \simeqq 2.287$ mb as long as δ_0 is neglected. But in our previous estimation

$$\sigma_{l=0}^{scatt}=4.91 \text{ mb}, \quad \text{and} \quad \sigma_{l=0}^{prod}=1.79 \text{ mb}. \tag{3}$$

Therefore it may be impossible to say that our previous results agree fairly well with the experimental ones although we obtained the result of $\sum_{l=0}^{6}{}' \sigma_l^{scatt}=9.1$ mb which is nearly equal to the observed one $\sigma^{scatt}=(10.0\pm 0.8)$ mb.[3] With respect to s-wave it will be natural for us to assume $\eta_0=0$. Under this assumption,

$$\sigma_{l=0}^{scatt}=\sigma_{l=0}^{prod}=\pi/k^2 \simeqq 2.287 \text{ mb}. \tag{4}$$

Comparing these values with those in Eq. (3) and considering the result $\sum_{l=0}^{6} \sigma_l^{scatt}=9.1$ mb and $\sum_{l=0}^{6}{}' \sigma_l^{prod}=25.1$ mb we may say as follows. The incorrect estimation for η_0 has a large influence upon $\sigma_{l=0}^{scatt}$ and $\sum' \sigma_l^{scatt}$, while it has not a large influence upon $\sigma_{l=0}^{prod}$ and $\sum' \sigma_l^{prod}$. Since the experimental results for inelastic cross sections were able to be explained by our previous study, it may be possible to regard the η_l's estimated previously as the suitable ones if only a necessary correction is made with respect to η_0.

Since $\sigma^{prod} > 2\sigma^{scatt}$, it may be reasonable to assume the following condition. The magnitude of the correction $\Delta\sigma_l^{scatt}$ obtained by taking the effect of δ_l into account should not be larger than that of σ_l^{scatt} estimated without the effect of δ_l, that is, for each partial wave

$$2\eta_l(1-\cos 2\delta_l) \leqq (1-\eta_l)^2. \tag{5}$$

For the case of $\eta_l=0$ there is no need to estimate the magnitude of δ_l, so we hereafter take the case of $\eta_l \neq 0$ into consideration. From Eq. (5) it follows that

$$\cos 2\delta_l \geqq 1-(1-\eta_l)^2/(2\eta_l). \tag{6}$$

On the other hand, if the imaginary part of forward scattering amplitude due to l-wave is n_l times as much as the real part,

$$1-\eta_l \cos 2\delta_l = n_l \eta_l \sin 2\delta_l . \tag{7}$$

Then

$$\cos 2\delta_l = \frac{1 \pm n_l \sqrt{\eta_l^2(1+n_l^2)-1}}{\eta_l(1+n_l^2)}, \tag{8}$$

where

$$(n_l^2+1) \geqq 1/\eta_l^2 . \tag{9}$$

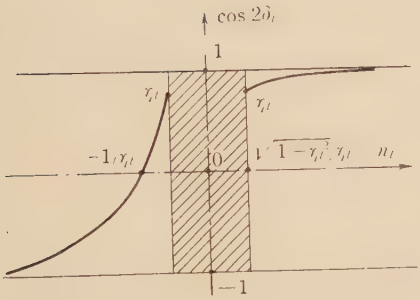

Fig. 1. $\cos 2\delta_l = \dfrac{1+n_l\sqrt{\eta_l^2(1+n_l^2)-1}}{\eta_l(1+n_l^2)}$

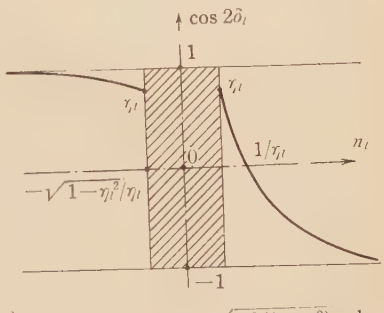

Fig. 2. $\cos 2\delta_l = \dfrac{1-n_l\sqrt{\eta_l^2(1+n_l^2)-1}}{\eta_l(1+n_l^2)}$

The relations between $\cos 2\delta_l$ and n_l are illustrated in Fig. 1 and Fig. 2. We now choose the solution of phase shift so that our previous result[1] (η_l-1) for R-matrix may be reproduced when the real part of forward scattering amplitude due to l-wave turns out to be zero. Then $\cos 2\delta_l$ should tend to 1 as n_l approaches to $\pm\infty$.* From this it follows that

$$\cos 2\delta_l = \frac{1+n_l\sqrt{\eta_l^2(1+n_l^2)}-1}{\eta_l(1+n_l^2)} \quad \text{for} \quad n_l > 0, \tag{10}$$

$$\cos 2\delta_l = \frac{1-n_l\sqrt{\eta_l^2(1+n_l^2)}-1}{\eta_l(1+n_l^2)} \quad \text{for} \quad n_l < 0. \tag{11}$$

Since $\cos 2\delta_l$ in Eq. (10) is an increasing function and $\cos 2\delta_l$ in Eq. (11) is a decreasing function, it can easily be seen that the minimum value of $\cos 2\delta_l$ is η_l.

$$\cos 2\delta_l \geqq \eta_l . \tag{12}$$

Let us compare the condition of Eq. (12) with that of Eq. (6).

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* If $\cos 2\delta_l$ tends to -1 as n_l approaches to $\pm\infty$, the value of $\sigma_l^{scatt} = (\pi/k^2)(2l+1)(1+\eta_l)^2$ for arbitrary l-state $(l \geq 1)$ turns out to be larger than the experimental cross section $\sigma^{scatt} \cong 10$ mb (cf. η_l in Table I).

$$\left[1-\frac{(1-\eta_l)^2}{2\eta_l}\right]-\eta_l=\frac{(1-\eta_l)(3\eta_l-1)}{2\eta_l}. \tag{13}$$

If $\eta_l > (1/3)$, the allowable region of $\cos 2\delta_l$ is determined by Eq. (6), while if $\eta_l < (1/3)$, it is determined by Eq. (12). As is seen from Table I, the values of η_l's are larger than 1/3. Thus the relation of Eq. (6) plays an important role in our study.

§ 3. Results of analysis

In order to reproduce the experimental value $\sigma^{scatt}=(10.0\pm0.8)$ mb, the magnitude of $\sum_{l=1}^{\infty}\sigma_l^{scatt}$ must be nearly equal to $(7.7+0.8)$ mb because $\sigma_{l=0}^{scatt}\simeq 2.29$ mb in our estimation. Comparing this with $\sum_{l=1}^{6}\sigma_l^{scatt}=4.19$ mb mentioned in Table I,* we can get the following relation,

$$\sum_{l=1}^{6}(2l+1)(1+\eta_l^2-2\eta_l\cos 2\delta_l)\cong 2\sum_{l=1}^{6}(2l+1)(1-\eta_l)^2. \tag{14}$$**

It must be noted that the contribution from $l=0$ wave is excluded in the summation with respect to l and that we cannot of course estimate the magnitude of $\cos 2\delta_l$ for each partial wave only from Eq. (14).

Considering the relation in Eq. (14) together with that in Eq. (5) we get

$$2\eta_l(1-\cos 2\delta_l)\cong(1-\eta_l)^2 \tag{15}$$

for each partial wave. Thus we can determine the value of $\cos 2\delta_l$. Now let us examine whether the experimental results for angular distribution can be reproduced or not. But there is another problem to be discussed. Even if the value of each $\cos 2\delta_l$ is determined, the sign of $\sin 2\delta_l$ is still unknown to us. Sternheimer[4] has obtained the following result by means of dispersion relation,

$$|\mathrm{Im}f(0^\circ)|^2/|\mathrm{Re}f(0^\circ)|^2=(10\sim 14). \tag{16}$$

We now try to choose the relative sign of $\sin 2\delta_l$ so that the relation of Eq. (16) may be satisfied. In this determination there are several possible solutions and the angular distributions in the backward direction are affected more or less by the choice of relative sign of each $\sin 2\delta_l$. We show one set of relative sign which may be regarded as the most suitable one for reproducing the experimental angular distribution (particularly the small bump in the back-

* Note that the contributions from $l\geqq 7$ waves to elastic scattering are negligibly small in our previous analysis.

** The relation of (14) shows that the effects of phase shifts δ_l's ($l\geqq 1$) are not so small and that the magnitude of correction due to δ_l corresponds nearly to that in the limiting case mentioned in Eq. (5).

ward hemisphere). The values of $\cos 2\delta_l$, n_l and σ_l^{scatt} are mentioned in Table II, and the angular distributions for elastic scattering are illustrated in Fig. 3.*

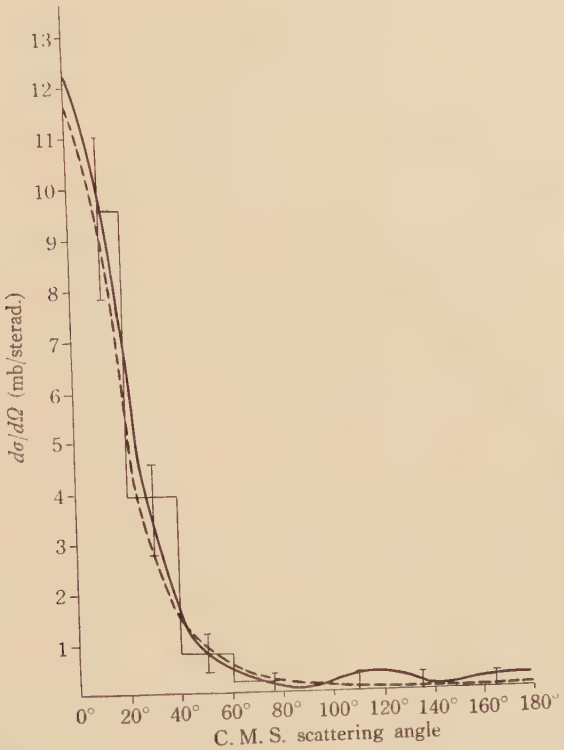

Fig. 3. Angular distributions for π^--p elastic scattering at 1.4 Bev. The solid curve represents our present result and the dotted one shows the form of $0.2/(1.13-\cos\theta)^2$ which was assumed in our previous study.

It must be noted that the values of n_l in Table II are not so large. Thus we may conclude as follows. The approximation in which the real parts of scattering amplitudes are neglected may not necessarily be good.

In order to improve roughness of our analysis it is of course necessary to take into account the spin dependence of R-matrix. We will study this problem by making use of the experimental results for polarization of recoil nucleon when the measurement will be made.

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* When the contributions from $l \geqq 7$ states are taken into consideration, we may expect that the angular distributions in the backward direction will agree very well with the experimental ones although the magnitude of $\sum_{l=7}^{\infty} \sigma_l^{scatt}$ will be negligibly small.

Table II. (n_l means the ratio of the imaginary part of scattering amplitude due to l-wave to the real one.)

l	$\cos 2\delta_l$	n_l	σ_l^{scatt}(mb)
0		$\pm\infty$	2.287
1	0.5285	± 2.382	5.072
2	0.9369	± 1.393	2.027
3	0.9857	∓ 1.177	0.773
4	0.9962	± 1.090	0.290
5	0.9989	± 1.049	0.108
6	0.9996	∓ 1.020	0.041

$|\mathrm{Im}\, f(0^\circ)|^2/|\mathrm{Re}\, f(0^\circ)|^2 \cong 14.0$, and $\sigma^{scatt} \cong \sum_{l=0}^{6} \sigma_l^{scatt} = 10.6$ mb.

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Remarks on the Lowlying Baryon Isobars

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The relative energies and widths of the lowlying baryon isobars have been examined from the point of view of the global symmetry hypothesis. In particular, the energy shifts of the hyperon isobars Λ^* and Σ^* due to the mass differences between the baryons have been calculated independent of the dynamics.

§ 1. Introduction

The lowest nucleon isobar N_1^* is identified with the well-known $I=3/2$, $J=3/2$ pion-nucleon resonance and lies at an excitation energy[1] $E(N_1^*)-m_N \simeq 2.12\mu$ (μ and m_N are the pion and nucleon masses respectively) with a total width $\Gamma(N_1^*)\simeq 0.90\mu$.† The excitation energy is small compared with the nucleon rest mass and we say that N_1^* is a lowlying nucleon isobar. The next nucleon isobar N_2^* already lies at a sufficiently high excitation energy $E(N_2^*)-m_N\simeq 3.85\mu$ that the term "lowlying" can hardly be applied to it. For the purpose of this discussion we shall only consider N_1^*. The physics underlying this limitation may perhaps be expressed by saying that the properties of N_1^* are determined chiefly by the strong p-wave pion-nucleon interaction whereas this need not be true of N_2^* and higher nucleon isobars (where the kaon-baryon interaction may play a role, etc.).

The other known baryons, Λ, Σ and Ξ may also exist in lowlying isobaric levels and indeed there is now rather conclusive experimental evidence[2] for a lowlying Λ isobar, Λ_1^*, having an excitation energy $E(\Lambda_1^*)-m_\Lambda=1.92\mu$ and a total width $\Gamma(\Lambda_1^*)=0.45\mu$. Further, if some form of global symmetry holds, lowlying Σ and Ξ isobars (Σ_1^* and Ξ_1^*) should also exist. Whether there are additional "lowlying" hyperon isobars (i.e. having excitation energies small compared with hyperon rest masses) is an open question and will depend on the complicated interplay of pion and kaon couplings. However, if a universally

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† This value has been estimated from the data given by S. J. Lindenbaum and L. C. L. Yuan, Phys. Rev. **111**, 1380 (1958). The Chew-Low theory seems to predict a higher value $\Gamma\simeq 1.40\mu$, if we use their formula (see G. F. Chew, *Proceedings of the 7th Rochester Conference*), Interscience Publishers, 1957).

strong p-wave pion-baryon dominates the situation for the lowest lying hyperon isobars, it should be possible to establish some (necessarily approximate) relations among the energies and half-widths of N_1^*, Λ_1^*, Σ_1^* and Ξ_1^* by treating the mass differences of the baryons as small perturbations. We note that the neglect of the kaon interaction for the $J=3/2$ isobar states may be reasonable since the p-wave kaon-nucleon interaction appears to be small in the low energy region. However, for the study of $J=1/2$ levels, it may be important.

The essential results of this note are already contained in the papers of Amati, Vitale and Stanghellini[3] and of Lee and Yang.[4] However, we believe that our approach is physically more perspicuous than the methods given in references 3) and 4). In § 2, we derive the formulas which connect the energies of the lowlying baryon isobars N^*, Λ^*, Σ^* and Ξ^* (hereafter we drop the subscript 1) and compare our results with references 3) and 4). In § 3, we briefly discuss the half-widths of the lowlying baryon isobars.

§ 2. Energies of the lowlying baryon isobars

We derive the displacements of the energies of the lowlying baryon isobars (with respect to their global symmetric value) by treating the mass differences of the baryons as a small perturbation on the globally symmetric interaction. It is plausible that this method will become increasingly bad for the higher baryon isobars.

We first derive the masses of Λ^* and Σ^* by writing for the total Hamiltonian

$$H=H_0+H_1$$

where H_0=globally symmetric Hamiltonian (which need not be specified) and

$$H_1=\int d^3\boldsymbol{x}\{(m_\Lambda-m_N)(\bar{\Lambda}\Lambda)+(m_\Sigma-m_N)(\bar{\boldsymbol{\Sigma}}\cdot\boldsymbol{\Sigma})\}. \tag{1}$$

We have omitted terms containing Ξ because obviously these terms do not affect this calculation. If we define Y and Z doublets as follows:

$$Y^+=\Sigma^+,\quad Z^0=\frac{1}{\sqrt{2}}(\Lambda+\Sigma^0),$$

$$Y^0=\frac{1}{\sqrt{2}}(\Lambda-\Sigma^0),\quad Z^-=\Sigma^-,$$

H_1 is transformed into

$$H_1=\int d^3\boldsymbol{x}\left\{\left[\frac{1}{4}(m_\Lambda+3m_\Sigma)-m_N\right](\bar{Y}Y+\bar{Z}Z)\right.$$

$$\left.+\frac{1}{4}(m_\Sigma-m_\Lambda)(\bar{Y}\tau_3 Y-\bar{Z}\tau_3 Z)\right.$$

$$-\frac{1}{4}(m_\Sigma - m_\Lambda)[\bar{Y}(\tau_1 - i\tau_2)Z + \bar{Z}(\tau_1 + i\tau_2)Y]\Big\}. \tag{1'}$$

We start with the eigenfunctions $|Y^*\rangle$, $|Z^*\rangle$ of H_0, which have isospin $I=3/2$ corresponding to the (3/2, 3/2) nucleon isobar N^*. We then can calculate the new eigenvalues of the energy resulting from switching on H_1. This is done by using standard degenerate perturbation theory.

Let $|Y^*, M\rangle$, $|Z^*, M\rangle$ be the eigenfunctions of H_0 corresponding respectively to Y^* and Z^* with the third component of isospin $I_z=M$. Hence M takes on the values 3/2, 1/2, $-1/2$, $-3/2$. However, in what follows, it is convenient to include $M=\pm 5/2$ by putting $|Y^*, M=\pm 5/2\rangle = |Z^*, M=\pm 5/2\rangle = 0$.

Now, inspection of the perturbation H_1 shows that the eigenfunctions of (H_0+H_1) must have the following form,

$$\Psi_M{}^* = \alpha_M |Y^*, M\rangle + \beta_M |Z^*, (M+1)\rangle \tag{2}$$

where the constants α_M, β_M are determined by

$$\begin{aligned} &\langle Y^*, M|H_1|Y^*, M\rangle \alpha_M + \langle Y^*, M|H_1|Z^*, M+1\rangle \beta_M = \Delta E^* \cdot \alpha_M \\ &\langle Z^*, M+1|H_1|Y^*, M\rangle \alpha_M + \langle Z^*, M+1|H_1|Z^*, M+1\rangle \beta_M = \Delta E^* \cdot \beta_M \end{aligned} \tag{3}$$

with ΔE^* the energy shift.

If we now use global symmetry and the isospin invariance of the theory, we may put

$$\begin{aligned} &\langle Y^*, M|\int d^3x(\bar{Y}Y + \bar{Z}Z)|Y^*, M\rangle = A, \\ &\langle Z^*, M+1|\int d^3x(\bar{Y}Y + \bar{Z}Z)|Z^*, M+1\rangle = A, \end{aligned} \tag{4a}$$

$$\begin{aligned} &\langle Y^*, M|\frac{1}{2}\int d^3x(\bar{Y}\tau_3 Y - \bar{Z}\tau_3 Z)|Y^*, M\rangle = M \cdot B, \\ &\langle Z^*, M+1|\frac{1}{2}\int d^3x(\bar{Y}\tau_3 Y - \bar{Z}\tau_3 Z)|Z^*, M+1\rangle = -(M+1)B, \end{aligned} \tag{4b}$$

$$\langle Y^*, M|\frac{1}{2}\int d^3x\, \bar{Y}(\tau_1 - i\tau_2)Z|Z^*, M+1\rangle = \left[\left(\frac{3}{2} - M\right)\left(\frac{5}{2} + M\right)\right]^{1/2} B,$$

$$\langle Z^*, M+1|\frac{1}{2}\int d^3x\, \bar{Z}(\tau_1 + i\tau_2)Y|Y^*, M\rangle = \left[\left(\frac{3}{2} - M\right)\left(\frac{5}{2} + M\right)\right]^{1/2} B, \tag{4c}$$

$$\begin{aligned} &\langle Y^*, M|\frac{1}{2}\int d^3x \bar{Z}(\tau_1 + i\tau_2)Y|Z^*, M+1\rangle = 0, \\ &\langle Z^*, M+1|\frac{1}{2}\int d^3x \bar{Y}(\tau_1 - i\tau_2)Z|Y^*, M\rangle = 0, \end{aligned} \tag{4d}$$

where A and B are constants independent of M. If, furthermore, we define

$$a=-\frac{1}{4}(m_\Lambda+3m_\Sigma-4m_N)A,$$

$$b=\frac{1}{2}(m_\Sigma-m_\Lambda)B, \tag{5}$$

then Eq. (3) becomes

$$(a+bM-\Delta E^*)\alpha_M-\left[\left(\frac{3}{2}-M\right)\left(\frac{5}{2}+M\right)\right]^{1/2}b\beta_M=0,$$

$$-\left[\left(\frac{3}{2}-M\right)\left(\frac{5}{2}+M\right)\right]^{1/2}b\,\alpha_M+[a-b(M+1)-\Delta E^*]\beta_M=0. \tag{6}$$

The two solutions of Eq. (6) are given by

$$\Delta E_2^*=a+\frac{3}{2}b,\ \alpha_M^{(2)}=\frac{1}{2}\left(\frac{5}{2}+M\right)^{1/2},\ \beta_M^{(2)}=-\frac{1}{2}\left(\frac{3}{2}-M\right)^{1/2}, \tag{7a}$$

$$\Delta E_1^*=a-\frac{5}{2}b,\ \alpha_M^{(1)}=\frac{1}{2}\left(\frac{3}{2}-M\right)^{1/2},\ \beta_M^{(1)}=+\frac{1}{2}\left(\frac{5}{2}+M\right)^{1/2}, \tag{7b}$$

and, hence, the corresponding wave functions by

$$\Psi_2^*(M)=\frac{1}{2}\left[\frac{5}{2}+M\right]^{1/2}|Y^*,M\rangle-\frac{1}{2}\left[\frac{3}{2}-M\right]^{1/2}|Z^*,M+1\rangle \tag{8a}$$

$$\Psi_1^*(M)=\frac{1}{2}\left[\frac{3}{2}-M\right]^{1/2}|Y^*,M\rangle+\frac{1}{2}\left[\frac{5}{2}+M\right]^{1/2}|Z^*,M+1\rangle. \tag{8b}$$

From Eqs. (8a) and (8b), it follows that Ψ_2^* has five non-zero components corresponding to $M=3/2, 1/2, -1/2, -3/2, -5/2$ and Ψ_1^* has three non-zero components corresponding to $M=1/2, -1/2, -3/2$. It is not difficult to check that if one uses the usual isospin assignments instead of the Y and Z doublet formalism, then $\Psi_2^*(M)$ behaves as $\mathbf{y}^{(2)}_{(M+1/2)}$, and $\Psi_1^*(M)$ as $\mathbf{y}^{(1)}_{(M+1/2)}$, apart from a common numerical factor. Hence, $\Psi_2^*(M)$ corresponds to $I=2$ with $I_z=M+1/2$ and $\Psi_1^*(M)$ to $I=1$ with $I_z=M+1/2$ in the usual isospin notation. We identify the states represented by the Ψ_2^* and Ψ_1^* wave functions with the Σ^* and Λ^* isobars respectively, namely

$$|\Sigma^*,I_z\rangle=\Psi_2^*\left(I_z-\frac{1}{2}\right),$$

$$|\Lambda^*,I_z\rangle=\Psi_1^*\left(I_z-\frac{1}{2}\right). \tag{9}$$

We believe that this notation is more consistent than that given in references 3) and 4) where Y^* and Z^* are taken as the baryon isobars. The quantities ΔE^* given by Eqs. (7a) and (7b) then represent $E(\Sigma^*)-E(N^*)$ and $E(\Lambda^*)-E(N^*)$ respectively.

Up to this point, the results contained in Eqs. (7a) and (7b) are correct for the relativistic theory. However, the quantities A and B (entering into Eqs. (7a) and (7b) through the definitions (4a) and (4b)) are very difficult to compute in general and it is necessary to make certain approximations. (We shall attempt to justify these approximations below by repeating our perturbation calculation for the ground states, i.e. rest masses of the Λ and Σ hyperons.) If we suppose that baryon pairs do not make an appreciable contribution to the positions of the lowlying nucleon isobars (the static theory is the limiting case of this approximation), then we may replace the expression $(\bar{Y}Y+\bar{Z}Z)$ by (Y^+Y+Z^+Z) in Eq. (4a) and we obtain

$$A \simeq \langle Y^*, M | \int d^3\boldsymbol{x}(Y^+Y+Z^+Z) | Y^*, M \rangle = 1. \tag{10}$$

Eq. (10) follows since $\int d^3\boldsymbol{x}(Y^+Y+Z^+Z)$ is the baryon number operator.

In order to evaluate B, we do not only neglect the contribution of the baryon pairs but we also neglect the pion contribution to the third component of the total isospin. More explicitly, the neglect of the baryon pair contribution permits us to rewrite Eq. (4b) as follows,

$$\begin{aligned} MB &\simeq \frac{1}{2} \langle Y^*, M | \int d^3\boldsymbol{x}(Y^+\tau_3 Y - Z^+\tau_3 Z) | Y^*, M \rangle \\ &\simeq \frac{1}{2} \langle Y^*, M | \int d^3\boldsymbol{x}(Y^+\tau_3 Y + Z^+\tau_3 Z) | Y^*, M \rangle. \end{aligned} \tag{11}$$

The second step in Eq. (11) is already permissible if we neglect the Z baryon loops (a condition less stringent than the complete neglect of baryon pair contributions), since, then,

$$\langle Y^*, M | \int d^3\boldsymbol{x} Z^+\tau_3 Z | Y^*, M \rangle = 0. \tag{12}$$

Furthermore, the third component of the total isospin operator is given by

$$I_z = \frac{1}{2} \int d^3\boldsymbol{x}(Y^+\tau_3 Y + Z^+\tau_3 Z) + i \int d^3\boldsymbol{x}(\phi^+ \cdot \pi^+ - \pi \cdot \phi) \tag{13}$$

where ϕ is the pion field operator and π is the canonically conjugate operator of ϕ. If we now neglect the pion contribution to (13), we obtain from Eq. (11) that

$$B \simeq 1. \tag{14}$$

Thus, neglecting the contribution of baryon pairs and the pion contribution to the third component of the total isospin, Eqs. (7a) and (7b) reduce to

$$\Delta E^*(I=2) \equiv E(\Sigma^*) - E(N^*) \simeq (m_\Lambda - m_N) + \frac{3}{2}(m_\Sigma - m_\Lambda) \tag{15a}$$

$$\Delta E^*(I=1) \equiv E(\Lambda^*) - E(N^*) \simeq (m_\Lambda - m_N) - \frac{1}{2}(m_\Xi - m_\Lambda) \qquad (15b)$$

which are precisely the formulas given in references 3) and 4). It is easy to show further that in our approximation

$$E(\Xi^*) - E(N^*) \simeq m_\Xi - m_N. \qquad (15c)$$

The advantage of our derivation has been its straightforwardness and the clear identification of the approximations which must be made to lead to the simple result (15). In the derivation of reference 3) the features of the Chew-Low approximation responsible for the final result are not too transparent (particularly with regard to the pion contribution to the third component of the total isospin) while in reference 4) [Eq. (22)], an assumption must be made which is not easily justified.

In assessing the validity of the approximations which led to Eqs. (15a) and (15b) we would add the following remarks. It seems rather plausible that the relative positions of the lowlying baryon isobars will be only slightly affected by the baryon pair (or loop) contributions. However, at first sight, it would appear that the neglect of the pion contribution to the third component of total isospin [cf. Eq. (13)] would be a poor approximation even for the lowlying baryon isobars. We justify this approximation essentially "ad hoc" by computing the "ground states" of the Λ and Σ hyperons by means of the identical perturbation procedure which we have employed for computing the first excited states (i.e. Λ^* and Σ^*). We find, in analogy to Eqs. (8a) and (8b),

$$\Psi_1(M) = \left[\frac{3}{2}+M\right]^{1/2} |Y, M\rangle - \left[\frac{1}{2}-M\right]^{1/2} |Z, M+1\rangle, \qquad (16a)$$

$$\Psi_0(M) = \left[\frac{1}{2}-M\right]^{1/2} |Y, M\rangle + \left[\frac{3}{2}-M\right]^{1/2} |Z, M+1\rangle, \qquad (16b)$$

where $\Psi_1(M)$ has three non-zero components $M=1/2, -1/2, -3/2$ and $\Psi_0(M)$ one non-zero component $M=-1/2$. The analogue of Eq. (9) becomes

$$|\Sigma, I_z\rangle = \Psi_1\left(I_z - \frac{1}{2}\right),$$

$$|\Lambda, I_z\rangle = \Psi_0\left(I_z - \frac{1}{2}\right) \qquad (17)$$

and, finally, Eqs. (7a) and (7b) are converted into

$$\Delta E(I=1) \equiv E(\Sigma) - E(N) = a + \frac{1}{2}b, \qquad (18a)$$

$$\Delta E(I=0) \equiv E(\Lambda) - E(N) = a - \frac{3}{2}b \qquad (18b)$$

where a and b are defined by Eq. (5), as before, and A and B are, in turn, defined by Eqs. (4a) and (4b) with Y and Z replacing Y^* and Z^* respectively. The approximations of neglecting the baryon pair (or loop) contributions and the pion contribution to the third component of the total isospin permit us to set again $A\simeq 1$, $B\simeq 1$. With these values of A and B, Eqs. (18a) and (18b) reduce to

$$E(\Sigma)-E(N)\simeq m_\Sigma-m_N, \tag{19a}$$

$$E(\Lambda)-E(N)\simeq m_\Lambda-m_N \tag{19b}$$

which are exactly correct. The same approximations lead to

$$E(\Xi)-E(N)\simeq m_\Xi-m_N. \tag{19c}$$

It should be pointed out that B can be calculated in the static limit; Miyazawa[5] obtains the value $B\sim 0.37$ whereas Fubini and Thirring[6] get $B\sim 0.28$. This is small compared to 1 so that the Λ and Σ masses come out quite incorrect, since in the same static limit $A\simeq 1$. The fact that Eqs. (19a) and (19b) are exact encourages us to believe that the same values $A\simeq 1$ and $B\simeq 1$ leading to Eqs. (15a) and (15b) may not be too bad.

The above comments are based on the usual view that the mass differences among the baryons are due to the kaon-baryon couplings. Thus, as far as the pure kaon interactions are concerned, m_Λ, m_Σ and m_Ξ in our original Hamiltonian can be regarded as the renormalized masses instead of the bare ones. However, we are still not taking account of the renormalization effects of the combined pion and kaon interactions, so that the baryon masses are still not the observed renormalized masses, unless these combined renormalization effects are negligible. One way out of this difficulty is to suppose that the quantities A and B, which may be different from unity (thus representing the combined pion-kaon renormalization effects) are the same respectively for the Y, Z doublets and the lowlying Y^*, Z^* doublets. It would follow then that the formulae Eqs. (19a), (19b), (19c) and Eqs. (15a), (15b) are still true provided that we replace the masses m_Λ, m_Σ and m_N by their corresponding renormalized values. Roughly speaking, this assumption corresponds to the idea that the pion clouds in the lowlying Y^* and Z^* doublets do not so much affect the properties of the cores represented by the Y and Z doublets.

It is interesting to note that this statement is equivalent to the assumption that $\alpha'=\alpha$, $\beta'=\beta$, $\gamma'=\gamma$ in the formulation of Lee and Yang [cf. Eqs. (19) and (20) of their paper[4]]. Indeed, we can easily write down the following empirical formulas for the "ground states" and "first excited states" of the baryons:[7]

$$E=a+bI(I+1)+cS+dS^2, \qquad (N,\ \Lambda,\ \Sigma,\ \Xi) \tag{20a}$$

$$E=a^*+bI(I+1)+cS+dS^2, \qquad (N^*,\ \Lambda^*,\ \Sigma^*,\ \Xi^*) \tag{20b}$$

with the same b, c, d. In Eqs. (20a) and (20b), I=isospin, S=strangeness number and

$$\begin{aligned} a&=m_N-\frac{3}{8}(m_\Sigma-m_\Lambda), \quad a^*=E(N^*)-\frac{15}{8}(m_\Sigma-m_\Lambda),\\ b&=\frac{1}{2}(m_\Sigma-m_\Lambda),\\ c&=\frac{1}{2}(m_\Xi-m_N)-2(m_\Lambda-m_N)-\frac{3}{4}(m_\Sigma-m_\Lambda),\\ d&=\frac{1}{2}(m_\Xi-m_N)-(m_\Lambda-m_N)-\frac{3}{8}(m_\Sigma-m_\Lambda). \end{aligned} \tag{21}$$

Eqs. (20a) and (20b) can be translated into Eqs. (19) and (20) of reference 4) provided we choose $\alpha'=\alpha$, $\beta'=\beta$, $\gamma'=\gamma$.

Finally, we remark that once we know the wave functions for the lowlying baryon isobars [as given, for example, by Eqs. (8) and (9)], we can readily calculate the weight factors for the various partial transitions, namely:

Type	Weight
$(N^*)^{++}\to p+\pi^+$	1
$(\Lambda^*)^+\to \Lambda+\pi^+$	2/3
$(\Lambda^*)^+\to \Sigma^0+\pi^+$	1/6
$(\Lambda^*)^+\to \Sigma^++\pi^0$	1/6
$(\Sigma^*)^{++}\to \Sigma^++\pi^+$	1
$(\Xi^*)^+\to \Xi^0+\pi^+$	1.

These weight factors are independent of the values of A and B (i.e. the approximations of neglecting baryon pairs and the pion contribution to the third component of the total isospin) and depend only on the structure of H_1 [cf. Eq. (1)]. It is interesting that they are identical with the weight factors obtained by Lee and Yang[4] on the basis of group theoretic considerations and, while not at all surprising, perhaps the physical basis of their results is now more clear.

§ 3. Widths of lowlying baryon isobars

In this section, we briefly discuss the half-widths of the "first excited states" of the baryons. The intention is to spell out in an intuitive fashion the essential parameters which contribute to this width and thereby to make some crude estimates of the relative widths of the lowlying baryon isobars. For this purpose, we have recourse to a simple formula for the width of a metastable level of a baryon isobar B^* derived on the basis of classical strong coupling theory,[8] namely;

$$\Gamma_{B^*} \simeq \frac{C_1 a_B p_{B^*}^3}{[E(B^*) - m_B]} \tag{22}$$

where a_B is the baryon "source size", p_{B^*} is the momentum of the pion in the rest system of B^*, $[E(B^*) - m_B]$ is the excitation energy of B^*, and C_1 is a constant depending on the quantum numbers of B^* (i.e. I, J, etc.) which must be assigned in accordance with the "correspondence principle". Eq. (22) follows in classical strong coupling theory for a strong p-wave pion-baryon interaction provided that $a_B \mu \ll 1$, i.e. the "source size" is small compared with the pion Compton wavelength. Eq. (22) can also be derived quantum mechanically in the strong coupling approximation[9] and the only difference is a slight change in the constant C_1 (which cancels out when the relative widths are compared). Indeed, Eq. (22) maintains its essential form in the Chew-Low theory[10] if one identifies the "source size", a_B, with the inverse of the cutoff energy ω_{max} of Chew and Low [cf. their equations (52) – (53)]; the "natural" choice of the baryon Compton wavelength for a_B also leads to good quantitative accord with the Chew-Low theory. This simple physical picture yields the crude estimate that

$$a_B \sim \frac{1}{\omega_{max}(B)} \sim \frac{1}{m_B}.$$

The strong coupling theory also makes a prediction concerning the excitation energy of the baryon isobar, namely

$$E(B^*) - m_B = C_2 \frac{a_B}{f_B^2} \tag{23}$$

where C_2 is a constant like C_1 and f_B^2 is the renormalized coupling constant for the p-wave pion-baryon interaction. The Chew-Low theory again yields a similar equation to (23) if a_B is replaced[11] by $\omega_{max}^{-1}(B)$. Combining Eqs. (22) and (23), we obtain

$$\Gamma_{B^*} \simeq C f_B^2 p_{B^*}^3. \tag{24}$$

From Eq. (24), it is evident that global symmetry (f_B^2 = constant for all baryons) implies that

$$\Gamma_{B^*} \sim p_{B^*}^3 \tag{25}$$

which is essentially the p-wave phase space factor used by Lee and Yang[4] [cf. their Eq. (18)]. By the same token, it is understandable from Eq. (23) (since $a_B \sim 1/m_B$) that the heavier baryons will have somewhat smaller excitation energies for their lowlying isobars. When this is combined with the subtractive term in Eq. (15b), it is easy to explain a reduction by a factor 2 in the half-width of Δ^* compared to that of N^*. On the other hand, the additive term in Eq. (15a) [even when one measures $E(\Sigma^*)$ with respect to m_Σ] leads to the

expectation that the half-width of Σ^* will be as large, or even larger, than that of N^*. Similar qualitative arguments predict that the half-width of Ξ^* will again be comparable to (perhaps smaller than) that of N^*. These statements are rougly in accordance with the numbers given in Table III of reference 4).

In conclusion, it should be emphasized that the purpose of this note was to obtain a more intuitive understanding of the relative energies and widths of the lowlying baryon isobars within the framework of global symmetry. We find, in agreement with references 3) and 4), that the observations concerning the excitation energy and half-width of Λ^* can readily be understood on this basis. If it turns out that the observed lowlying Λ isobar does not have the quantum number[12] $J=3/2$, the globally symmetric starting point for our calculations [as well as those of references 3) and 4)] would then become doubtful because opposite parity of Λ and Σ' would be likely[2),12)].

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4) T. D. Lee and C. N. Yang, preprint (Institute for Advanced Study, Princeton), to be published in Physical Review.
5) H. Miyazawa, Phys. Rev. **101** (1956), 1564.
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7) It can be shown, on the basis of the symmetrical Sakata model (see e. g. Y. Yamaguchi, Prog, Theor. Phys. Suppl. **11** (1959), 1, 37 and M. Ikeda, S. Ogawa and Y. Ohnuki, Prog. Theor. Phys. **22** (1960), 1073), that the mass formula for particles in the same representation must have the form of Eq. (20a); however, the symmetrical Sakata model further demands that $d=-b/4$, a condition which is not fulfilled by our coefficients. Of course, one can always argue that the particles represented by such a formula do not belong to the same representation or that the symmetrical Sakata model is only true at high energies. Details concerning these subjects will be published separately by one of the authors (S. O.).
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A Simple Model for Two-Channel Reaction in Field Theory and Its Application to the Y^* Problem

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A simple two-channel model in field theory is presented in order to know characteristic features of the problems concerning Y^* recently observed. Four particles, K_-, N, π and Λ, all assumed to be spinless, are considered, and the S-state amplitudes are calculated in the chain approximation. The calculated amplitudes satisfy the unitarity condition and reproduce bound states, resonances, and other features of the reactions. From the examples treated it is concluded that a strong attractive force between K_- and N such that there is a bound state and a weak coupling between K_-N and $\pi\Lambda$ channels are essential for the appearance of a sharp peak both in the K_-N total cross section and the $K_-+N\to\pi+\Lambda$ cross section in the unphysical region. The relation between height and width is such that the peak tends to a δ-function if the width tends to an infinitesimal. This conclusion remains unchanged even if a force between π and Λ is varied over a wide range. According to the variety of a $\pi\Lambda$ force, the $\pi\Lambda$ cross section exhibits either a peak or a dip near the energy at which the K_-N total cross section exhibits a peak. Furthermore an attractive or a repulsive $\pi\Lambda$ force contributes to make a peak broader or narrower, respectively.

§ 1. Introduction

In a recent experiment with the Berkeley K^--beam, an interesting fact was reported.[1),2)] Namely an outgoing pion (positive or negative) in the reaction $K^-+p\to\Lambda+\pi^++\pi^-$ distributes along a straight line in the Dalitz plot with a fairly large probability. This suggests an existence of some kind of excited state composed of a Λ and a pion (Y^*), the mass of which is nearly equal to 1380 Mev. A remarkable fact is that the half-width of that state is very narrow: about 15 Mev, as reported most recently.[2)] This value is much smaller than the half-width of the 3-3 resonance in pion-nucleon scattering (about 50 Mev), though a similarity to the 3-3 resonance was stressed in the earlier analysis.[1)]*

If we assume an interaction of Y^* weak enough to give such a narrow width, we could not explain a "large" cross section for the production of Y^*. In order to overcome this dilemma Takeda proposed a kind of "even-odd rule" similar to that proposed earlier to account for the copious production and long life of strange particles.[4)]

* See also a paper by Amati et al.[3)]

We may introduce Y^* as a new elementary particle similar to Λ and Σ. It is, however, worthwhile to note that we can consider Y^* as a bound state of the K_-N system with a binding energy ~ 50 Mev.[5),6)]* This conjecture will be supported further if Y^* turns out to decay into π and Λ in an S-state, though this has not yet been unambiguously confirmed. This is because we can hardly expect a *sharp* resonance in an S-state due to the lack of a centrifugal barrier.

If this conjecture is valid, the Y^* phenomena will provide an actual example to which we can apply a field theory of unstable particles (or bound states), which have been discussed by many authors.[7)] It is also necessary to develop a field theoretical multi-channel problem (K_-N, $\pi\Lambda$, and $\pi\Sigma$ channels in the case of Y^*), which has been considered partly by Igi,[8)] and by Matthews and Salam.[9)]**

Several authors have already attempted to make phenomenological analyses in terms of complex scattering lengths.[5),6),11)] But, since these problems are rather new to us, we consider it useful first to make more general and fundamental considerations before entering into detailed comparisons with experimental results. From this standpoint we shall investigate here a simplified two-channel model in field theory.

We shall consider four particles, K, N, π and Λ, neglecting isospins, all assumed to be spinless, and such that

$$M_\pi + M_\Lambda < M_K + M_N .***$$

We shall calculate various amplitudes in the chain approximation method in an S-state, by assuming appropriate four-vertex interactions. A generalization so as to include Yukawa-type interactions is easy. An analogous treatment in a P-state is also possible.

This model is too simple to draw detailed quantitative conclusions on the actual Y^* problem, because of the neglect of isospins, spins, parities and actual mass ratios. But we could expect to learn a number of qualitative features which are characteristic of a two-channel (S-state) reactions involving an unstable bound state. On the other hand, this model is, of course, not very general; rather it carries some restrictive properties. At the sacrifice of complete generality, however, we can reproduce bound states, resonances, and other features of reactions without any further approximation. Therefore we may expect to give instructive examples of more general considerations of the theory.

In § 2, the interaction Hamiltonian is given and the S-state amplitudes are calculated in the chain approximation. The connection between these amplitudes and the conventional quantities are established. In § 3, the general proper-

* K_+ denotes K^+ or K^0, and K_- denotes K^- or $\widetilde{K}^0$.

** Multi-channel problems in terms of potentials have been investigated by Ross and Shaw.[10)]

*** Hereafter K stands for K_-, namely K^- or $\widetilde{K}^0$.

ties of this model are discussed. Among them the unitarity condition plays particularly important roles in the following discussions. In § 4, several typical cases are discussed. From the examples treated we find that a strong attractive force between K_- and N such that there is a bound state, and a weak coupling between two channels are essential for the appearance of a sharp peak. Natures of the peaks are discussed: i.e. relations between height and width, the possible appearance of an "imaginary resonance". § 5 is devoted to the summary and concluding discussions.

§ 2. S-state amplitudes in the chain approximation

We assume the interaction Hamiltonian

$$H=g_1\bar{K}\bar{N}KN+g_2\bar{\pi}\bar{\Lambda}\pi\Lambda+g_3(\bar{K}\bar{N}\pi\Lambda+\bar{\pi}\bar{\Lambda}KN), \tag{2·1}$$

for the interactions between four particles, K, N, π and Λ, which are all assumed to be spinless for simplicity. We assume the simplified mass ratios given by

$$M_K=M_N=M_1,$$
$$M_\pi=M_\Lambda=M_2<M_1.$$

Fig. 1. The simple chain diagram for KN-KN scattering amplitude T_1.

First we shall calculate the simple chain diagrams illustrated in Fig. 1, which result if we switch off the coupling between the KN and the $\pi\Lambda$ channels (the g_3-interaction). Denoting the scattering amplitudes of KN-KN and $\pi\Lambda$-$\pi\Lambda$ as T_1 and T_2, respectively, we have

$$T_1(\xi)=\frac{g_1}{1-g_1R_1(\xi)}=\frac{g_1}{\mathfrak{D}_1(\xi)},$$
$$T_2(\xi)=\frac{g_2}{\mathfrak{D}_2(\xi)}, \tag{2·2}$$

where

$$\xi=W^2=-p^2, \quad (p: \text{ total momentum})$$
$$R_r(\xi)=i(2\pi)^{-4}\int dq\ \frac{1}{(q^2+M_r^2)[(p-q)^2+M_r^2]}$$
$$=-\frac{1}{16\pi^2}\int_{4M_r^2}^{\infty}\frac{dm^2}{m^2-\xi-i\varepsilon}\sqrt{1-\frac{4M_r^2}{m^2}}. \tag{2·3}$$
$$(r=1,\ 2)$$

The latter expression has been obtained using either the technique developed by Nakanishi[12] or the standard method in a dispersion relation theory.[13] If we divide R_r into real and imaginary parts,

$$R_r = A_r + iB_r,$$

we have

$$A_r(\xi) = -\frac{1}{16\pi^2} P \int_{4M_r^2}^{4\Lambda_r^2} \frac{dm^2}{m^2-\xi} \sqrt{1-\frac{4M_r^2}{m^2}}, \tag{2·4a}$$

$$B_r(\xi) = -\frac{1}{16\pi} \theta(\xi-4M_r^2) \sqrt{1-\frac{4M_r^2}{\xi}}. \tag{2·4b}$$

We have cut off the integration with respect to m^2 at $4\Lambda_r^2$.

Next we take into account a coupling between the KN and the $\pi\Lambda$ channels. As illustrated in Fig. 2, we can express the amplitudes as power series in g_3^2, which can be summed to yield

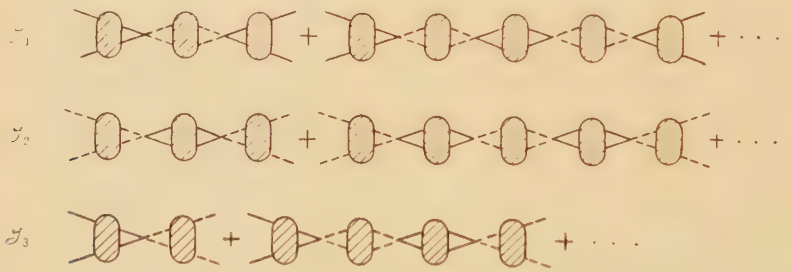

Fig. 2. Compound chain diagrams for $\mathcal{T}$'s. A solid line represents K or N, and a broken line π or Λ. Appropriate simple chain diagrams (with $g_3=0$) should be inserted for the shaded parts.

$$\mathcal{T}_1 = \frac{g_1+hR_2}{\mathcal{D}},$$

$$\mathcal{T}_2 = \frac{g_2+hR_1}{\mathcal{D}}, \tag{2·5}$$

$$\mathcal{T}_3 = \frac{g_3}{\mathcal{D}},$$

where $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are the amplitudes for the processes KN-KN, $\pi\Lambda$-$\pi\Lambda$ and KN-$\pi\Lambda$, respectively, and

$$\begin{aligned} h &= g_3^2 - g_1 g_2, \\ \mathcal{D} &= \mathcal{D}_1 \mathcal{D}_2 - g_3^2 R_1 R_2. \end{aligned} \tag{2·6}$$

According to the chain approximation we have included only diagrams which

are serial combinations of "bubbles". Therefore Eqs. (2·5) do not satisfy the requirement of crossing symmetry. The denominator $\mathcal{D}(\xi)$, common to the three amplitudes in Eqs. (2·5), can be divided into real and imaginary parts as follows:

$$\mathcal{D}=\mathcal{A}-i\mathcal{B},$$

where

$$\begin{aligned}\mathcal{A}&=D_1D_2-g_3^2A_1A_2+hB_1B_2,\\ \mathcal{B}&=(g_2+hA_1)B_2+(g_1+hA_2)B_1,\end{aligned}\tag{2·7}$$

and D_r is the real part of $\mathcal{D}_r$ given by

$$D_r=1-g_rA_r.\tag{2·8}$$

Now we establish the connection between the $\mathcal{J}$'s given by Eqs. (2·5) and conventional quantities. $\mathcal{J}_1$ and $\mathcal{J}_2$ are simply the scattering amplitudes. Using the complex phase shifts δ_r, we have

$$\mathcal{J}_r=(1/B_r)\,e^{i\delta_r}\sin\delta_r=-8\pi(W/k_r)\,e^{i\delta_r}\sin\delta_r,\tag{2·9}$$

for the physical region of each reaction, where W and k_r are the energy and momentum in the center of mass system, respectively, given by

$$\begin{aligned}W&=\sqrt{\xi},\\ k_r&=\sqrt{\frac{1}{4}\xi-M_r^2}.\end{aligned}\tag{2·10}$$

The imaginary parts of the $\mathcal{J}_r$ are related to the total cross section as follows:

$$\begin{aligned}\mathrm{Im}\,\mathcal{J}_r&=-2Wk_r\sigma_r,\\ \sigma_r&=(\pi/k_r^2)\,|1-e^{2i\delta_r}|^2.\end{aligned}\tag{2·11}$$

The quantity $|\mathcal{J}_3|^2$ obviously gives the cross section for the reaction

$$K+N\to\Lambda+\pi.$$

In addition to this reaction we are also interested in the reaction

$$K+N\to\Lambda+\pi+\pi.$$

It is expected that the cross section of this reaction is intimately related $|\mathcal{J}_3|^2$. In order to calculate the amplitude of the above reaction, however, we must introduce the interaction Hamiltonian for the emission of a single pion. Among many possibilities an example is given by

$$H=G\bar{N}N\pi,\tag{2·12}$$

in terms of which we can also construct a chain diagram. The calculation can be simplified further by assuming a five-vertex interaction Hamiltonian given by

$$H=g_4\bar{K}\bar{N}\pi\pi\Lambda+h.c. \qquad (2\cdot13)$$

Of course we can produce an effective Hamiltonian (2·13) by calculating the closed loop diagram illustrated in Fig. 3. The coupling constant g_4, thus interpreted, is proportional to Gg_3. Assuming, for the moment, the interaction Hamiltonian (2·13), we can calculate $\mathcal{J}_4$, the amplitude for $K+N\to\Lambda+\pi+\pi$, from the diagrams in Fig. 4. If we substitute $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$ calculated before, for the shaded parts, these diagrams include all possible chain diagrams of lowest order in g_4.

Fig. 3. A closed loop diagram resulting from the interaction Hamiltonian (2·12).

Fig. 4. Chain diagrams for the reaction $K+N\to\Lambda+\pi+\pi$. Appropriate compound chain diagrams illustrated in Fig. 2, should be inserted for the shaded parts.

These too can be summed to yield

$$\mathcal{J}_4=-\frac{g_4}{\mathcal{D}(\xi_i)\,\mathcal{D}(\xi_f)}[g_3^2+\mathcal{D}_2(\xi_i)\,\mathcal{D}_1(\xi_f)], \qquad (2\cdot14)$$

where ξ_i and ξ_f respectively denote the squared momentum of the initial state and that of the "final" state excluding the pion emitted directly by the g_4-interaction.

In the experimental situation of reference 1), the cross section is obtained as a function of ξ_f, with a specified ξ_i. In the vicinity of the "peak", the behavior of $|\mathcal{J}_4|^2$ as a function of ξ_f can be considered to be given by

$$\sigma\sim|\mathcal{J}_4|^2\sim\frac{g_4^2}{|\mathcal{D}(\xi_f)|^2}\sim\frac{g_3^2}{|\mathcal{D}(\xi_f)|^2}=|\mathcal{J}_3(\xi_f)|^2. \qquad (2\cdot15)$$

On expectation that this relation will hold also in more general cases, we shall discuss $|\mathcal{J}_3|^2$ in the following.

§ 3. General properties of the model

In this section we summarize the general properties of the amplitudes calculated in § 2.

(i) Properties of the simple chain amplitude T_r

We consider T_1, from which T_2 can be easily inferred. In the right-hand side of Eq. (2·4a) the integration can be performed explicitly as follows

$$-16\pi^2A_1(\eta)=L_1-\begin{cases}\sqrt{1-\dfrac{1}{\eta}}\,\log\dfrac{\sqrt{1-(1/\lambda_1^2)}+\sqrt{1-(1/\eta)}}{\sqrt{1-(1/\lambda_1^2)}-\sqrt{1-(1/\eta)}}, & \text{for } \eta>1,\\ 2\sqrt{\dfrac{1}{\eta}-1}\,\tan^{-1}\sqrt{\dfrac{1-(1/\lambda_1^2)}{(1/\eta)-1}}, & \text{for } \eta<1,\end{cases} \qquad (3\cdot1)$$

where

$$\eta=\xi/4M_1^2, \quad \lambda_1^2=4.1_1^2/4M_1^2,$$

$$L_1=\log\frac{\lambda_1+\sqrt{\lambda_1^2-1}}{\lambda_1-\sqrt{\lambda_1^2-1}}.$$

The function $-16\pi^2A_1(\eta)$ given by Eq. (3·1) diverges logarithmically at $\eta=\lambda_1^2$. As easily noted, however, the behavior of the function near the cutoff energy changes depending strongly on the method of cutoff. Therefore we should be interested mainly in the energy region far below the cutoff, since we know at present no definite method of cutoff. In order to see the general behavior of $-16\pi^2A_1(\eta)$ in the energy region far below the cutoff, we may put $\lambda_1\to\infty$ in the right-hand side of Eq. (3·1), except for the *constant* term L_1. In this approximation we can prove that

$$\frac{d}{d\eta}[-16\pi^2A_1(\eta)]\gtrless 0, \quad \text{for } \eta\lessgtr 1. \tag{3·2}*$$

In the neighbourhood of the threshold we find

$$-16\pi^2A_1(\eta)\approx -16\pi^2A_1(1)-\begin{cases}\pi\dfrac{\kappa_1}{M_1}, & \text{for } \eta<1,\\[2ex] 2\dfrac{k_1^2}{M_1^2}, & \text{for } \eta>1,\end{cases} \tag{3·3}$$

where k_1 is the relative momentum in the center of mass system given by Eq. (2·10) and κ_1 is its analytically continued quantity given by

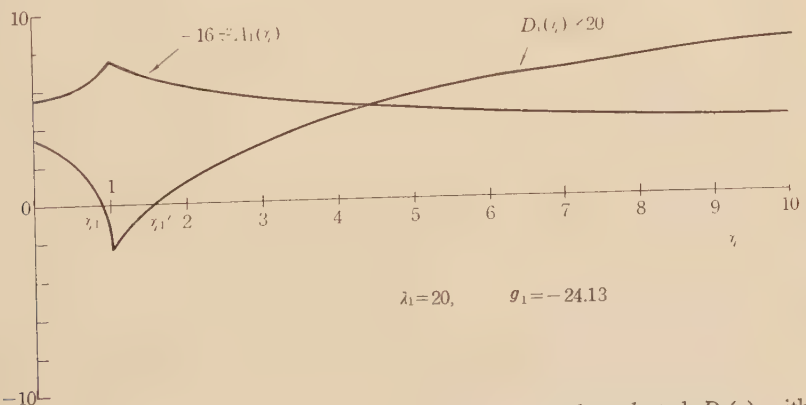

Fig. 5. A plot of $-16\pi^2A_1(\eta)$ with $\lambda_1=20$. There is also plotted $D_1(\eta)$ with $g_1=-24.13$, which will be used in Case 1, Case 5 and Case 6 (§ 4).

* The inequality for $\eta<1$ can be proved directly from Eq. (2·4a), since the denominator $m^2-\xi$ never vanishes below the threshold.

$$\kappa_1=\sqrt{M_1^2-\frac{1}{4}\xi}. \tag{3·4}$$

Eqs. (3·3) show that $-16\pi^2 A_1(\eta)$ exhibits a "cusp" at the threshold. In Fig. 5 we have plotted $-16\pi^2 A_1(\eta)$ with $\lambda_1=20$, in which we can see all the general features implied in Eqs. (3·2) and (3·3).

On the other hand, $D_1(\eta)$ exhibits several distinguished behaviors according to the different values of g_1.

For $g_1>0$, corresponding to a repulsive force, $D_1(\eta)$ never vanishes and is larger than unity for $\eta<1$. If we confine ourselves to the energy region far below the cutoff such that

$$-16\pi^2 A_1(\eta)>0,$$

we have

$$D_1(\eta)>1,$$

for $\eta>1$ also. A schematic plot is given in Fig. 6(a).

For $-16\pi^2/L_1<g_1<0$, we have

$$0<D_1(\eta)<1,$$

in the region mentioned above. This case corresponds to such a weak attractive force that there is neither bound state nor resonance (Fig. 6(b)).

For $g_1=-16\pi^2/L_1$, the bottom of the "cusp" touches the abscissa which represents the zero of $D_1(\eta)$ (Fig. 6(c)). This case corresponds to a zero energy bound state (or resonance).

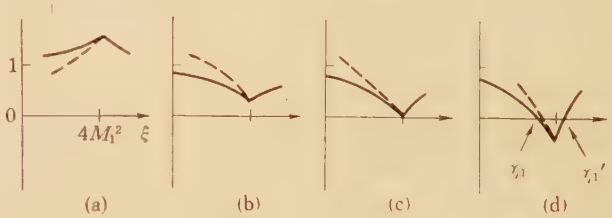

Fig. 6. Schematic plots of $D_1(\eta)$ for various values of g_1. (a): $g_1>0$, (b): $-16\pi^2/L_1<g_1<0$, (c): $g_1=-16\pi^2/L_1$, (d): $g_1<-16\pi^2/L_1$. The points η_1 and η_1' represent the energy of a bound state and that of a resonance respectively. Broken curves represent the extrapolations of $D_1(\eta)$ in the scattering length approximation.

For $g_1<-16\pi^2/L_1$, $D_1(\eta)$ vanishes at two points, one below the threshold (η_1) and one above the threshold (η_1') (Fig.6(d)). The points η_1 and η_1' can be considered to correspond to a bound state and a resonance, respectively.

If we arbitrarily make g_1 negatively large we would have a negative η_1, which is a kind of "ghost". In order to exclude such a ghost, which may be related with the lack of crossing symmetry in our model, we must assume a lower limit on g_1 as

$$g_1\geq-\frac{16\pi^2}{L_1-2\lambda_1\sqrt{1-(1/\lambda_1^2)}}.$$

Summarizing we have the following theorems:

Theorem I

The inequality

$$-\frac{1}{L_1-2\sqrt{1-(1/\lambda_1^2)}}\leqq\frac{g_1}{16\pi^2}\leqq-\frac{1}{L_1},$$

must be satisfied if a bound state is to exist.

Theorem II

There exists either one bound state or no. If there exists one (no) bound state, we have necessarily one (no) resonance also. This is a kind of Levinson's theorem.[14] If the binding energy tends to zero, then the resonance energy also tends to zero (a zero energy resonance).

We can further prove the following theorem from the monotonic decrease (increase) of $-16\pi^2 A_1(\eta)$ at $\eta>1$ $(\eta<1)$ (Eq. (3·2)).

Theorem III

η_1' has a maximum which is obtained when η_1 is a minimum. This maximum, which depends on the cutoff, becomes largest for $\lambda=\infty$.

We shall add a remark on the resonance here obtained. As a consequence of the inequality in Eq. (3·2), $D_1(\eta)$ changes its sign *from negative to positive* as η increases through η_1. This, combined with Eq. (2·9), leads to the conclusion that the (real) phase shift δ_1 goes through $\pi/2$ *decreasing*. In examining more in detail we find that the cross section exhibits no sharp peak near the resonance energy. These facts are in agreement with what we know about an S-wave potential scattering.[15]

(*ii*) *Unitarity condition*

Using the expression given by Eqs. (2·5) we can derive the following formulae :

$$\mathrm{Im}\,\mathcal{J}_1=B_1|\mathcal{J}_1|^2+B_2|\mathcal{J}_3|^2, \tag{3·5-1}$$

$$\mathrm{Im}\,\mathcal{J}_2=B_2|\mathcal{J}_2|^2+B_1|\mathcal{J}_3|^2, \tag{3·5-2}$$

$$\mathrm{Im}\,\mathcal{J}_3=B_1\mathcal{J}_3\mathcal{J}_1^*+B_2\mathcal{J}_2\mathcal{J}_3^*. \tag{3·5-3}$$

These are combined to the equation

$$\mathrm{Im}\langle f|\mathcal{J}|i\rangle=\sum_{n=1,2}\langle f|\mathcal{J}^+|n\rangle B_n\langle n|\mathcal{J}|i\rangle,$$

which turns out to imply the unitarity condition for $\mathcal{J}$, if we note that

$$-B_r=k_r/8\pi W,$$

from Eqs. (2·4b) and (2·10). From Eqs. (3·5-1) and (3·5-2) we find

$$\begin{aligned}\mathrm{Im}\,\mathcal{J}_1&\leqq 0,\\ \mathrm{Im}\,\mathcal{J}_2&\leqq 0,\end{aligned} \tag{3·6}$$

in both the physical and unphysical regions.

(iii) Scattering lengths

We can prove that $\mathcal{J}_1$ and $\mathcal{J}_2$ as given by Eqs. (2·5) behave as

$$\mathcal{J}_r(\xi)\approx\mathcal{J}_r(4M_r^2)\begin{cases}\left[1+i\dfrac{1}{16\pi}\dfrac{k_r}{M_r}\mathcal{J}_r(4M_r^2)\right]^{-1}, & \text{for } \xi\geqq 4M_r^2,\\ \left[1-\dfrac{1}{16\pi}\dfrac{\kappa_r}{M_r}\mathcal{J}_r(4M_r^2)\right]^{-1}, & \text{for } \xi\leqq 4M_r^2.\end{cases}\tag{3·7}$$

This equation leads to the introduction of a scattering length defined by

$$a_r\doteq\frac{1}{16\pi M_r}\mathcal{J}_r(4M_r^2),\tag{3·8}$$

in terms of which we can write

$$\mathcal{J}_r(\xi)\approx 16\pi M_r a_r\begin{cases}\dfrac{1}{1+ik_r a_r}, & \text{for } \xi\geqq 4M_r^2,\\ \dfrac{1}{1-\kappa_r a_r}, & \text{for } \xi\leqq 4M_r^2.\end{cases}\tag{3·9}*$$

When the two channels are coupled $\mathcal{J}_r$ is in general complex, therefore a_r is also complex and may be decomposed as

$$a_r=b_r+ic_r.$$

The imaginary part c_r is negative from (3·6). The behavior of $\mathcal{J}_r$ near the threshold, given by (3·9), is schematically illustrated in Fig. 7.

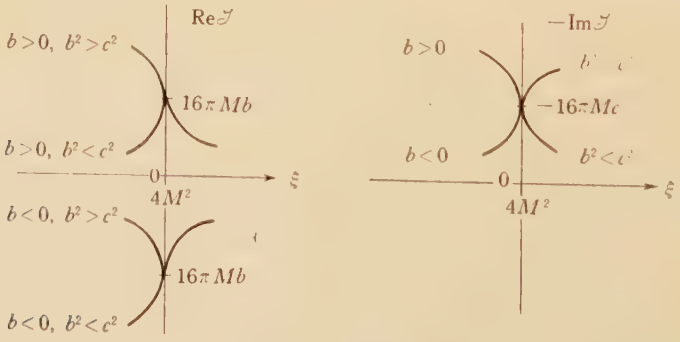

Fig. 7. Behaviors of scattering amplitudes near the threshold.

* This definition of a corresponds to

$$k\cot\delta=-\frac{1}{a}+\cdots.$$

(*iv*) *Renormalization*

In the simple chain case the renormalization procedure is easy. Namely we can write

$$\mathfrak{D}_r(\xi)=1-g_r R_r(\xi)=D_r(4M_r^2)-g_r[R_r(\xi)-A_r(4M_r^2)],$$

then we have

$$T_r(\xi)=\frac{\tilde{a}_r}{1-\tilde{a}_r R_{rf}(\xi)}, \tag{3·10}$$

where

$$\tilde{a}_r=\frac{g_r}{D_r(4M_r^2)}=\frac{g_r}{1+g_r L_r/16\pi^2}, \tag{3·11}$$

and the *finite* part of R_r is given by

$$R_{rf}(\xi)=R_r(\xi)-A_r(4M_r^2)=A_r(\xi)-A_r(4M_r^2)+iB_r(\xi). \tag{3·12}$$

From Eq. (3·8) we find

$$\tilde{a}_r=16\pi M_r a_r. \tag{3·13}$$

In the compound chain case the situation is not as simple. We can prove that the divergence can be absorbed into two scattering lengths, b_1 and b_2, and that the *behavior* of the functions is insensitive to the cutoff. In the following, however, we shall not try to apply the renormalization procedure consistently. It is worthwhile to observe that a cutoff gives some kind of inverse length as seen from Eq. (3·11).

§ 4. Special cases

In this section we shall consider several special cases which serve to investigate the general features of the Y^* problem. Numerical values, i.e. mass ratios, cutoff momenta, binding energies, etc., are chosen for the sake of easiness of computation only. The parameters used are summarized in Table I.

It is convenient to divide the energy region into three parts:

Region I : $\xi \leqq 4M_2^2$, or $\eta \leqq \eta_0$,

Region II : $4M_2^2 \leqq \xi \leqq 4M_1^2$, or $\eta_0 \leqq \eta \leqq 1$,

Region III : $4M_1^2 \leqq \xi$, or $1 \leqq \eta$.

$$(\eta_0=4M_2^2/4M_1^2)$$

We shall mainly consider Region II. The behavior of the *KN-KN* scattering amplitude near and *above* the threshold (Region III) can be inferred from the scattering lengths by using Eq. (3·7). In Region II,

$$B_1=0,$$

Table I.

Case	1	2	3	4	5	5	5	6	
Example	1	2	3	4	5-1	5-2	5-3	6	
Figure	8	9	10	12	13	14	15	16	
η_0	0.5	0.5	0.5	0.5	0.8	0.8	0.8	0.8	M_2^2/M_1^2
λ_1	20	20	20	20	20	20	20	20	Λ_1/M_1
λ_2	20	20	20	20	25	25	400	25	Λ_2/M_2
g_1	−24.13	+24.13	0	0	−24.13	−24.13	−24.13	−24.13	
g_2	0	0	0	+104.2	−20.18	−21.36	−13.24	+20.00	
$g_3^2/4\pi$	2.00	2.00	48.12	200.0	0.04	0.04	0.04	0.04	
η_1	0.900	—	—	—	0.900	0.900	0.900	0.900	$D_1=0$
η_1'	1.573	—	—	—	1.573	1.573	1.573	1.573	$D_1=0$
η_2	—	—	—	—	0.800	0.781	0.48	—	$D_2=0$
η_2'	—	—	—	—	0.800	1.000	1.800	—	$D_2=0$
η_1^*	0.802	—	—	—	—	—	0.930	0.899	$\mathcal{A}=0$
η_1^{**}	—	—	—	—	0.901	0.901	0.903	0.898	$\mathcal{B}=0$
$\bar{\eta}_1$	0.874	−19.42	0.793	—	0.929	0.931	0.938	0.929	Eq. (5·2)
η_2^*	—	—	—	—	0.793	0.768	0.33	—	$\mathcal{A}=0$
η_3	—	—	0.660 0.900	—	—	—	—	—	$\mathcal{A}=0$
η_4	—	—	—	0.900	—	—	—	—	$\mathcal{B}=0$
$\tilde{b}_1$	+140.8	+11.12	+33.72	−199.3	+188.5	+190.8	+201.1	+188.9	
$-\tilde{c}_1$	11.73	0.0816	50.90	130.6	7.740	7.770	15.00	0.0724	$a=\tilde{a}/16\pi M_1$
$\tilde{b}_1^{(0)}$	+189.6	+189.6	0	0	+189.6	+189.6	+189.6	+189.6	Values with $g_3=0$
$\tilde{b}_2^{(0)}$	0	0	0	+17.76	—	+366.5	+109.1	+10.05	Values with $g_3=0$

so that $\mathcal{A}$ and $\mathcal{B}$ in Eq. (2·7) can be simplified to

$$\mathcal{A}=D_1D_2-g_3^2A_1A_2, \tag{4·1a}$$

(Region II)

$$\mathcal{B}=(g_2+hA_1)B_2, \tag{4·1b}$$

and the unitarity condition (3·5) becomes

$$\mathrm{Im}\,\mathcal{J}_1=B_2|\mathcal{J}_3|^2,$$
$$\mathrm{Im}\,\mathcal{J}_2=B_2|\mathcal{J}_2|^2,$$
$$\mathrm{Im}\,\mathcal{J}_3=B_2\mathcal{J}_2\mathcal{J}_3^*.$$

Case 1. g_1 is negative and large such that there is a bound state of the KN system in Region II; $g_2=0$, and g_3^2 is small.

In this case Eqs. (4·1) can be simplified further to

$$\mathcal{A}=D_1-g_3^2A_1A_2, \tag{4·2a}$$
$$\mathcal{B}=g_3^2A_1B_2. \tag{4·2b}$$

If

$$g_3^2 \ll |g_1|,$$

we have the approximate forms for $\mathcal{A}$ and $\mathcal{B}$ near η_1:

$$\begin{aligned} \mathcal{A} &\approx -\alpha(\eta-\eta_1^*), \\ \mathcal{B} &\approx \frac{g_3^2}{g_1} B_2(\eta_1) \sim -\frac{1}{16\pi}\frac{g_3^2}{g_1}, \end{aligned} \tag{4·3}$$

where

$$\alpha \approx \theta \frac{-\tilde{b}_1}{1-\eta_1} > 0, \quad (0<\theta<1) \tag{4·4}$$

is the slope of $\mathcal{A}$ near η_1 and is insensitive to g_3^2. From Eqs. (4·3) we have a one-level formula for $\mathcal{J}_3$,

$$\mathcal{J}_3 \approx -\frac{g_3}{\alpha}\frac{1}{\eta-\eta_1^*+i\Gamma/2}, \tag{4·5}$$

where the half-width $\Gamma/2$ is given by

$$\frac{\Gamma}{2} \approx -\frac{1}{16\pi\alpha}\frac{g_3^2}{g_1} > 0. \tag{4·6}$$

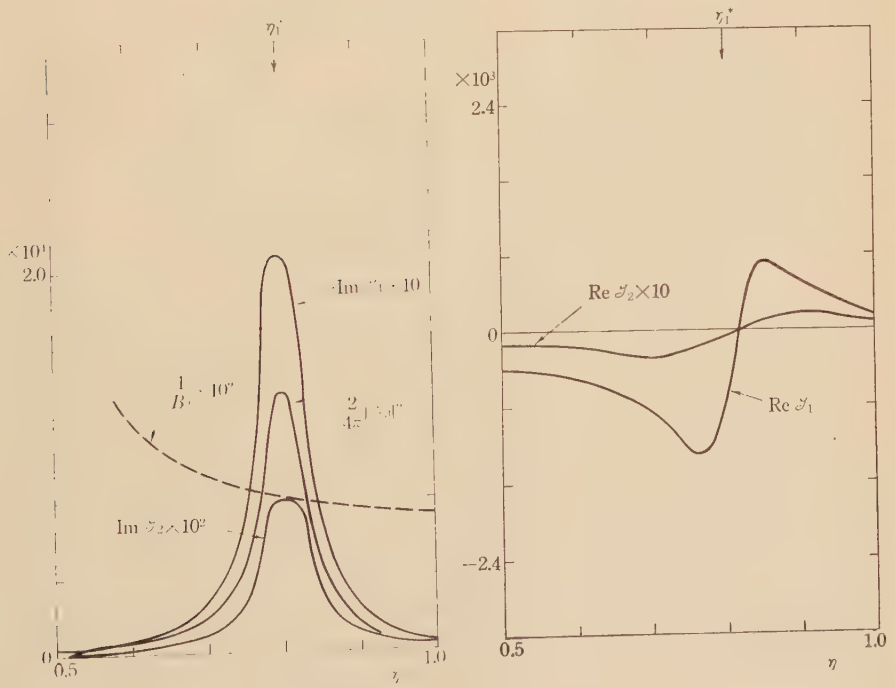

Fig. 8. Example 1. $-\mathrm{Im}\,\mathcal{J}_1$, $-\mathrm{Im}\,\mathcal{J}_2$ and $|\mathcal{J}_3|^2$ exhibit a sharp peak. The curve $-1/B_2$ represents the kinematical upper bound for $-\mathrm{Im}\,\mathcal{J}_2$. The behavior of $\mathrm{Re}\,\mathcal{J}_2$ shows that δ_2 goes through $\pi/2$ increasing as η increases through η_1^*.

Both the width Γ and the difference $\eta_1-\eta_1^*$ are of order g_3^2. We have further

$$|\mathcal{J}_3|^2=\frac{1}{B_2}\operatorname{Im}\mathcal{J}_1\approx\frac{g_3^2}{\alpha^2}\frac{1}{(\eta-\eta_1^*)^2+\Gamma^2/4}, \tag{4·7}$$

$$\mathcal{J}_2=g_3A_1\mathcal{J}_3\approx-\frac{g_3^2}{g_1\alpha}\frac{1}{\eta-\eta_1^*+i\Gamma/2} \tag{4·8}$$

An example is shown in Fig. 8. The following points should be noted:
(i) The second term of $\mathcal{A}$ given by Eq. (4·2a) makes the zero point η_1 of D_1 shift to the left (η_1^*). This means that the g_3-interaction is some kind of attractive force. The difference $\eta_1-\eta_1^*$ is, of course, of order g_3^2.
(ii) The coupling between two channels (the g_3-interaction) produces a width of order g_3^2. The relation between the width and the height of the peak is of special interest. In the limit where g_3^2 tends to an infinitesimal, we have

$$\operatorname{Im}\mathcal{J}_1=B_2|\mathcal{J}_3|^2\to\sim-\delta(\eta-\eta_1).$$

We can naturally expect that $\operatorname{Im}\mathcal{J}_1$ tends to a δ-function, since a *stable* Y^* appears if the coupling is switched off leaving g_1 unchanged. In this case $\mathcal{J}_1$ contains a term like

$$\frac{1}{\xi-M_0^2+i\varepsilon},$$

where M_0^2 is the square of the Y^* mass.

On the other hand, it seems strange that $|\mathcal{J}_3|^2$ also contains a δ-function, since the reaction $K+N\to\pi+\Lambda$ will not occur if the coupling is switched off. But we must include a stable Y^* into the complete set of states for $g_3=0$.* The unitarity condition (3·5-1) should be modified to

$$\operatorname{Im}\mathcal{J}_1=-\pi g_5^2\delta(\xi-M_0^2)+B_1|\mathcal{J}_1|^2, \tag{4·9}$$

where g_5 is the "coupling constant" between K, N and Y^*. The attenuation of the KN beam ($-\operatorname{Im}\mathcal{J}_1$) below the threshold now results from the reaction

$$K+N\to Y^*.**$$

In our model a Y^* is detected either by observing a peak in the energy spectrum of π which has been produced together with Y^*, or by observing π and Λ which have decayed from Y^*. The latter observation is replaced by the observation of a stable Y^* if the g_3-interaction is switched off. We can consider, therefore, the right-hand side of Eq. (3·5-1) or Eq. (4·9) as representing the squared amplitude for detecting a Y^* in the final state, irrespective of

* See the paper by T. Goto for the proof that the complete set of states can be formed from the scattering states alone, not including the unstable *particle* state.[16]

** Of course other particles, e. g. pions, must be emitted in order that this reaction actually occurs.

whether g_3^2 is zero or not. Then we can easily understand that the integral of the squared amplitude does not depend on g_3^2, because we should detect "one" Y^* in any way, if "one" Y^* is produced at all.

We can make another interpretation. The height of the peak of $|\mathcal{J}_3|^2$ is $\sim g_3^2/\Gamma'^2 \sim 1/\Gamma'$, which, multiplied by Γ', has given a value independent of g_3^2. The width Γ' represents a probability of Y^*-decay *per unit time*, while the height ($\sim 1/\Gamma'$) represents a *duration* of a Y^*. The total probability, integrated with respect to time, is a product of the above two quantities, and then independent of Γ'.

(iii) $-\mathrm{Im}\,\mathcal{J}_2$ exhibits also a peak, but its height is bounded by a factor which is purely kinematical and independent of g's and the cutoff. In fact we have

$$-\mathrm{Im}\,\mathcal{J}_2(\eta_1^*) = -\frac{1}{B_2(\eta_1^*)} = 8\pi \frac{W}{k_2}\Big|_{\eta=\eta_1^*}, \tag{4·10}$$

which turns out to be equal to the kinematical upper limit in an S-state scattering. In examining $\mathrm{Re}\,\mathcal{J}_2$ we find that the phase shift of $\pi\Lambda$-$\pi\Lambda$ scattering goes through $\pi/2$ *increasing* as η increases through η_1^*.

(iv) We find

$$b_1 > 0, \quad b_1^2 > c_1^2.$$

Therefore, $-\mathrm{Im}\,\mathcal{J}_1$ exhibits a "cusp" at the threshold of the KN channel (see Fig. 7).

Case 2. g_1 is positive, $g_2=0$, and g_3^2 is small.

In the single-channel scattering we have the following three cases from Eqs. (3·11) and (3·13):

$$b_1 > 0 \quad \text{for} \quad g_1 > 0,$$
$$b_1 < 0 \quad \text{for} \quad -16\pi^2/L_1 < g_1 < 0,$$
$$b_1 > 0 \quad \text{for} \quad g_1 < -16\pi^2/L_1.$$

For each case $D_1(\tilde{s})$ has already been plotted in Fig. 6. A positive b_1 implies that the *absolute value* of $D_1(\tilde{s})$ decreases when $\tilde{s}$ decreases from $4M_1^2$. Therefore, we cannot necessarily conclude that there is a bound state only because b_1 is positive.

The situation will be similar if a weak channel coupling is taken into account. In Fig. 9, we shall show an example in

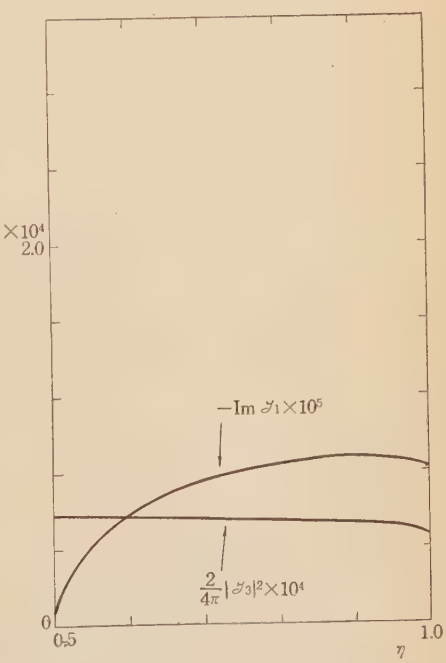

Fig. 9. Example 2. $-\mathrm{Im}\,\mathcal{J}_1$ exhibits no sharp peak in spite of the cusp-type behavior at the threshold.

which $g_1>0$. The inequality $b_1^2>c_1^2$ is still satisfied and $-\mathrm{Im}\mathcal{J}_1$ exhibits a cusp at the threshold. But $\mathcal{A}$ never vanishes in Regions I and II, and consequently neither $-\mathrm{Im}\mathcal{J}_1$ nor $|\mathcal{J}_3|^2$ exhibits any sharp peak. On extrapolating these quantities by the rigid use of the scattering length approximation (see § 5, (v)), we obtain a peak in the space-like region of ξ.

Case 3. $g_1=g_2=0$, $g_3^2\neq 0$

As noted in Case 1, (i), the g_3-interaction is an attractive one. Thus we may expect the possibility that the g_3-interaction causes both the formation of a bound state and its decay into π and Λ. We find, however, that the width is essentially broad in this case. Examples are shown in Fig. 10. Furthermore, $\mathcal{A}(\eta)$ is found not to behave as simply as in Case 1. In the present example, $\mathcal{A}(\eta)$ behaves somewhat like a parabola, and two peaks appear.

Now we shall take into account the effect of the g_2-interaction. If g_2 is very small its effect is simple. A small positive g_2 shifts η_1 to the right and a small negative g_2 to the left. If g_2 is of the comparable order of magnitude as g_1, however, the behaviors of the amplitudes are so complicated that general treatments are rather difficult. In the follwing we shall, at first, consider the case in which only g_2 and g_3 are effective, neglecting g_1, and subsequently con-

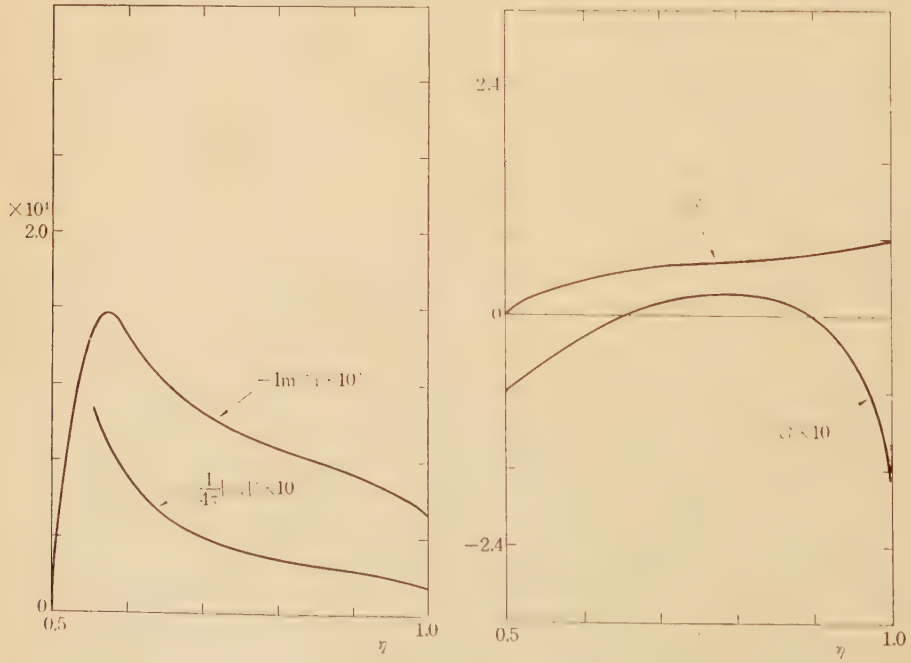

Fig. 10. Example 3. $\mathcal{A}(\eta)$ behaves like a parabola and much smaller than $\mathcal{B}(\eta)$. $|\mathcal{J}_3|^2$ exhibits no sharp peak.

sider some interesting cases in which g_1 and g_2 are of the comparable order of magnitude.

Case 4. $g_1=0$, $g_2>0$, $g_3^2\neq 0$

We are interested in this case because there is some evidence that the K-meson interaction (here the g_1-interaction) is weaker than the pion interaction (here the g_2-interaction). We shall consider only the case of $g_2>0$, since a negative g_2 turns out not to produce any resonance-like behavior.

We have

$$\mathcal{A}=D_2-g_3^2 A_1 A_2, \tag{4·11a}$$

$$\mathcal{B}=(g_2+g_3^2 A_1)B_2. \tag{4·11b}$$

It should be noted that $\mathcal{B}$ is no longer of definite sign. In fact it is possible for $\mathcal{B}(\eta)$ to change sign in Region II, if g_2 is positive and large enough. This is illustrated schematically in Fig. 11, where η_4 is the zero point of $\mathcal{B}(\eta)$ given by

$$A_1(\eta_4)=-g_2/g_3^2. \tag{4·12}$$

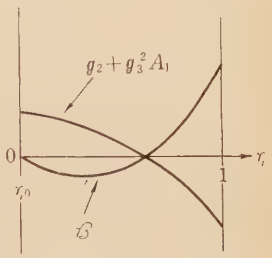

Fig. 11. It is possible for $\mathcal{B}(\eta)$ (Eq. (4·11b)) to change sign.

If the slope of $\mathcal{B}(\eta)$ near η_4 is large enough, $\mathcal{B}(\eta)$ can be approximated by

$$\mathcal{B}(\eta)\approx\beta(\eta-\eta_4), \tag{4·13b}$$

where

$$\beta\sim-\frac{1}{16\pi}g_3^2\frac{dA_1}{d\eta}\Big|_{\eta=\eta_4}\sim\frac{1}{64\pi^2}\frac{g_3^2}{4\pi}L_1>0. \tag{4·14}$$

Also we have

$$\mathcal{A}(\eta)\sim\mathcal{A}(\eta_4)=1. \tag{4·13a}$$

The latter equality is obtained by substituting Eq. (4·12) in Eq. (4·11a). From Eqs. (4·13a) and (4·13b) we have

$$\mathcal{Y}_3\approx i\frac{g_3}{\beta}\frac{1}{\eta-\eta_4+i\Gamma/2}, \tag{4·15}$$

where Γ is given by

$$\Gamma/2=1/\beta>0. \tag{4·16}$$

The right-hand side of Eq. (4·15) is of a form of the one-level formula, with the real and imaginary parts interchanged in the denominator. This can also be considered to be a kind of resonance, which we shall call an *imaginary resonance.*

We have

$$|\mathcal{J}_3|^2 \approx \frac{g_3^2}{\beta^2} \frac{1}{(\eta-\eta_4)^2+\Gamma^2/4} \tag{4·17}$$

and we can expect that $|\mathcal{J}_3|^2$, and consequently $-\mathrm{Im}\,\mathcal{J}_1$, exhibit a peak.

On the other hand we have

$$\begin{aligned} \mathcal{J}_2 &\approx i \frac{1}{B_2(\eta_4)} \frac{\eta_i-\eta_4}{\eta-\eta_4+i\Gamma/2}, \\ \mathrm{Im}\,\mathcal{J}_2 &\approx \frac{1}{B_2(\eta_4)} \frac{(\eta-\eta_4)^2}{(\eta-\eta_4)^2+\Gamma^2/4}. \end{aligned} \tag{4·18}$$

It is noted that $-\mathrm{Im}\,\mathcal{J}_2$ exhibits a "dip" at $\eta=\eta_4$ instead of a peak. At this energy we find

$$\delta_2=0.$$

Now we shall examine more specifically whether we can really expect a sharp resonance in the present case. From Eq. (4·16) and Eq. (4·14) we must assume a large g_3^2 in order to have a small Γ. This leads necessarily to a large g_2 from Eq. (4·12). These large constants cause $\mathcal{A}(\eta)$ also to have a zero point (sometimes more than one) near η_4 so that a simple one-level formula as in Eq. (4·15) is no longer valid. Moreover we find

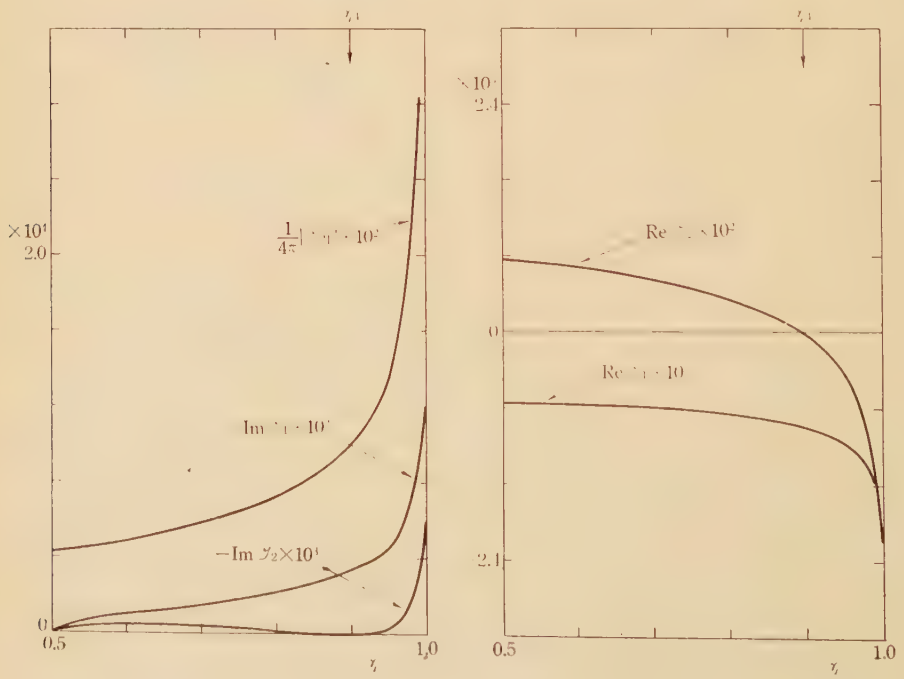

Fig. 12. Example 4. An imaginary resonance occurs at η_4 where $\mathcal{B}(\eta_4)=0$. $-\mathrm{Im}\,\mathcal{J}_2$ exhibits a dip at η_4. But the behaviors of various quantities are too complicated to apply simple one-level formulae. Consequently there occurs no sharp peak.

$$|\mathcal{B}| \ll \mathcal{A}(\eta_4) = 1,$$

unless very extreme values are chosen for the parameters. These facts make the problem so complicated that we can hardly expect a sharp peak as simply as in Case 1. An example is shown in Fig. 12.

Case 5. Case 1 is modified by assuming a large negative g_2 such that there is a bound state in the $\pi\Lambda$ system.

Eq. (4·1b) can be put into the form

$$\mathcal{B} = (g_2 D_1 + g_3^2 A_1) B_2 = g_2 \left(D_1 + \frac{g_3^2}{g_2} A_1 \right) B_2, \tag{4·19}$$

which shows that an imaginary resonance can occur irrespective of the sign of g_2, if D_1 changes sign in the region considered.

The energy η_2^* of a bound state of $\pi\Lambda$ system, which we may identify as a Σ, is determined by

$$D_2 - g_3^2 (A_1 A_2 / D_1) = 0, \tag{4·20}$$

if D_1 does not vanish in Region I. In Case 1 the second term on the right-hand side of Eq. (4·20) is negative because D_1 is positive in Region I. Therefore, the energy of Σ, denoted by η_2^*, shifts to the left of η_2, the zero point of $D_2(\eta)$.

We shall consider three examples in which η_2', another zero point of $D_2(\eta)$ which corresponds to a resonance in $\pi\Lambda$ scattering, lies at η_0, 1, and to the right of 1.

In Example 5-1, in which

$$\eta_2 = \eta_2' = \eta_0,$$

g_2 is given by

$$g_2 / 16\pi^2 = -1/L_2. \tag{4·21}$$

Even with this condition (zero binding energy), there is a bound Σ because $\eta_2^* < \eta_2$. In this case the product $D_1 D_2$ is small in Region II and

$$|D_1 D_2| \ll g_3^2 |A_1|$$

can be satisfied unless g_3^2 is extremely small. If the above inequality is satisfied we can approximate $\mathcal{A}(\eta)$ as

$$\mathcal{A} \approx -g_3^2 A_1 A_2, \tag{4·22a}$$

which varies slowly except near both edges of Region II. On the other hand we can approximate $\mathcal{B}$, given by (4·19), as

$$\mathcal{B} \approx \alpha' (\eta - \eta_1^{**}), \tag{4·22b}$$

where

$$\eta_1^{**} - \eta_1 = +0(g_3^2 / |g_2|),$$

and

$$\alpha' \approx g_2 B_2 \alpha \sim -(g_2/16\pi)\alpha > 0. \tag{4·23}$$

(α is given by Eq. (4·4).)

From

$$A_1(\eta_1) = 1/g_1,$$

and Eq. (4·22a) we have

$$\mathcal{A}(\eta_1^*) \approx \mathcal{A}(\eta_1) = -(g_3^2/g_1) A_2(\eta_1). \tag{4·24}$$

Thus we have

$$\mathcal{D} \approx -i\alpha'(\eta - \eta_1^{**} - i\Gamma/2), \tag{4·25}$$

which exhibits an imaginary resonance with a half-width

$$\Gamma/2 = (g_3^2/\alpha' g_1) A_2(\eta_1). \tag{4·26}$$

$|\mathcal{J}_3|^2$ takes the form

$$|\mathcal{J}_3|^2 \approx \frac{g_3^2}{\alpha'^2} \frac{1}{(\eta - \eta_1^{**})^2 + \Gamma^2/4}. \tag{4·27}$$

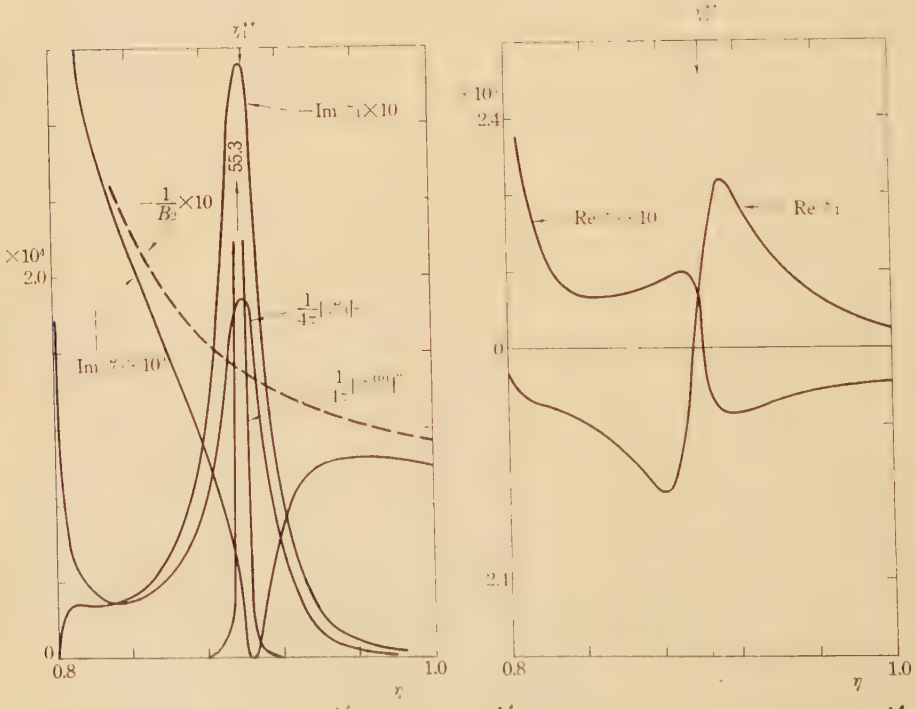

Fig. 13. Example 5-1. Both $|\mathcal{J}_3|^2$ and $-\mathrm{Im}\,\mathcal{J}_1$ exhibit a sharp peak, while $-\mathrm{Im}\,\mathcal{J}_2$ exhibits a dip. $|\mathcal{J}_3|^2$ with $g_2=0$ ($|\mathcal{J}_3^{(0)}|^2$) is also plotted for the sake of comparison. Peaks at $\eta=\eta_0$ come from the zero energy resonance in $\pi\Lambda$-$\pi\Lambda$ scattering with $g_3=0$.

The relation between the width and the height of the peak is again similar to that of a δ-function if g_3^2 tends to an infinitesimal.

An example is shown in Fig. 13. In the Figure, we have also plotted $|\mathcal{J}_3^{(0)}|^2$, with $g_2=0$. The half-width $\Gamma/2$ of $|\mathcal{J}_3^{(0)}|^2$ calculated from Eq. (4·6) is 2.56×10^{-4}. Comparing this with $\Gamma/2$ of $|\mathcal{J}_3|^2$, about ~0.015, we find that the g_2-interaction here considered makes the peak appreciably broader. This feature is common to the three examples in Case 5. $-\mathrm{Im}\,\mathcal{J}_2$ exhibits a dip at η_1^{**}.

We see that $|\mathcal{J}_3|^2$ exhibits another peak at $\eta=\eta_0$. This comes from the fact that $\mathcal{A}(\eta)$ becomes a minimum due to the vanishing of the first term of the right-hand side of Eq. (4·1a). This peak, however, will hardly be observed in the actual processes, e.g. $K^-+p\to\Lambda+\pi+\pi$, since this peak appears at a threshold and a vanishing phase volume of the final products should be multiplied.

Example 5-2, in which

$$\eta_2'=1,$$

is the next typical example. This example is, however, found to be very similar to Example 5-1 except that there is no peak at $\eta=\eta_0$. An example is shown in Fig. 14. At $\eta\sim1$, $-\mathrm{Im}\,\mathcal{J}_2$ is large (close to the kinematical upper bound

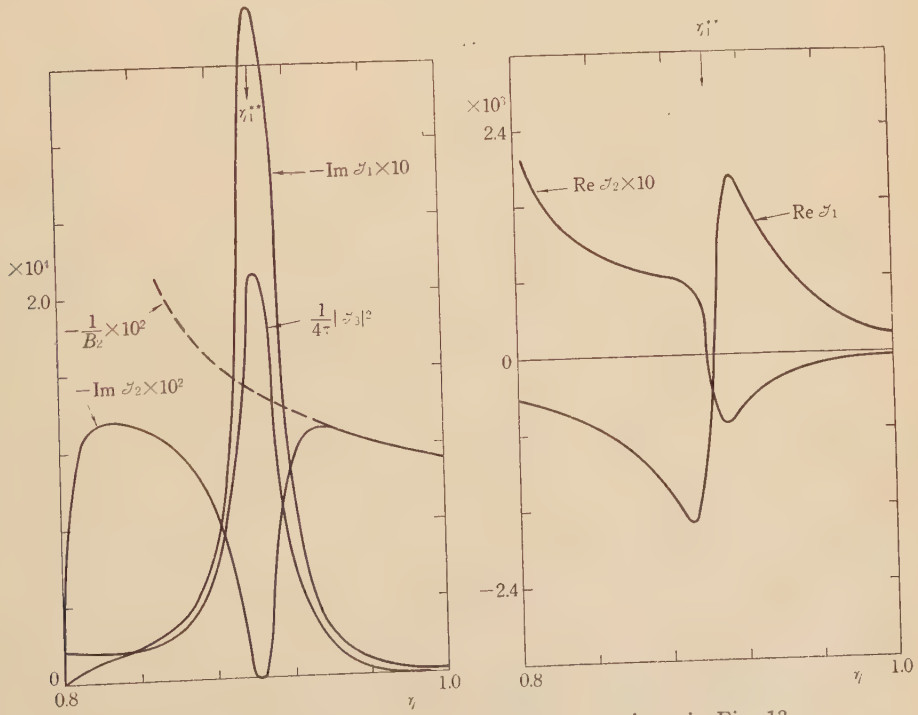

Fig. 14. Example 5-2. Curves are similar to those in Fig. 13.

$-1/B_2$, also plotted in the Figure), evidently due to the assumed vanishing of $D_2(\eta)$ at $\eta=1$.

In Example 5-3, parameters were chosen in order that

$$\eta_2' \gg 1.$$

In our model, this condition is, at the same time, apt to give a very light Σ. This is connected to the restriction stated in Theorem III in § 3.

There appears again an imaginary resonance similar to those in Examples 5-1 and 5-2. It is noted that a conventional resonance appears also. Namely Eq. (4·1a) can be written as

$$\mathcal{A}=D_2\left(D_1-g_3^2\,\frac{A_1A_2}{D_2}\right), \tag{4·28}$$

since D_2 does not vanish (and is negative) in Region II. Eq. (4·28) shows that η_1^* lies to the right of η_1, the difference being proportional to g_3^2. In the curve in Fig. 15, the two peaks at η_1^* and η_1^{**}, respectively, overlap and form a peak much broader than that with $g_2=0$. $-\mathrm{Im}\,\mathcal{J}_2$ exhibits both a dip at η_1^{**} and a peak at η_1^*.

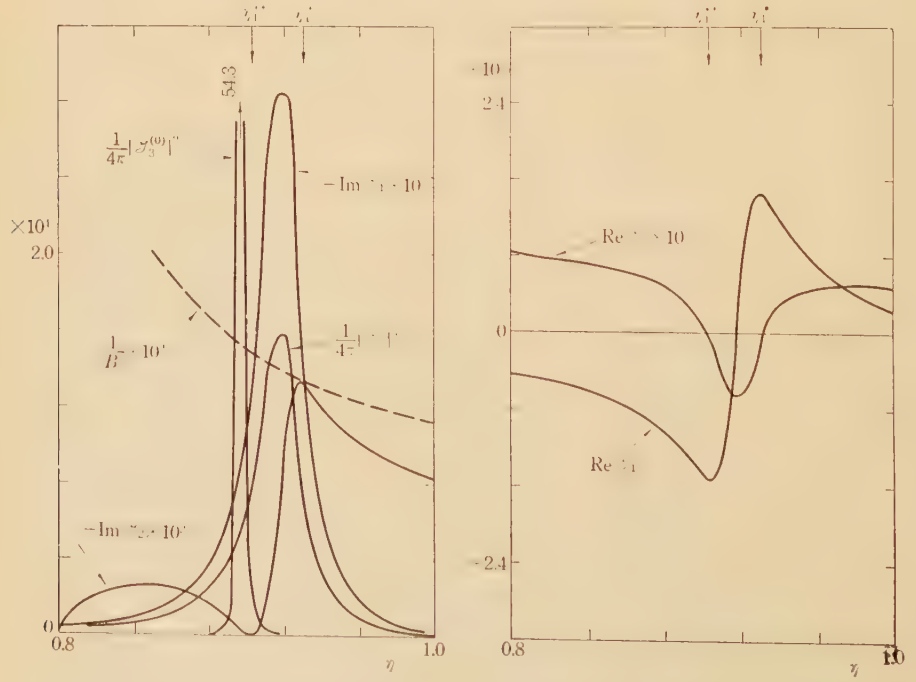

Fig. 15. Example 5-3. There occur both a conventional and an imaginary resonances. Two peaks each in $|\mathcal{J}_3|^2$ and $-\mathrm{Im}\,\mathcal{J}_1$ overlap with each other to form a single peak.

Case 6. Case 1 is modified by a large positive g_2.

We shall choose g_2 large enough to give a zero point of $\mathcal{B}$ in Region II. Eq. (4·19) shows that η_1^{**} lies to the left of η_1, the difference being proportional to g_3^2/g_2. Also a conventional resonance occurs at η_1^*, which is shifted to the left of η_1 by a distance proportional to g_3^2. At these energies we find

$$\begin{aligned} &\mathcal{A}(\eta_1^*)=0, \\ &\mathcal{B}(\eta_1^*)=\frac{g_3^2}{g_1+hA_2}B_2=\frac{g_3^2}{g_1D_2+g_3^2A_2}B_2, \end{aligned} \tag{4·29}$$

and

$$\begin{aligned} &\mathcal{A}(\eta_1^{**})=\frac{g_3^2}{h}, \\ &\mathcal{B}(\eta_1^{**})=0. \end{aligned} \tag{4·30}$$

We note that D_2 is positive and "large" (i.e. not so small as in the $g_2<0$ case) for $g_2>0$. This has the two effects of increasing the slopes of $\mathcal{A}(\eta)$ at $\eta=\eta_1^*$ as seen from Eq. (4·28), and of reducing $\mathcal{B}(\eta_1^*)$ as seen from Eqs.

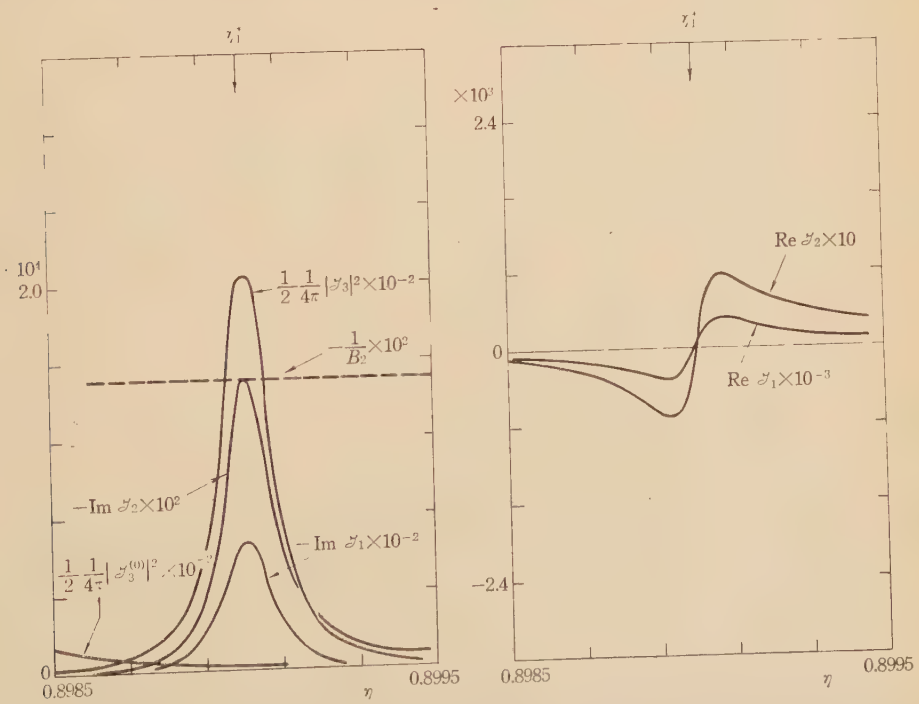

Fig. 16. Example 6. There occur both a conventional and an imaginary resonances. The peak at $\eta=\eta_1^*$ each in $|\mathcal{J}_3|^2$ and $-\mathrm{Im}\,\mathcal{J}_1$ is much more prominent than that at $\eta=\eta_1^{**}$ and there results practically a single peak. The scale of abscissa has been enlarged because peaks are very narrow.

(4·29). The combination of these two effects produces a very sharp peak in $|\mathcal{D}|^{-2}$ at $\eta=\eta_1^*$. An example, shown in Fig. 16, really exhibits such a sharp peak. There occurs a sharp peak also at η_1^{**} (an imaginary resonance). In the present example the peak at η_1^* (a conventional resonance) is much more prominent than the peak at η_1^{**}, and there results practically a single peak. The half-width of the resultant peak is estimated as $\sim 0.75\times 10^{-4}$, about one-third of that in $|\mathcal{J}_3^{(0)}|^2$.

Also $-\mathrm{Im}\,\mathcal{J}_2$ exhibits a very sharp peak, with a dip closely adjacent.

§ 5. Summary and concluding discussions

(i) The cases considered in § 4 can be classified into two groups.

A. Case 1, Case 5, Case 6

In these cases we have assumed a strong attractive force between K and N such that there is a bound state in Region II, and a weak coupling between two channels. We have shown that one or two sharp peaks can occur in $|\mathcal{J}_3|^2$, $-\mathrm{Im}\,\mathcal{J}_1$ and $-\mathrm{Im}\,\mathcal{J}_2$, for $\pi\Lambda$ interactions varying over a fairly wide range. A peak occurs either as a result of a conventional or an imaginary resonance, or both.

B. Case 2, Case 3, Case 4

In these cases we have failed to present examples in which there occurs a sharp peak. This situation seems to be valid more generally.

We can, thus, conclude that the existence of a bound state in the K_-N system and a weak coupling between two channels are essential for a sharp peak (or peaks) in the various amplitudes. We can understand this very naturally:

Let us consider the process

$$K^-+p\to\Lambda+\pi^++\pi^-.$$

We can divide this process into two steps, say,

$$\begin{aligned} K^-+p &\to Y^{*+}+\pi^- \\ &\to \Lambda+\pi^+. \end{aligned}$$

The initial K^-p system falls into a bound state Y^{*+} after the emission of a π^-. The Y^* thus formed cannot decay into a K_- and an N from the law of energy conservation. It continues to exist until K_- and N bound in the Y^* change into π and Λ. If the probability of this change is small, the lifetime of Y^* is long, and we observe a narrow peak in the cross section.* The integrated value of the peak, which gives the total yield of the process, is

* This possibility was already suggested by G. Takeda.

determined essentially by the probability of the first step and does not depend on that of the second step, i.e. whether Y^* is long lived or not. Copious production and long life, which seem to contradict to each other at first sight, are, thus, ascribed to two distinct interactions giving rise to the formation and to the decay of a bound state, the former being strong and the latter weak.

(ii) In Case 5 and Case 6, we have found that an attractive $\pi\Lambda$-$\pi\Lambda$ force makes the peak broader and a repulsive one sharper. We could understand this effect, at least qualitatively, since an attractive (a repulsive) interaction in the *final* $\pi\Lambda$ state will make the transition into that state more probable (unprobable). If the $\pi\Lambda$-$\pi\Lambda$ force turns out to be a repulsive one with an appropriate strength, a channel coupling not very weak would be allowed in order to reproduce a narrow peak.

(iii) We should be particularly interested in the conclusion that a large negative g_1 and a small g_3^2 are required in our model; g_1^2/g_3^2 amounts to $10^2 \sim 10^3$, though these values depend possibly on the special choice of the numerical parameters and on the nature of a $\pi\Lambda$-$\pi\Lambda$ force. It seems strange that the K_-N interaction is very strong, because there is some evidence that the K_+N interaction is somewhat weaker than the πN interaction.[17] We should note, however, that the two interactions, K_+N and K_-N interactions, are not necessarily identical with each other only from general principles, e.g. charge conjugation invariance, or charge independence.

We can, of course, calculate the amplitudes for K_+N scattering from the graphs which are crossing symmetric to that in Fig. 1. But there is some doubt in applying our model to such graphs straightforwardly, since this model has been so invented as to be particularly suitable for the S-wave K_-N interaction.*

The interaction Hamiltonian (2·1) need not be considered to be a truly elementary one; rather it may be interpreted more reasonably as an effective Hamiltonian derived from more fundamental ones, e.g. an appropriate set of Yukawa interactions.

(iv) It may happen that the more correct inclusion of the Σ-particle would change the results appreciably. In particular, it may be troublesome if we are forced to assume a large g_3', the coupling constant between K_-N channel and $\pi\Sigma$ channel. In this connection it seems interesting that we have obtained the examples in which a Σ results as a bound state of the $\pi\Lambda$ system (Case 5). If we can reproduce similar cases with spins, parities, iso-spins, and mass ratios being taken into account correctly, then we would have another merit.

Namely, Σ, resulting as an S-state bound state of the $\pi\Lambda$ system, has a parity opposite to that of Λ. This leads to a conclusion that Y^* decays into Σ and π in a P-state, which will be consistent with the preliminary experi-

* Crossing symmetric graphs cause scatterings in higher wave states as well as an S-state.

mental result that the ratio of $Y^*\to\Sigma+\pi$ to $Y^*\to\Lambda+\pi$ is very small (~ 0.08).[1),2)]

In any case it will be necessary to develop a three-channel model including the $\pi\Sigma$ system for further investigations.

(v) We have listed the scattering lengths b_1 and c_1 in Table I. A direct comparison of these values with the realistic ones, e.g. Dalitz solution,[6)] may be meaningless, because of the crudeness of the model and the arbitrariness of parameters chosen, though they are found to give rather reasonable values and will provide some reference in more realistic considerations.* Here, we shall only summarize the results very briefly without discussing any more in detail.

We see that b_1 in Case 1, Case 5 and Case 6 are not much different from $b_1^{(0)}$, with $g_3=0$, while c_1 varies to some extent. In particular, c_1 in Example 6 is very small. This is evidently due to the repulsive $\pi\Lambda$ interaction (a positive g_2 makes D_2 large and gives a large $\mathcal{A}$) and will be valid generally.

If the "scattering length approximation" is valid, Im $\mathcal{J}_1$ below the threshold can be written as

$$-\mathrm{Im}\,\mathcal{J}_1 \simeq 16\pi M_1 \frac{-c_1}{1-2\kappa_1 b_1+\kappa_1^2|a_1|^2}, \tag{5·1}$$

from the second equation in Eqs. (3·9). We can calculate $\bar{\kappa}_1$, at which the right-hand side of Eq. (5·1) becomes maximum, as

$$\bar{\kappa}_1 = \frac{b_1}{b_1^2+c_1^2},$$

and correspondingly we have

$$\bar{\eta}_1 = 1-\frac{b_1^2}{M_1^2(b_1^2+c_1^2)^2}, \tag{5·2}$$

which are also listed in Table I. We find that they give positions of the peaks not inconsistent with the correct ones, at least in Case 1, Case 5 and Case 6, but are far from giving information about detailed features of the resonance, e.g. whether it is a conventional resonance or an imaginary one, whether the peaks, if there are two, overlap or not, and so on.

(vi) There is reported also an existence of K^*, which can be considered to be an excited state in the $K\pi$ system.[2)] The width of the observed peak is again so small (~ 10 Mev) that we cannot reproduce it in a P-state chain approximation applied to the $K\pi$ system.[18)] Moreover there are no appropriate two-particle system in which a bound state can appear and be identified as K^*, in ananlogy with the case of Y^* (the mass of K^* is 884 Mev). Therefore, the introduction of K^* as a new elementary particle may be required.

* Note that $\tilde{a}$ is a dimensionless quantity. In order to obtain a true scattering length a in unit of, say, the pion Compton wavelength, we must divide $\tilde{a}$ by a factor $8\pi(2M_1/M_\pi)$, which is about 258, if we replace $2M_1$ by M_N+M_K tentatively.

Acknowledgments

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Letters to the Editor

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Isotopic Abundance of Carbon in Primary Cosmic Rays

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July 3, 1961

Previously, one of the authors (K. I.) pointed out that not only the chemical composition but also the isotopic abundance in primary cosmic rays will inform us on the origin of the cosmic rays.[1] Especially the $^{13}C/^{12}C$-ratio should reflect the nature of the source. If the sources are either the general celestial objects in which the CN cycle process takes place, or the evolving ones such as the red giants, the $^{13}C/^{12}C$-ratio must be $1/4.6=22\%$ or less. On the other hand, the carbon 13 overabundance in the source can be expected only when the rapid CNO cycle process occurs there under the suitable physical condition. This condition may be satisfied at the earliest stage of the supernova explosion, according to the recent work by Hayakawa et al.[2]

Now we propose a method for determining this ratio. As the mass difference between the ^{13}C and ^{12}C is rather small, the usual technique for mass determination is not applicable. But the difference in the nuclear structure gives rise to the different results through the nuclear reaction.

Carbon 12 is a strongly closed nucleus while the binding of the last neutron of carbon 13 is considerably weak. For an incident ^{13}C, one neutron is easily stripped through the collision with the emulsion nuclei and ^{12}C emerges.

We denote the cross section for neutron missing process by $\sigma(^{13}C\rightarrow C)$. For an incident ^{12}C, because of its close packing, the cross section for neutron missing process $\sigma(^{12}C\rightarrow C)$ is much smaller than $\sigma(^{13}C\rightarrow C)$. The cross section $\sigma(^{12}C\rightarrow B)$ for the process in which incident ^{12}C turns into boron isotopes and that cross section $\sigma(^{13}C\rightarrow B)$ for incident ^{13}C are nearly equal to each other. Moreover, we may expect that $\sigma(^{12}C\rightarrow B)\fallingdotseq\sigma(^{12}C\rightarrow C)$.

Suppose that among the nuclear interactions due to the carbon nuclei of primary cosmic rays in the emulsion, there are $N(C)$ or $N(B)$ events in which a carbon nucleus or a boron nucleus emerges respectively. Then $N(C)\gg N(B)$ means the carbon 13 overabundance and $N(C)\fallingdotseq N(B)$ means

the small contents of carbon 13, or more precisely,

$$^{13}C/^{12}C\text{-ratio} = \frac{N(C)}{N(B)} \cdot \frac{\sigma(^{12}C \to B)}{\sigma(^{12}C \to C)} = \frac{\sigma(^{13}C \to C)}{\sigma(^{12}C \to C)} \cdot \frac{N(C)}{N(B)} \cdot \frac{\sigma(^{13}C \to B)}{\sigma(^{12}C \to C)} .$$

Since little information on the respective cross sections is obtained so far, we are obliged to infer as follows. From the heavy ion experiments,[3] for example, the cross section $\sigma(^{12}C + ^{103}Rh \to ^{11}C + X)$ is found to be about 15 mb at 120 Mev. Assuming the classical orbit theory for such a grazing collison, the above figure must be multiplied by factor 1.2 in the cosmic-ray energy region. Also $\sigma(^{12}C + p \to ^{11}C + pn)$ is well known in Gev region and found to be 23–30 mb.[4] Summarizing these data we estimate $\sigma(^{12}C \to C)$ through the collision with the emulsion nuclei to be $25(\pm 5)$ mb. No experimental information on ^{13}C reactions is available. Therefore we assumed that the carbon 13 is composed of ^{12}C and one neutron, and we calculated the cross section $\sigma(^{13}C \to C)$ in the same manner as that in the calculations of the stripping and Coulomb disintegration cross sections of deuteron. The result is about 75 mb.

There has been no systematic scanning for this purpose. But we would mention the tentative values of $^{13}C/^{12}C$-ratio. We collected the data so far published,[5] and obtained $N(B) = 6$ and $N(C) = 12$. Adopting the above values for the cross sections and allowing the uncertainties $\sigma(^{12}C \to B)/\sigma(^{12}C \to C) = 1 \pm 0.1$ and $\sigma(^{13}C \to B)/\sigma(^{12}C \to C) = 1 \pm 0.1$, we get $^{13}C/^{12}C$-ratio $1.0^{+0.5}_{-0.3}$ (see Fig. 1). If this value is not far from

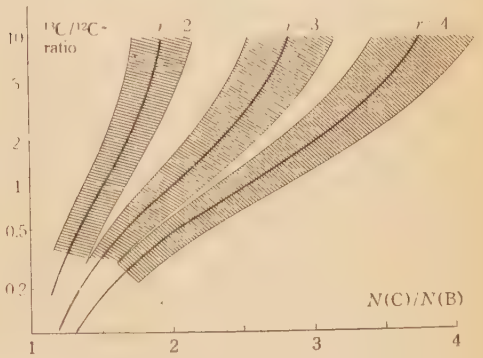

Fig. 1. Diagram for determining the $^{13}C/^{12}C$-ratio from experimental data $N(C)/N(B)$ for various cross section ratios $r = \sigma(^{13}C \to C)/\sigma(^{12}C \to C)$. The shaded domains along the respective solid lines show the range of allowance due to the uncertainties of 10% in the cross section ratios $\sigma(^{12}C \to B)/\sigma(^{12}C - C) = 1$ and $\sigma(^{13}C \to B)/\sigma(^{12}C \to C) = 1$.

reality, we may take part with the theory of the supernova origin of the cosmic rays. It is also noticeable that, under the condition responsible for the carbon 13 overabundance, the amount of carbon itself is larger than the amount of the oxygen,[6] which is one of the qualifications for the cosmic-ray source.

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The Shielding of a Fixed Charge in a Metal

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March 25, 1961

When a few impurities are melted in a monovalent metal and are ionized, the conduction electron density of the metal changes due to the static charges of the ionized impurities and to the Coulomb interaction of the electron with each other, and the conduction electrons are rearranged in the charge distribution. As a result, the charges of impurity ions are shielded by these displaced electrons. There have already been made several calculations of the potential and charge distribution of the electrons. For example, J. Friedel attempted to deal with this problem in a self-consistent manner, and J. S. Langer and S. H. Vosko solved it in the quantum mechanical many-body theory. But most of these calculations include only the Coulomb interaction, and not the exchange one.[1)~4)]

The author and T. Toya treated this problem in 1955 including both interactions in the linear approximation of the perturbation theory.[5)] For the purpose of deciding the Coulomb and the exchange potential self-consistently, however, they had to use a drastic approximation, and their results have the difficult point that the displaced charge density of electrons diverges at the origin. It is the object of this note to present a somewhat improved calculation on the same problem.

The model with which we shall be concerned is an electron gas with a uniform positively charged back ground, so that the whole system is neutral. Into this system we insert a fixed charge, $(Z+1)e$, at the origin. The conduction electrons are redistributed due to the Coulomb potential of this impurity charge, and to the Coulomb and exchange interaction with each other. We shall determine the charge density of these displaced electrons self-consistently in the following way.

We write the Hamiltonian of the system as

$$H=H_0+V_1+V_2+A. \qquad (1)$$

The perturbation terms consist of three parts, that is, the Coulomb in-

teraction energy V_1, the exchange one A of the conduction electrons, and the Coulomb interaction energy V_2 between the ionized impurity and the conduction electrons. V_1 and A are operators defined for any one-electron wave function $\psi(\boldsymbol{r})$ as[6]

$$V_1\psi(\boldsymbol{r}) = \sum_{k,spin}\int \varphi_k^*(\boldsymbol{r}_2)\varphi_k(\boldsymbol{r}_2) \times \frac{e^2}{|\boldsymbol{r}-\boldsymbol{r}_2|}\,d\boldsymbol{r}_2\psi(\boldsymbol{r}), \tag{2}$$

$$A\psi(\boldsymbol{r}) = -\sum_{k}\int \varphi_k^*(\boldsymbol{r}_2)\psi(\boldsymbol{r}_2) \times \frac{e^2}{|\boldsymbol{r}-\boldsymbol{r}_2|}\,d\boldsymbol{r}_2\varphi_k(\boldsymbol{r}), \tag{3}$$

where $\varphi_k(\boldsymbol{r})$ are the wave functions of occupied one-particle states, which can be written in the first order perturbation as

$$\varphi_k(\boldsymbol{r}) = \left(\frac{1}{\Omega}\right)^{1/2} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\chi_s\left[1+\frac{1}{N}\times\sum_{\boldsymbol{q}\neq 0}\frac{V_1(\boldsymbol{q})+V_2(\boldsymbol{q})+A(\boldsymbol{q})}{E_k-E_{k+q}}e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\right]. \tag{4}$$

χ_s is the spin function. The expression (4) contains the matrix element of V_1, V_2 and A, namely

$$V_1(\boldsymbol{q}) = \frac{N}{\Omega}\int e^{-i(\boldsymbol{k}+\boldsymbol{q})\cdot\boldsymbol{r}}V_1 e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,d\boldsymbol{r} \tag{5}$$

and similar relations for V_2 and A. Therefore (2)–(5) provide a set of equations to determine the charge distribution self-consistently. First we can solve $V_1(\boldsymbol{q})$ and $A(\boldsymbol{q})$ in terms of $V_2(\boldsymbol{q})$. The results are

$$V_1(\boldsymbol{q}) = -V_2(\boldsymbol{q}) \times \frac{\frac{2k_0}{\pi a_H}f(k_0q)}{q^2+\frac{2k_0}{\pi a_H}f(k_0q)-\frac{2q}{\pi a_H}F(k_0q)}, \quad q\neq 0 \tag{6}$$

and

$$A(\boldsymbol{q}) = V_2(\boldsymbol{q}) \times \frac{\frac{2q}{\pi a_H}F(k_0q)}{q^2+\frac{2k_0}{\pi a_H}f(k_0q)-\frac{2q}{\pi a_H}F(k_0q)}, \quad q\neq 0 \tag{7}$$

where

$$f(k_0q) = 1+\frac{4k_0^2-q^2}{4k_0q}\log\left|\frac{2k_0+q}{2k_0-q}\right|, \tag{8}$$

$$F(k_0q) = \int_0^{k_0}\frac{dk}{k}\left[\log\left|\frac{2k+q}{2k-q}\right| - \log\left|\frac{k+q}{k-q}\right|\right], \tag{9}$$

and a_H is the Bohr radius and k_0 the magnitude of the Fermi wave vector.

The charge density due to the displaced electrons is given by[7]

$$\Delta n(\boldsymbol{r}) = -\frac{mk_0}{2N\pi^2\hbar^2}\sum_{q}\left(V_1(\boldsymbol{q})+V_2(\boldsymbol{q}) + A(\boldsymbol{q})\right)e^{i\boldsymbol{q}\cdot\boldsymbol{r}}f(k_0q). \tag{10}$$

Substituting (6) and (7) in (10), and using the Fourier component of Coulomb interaction between the impurity ion and electron $V_2(\boldsymbol{q}) = -4\pi Ze^2/|\boldsymbol{q}|^2$, we obtain the displaced charge density

$$\Delta n(\boldsymbol{r}) = \frac{Zk_0}{\pi^3 a_H} \int_0^\infty dq \frac{q \sin qr}{r} \times \frac{f(k_0 q)}{q^2 + \frac{2k_0}{\pi a_H} f(k_0 q) - \frac{2q}{\pi a_H} F(k_0 q)} . \tag{11}$$

The third term of the denominator of the right-hand side in this formula represents the exchange effect of the conduction electrons.

Without the exchange effect, it is shown that the number of electrons in the range from the origin to the first node of the displaced charge density is equal to about Z, and therefore the positive charge of the impurity is nearly shielded. On the other hand, as Eq. (11) shows, the attractive force of the exchange interaction makes the number of electrons in the same range increase more than Z, leaving positive charges outside it. Accordingly, the displaced charge density becomes more far reaching if the exchange effect is included.

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Remarks on Interacting Many-Fermion System

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The correlation characters of interacting many-fermion system have been investigated by many authors. In this short note dynamical correlation behavior of nuclear matter will be considered.

Independent particle motion of nuclear matter, which is the foundation of nuclear shell model, does not seem to have been made so clear. K. A. Brueckner et al.[1] or K. Sawada[2] more completely predicted that total energy of nuclear matter could be obtained by the sum of one particle energy defined appropriately. Whether the one particle eigenmode is good or not may be a different problem from the above conclusion. One suggestion on this point comes from the calculation of self-energy of a particle in V. M. Galickij's argument.[3] This self-energy became complex, and a particle level will be dissipative generally. As is well known, there is Weisskopf's argument on independent particle model of nuclear matter, which is the following: If we take into account only the particle-particle interactions the independent particle model is valid where the two particles are distant apart.[4] But when we consider the other cor-

relations, it becomes an open question whether the independent particle model would be valid even in Weisskopf's sense.

For example, let us consider two-particle scattering mode including the hole motion. In this case the Bethe-Goldstone equation is[5]

$$(E-T_1-T_2)\psi(\boldsymbol{r}_1, \boldsymbol{r}_2) = (Q-P)\,V(\boldsymbol{r}_1, \boldsymbol{r}_2)\psi(\boldsymbol{r}_1, \boldsymbol{r}_2), \quad (1)$$

where, Q and P are the projection operators keeping both particles outside and inside the Fermi sea, respectively. Following the treatment of Weisskopf et al., at first we consider the inter-nucleon potential having only a repulsive core, and the nucleon pair being in the s-state and having zero total momentum. Then writing the relative wave function as $\psi(r)=u(r)/r$, where $r=|\boldsymbol{r}_1-\boldsymbol{r}_2|$ and solving Eq. (1), one obtains $u(r)$ as follows,

$$u_\kappa(x)=\sin\kappa x-[\sin\kappa c/F_\kappa(c)]\cdot F_\kappa(x), \quad (2)$$

with

$$\begin{aligned} F_\kappa(x)=&\sin\kappa(x-c)\,\{\pi/2-Si[(1-\kappa)\\ &\times(x-c)]-Si[(1+\kappa)(x-c)]\}\\ &-\sin\kappa(x+c)\,\{\pi/2-Si[(1-\kappa)(x+c)]\\ &-Si[(1+\kappa)(x+c)]\}\\ &+\cos\kappa(x-c)\,\{Ci[(1-\kappa)(x-c)]\\ &-Ci[(1+\kappa)(x-c)]\}\\ &-\cos\kappa(x+c)\,\{Ci[(1-\kappa)(x+c)]\\ &-Ci[(1+\kappa)(x+c)]\} \end{aligned}$$

where, $\kappa=k/k_F$, $x=k_Fr$, $c=k_Fr_c$, k is relative momentum, k_F the Fermi momentum and r_c the core radius. The wave function with $\kappa=1/2$ is shown in Fig. 1

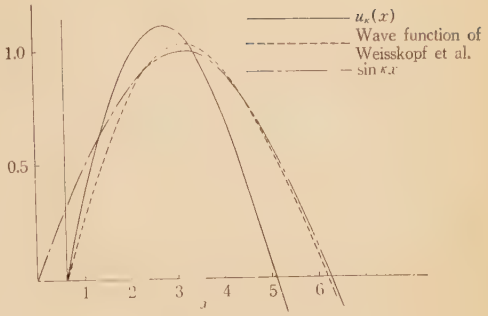

Fig. 1. The wave function for a pair with $\kappa=1/2$ $k_F=1.48\times10^{+13}$ cm^{-1} and $r_c=0.4\times10^{-13}$ cm.

As can be seen, the above wave function has a finite phase shift. And it is very interesting that this phase shift is positive. This means that the wave function is affected under the effective attractive interaction due to the Pauli principle in spite of the original interaction being only a repulsive core. The momentum dependence of phase shift is exhibited in Fig. 2.

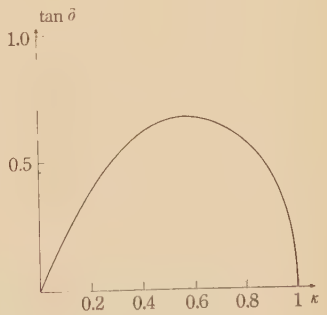

Fig. 2. The phase shift as the function of κ.

From this graph, one can see that the mode of the independent particle motion under the above situation may not be certified in the Fermi sea except as the very neighborhood of

Fermi surface. At the latter place this mode seems to be certified at a glance, since the phase shift becomes very close to zero. But there exists a difficulty. That is, taking $\kappa \to 1$ in Eq. (1), one can easily see $u(x) \to 0$ or the wave function vanishes at the Fermi surface. Of course, this occurs in Weisskopf's case.

In terms of the above limiting process, one can see that this catastrophe may be caused by the sharp-cut Fermi surface. So we may consider that even in the case of repulsive interaction, two-particle state may be formed in the distorted Fermi distribution. But the physical meaning of such distribution may not be so clear. On the other hand, it is well known that if we apply Bogoliubov's method[6] for the repulsive interaction, the system may be stable in normal state. These arguments may be very close to L. van Hove.[7] Detailed report will be published in near future.

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On the Validity of the Adiabatic Approximation in the Theory of Electron Broadening in a Plasma

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It is well known that spectral lines from radiating atoms in a plasma play an important role as probes of the microfields in the interior of the plasma. Theories of pressure broadening have been developed by many authors, and experimental results have been explained fairly well by treating perturbing ions and perturbing electrons from the standpoint of statistical theory and impact theory, respectively.

In the usual case where temperature of the plasma is of the order of magnitude $10^4 \sim 10^5$ °K, perturbing ions can be treated exactly by classical theory. On the other hand, it is desirable to describe perturbing electrons quantum-mechanically, since the de Broglie wavelength of the electrons considered to be at the average velocity become comparable to the atomic radius.

The main part of electron-impact broadening arises from the potential difference between the initial and final states of a radiating atom. In the previous theories the effective poten-

tial on a perturbing electron has been calculated by the adiabatic approximation.[1] It is the purpose of the present note to verify the validity of this adiabatic approximation. The effective potential of perturbing electrons for the atom in a stationary state n is generally given by the following formula:

$$\mathcal{V}_n(R) = -\frac{Ze^2}{R} + \langle n|\, \Sigma_i V_i(\boldsymbol{R}, \boldsymbol{r}_i)\, |n\rangle + \mathcal{V}_n^{P} + \cdots, \tag{1}$$

$$\mathcal{V}_n^{P} = \langle n|\, \Sigma_i V_i(\boldsymbol{R}, \boldsymbol{r}_i)\, \frac{1}{d} \times P \Sigma_i V_i(\boldsymbol{R}, \boldsymbol{r}_i)\, |n\rangle, \tag{2}$$

where

$$d = (\varepsilon_n - \varepsilon_{n'}) + (W_n - W_{n'}) + i\eta,$$

$$\varepsilon_n = T + \mathcal{V}_{n'}(R),\ \varepsilon_{n'} = T' + \mathcal{V}_{n'}(R)$$

and η is a positive infinitely small number, $V_i(\boldsymbol{R}, \boldsymbol{r}_i)$ the Coulomb interaction between the perturbing electron and the i-th electron in the radiating atom, $\boldsymbol{R}$ the position vector of the perturbing electron, $\boldsymbol{r}_i$ the position vector of the i-th electron in the radiating atom, W_n and $W_{n'}$ the energy levels of the radiating atom in the initial state n and the intermediate state n', respectively. T and T' are the kinetic energies of the perturbing electron in the initial and intermediate states, respectively, and P means that all initial and final states should differ from each other.

The first two terms in Eq. (1) represent the shielding potential of the radiating atom, and $\mathcal{V}_n^{P}$ the polarization potential including higher order terms.

The adiabatic approximation corresponds to taking

$$d = (W_n - W_{n'}) + i\eta.$$

Now $1/d$ can be written exactly in the following way:

$$\frac{1}{d} = \frac{1}{W_n - W_{n'}} + \frac{1}{(W_{n'} - W_n)\{(W_{n'} - W_n) + (\varepsilon_{n'} - \varepsilon_n) - i\eta\}}. \tag{3}$$

Then the leading term of the non-adiabatic correction to the polarization potential is given by

$$\Delta\mathcal{V}_n^{P} = \sum_{n' \neq n} \frac{\langle n|\, \Sigma_i V_i |n'\rangle\langle n'|\, [\varepsilon_{n'}, \Sigma_i V_i]\, |n\rangle}{(W_{n'} - W_n)\{(W_{n'} - W_n) + (\varepsilon_{n'} - \varepsilon_n) - i\eta\}}, \tag{4}$$

where $[\varepsilon_{n'}, \Sigma_i V_i]$ is the commutator of two operators $\varepsilon_{n'}$ and $\Sigma_i V_i$.

As $\mathcal{V}_n$ and $\Sigma_i V_i$ are commutable in the extreme adiabatic limit,

$$[\varepsilon_{n'}, \Sigma_i V_i] = -\frac{\hbar^2}{2m}\{\nabla^2 \Sigma_i V_i + 2(\nabla \Sigma_i V_i)\cdot\nabla\}, \tag{5}$$

where m is the mass of the perturbing electron. When the perturbing electron is outside the radiating atom,

$$[\varepsilon_{n'}, \Sigma_i V_i] \simeq (\hbar/m)\,(K/R)\cdot \Sigma_i V_i, \tag{6}$$

where K is the momentum of the perturbing electron. Hence

$$\Delta\mathcal{V}_n^{P} \simeq \mathcal{V}_n^{P}\cdot(\hbar/m)\,(K/R) \times \langle W_{n'} - W_n\rangle_{Av}^{-1}, \tag{7}$$

where $\langle W_{n'} - W_n\rangle_{Av}$ means an average

over neighboring energy levels. Eq. (7) leads to the following condition for the validity of the adiabatic approximation:

$$(\hbar/R)(K/m) \ll \langle W_{n'} - W_n \rangle_{Av}. \quad (8)$$

For example, we consider the Lyman-α line, which provides convenient data for examining the electron broadening effect since it does not shift under the influence of static ion fields, while its Stark components have a central line. For numerical estimation we take the plasma in the thermal equilibrium with $T=2\times10^4$ °K and $n=10^{15}\,\mathrm{cm}^{-3}$. Then the thermal average of $(\hbar/R)(K/m)$ being of the order of magnitude of 6×10^{-3} ev, the validity of the adiabatic approximation holds in the case of the final state given by Eq. (8). On the other hand, the initial state can be separated by static ion fields as

$$-\frac{e^2}{8a_0} \pm 3ea_0F,$$

where a_0 is the Bohr radius, F the static ion field, and the average of $3ea_0F$ gives 6×10^{-5} ev. Therefore, Eq. (8) does not hold in the initial state, and the high energy approximation[2] should be used instead of the adiabatic approximation.

Thus we have shown by means of the Lyman-α line that the adiabatic approximation should not be used for all effective potentials in electron impact broadening. Detailed calculations based upon these considerations in which the effect of the collective motion of the perturbers will be taken into account shall be published later.

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Effet de la polarisation chargée sur l'interaction entre ions lourds

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Sur l'interaction électrique entre ions lourds l'importance de l'effet de distorsion polarisée a été déjà signalée et étudiée théoriquement par Breit et ses collaborateurs.[1] Ils ont trouvé que, pour la collision de O^{16} avec U^{238}, une assez grande distorsion était provoquée. Cette distorsion est conforme au changement en barrière coulombienne de -1.4 Mev.

Nous voulons examiner, d'un point de vue différent, les effets de la polarisation chargée provoquée sur la pure interaction coulombienne sans l'emploi de la fonction d'onde nucléaire et nous les remplaçons par les données connues empriques pour éviter l'ambiguité de faire usage du modèle nucléaire.

Supposons que les deux noyaux "a" et "b" avec leurs charges électriques Z^ae et Z^be soient séparés par une distance R, avec comme protons à r_i^a et r_j^b de leurs centres de masse. En ce cas leur l'interaction électrique V est donnée par

$$V=\sum_{i=1}^{z^a}\sum_{j=1}^{z^b}\frac{e^2}{|\boldsymbol{R}-\boldsymbol{r}_i{}^a+\boldsymbol{r}_j{}^b|}. \quad (1)$$

Et si $r_i{}^a$ et $r_j{}^b$ sont assez petits en comparaison de R nous pouvons la développer dans la série de Taylor.

Ensuite nous estimerons la valeur d'expectation de l'énergie potentielle de ces systèmes en employant Eq. (1) comme une perturbation. A l'approximation de premier ordre le résultat donne immédiatement

$$\Delta_1 E_{a,b}=\frac{Z^aZ^be^2}{R}+\frac{3e^2}{2R^3}(Z^bQ_0{}^a+Z^aQ_0{}^b), \quad (2)$$

où $Q_0{}^a$ et $Q_0{}^b$ sont les moments quadripolaires. A l'approximation de deuxième ordre si nous nous rappelons la relation entre l'élément de matrice dipolaire et la section efficace photonucléaire σ_{ph} donnée par Levinger-Bethe,[2] et la relation entre l'élément de matrice quadripolaire et le temps de la vie moyenne radiative τ,[3] nous avons

$$\begin{aligned}\Delta_2 E_{a,b}=&\frac{1}{4\pi^2}\frac{\hbar c}{e^2}\left(\frac{e^2}{R^2}\right)^2\\&\times\left\{(Z^b)^2\int\frac{\sigma_{ph}{}^a(E_1{}^a)}{(E_1{}^a)^2}dE_1{}^a+(a\rightleftarrows b)\right\}\\&+\frac{75}{16\pi}\frac{\hbar}{e^2}\left(\frac{e^2}{R^3}\right)^2\left\{(Z^b)^2(2j_a+1)\right.\\&\times\int\left(\frac{\hbar c}{E_2{}^a}\right)^5\frac{1}{\tau^a}\frac{\rho(E_2{}^a)}{E_2{}^a}dE_2{}^a\\&\left.+(a\rightleftarrows b)\right\}\\&+\left(\frac{1}{4\pi^2}\frac{\hbar c}{e^2}\frac{e^2}{R^3}\right.\\&\left.\times\frac{C_\xi{}^aC_\eta{}^b+C_\eta{}^aC_\xi{}^b+2C_z{}^aC_z{}^b}{C_z{}^aC_z{}^b}\right)^2\\&\times\iint\frac{\sigma_{ph}{}^a(E_1{}^a)\sigma_{ph}{}^b(E_1{}^b)}{(E_1{}^a+E_1{}^b)E_1{}^aE_1{}^b}dE_1{}^adE_1{}^b\end{aligned} \quad (3)$$

où $E_{1(2)}$ signifie une énergie d'excitation, j est son moment cinétique angulaire à l'état normal et $\rho(E)$ est la quantité des états à l'unité d'intervalle d'énergie. C_ξ, C_η et C_z sont les coefficients d'élément de matrice réduite et ils s'expriment avec les coefficients de Clebsch-Gordan.

Pour estimer Eq. (3) d'une manière concrète supposons qu'un seul état excité spécial ait un élément de matrice de transition considérable à l'état normal et que les autres soient négligeables par comparaison. En cas de la $E1$ excitation le niveau de résonance géante convient à cette supposition et de la $E2$ excitation la probabilité de transition à l'état normal est la plus grande cette fois-ci sur le premier niveau admissible.

Concernant l'évaluation numérique concrète nous employons les valeurs suivantes pour oxygène-oxygène ; $\sigma_{ph}=33$ mb,[4] $E_1=22$ Mev,[5] $E_2=6.9$ Mev,[6] $\tau=1.2\times10^{-14}$ sec :[6] et pour carbone-carbone ; $\sigma_{ph}=30$ mb,[7] $E_1=23$ Mev,[7] $E_2=4.4$ Mev,[6] $\tau=3.3\times10^{-14}$ sec.[6] Donc Eq. (3) s'écrit :

$$\begin{aligned}\Delta_2 E&=-\frac{140}{R^4}-\frac{140}{R^6}\ \text{(Mev)}\\&\qquad\text{oxygène-oxygène,}\\\Delta_2 E&=-\frac{80}{R^4}-\frac{180}{R^6}\ \text{(Mev)}\\&\qquad\text{carbone-carbone.}\end{aligned} \quad (4)$$

Si nous adoptons une distance critique R_0 qui rend le développement (1)

valide comme deux fois le rayon de O^{16} et C^{12}, la réduction susmentionnée est moins grande d'environ 0.5% pour $O^{16}-O^{16}$ et environ 0.7% pour $C^{12}-C^{12}$. Ce procédé pour évaluer l'énergie par distorsion s'accorde avec le calcul fait de Breit pour le cas de $O^{16}-U^{238}$.

Une autre estimation est possible si nous remplaçons les éléments de matrice par les données expérimentales de la diffusion d'électron après avoir employé l'approximation de "closure". En ce cas $\Delta_2 E$ s'écrit approximativement :

$$|\Delta_2 E| < \frac{1}{3}\left(\frac{e^2}{R^2}\right)^2 \left\{(Z^b)^2 \frac{\langle (r^a)^2\rangle_{av}}{\langle E_1^a\rangle_{av}} + (a \rightleftarrows b)\right\} + \frac{45}{64\pi}\frac{(\hbar c)^2}{Mc^2}\left(\frac{e^2}{R^3}\right)^2 \times \left\{A^a (Z^b)^2 \frac{\langle (r^a)^2\rangle_{av}}{\langle E_2^a\rangle_{av}^2} + (a \rightleftarrows b)\right\} + \left(\frac{C_\xi^a C_\eta^b + C_\eta^a C_\xi^b + 2C_z^a C_z^b}{3C_z^a C_z^b}\frac{e^2}{R^3}\right)^2 \times \frac{\langle (r^a)^2\rangle_{av}\langle (r^b)^2\rangle_{av}}{\langle E_1^a + E_1^b\rangle_{av}} \qquad (5)$$

et
$$< 0.25 \text{ Mev} \quad \text{pour } O^{16}-O^{16}$$
$$< 0.27 \text{ Mev} \quad \text{pour } C^{12}-C^{12}.$$

En tout cas, cet effet pourra influer sur les particules qui passent *près* de l'ion de la cible parce que le potentiel additif est rapidement amorti que celui de Coulomb, et l'effet sur la distribution angulaire serait un peu visible plutôt en arrière.

Une expérience intéressante est désirée dans la région des noyaux déformés parce que l'effet de son moment quadripolaire est bien efficace sur l'interaction coulombienne. Par exemple, dans le cas de $O^{16}-Er^{167}$ le second terme de Eq. (2) est presque 24% de celui de pur Coulomb.

Nous remercions le Professeur M. Nogami, avec qui nous avons échangé une correspondance qui nous a beaucoup stimulé. Nous remercions également pour d'intéressantes discussions les D^rs A. Arima, M. Kawaï et A. Fukuhara.

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Errata

On the Kinetic Equation for Electrons in Metals at Low Temperatures

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Prog. Theor. Phys. **25** (1961), 845

At the 7th and 8th rows above Eq. (1), "$(\rho\lambda)^n\sim r_s$, $\lambda(\rho\lambda)^n\sim r_s^2$," should read "$(\rho\lambda)^n\sim 1$, $\lambda(\rho\lambda)^n\sim r_s$,".

The second row in Eq. (3): " $+\partial/\partial \boldsymbol{x}\cdot A^*(\boldsymbol{x},t)\cdot 1/\hbar\cdot\partial/\partial\boldsymbol{p}\cdot f_0(\boldsymbol{p})$ " should read

$$" +\frac{\partial}{\partial \boldsymbol{x}}\left\{A^*(\boldsymbol{x},t)+\frac{1}{2\Omega}\Sigma_{\boldsymbol{q}}V(\boldsymbol{q})\,f_1(\boldsymbol{x},\boldsymbol{p}-\boldsymbol{q},t)\right\}\cdot\frac{1}{\hbar}\,\frac{\partial}{\partial\boldsymbol{p}}\,f_0(\boldsymbol{p})\ ".$$

The first and second rows below Eq. (4) "$\varepsilon_{\boldsymbol{p}}$ and···vector" should read "$\varepsilon_{\boldsymbol{p}}=\hbar^2\boldsymbol{p}^2/2m^*+1/2\Omega\cdot\Sigma_{\boldsymbol{q}}V(\boldsymbol{q})f_0(\boldsymbol{p}-\boldsymbol{q})$, $\omega_{\boldsymbol{p}}$ is the energy of a phonon of wave vector".

At the 15th row on the left column of p. 847, "potential, the equation···" should read "potential as well as the correction, due to the exchange effect, added to $A^*(\boldsymbol{x},t)$ in the second term, the equation···".

JOURNAL OF THE PHYSICAL SOCIETY OF JAPAN
Volume 16, Number 7, July 1961

CONTENTS

Page

Plane Polarization of 8.5 MeV Bremsstrahlung ..... Katsufusa SHODA 1271
Decay of Rh^{102} and Levels in Ru^{102} ..... Kazuo HISATAKE 1280
Ferromagnetic Crystalline Anisotropy of $MeFe_2O_4$-Fe_3O_4 Ferrite Solid Solution ....
..... Nahonori MIYATA 1291
Polarization Waves in Solids ..... Tatsumi KUROSAWA 1298
Cleavage Fracture in Single Crystals of Silicon Iron ..... Ryūkichi HONDA 1309
Some Remarks on the Phenomenological Theory of Ferroelectricity .....
..... Shozo SAWADA and Ikushi YOSHIDA 1322
Fermi Momentum for Free Electron Metals Perturbed by Localized Imperfections
..... Akira SUGIYAMA 1327
Antiferromagnetic Resonance in $CoCl_2 6H_2O$ ..... Muneyuki DATE 1337
Thermal Properties of $NaNO_2$ ..... Shoichiro NOMURA 1352
Notes on the Lustre of Moiré Patterns ..... Jien NISHIWAKI 1358
Effect of Incoherent Waves on the Electron Microscopic Images of Crystals .....
..... Yoshihiro KAMIYA and Ryozi UYEDA 1361
Flow Birefringence of Polymer Solutions ..... Ryuzo KOYAMA 1366
An Expression for Orientation in Cellulose Fibres ..... V. D. GUPTA 1377
The Confinement of Plasma by the Heliotron Magnetic Field ..... Kōji UO 1380
Microwave Study of Non-ionizing Shock Waves ..... Susumu TAKEDA 1395
Scattering of Microwaves from a Cylindrical Plasma in the Born Approximation. II.
..... Yukio MIDZUNO 1403
Temperature Dependences of the Fundamental Absorption Tail of KCl and the
Absorption due to Bromines in KCl .....
..... Koichi KOBAYASHI and Tetsuhiko TOMIKI 1417
On the Commencement of the Hartmann Flow of Conducting Fluid .....
..... Akira OGAWA and Yoshio SONE 1423
The Hall Effect in the Viscous Flow of Ionized Gas between Parallel Plates under
Transverse Magnetic Field ..... Hiroshi SATO 1427
Hydromagnetic Oseen Flow past an Elliptic Cylinder in a Uniform Magnetic Field
..... Tosio MIYAGI 1434
Molar Sound Velocity in Inorganic Melts and Solutions .....
..... S. V. SUBRAHMANYAM and J. BHIMASENACHAR 1447
Experimental Studies on Sonoluminescence and Ultrasonic Cavitation .. Katsuo NEGISHI 1450
Experimental Evidence for Frank's Theorem on Dislocations by X-ray Diffraction
Micrography ..... Mitsuru YOSHIMATSU 1465

SHORT NOTES

Antiferromagnetism of MnHg ..... Yasuaki NAKAGAWA and Tomiei HORI 1470
On the Faraday Effect in Many-Valley Semiconductors ..... Louis GOLD 1471
Passage Effect in the Electron Spin Resonance of Fe-Doped Silicon ..... Takeo IGO 1472
Logarithmic Growth of the Cesium Antimonide Films on Antimony .. Kiyoshi MIYAKE 1473
Electron Distribution in $KMnF_3$, $KFeF_3$, $KCoF_3$, and $KNiF_3$ .....
..... Atsushi OKAZAKI and Yasutaka SUEMUNE 1474
A Note on the Magnetoresistance of PbTe at High Magnetic Fields .....
..... Eizaburo YAMADA and Kisaburo SHOGENJI 1475
Optical Constants of LiF in the Extreme Ultraviolet ..... Riso KATO 1476

Temporary Bleaching of Color Centers in KCl Crystal ..... Masamitsu HIRAI, Mikihiko IKEZAWA and Masayasu UETA 1477
Magnetic Annealing Effect in Cobalt ..... Takashi SAMBONGI and Tadayasu MITUI 1478
Mössbauer Effect of the Intermetallic Compound UFe_2 ..... Shigehiro KOMURA, Nobuhiko KUNITOMI, Poh-Kun TSENG, Naomoto SHIKAZONO and Hidekuni TAKEKOSHI 1479
On Radiations from Plasma in a Static Magnetic Field ..... Kenji MITANI and Hiroshi KUBO 1480
Magnetic Annealing of Cobalt ..... C. D. GRAHAM, JR. 1481
Drift Mobility of Photo-Injected Electrons in Hexane-Silicone Mixture ..... Pyon-un CHONG and Yoshio INUISHI 1482
Magnetic Domain Observation of $BaFe_{12}O_{19}$ by Faraday Effect ..... Hiroshi KOJIMA and Kimiyoshi GOTO 1483
The Growth of Zinc Sulphide Single Crystals from Flux ..... Yo MITA 1484
Effect of Neutron Irradiation on Carrier Lifetime in Si ..... Kenji MATSUURA and Yoshio INUISHI 1485
Ferromagnetism of the Intermetallic Compound UFe_2 ..... Shigehiro KOMURA, Nobuhiko KUNITOMI, Yoshikazu HAMAGUCHI and Masanobu SAKAMOTO 1486
Oscillations in a Plasma in a Weak Magnetic Field ..... Shoji KOJIMA, Eiichi KAWASAKI and Kiyoe KATO 1487
ESR Spectra of Irradiated Polyvinylalcohol and Its Related Compounds ..... Seiji OGAWA 1488
Emission Spectra of Potassium Bromide-Thallium Phosphors ..... Tokumichi TAMAI and Eiichi MATSUI 1489
Tetragonal Distortion in Copper Manganite-Chromite System ..... Syôhei MIYAHARA, Kiyoshi MURAMORI and Naoki TOKUDA 1490

Errata

A New Method to Determine Hydration Energy of Ion ..... Yoshio ISHIDA 1469

JOURNAL OF THE PHYSICAL SOCIETY OF JAPAN
Volume 16, Number 8, August 1961

CONTENTS

Page
The Second Excited State of Sn^{120} ..... Mitsuhiro KAWAMURA, Atsushi AOKI and Hidetsugu IKEGAMI 1493
Nuclear Ground-State Energies ..... Masami YAMADA and Zyun-itiro MATUMOTO 1497
The $N^{14}(d, \alpha)C^{12}$ Ground-State Reaction in the Energy Range of Deuteron from 1.5 to 3.0 MeV ..... Toshiyuki ISHIMATSU 1529
Level Structure of Nd^{144} ..... Kazusuke SUGIYAMA, Akira FURUSAWA, Shogo HAYASHIBE and Motoharu KIMURA 1538
Magnetic Susceptibilities of Cr and Cr-Fe Alloy ..... Masao SHIMIZU and Takeshi TAKAHASHI 1544
Interaction Effects in Potassium Ferricyanide ..... Taiichiro OHTSUKA 1549
Theory of Seebeck Effect in Plastically Deformed Semiconductors ..... Tokio OHTA 1561
Measurement of Seebeck Effect in Plastically Bent Germanium ..... Tadayoshi YAMASHITA and Tokio OHTA 1565

On the Interference Phenomena of Compton Scattering from Diamond ..... Yasuharu YONEDA 1570
The Crystal Structure of Nickel Chloride Hexahydrate, $NiCl_2 \cdot 6H_2O$ ..... Joji MIZUNO 1574
The Dynamic Bulk and Shear Viscosity of High Polymers, 1 ..... Ryusuke KONO 1580
Ionization Efficiency Curves for A^+, Kr^+, N_2^+, and CO^+ by Electron Impact ..... Yozaburo KANEKO 1587
Three-Level Maser Action in Gas I. Theory of Multiple Quantum Transition and Doppler Effect in Three-Level Gas Maser ..... Tatsuo YAJIMA 1594
Some Optical Properties of Lead Activated Potassium Halide Phosphors ..... Yozo KAIFU 1605
Local Description of the Flow behind a Steady, Two-Dimensional, Curved Shock Wave ..... Zirō HASIMOTO and Shigeki MORIOKA 1616
Two-Dimensional Supersonic Jet of a Conducting Inviscid Gas in the Presence of a Uniform Magnetic Field ..... Shigeki MORIOKA 1625
Steady Temperature in a Composite Elliptic Cylinder ..... Václav VODIČKA 1630
Ultrasonic Dispersion in Aceton ..... C. Raghupathi RAO 1636

SHORT NOTES

Paramagnetic Resonance of Cobalt in Cadmium Chloride and Cadmium Bromide .... Kazuo MORIGAKI 1639
Oscillator Strength of F^--Absorption Band in KCl Crystal ..... Takehiko ISHII 1640
Magnetic After Effects in Manganese Zinc Ferrites ..... Tsuneo AKASHI and Takeshi OKADA 1641
Some Properties of Complex Oxides $AB_{0.5}B'_{0.5}O_3$ of Perovskite Type ..... Shoichiro NOMURA and Tatsuyuki KAWAKUBO 1642
Cross Modulation Line Profiles in Cyclotron Resonance Experiments ..... Louis GOLD 1643
Temperature Distribution around the Rest or Moving Solid-Liquid Interface in a Rod ..... Mikio YAMAMOTO 1644
Electron Spin Resonance in Neutron-Irradiated LiF ..... Kazuo MORIGAKI 1645
Linewidth of Cyclotron Resonance of Pure Germanium ..... Hazimu KAWAMURA, Masakazu FUKAI, Yoshikazu HAYASHI and Tamotsu HASHIMURA 1646
Etch Pits on X-Ray Damaged Rochelle Salt ..... Kenkichi OKADA 1647
Spin Echoes of Co^{59} in fcc Cobalt Metal ..... Kiyoshi SUGIBUCHI, Motohiro MATSUURA and Tsuneo HASHI 1648
Group Velocity of Moving Striations in Hg-A Discharge ..... Hiroshi YOSHIMOTO and Yoshihiko YAMASHITA 1649
The Colloidal Band in KCl Containing NaCl ..... Shinichi NAKASHIMA 1650

Errata

A Direct Determination of the Crystal Structure of Ethylenediammonium Sulphate ..... Kiichi SAKURAI 1651

SUPPLEMENT
OF THE
PROGRESS OF THEORETICAL PHYSICS

No. 17 (1961)

Biophysics ..... Syoten Oka et al.

No. 18 (1961)

Parametric Integral Formulas and Analytic Properties in Perturbation Theory ..... Noboru Nakanishi

Wightman Functions, Retarded Functions and Their Analytic Continuations ..... Huzihiro Araki

No. 19 (1961)

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No. 20 (1961)

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On a Model of Nucleon Structure and Meson-Nucleon Interaction. II

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(Received May 12, 1961)

The elastic and inelastic widths of the second and third resonances of π-nucleon collision are calculated using our model.[1] Though some parts of this calculation could be translated to that of Takeda's ρ-meson theory[2] but it is shown that the elastic width depends on the corresponding ρ-nucleon wave function sharply. The ρ-π coupling constant $F^2/4\pi$ is estimated to be 1~2 from the experimental value of the inelastic width.

We also calculate two pion exchange ρ-nucleon potential and find that this is attractive in the second region $(0.7\mu^{-1} \leq r \leq 1.5\mu^{-1})$ and strong enough to construct ρ-nucleon bound states.

As a straightforward application of our model, 5 Bev π-nucleon scattering is analysed. This analysis gives rather large ρ-meson mass $5\sim6\mu$ and ρ-π coupling constant ~3.

§ 1. Introduction

Within the last years several investigations and analyses[3)-6)] have been carried out in an attempt to discover the character of the second and third resonances of π-nucleon scattering in the energy region between 0.4 Bev and 1.6 Bev. In particular, the investigation of the differential cross sections made by Wood et al.[6] indicated that as long as higher waves are negligible, the second and third resonances will be $D_{3/2}$ and $F_{5/2}$ state respectively.

Recently, many authors proposed various models to interpret these curious energy levels of nucleon.[1),2),7),8)] Takeda et al. showed in their recent work[2] that these levels could be explained by suitable introduction of π-π resonance of the iso-spin 1 and angular momentum 1. Their theory is summarized essentially in the following two points.

i) The assumed π-π resonance state can be approximately treated as an actual particle $\rho^{\pm 0}$. This is, of course, an iso-vector and ordinary vector meson.

ii) The interaction between ρ and π,

$$\rho \rightleftarrows \pi + \pi, \tag{1·1}$$

produces an attractive ρ-nucleon potential from which a kind of meta-stable ρ-nucleon bound states are constructed which correspond to the second and third resonances of π-nucleon collision.

It may be the main point of their description that the ρ-nucleon meta-stable

states are constructed and destroyed by the same interaction (1·1) simultaneously. In the previous paper,[1] we also proposed another model which was based on the following assumptions.

A) There are very strong S-wave bound states (i, j) between a nucleon and an antinuleon (which we call *nucleoneum*) were i and j represent the isospin and spin states. These are meta-stable states in general and decay into several pions.

B) There are some weak meta-stable bound states between a nucleon and nucleoneums which are something like compound nucleons and these induce the second and third resonances for π-nucleon collision.

The latter assumption indicates that each pair of the inner nucleons and antinucleons forms a well defined two-body cluster (i, j) in the possible bound states of $n+1$ nucleons and n antinucleons. Since (1, 1) nucleoneum is an iso-vector vector state, our description becomes essentially the same as Takeda's, if we identify (1, 1) as ρ-meson, and we also reach the same conclusion, viz. the second resonance should be $D_{3/2}$. As to the third resonance, we considered in the previous paper that this corresponds to pure S-wave 2 nucleoneums-nucleon bound state while it is a P-wave ρ-nucleon bound state in their description. Although one can conclude that the third resonance should be $F_{5/2}$ from both descriptions, the three-body character of the former complicates the problems considerably. Therefore, for simplicity, we may take the latter description in the following, i.e. the second and third resonances correspond to the $n=1$ case.

Though these similarities indicate that Takeda's theory is applicable directly to our model, there are some different points still. First, in our model, the decay interaction of (1, 1) state to the two-pion state is different from the interactions which produce nucleoneum-nucleon potential through pion exchange such as (1·2) and (1·3)

$$(i, j) \rightleftarrows (k, h) + \pi, \tag{1·2}$$

$$(i, j) \rightleftarrows (k, h) + 2\pi, \tag{1·3}$$

and this enables us to describe the phenomena as something like compound nucleus model on the condition that the decay interaction is weak enough. Secondly, ρ-meson is an ideal particle which is introduced as a means of calculation but (1, 1) state is an actual particle. This difference arises from the fact that Takeda's model is based on Frazer and Fulco's theory,[9] where they handled the electromagnetic form factor of nucleon, and their expression of the iso-vector weight function $g^{n}_{1,2}(t)$ becomes different, if we treat ρ-meson as an actual particle. We shall show in the following paper that our expression includes three additional parameters Z, m_1, m_2 due to the nature of actual particle and the special case, $Z=0$ and $m_1=m_2=\infty$, corresponds to theirs. But there is not any serious difference between these two descriptions as far as it is concerned with π-nucleon resonances which we shall treat here. There-

fore, we denote the (1, 1) state by ρ in this paper and do not distinguish these particles.

We remark that we cannot distinguish them directly whether this iso-vector vector meson is a new elementary particle or composite particle, even if it is an actual particle. Since our model suggests the possible existence of the other states, (0, 1), (1, 0) and (0, 0), we may indirectly examine the reality of our model by the detection of these particles. Perhaps, the analysis of nucleon-anti-nucleon annihilation is powerful for this purpose as we mentioned in the previous paper. We also examine the effect of the (0, 1) state on the iso-scalar part of the electromagnetic structure of nucleon for this purpose, because this state is a neutral vector meson and plays the same role for the iso-scalar part just as ρ-meson in the iso-vector part.

In the previous paper, our discussion was limited only to the qualitative feature as the calculation was based on non-relativistic theory and the quantitative results were hardly reliable. In the present paper, we may enter into more detailed discussion as we apply relativistic theory to our model.

Our fundamental expression with respect to π-nucleon resonances is the well-known Breit-Wigner's one level formula

$$\sigma^{el}=\frac{2J+1}{2}\frac{\pi}{k^2}\left|\frac{i\Gamma^{el}}{E-E_0+i\Gamma^{tot}/2}+A_{pot}\right|^2, \tag{1·4}$$

$$\sigma^{in}=\frac{2J+1}{2}\frac{\pi}{k^2}\frac{\Gamma^{el}\Gamma^{in}}{(E-E_0)^2+(\Gamma^{tot}/2)^2}, \tag{1·5}$$

$$\Gamma^{tot}=\Gamma^{el}+\Gamma^{in},$$

where k, E and J represent the momentum of incident pion, total energy and total angular momentum respectively. Therefore, our object is to calculate the elastic and inelastic widths $\Gamma^{el}/2$, $\Gamma^{in}/2$ and the π-nucleon potential A_{pot}. We must solve a kind of Bethe-Salpeter equation which includes some imaginary parts for this purpose. Although it is difficult to obtain the correct solution, we could prove that $\Gamma^{in}/2$ equals the natural decay width of ρ-meson except the trivial kinematic factor, if we neglect the rescattering correction, and $\Gamma^{el}/2$ sharply depends on the iso-spin T, the angular momentum J and the corresponding ρ-nucleon wave function $g_{TJ}(r)$. Using this strong correlation between $\Gamma^{el}/2$ and $g_{TJ}(r)$, we can draw some conclusions with respect to the properties of $g_{TJ}(r)$ from the experimental values of $\Gamma^{el}/2$. Though this analysis is not so much reliable on account of the rough approximation, the presumed radius of $g_{TJ}(r)$ is considerably large ($\sim 1\mu^{-1}$). Since this result indicates that the wave function $g_{TJ}(r)$ is related not only to the short-range part of ρ-nucleon potential but also to the relatively long-range part, it may be an interesting problem to calculate the long-range part using pion physics. We divide ρ-nucleon potential into the so-called three regions just as nucleon-nucleon case according to

Taketani.* In nucleon-nucleon case, there is not any question about the validity of the theoretical approach in the first region, but the ability of the pion theory to describe quantitatively the potential in the second region is still uncertain. But we assume in this paper that one and two pion exchange potentials give the main part of ρ-nucleon potential in the first and second regions. § 3 is devoted to this calculation. Of course, we should use only phenomenological approach concerning the third region. In the latter half of § 2, we shall discuss the property of the potential in the inner part of the third region using the experimental values of $\Gamma^{el}/2$.

Recently, Anderson et al.[10] reported the experimental results on the π-π cross section which they obtained by applying the Chew-Low extrapolation method[11] to 1.03 Bev/c π^--proton inelastic scattering. These results were consistent with the presence of ρ-meson. In § 4 we analyze the data 5 Bev π^--proton inelastic collision and examine the properties of ρ-meson in high energy phenomena.

§ 2. Calculation of Γ^{in} and Γ^{el}

As we remarked in the Introduction, ρ-nucleon meta-stable bound state is expressed by a function $F(q, k)$ which satisfies a kind of Bethe-Salpeter equation as follows.

$$F(q,k)=\frac{1}{q^2+m^2-\Sigma(-q^2)}\frac{-i\gamma(k-q)+M}{(q-k)^2+M^2}\,i\lambda\int d^4p\,V(q,p,k)F(p,k)\,,$$

$$V=V^R+V^I,\quad (m\,;\ \rho\text{-meson mass}) \tag{2·1}$$

where k and q represent total and relative momentum-enegy respectively (Fig. 1) Σ is the self-energy part of ρ-meson and this includes two pion decay imaginary part (Fig. 2), which corresponds to the natural decay of ρ-meson. V^R and V^I represent ρ-nucleon potential and are distinguished by whether they include the imaginary part or not. Fig. 3 shows a kind of V^R and Figs. 4, 5 and 6 are some typical V^I which include one or two pion production imaginary parts. In general, k is a complex vector and the imaginary part of k_0 is the total width $\Gamma^{tot}/2$.

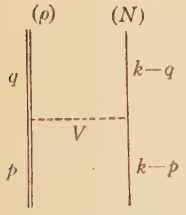

Fig. 1.

Fig. 2.

Before going into the details, we remark that in order to construct a meta-

* The conventional definition of the three regions is given as follows.
The first region; $1.5\mu^{-1}\leq r$.
The second region; $0.7\mu^{-1}\leq r\leq 1.5\mu^{-1}$.
The third region; $r\leq 0.7\mu^{-1}$.
For details, see Supplement of Prog. Theor. Phys, No. 3 (1956).

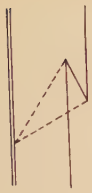

Fig. 3.

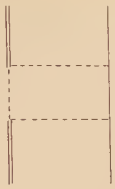

Fig. 4.

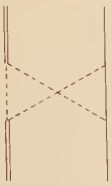

Fig. 5.

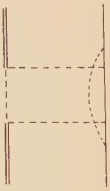

Fig. 6.

stable state by some pion exchange potential the decay interaction must not disturb the pion exchange. This condition is expressed as

$$\Gamma^{tot}/2 \ll |E^I - E|, \qquad (2\cdot2)^{12)}$$

where E and E^I represent the total energy and the energy of the intermediate state respectively. Since the effective region of V is the second or third region as will be shown soon, $|E^I-E|$ is larger than 2μ in general. Experimentally, $\Gamma^{tot}/2$ are estimated to be $\sim 0.4\mu$ for both resonances. Therefore, we may regard that this condition is satisfied.

Taking into account the smallness of the binding energy (Assumption B), we use here the following approximation.

i) The contribution from the pole of V is small as compared with that of the propagators.

First, we estimate $\Gamma^{in}/2$ on the following assumption:

ii) Diagram Fig. 7 makes the main contribution to $\Gamma^{in}/2$ and the other higher order diagrams such as Figs. 8, 9 are negligible, viz. the rescattering correction is small.

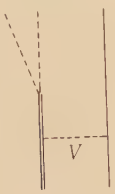

Fig. 7.

Fig. 8.

Fig. 9.

Then, (2·1) becomes

$$f(\boldsymbol{q}) = -\lambda\left(\frac{2\pi}{\boldsymbol{q}^2+m^2-(k_0-E_q)^2-\mathrm{Re}\,\bar{\Sigma}-i\,\mathrm{Im}\,\bar{\Sigma}}\Lambda_q^+\gamma_4+C(\boldsymbol{q})\right)\int d^3\boldsymbol{q}\,\bar{V}(\boldsymbol{q},\boldsymbol{p},k)f(\boldsymbol{q}), \qquad (2\cdot3)$$

where

$$f(\boldsymbol{q}) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dq_0 F(q,k),$$

$$\bar{V}(\boldsymbol{q},\boldsymbol{p},k) = V(q,p,k)\Big|_{\substack{p_0=-k_0+E_p \\ q_0=-k_0+E_q}}, \qquad \bar{\Sigma}(\boldsymbol{q}) = \Sigma(-q^2)_{q_0=-k_0+E_q},$$

$$C(\boldsymbol{q})=\int_{\omega}^{\infty}dq_0\frac{2\,\mathrm{Im}\,\Sigma}{(q^2+m^2-\mathrm{Re}\,\Sigma)^2+(\mathrm{Im}\,\Sigma)^2}\frac{-i\gamma(k-q)^2+M}{(k-q)^2+M^2},$$

$$E_q=\sqrt{\boldsymbol{q}^2+M^2},\quad \omega=\sqrt{\boldsymbol{q}^2+4\mu^2},\quad \boldsymbol{k}=0\ \text{(barycentric system)}$$

Λ_q^+ is the projection operator of nucleon to the positive energy states. The $\mathrm{Re}\Sigma$ is related to the renormalization procedure of ρ-meson but we will discuss this problem in part III and neglect this term in this paper. In this equation, only the first term includes the imaginary part by assumption ii). If we put

$$\mathrm{Im}\,\Sigma(\boldsymbol{q})=\Sigma^0+a\boldsymbol{q}^2,$$

$$k_0=m+M-B-i\Gamma^{in}/2,\quad (B\text{: binding energy})$$

$$E_q=M+\boldsymbol{q}^2/2M, \tag{2·4}$$

the denominator becomes

$$m(1/m+(1-B/m)/M-i(\Gamma^{in}/2M+a)/m)q^2+2mB-B^2+\Gamma^{in2}/4$$
$$+i((m-B)\Gamma^{in}-\Sigma^0). \tag{2·5}$$

Therefore, we get

$$\Gamma^{in}/2=\frac{\Sigma^0}{2(m-B)}, \tag{2·6}$$

on the condition

$$(\Gamma^{in}/2M)+a\sim 0. \tag{2·7}$$

This indicates that the lifetime of this meta-stable state coincides with that of ρ-meson itself except the kinematic factor.

Using effective interaction Hamiltonian between the pion field (notation Φ^α) and the ρ-field (notation χ_μ^α)

$$H_{\rho\pi\pi}=F\chi_\mu^{[\gamma}\partial_\mu\Phi^\alpha\Phi^{\beta]}, \tag{2·8}$$

(2·6) becomes

$$\Gamma^{in}/2=\frac{F^2}{4\pi}\frac{((m-B)^2-4\mu^2)^{3/2}}{24(m-B)^2}. \tag{2·9}$$

On the other hand, from the formula

$$\Gamma^{in}/2=\frac{\Gamma^{tot}}{2}\frac{\sigma^{in}}{\sigma^{tot}}, \tag{2·10}*$$

the experimental value is estimated to be

* We regarded the second and third resonances as one state resonance and neglected the non-resonant part of σ^{tot} and σ^{in}.

The possibility of being the coincidence of various resonating states will be discussed soon.

$$\Gamma^{in}/2=\begin{cases}0.16\sim0.21\,\mu & \text{(for the second resonance)}\\ 0.19\sim0.23\,\mu & \text{(for the third resonance)},\end{cases} \tag{2·11}$$

where σ^{tot} and σ^{in} represent total and inelastic cross sections in the resonant energy. From (2·9) and (2·11) we get

$$F^2/4\pi=\begin{cases}1.4\sim1.8 & \text{(for the second resonance)}\\ 1.1\sim1.3 & \text{(for the third resonance)},\end{cases} \tag{2·12}$$

while Takeda's theory gives $F^2/4\pi=\sim5$. But this poor agreement is not curious because, concerning electromagnetic form factor, our model gives different results with theirs as we remarked in the Introduction.

Using (2·12), we get

$$(\Gamma^{in}/2M)+a\leq0.27.$$

Since the corresponding real part in (2·5) is given by

$$1+\frac{m-B}{M}=\begin{cases}1.6 & \text{(for the second resonance)}\\ 1.8 & \text{(for the third resonance)},\end{cases}$$

we may consider that condition (2·7) is satisfied.

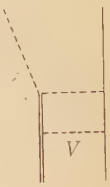

Fig. 10.

Next, we shall calculate $\Gamma^{el}/2$ which corresponds to one pion production diagram (Fig. 10) on the following assumption:

iii) With respect to $\Gamma^{el}/2$, perturbation theory gives good approximation.

The Feynman amplitude is given by

$$-\frac{1}{(2\pi)^4}\int d^4q\frac{1}{(q-h)^2+\mu^2}\Gamma_\pi\Gamma_N F(q,k), \tag{2·13}$$

where Γ_π and Γ_N represent ρ-π and nucleon-π vertexes respectively and h is the momentum-energy of outgoing pion. This formula is also written as

$$-\frac{1}{(2\pi)^4}\int_0^1 dx\int d^4q\,\frac{1}{(q-h)^2+\mu^2}\frac{1}{((q-xk)^2+J)^2}\Gamma_\pi\Gamma_N(-i\gamma(k-q)+M)\Gamma,$$
$$J=-x(1-x)k_0^2+(1-x)m^2+xM^2,$$
$$k_0=M+m-B,\quad \boldsymbol{k}=0,\quad \text{(c.m.s.)} \tag{2·14}$$

where Γ represents the vertex of ρ, nucleon and this ρ-nucleon composite particle. Since the minimum point $\bar{x}$ of J is given by

$$\bar{x}=\frac{k_0^2+m^2-M^2}{2k_0^2}, \tag{2·15}$$

we may localize the region of the integral within two fairly separated domains $q\sim\bar{x}k$ (notation D_1) and $q\sim h$ (notation D_2) in the approximation.* Therefore,

* $\bar{x}k_0-h_0\sim0$, $|\bar{x}\boldsymbol{k}-\boldsymbol{h}|\sim3\mu$.

in the case of S-wave bound state, we can evaluate (2·13) as follows:

$$-\frac{1}{(2\pi)^4}\left(\frac{1}{(q-h)^2+\mu^2}\Gamma_\pi\Gamma_N\right)_{q=\bar{x}k}\int_{D_1}d^4qF(q,k)$$
$$-\frac{1}{(2\pi)^4}\int_{D_2}d^4q\frac{1}{(q-h)^2+\mu^2}\Gamma_\pi\Gamma_NF(h,k). \qquad (2\cdot16)$$

As we mentioned in the previous paper, the effective part of the factor $\Gamma_\pi\Gamma_N/(q-h)^2+\mu^2$ is a kind of tensor force. So, the second term of (2·16) represents the contribution from the pole of a potential. Though the evaluation of $F(h,k)$ is difficult, we neglect this term in the following by assumption i). As to the evaluation of the first term, it is convenient to represent the wave function by ordinary three-dimensional space. For this purpose, we extend D_1 to infinite with respect to the energy variable q_0. Then, if we represent the domain D_1 by weight function and write the Fourier component of this weight function as $D(\boldsymbol{r})$, we get the following results:

$$\Gamma^{el,s}_{T,J}/2=\frac{\delta_T}{3}\frac{F^2}{4\pi}\frac{f^2}{4\pi}(\boldsymbol{h}^2/\bar{m}^2)^2\frac{2}{\mu^2}\frac{|\boldsymbol{h}|M}{k_0^2\bar{x}}4\pi|\bar{g}^s_{T,J}|^2, \qquad (2\cdot17)$$
$$\bar{g}^s_{T,J}=\int g^s_{T,J}(\boldsymbol{r})D(\boldsymbol{r})d^3\boldsymbol{r}, \quad \left(\int D(\boldsymbol{r})\,d^3\boldsymbol{r}=1\right)$$
$$\bar{m}=\sqrt{(\bar{x}k_0-\bar{h})^2+\mu^2},$$
$$\delta_{1/2}=4, \quad \delta_{3/2}=1,$$

for possible S-wave bound states (second resonance), where T and J represent the total iso-spin and the total angular momentum respectively and $g^s_{T,J}(r)$ represent normalized S-state wave functions of these bound states. f is pseudo-vector pion-nucleon coupling constant.

By the same calculation, we get

$$\Gamma^{el,p}_{T,J}/2=\delta_T\delta'_J(4/15)^2\frac{F^2}{4\pi}\frac{f^2}{4\pi}(\boldsymbol{h}^2/\bar{m}^2)^4\frac{M}{2|\boldsymbol{h}|k_0^2\bar{x}\mu^2}\frac{4\pi}{3}|\bar{g}^p_{T,J}|^2,$$
$$\bar{g}^p_{T,J}=\int\mathrm{div}\,\boldsymbol{g}^p_{T,J}(\boldsymbol{r})D(\boldsymbol{r})d^3\boldsymbol{r}, \qquad (2\cdot18)$$
$$\delta'_{1/2}=75/2, \quad \delta'_{3/2}=15/4, \quad \delta'_{5/2}=10,$$

for possible P-wave bound states (third resonance), where $\boldsymbol{g}^p_{T,J}(r)$ represent normalized P-state wave functions of these bound states. From the definition, the radius of $D(\boldsymbol{r})$ is expected to be smaller than that of $g^{(s,p)}_{T,J}(r)$. Therefore, $\bar{g}^{(s,p)}_{T,J}$ are related to the relatively inner part of $g^{(s,p)}_{T,J}(r)$.

Using the experimental values of $\Gamma^{el}/2$ and (2·12), (2·17), (2·18), we get

$$\frac{4\pi}{3}|\bar{g}^s_{1/2,3/2}|^2=0.4\sim0.7\mu^3,$$

$$\frac{4\pi}{3}|\bar{g}^{p}_{1/2,5/2}|^2=30\sim60\mu^5, \tag{2·19}$$

where we put $m=4.5\sim6.0\mu$ and assigned $T=1/2$, $J=3/2$ and $T=1/2$, $J=5/2$ states to the second and third resonances. If we substitute a simpler trial function such as of the exponential type or square-well type for $g^s_{1/2,3/2}(r)$ and $D(\boldsymbol{r})$ in (2·19), the radius of $g^s_{1/2,3/2}(r)$ is estimated to be $\sim1\mu^{-1}$. As to the $\bar{g}^p_{1/2,5/2}(r)$, the estimation of the radius is difficult because this depends on the radius of $D(\boldsymbol{r})$ more sharply. But the tentative value $\sim1\mu^{-1}$ is consistent also with (2·19). As we mentioned in the Introduction, these relatively large radii indicate that the effective regions of the potential V are not only the third region but also the second region. Therefore, the potential must be attractive in these regions and enough to produce the required binding energy. If we assume that the binding energy of the $T=1/2$, $J=5/2$ states is ~0 according to assumption B), that of the $T=1/2$, $J=3/2$ state must be $\sim1.3\mu$ and $m=\sim5.4\mu$. We shall prove in the following section that the two pion exchange potential is indeed attractive and strong enough in the second region but independent of the T and J. If this T and J independence holds also in the third region, the presence of such a bound state must be independent of T and J, viz. the resonances consist of various resonating states. We can examine the justice of this presumption by the evaluation of the ratios $|\bar{g}^s_{T,J}|^2/|\bar{g}^s_{1/2,3/2}|^2$ (notation $\theta^s_{T,J}$) and $|\bar{g}^p_{T,J}|^2/|\bar{g}^p_{1/2,5/2}|^2$ (notation $\theta^p_{T,J}$) for possible bound states using the experimental data, because they must be unit if this is the case.

The total cross section σ^{tot} (resonant part) and the ratio σ^{in}/σ^{el} (resonant part) at resonant energy are given by

$$\sigma^{tot(s,p)}_T=\sum_J 2\pi\lambda^2(2J+1)\frac{1}{1+\omega^{(s,p)}_{T,J}}, \tag{2·20}$$

$$\sigma^{in(s,p)}_T/\sigma^{el(s,p)}_T=\frac{\sum_J(2J+1)\dfrac{\omega^{(s,p)}_{T,J}}{(1+\omega^{(s,p)}_{T,J})^2}}{\sum_J(2J+1)\dfrac{1}{(1+\omega^{(s,p)}_{T,J})^2}}, \tag{2·21}$$

$$\omega^{(s,p)}_{T,J}=\Gamma^{in(s,p)}/\Gamma^{el(s,p)}_{T,J}=\frac{4\omega^{(s,p)}}{\delta_T\delta^{(s,p)}_J\theta^{(s,p)}_{T,J}},$$

$$\omega^s=\Gamma^{in,s}/\Gamma^{el,s}_{1/2,3/2},\quad \omega^p=\Gamma^{in,p}/\Gamma^{el,p}_{1/2,5/2},$$

$$\delta_J{}^s=1,\quad \delta_J{}^p=\delta_J'/10.$$

If we eliminate $\omega^{(s,p)}$ in these formulas, we get some conditions with respect to $\theta^{(s,p)}_{T,J}$.

i) $\theta^s_{1/2;1/2}$

In the case $\theta^s_{1/2,1/2}=1$, (2·21) becomes $\omega^s=\sigma^{in,s}_{1/2}/\sigma^{el,s}_{1/2}$. Using the experimental value $0.5\sim0.9$[3~6] and (2·20), we get $\sigma^{tot,s}_{1/2}=38\sim47$mb while the experimental value is ~32 mb. If we put $\theta^s_{1/2,1/2}=0$, we get a closer value $\sigma^{tot,s}_{1/2}=25\sim31$

mb. Therefore, our presumption might not hold in this state.

ii) $\theta^s_{3/2,\,3/2}$, $\theta^s_{3/2,\,1/2}$

Experimentally, any distinct $T=3/2$ peaks are not observed in this neighborhood. As the total cross section is estimated to be $\sim$18mb, the resonant part $\sigma^{tot,s}_{3/2}$ must be considerably smaller than 18 mb. By the assumption $\theta^s_{3/2,\,3/2}=\theta^s_{3/2,\,1/2}$, we get

$$\theta^s_{3/2,\,3/2}=\frac{8R\omega^s}{3(1+\omega^s)-2R},$$

$$R=\sigma^{tot,s}_{3/2}/\sigma^{tot,s}_{1/2,3/2}, \tag{2·22}$$

where $\sigma^{tot,s}_{1/2,\,3/2}$ represents the $J=3/2$ part of $\sigma^{tot,s}_{1/2}$. From this, we get $\theta^s_{3/2,\,3/2}\leqslant 0.7\sim 1.2$.

iii) $\theta^p_{1/2,\,3/2}$, $\theta^p_{1/2,\,1/2}$

It was already pointed out by Takeda that the third resonance might consist of two resonating states, $F_{5/2}$ and $P_{1/2}$. Indeed, using the tentative values $\theta^p_{1/2,\,3/2}=\theta^p_{1/2,\,1/2}=1$ and the experimental value $\sigma^{in,p}_{1/2}/\sigma^{el,p}_{1/2}=1.1\sim 1.7$, we get the following results:

$$\omega^p=1.3\sim 2.5,$$

$$\sigma^{tot,p}_{1/2}=26\sim 37\,\text{mb}.$$

This does not contradict with the experimental value $\sim$36 mb.

iv) $\theta^p_{3/2,\,5/2}$, $\theta^p_{3/2,\,3/2}$, $\theta^p_{3/2,\,1/2}$

If we put these quantities as unit, $\sigma^{tot,p}_{3/2}$ becomes $9\sim 14$ mb. Experimentally, the behavior of the total cross section in this energy region is consistent with the presence of this small peak.

From these results, we could conclude as follows:

a) In the case of S-wave, the ρ-nucleon potential should depend on the T and J considerably in the third region.

b) As to the P-wave case, it seems rather reasonable to assume the T and J independence.

Considering that the P-wave bound states are not so closely related to the inner part of the potential as S-wave bound state, we can explain these results by the extrapolation of the two pion exchange potential to the outer part of the third region and the suitable introduction of a T and J dependent core.

We remark that $D_{3/2}$ and $F_{5/2}$ states are, surely, the representative of the second and third resonances respectively in the case $\theta^{(s,\,p)}_{T,\,J}\sim 1$.

§ 3. ρ-nucleon potential

We shall use the following notation in this section.

$\chi_\mu{}^\alpha$: ρ-meson field.

χ_μ: (0, 1) field.

χ^α: (1, 0) field.

χ: (0, 0) field.
X: representation of these fields.
Φ^{α}: π-meson field.
m_{ij}: mass of (i, j).

Fig. 11. $n=2$

Fig. 12.

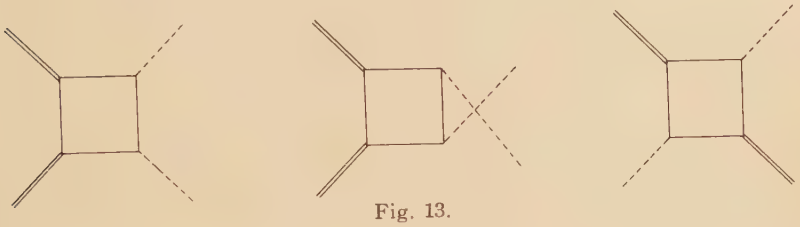

Fig. 13.

The first step of our discussion is to calculate the coupling constants of the interactions (I) $X\Phi^n$ (II) $XX\Phi$ (III) $XX\Phi\Phi$ which correspond to Fig. 11, Fig. 12, Fig. 13 respectively. Of course, ρ-π coupling (2·8) is a kind of (I). At first sight, it seems that there is not any essential difference between (I) and (II), (III). But in non-relativistic limit, it is easily shown that the coupling constants of (I) are proportional to $\psi_{ij}(0)$ where $\psi_{ij}(\boldsymbol{r})$ is the wave function of the inner nucleon and antinucleon of (i, j), while (II) and (III) are proportional to $\int \psi^{*}_{kh}(\boldsymbol{r})\psi_{ij}(\boldsymbol{r})\,d^3\boldsymbol{r}$ where (i, j) and (k, h) represent initial and final nucleoneums. Therefore, in the case $(i, j)=(k, h)$, the coupling constants of (II) and (III) are determined nearly independently of the shape of $\psi_{ij}(\boldsymbol{r})$ and, even if $(i, j)\neq(k, h)$, we can evaluate the upper limit, while (I) clearly has indefinite character. Probably, this is the case in the relativistic theory. The ρ-π coupling constant which is calculated from diagram Fig. 11 using suitable cutoff is fairly larger than the one deduced from the experiments but this is not curious because we should use only phenomenological speculation with respect to ρ-π interaction by the above stated reason. In the following, we restrict our calculation to (II) and (III) and assume that $\psi_{ij}(\boldsymbol{r})$ are independent of the i and j. Therefore, we may consider that the calculated coupling constants represent the upper limit in the case in which different nucleoneums take part.

Since the correct calculation is very troublesome, we regard all the four momentums of the external lines as zero for the sake of simplicity. We begin our detailed discussion by considering X-nucleon coupling. The main terms of these interactions are given by

$$\begin{aligned} &-iF_{11}\gamma_{\mu}\tau^{\alpha} \quad (\chi_{\mu}{}^{\alpha}\text{-nucleon}),\\ &-iF_{01}\gamma_{\mu} \quad (\chi_{\mu}\text{-nucleon}),\\ &-iF_{10}\gamma_{5}\tau^{\alpha} \quad (\chi^{\alpha}\text{-nucleon}),\\ &-iF_{00}\gamma_{5} \quad (\chi\text{-nucleon}), \end{aligned} \tag{3·1}$$

on account of the composite character of (i, j). By the normalization of the corresponding Bethe-Salpeter equation, we are able to evaluate the coupling con-

stants $F_{ij}^2/4\pi$ as follows:

$$F_{ij}^2/4\pi=\frac{2m_{ij}}{2M}\frac{\pi}{2}\frac{1}{D_{ij}},$$

$$D_{ij}=\frac{\omega(2a_{ij}^2+\omega^2-1)}{a_{ij}^2+\omega^2}+(1-2a_{ij}^2)\frac{\tan^{-1}\omega/a_{ij}}{a_{ij}},$$

$$a_{ij}=\sqrt{1-(m_{ij}/2M)^2},$$

$$\omega=\lambda/M, \qquad (3\cdot2)$$

where λ represents the space-like cutoff constant (Appendix).

Using (3·1) and pseudo-scalar π-nucleon coupling $-iG\gamma_5\tau^\alpha$, we get the following results:

(II)

$$\frac{iG_0}{M}\partial_{[\mu}\chi_\nu{}^\alpha\chi_\lambda\partial_{\sigma]}\Phi^\alpha,$$

$$G_1\chi_{[\mu}^\alpha\chi^\beta\partial_\mu\Phi^{\gamma]},$$

$$G_0=\frac{GF_{11}F_{01}}{\pi^2}B,\quad G_1=\frac{GF_{11}F_{10}}{\pi^2}C, \qquad (3\cdot3)$$

(III)

$$G_{11}\chi_\mu{}^\alpha\chi_\mu{}^\alpha\Phi^\beta\Phi^\beta+G_{11}'(\chi_\mu{}^\alpha\Phi^\alpha)(\chi_\mu{}^\beta\Phi^\beta),$$

$$G_{01}\chi_\mu\chi_\mu\Phi^\alpha\Phi^\alpha,$$

$$G_{10}\chi^\alpha\chi^\alpha\Phi^\beta\Phi^\beta+G_{10}'(\chi^\alpha\Phi^\alpha)^2,$$

$$G_{00}\chi\chi\Phi^\alpha\Phi^\alpha,$$

$$G_{11}=\frac{F_{11}^2G^2}{2\pi^2}A,\quad G_{11}'=-\frac{F_{11}^2G^2}{2\pi^2}C,\quad G_{01}=\frac{F_{01}^2G^2}{4\pi^2}B,$$

$$G_{10}=G_{10}'/2=\frac{F_{10}^2G^2}{4\pi^2}C,\quad G_{00}=\frac{3F_{00}^2G^2}{4\pi^2}C, \qquad (3\cdot4)$$

where the constants A, B, C are given by

$$A=\log(1+\bar\omega^2)-\frac{1}{2}\left(1-\frac{1}{1+\bar\omega^2}\right)-\frac{1}{4}\left(1-\frac{1}{(1+\bar\omega^2)^2}\right),$$

$$B=1-\frac{1}{1+\bar\omega^2}-\frac{1}{2}\left(1-\frac{1}{(1+\bar\omega^2)^2}\right),$$

$$C=\log(1+\bar\omega^2)-\left(1-\frac{1}{1+\bar\omega^2}\right),\quad \bar\omega=\bar\lambda/M. \qquad (3\cdot5)$$

$\bar\lambda$ is the relativistic cutoff constant. We consider in the following that $\bar\lambda$ is not so different from λ because both of them represent the radius of the form factor

of X-nucleon vertex.

On the other hand, these G_{ij} become $G^2 \cdot 2m_{ij}/2M$ (notation G_{ij}^N) in the non-relativistic limit. In Table I, we exihibit some values of A, B, C, D_{ij} and G_{ij}/G_{ij}^N.

Table I. Numerical values of A, B, C, D_{ij} and G_{ij}/G_{ij}^N

	A	B	C	$D_{ij}(a_{ij}=1)$	G_{11}/G_{11}^N	G_{01}/G_{01}^N	G_{10}/G_{10}^N	G_{00}/G_{00}^N
$\lambda=\bar{\lambda}=M$	0.256	0.125	0.1935	0.2146	1.19	0.29	0.45	1.35
$\lambda=\bar{\lambda}=2M$	0.969	0.320	0.809	0.893	1.09	0.18	0.45	1.36

From these results, we may consider that G_{ij} are nearly independent of the cutoff factor as we expected and, moreover, not so different from G_{ij}^N except the $(0, 1)$ case.

We now estimate one pion exchange (i, j)-nucleon potential V^1 from (3·3). By the same calculation as nucleon-nucleon case, we get the following results:

$$V^1(\boldsymbol{r}) = -i\frac{f^2}{4\pi}\frac{m_{11}}{M}\frac{2B}{\sqrt{D_{11}D_{01}}}(\boldsymbol{t}\cdot\boldsymbol{\tau})\left((\boldsymbol{\rho}\times\boldsymbol{\rho}')\cdot\boldsymbol{\sigma}\frac{e^{-\mu r}}{r}\right.$$
$$\left.+\frac{1}{3}(3((\boldsymbol{\rho}\times\boldsymbol{\rho}')\cdot\boldsymbol{n})(\boldsymbol{\sigma}\cdot\boldsymbol{n})-(\boldsymbol{\rho}\times\boldsymbol{\rho}')\cdot\boldsymbol{\sigma})\left(\frac{3}{\mu^2 r^2}+\frac{3}{\mu r}+1\right)\frac{e^{-\mu r}}{r}\right),$$
$$\boldsymbol{n}=\boldsymbol{r}/r, \quad \text{(for the process } (1, 1)\to(0, 1)) \tag{3·6}$$

$$V^1(r) = \frac{f^2}{4\pi}\frac{2C}{\sqrt{D_{11}D_{10}}}((\boldsymbol{t}\times\boldsymbol{t}')\cdot\boldsymbol{\tau})\left((\boldsymbol{\rho}\cdot\boldsymbol{\sigma})\frac{e^{-\mu r}}{r}\right.$$
$$\left.+\frac{1}{3}(3(\boldsymbol{\rho}\cdot\boldsymbol{n})(\boldsymbol{\sigma}\cdot\boldsymbol{n})-(\boldsymbol{\rho}\cdot\boldsymbol{\sigma}))\left(\frac{3}{\mu^2 r^2}+\frac{3}{\mu r}+1\right)\frac{e^{-\mu r}}{r}\right),$$
$$\text{(for the process } (1, 1)\to(1,0)) \tag{3·7}$$

where $\boldsymbol{\rho}$, $\boldsymbol{\rho}'$ are the unit polarization vectors of ρ and $(0, 1)$ and $\boldsymbol{t}$, $\boldsymbol{t}'$ are the unit iso-spin polarization vectors of ρ and $(1, 0)$ respectively. The second terms represent tensor force. Clearly, V^1 is non-diagonal with respect to the nucleoneum kind. As $2B/\sqrt{D_{11}D_{01}}\sim 1$ and $2C/\sqrt{D_{11}D_{10}}\sim 2$, the strengths of these potentials are of the same order with one pion exchange nucleon-nucleon potential in the case $m_{11}=m_{01}$ or $m_{11}=m_{10}$. If the mass difference $m_{kh}-m_{ij}$ are larger than the potentials, we can diagonalize the potential as follows:

$$\sim\sum_{kh\neq ij}\frac{-1}{m_{kh}-m_{ij}}V^{1*}\cdot V^1((i, j)\to(k, h)).$$

This indicates that the large mass difference gives small potential. Therefore, it may be difficult to produce required binding energy $\sim 1\mu$ for S-wave bound state by this potential.

Concerning the two pion exchange potential, we should calculate the diagrams Fig. 14, Fig. 15, Fig. 16 and Fig. 17. The first two diagrams become the

Fig. 14.

Fig. 15.

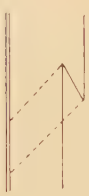
Fig. 16.

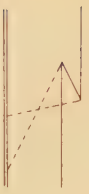
Fig. 17.

potentials V_4 and $V_{4,1}$ which were calculated by Lévy[13] and Klein[14] respectively except the numerical factor. Namely, the corresponding (i,j)-nucleon potential V^2 is given by

$$V^2=E_{ij}\left(V_4+\frac{1}{2}V_{4,1}\right),$$

$$V_4=-3\left(\frac{G^2}{4\pi}\right)^2\frac{\mu}{2M}\frac{1}{2Mr^2}\frac{2}{\pi}K_1(2\mu r),$$

$$V_{4,1}=3\left(\frac{G^2}{4\pi}\right)^2\left(\frac{\mu}{2M}\right)^2\frac{1}{2Mr^2}\left(1+\frac{1}{\mu r}\right)^2e^{-2\mu r},$$

$$E_{11}=\frac{2}{D_{11}}\left(A-\frac{C}{3}\right),\quad E_{01}=\frac{B}{D_{01}},$$

$$E_{10}=\frac{5}{3}\frac{C}{D_{10}},\qquad E_{00}=\frac{3C}{D_{00}}. \tag{3·8}$$

In nucleon-nucleon case, $V_{4,1}$ canceled the effect of V_4 almost completely but this cancelation is not so complete in our case, because we considered only S-wave $XX\Phi\Phi$ interaction. With respect to P-wave $XX\Phi\Phi$ interaction, the dia gram Fig. 14 becomes repulsive potential indeed but this contribution is only $\sim\mu/M$ as compared with V^2. Therefore, we can neglect this effect. In Fig. 19a, we have drawn the curve of ρ-nucleon potential when $\lambda=\bar{\lambda}=M$, $a_{11}=1$. Probably, the relatively strong attractive character in the second region and the outer part of the third region supports our description.

On the other hand, the calculation of the diagrams Fig. 16 and Fig. 17 are very troublesome just as nucleon-nucleon case. But it is easily seen that the contribution of these diagrams to ρ-nucleon potential becomes small, if $m_{01}\gg m_{11}$ and $m_{10}\gg m_{11}$. As we take into account the diagram Fig. 16 in the case $m_{01}\sim m_{11}$ or $m_{10}\sim m_{11}$, we should modify (3·8) as follows:

$$V^2=E_{11}\left(V_4+\frac{1+\lambda}{2}V_{4,1}\right),$$

$$\lambda=\begin{cases}\dfrac{2}{9E_{11}}\left(\dfrac{4B^2}{D_{11}D_{01}}\left(\dfrac{m_{11}}{M}\right)^2+\dfrac{4C^2}{D_{11}D_{10}}\right) & (m_{01}\sim m_{11},\ m_{10}\sim m_{11})\\ \dfrac{2}{9E_{11}}\dfrac{4B^2}{D_{11}D_{01}}\left(\dfrac{m_{11}}{M}\right)^2 & (m_{01}\sim m_{11},\ m_{10}\gg m_{11})\end{cases}$$

$$\frac{2}{9E_{11}}\frac{4C^2}{D_{11}D_{10}}. \qquad (m_{01}\gg m_{11},\ m_{10}\sim m_{11}) \qquad (3\cdot9)$$

Moreover, if we take into account the renormalization of $XX\Phi\Phi$ vertex, the E_{ij} becomes $E_{ij}Z_1^{-1}Z_2$ where Z_1 and Z_2 represent the renormalization constants of the vertex and X-propagator. We can easily see that $Z_1^{-1}Z_2>1$ by the calculation of the lowest order diagrams Figs. 18. Figs. 19b and c show the curves of these potentials when $\lambda=\bar{\lambda}=M$, $a_{ij}=1$, $m_{01}\gg m_{11}$, $m_{10}\gg m_{11}$, $Z_1^{-1}Z_2=2$ and $m_{01}=m_{11}=m_{11}Z_1^{-1}Z_2=1$ respectively, where we put $m_{11}=6\mu$. In our feeling, the $Z_1^{-1}Z_2=1$ case seems slightly weak to construct P-wave bound state suppressing the centrifugal potential. So, it may be a resonable result that $Z_1^{-1}Z_2>1$.

Fig. 18.

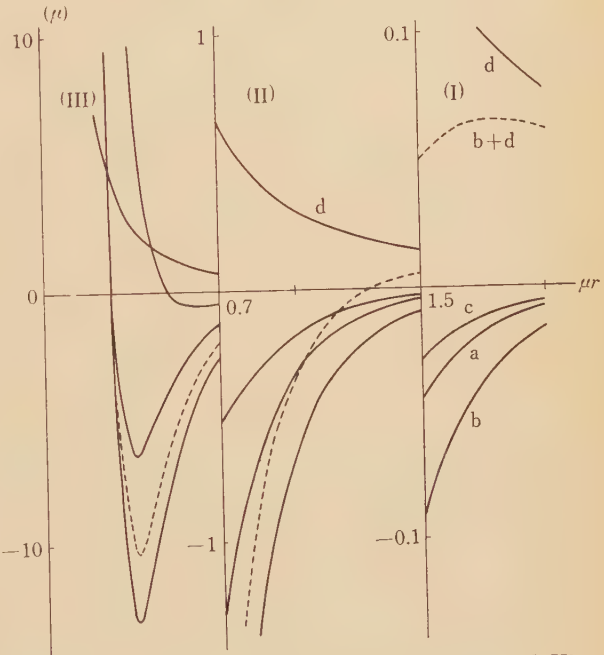

Fig. 19. Two pion exchange ρ-nucleon potential V^2
a: $m_{01}\gg m_{11}$, $m_{10}\gg m_{11}$, $Z_1^{-1}Z_2=1$.
b: $m_{01}\gg m_{11}$, $m_{10}\gg m_{11}$, $Z_1^{-1}Z_2=2$.
c: $m_{01}=m_{10}=m_{11}$, $Z_1^{-1}Z_2=1$.
d: centrifugal potential. (P-wave)

Concerning the case $m_{01}\ll m_{11}$ or $m_{10}\ll m_{11}$, the possible imaginary part which corresponds to the process $(1,1)\to(0,1)+\pi$ or $(1,1)\to(1,0)+\pi$ might disturb our description. Therefore, we shall omit this case in the following. We remark that, if we substitute the process $(1,1)\to\pi+\pi$ for these processes, our description becomes essentially the same as Takeda's.

Diagram Fig. 17 gives a complicated iso-spin and spin dependent potential; the complete calculation is difficult but we can easily show by comparing the coupling constant that this contribution is only $\sim(\mu/M)^2$ as compared with V^2.

We now summarize our results:

1) One pion exchange potential V^1 is negligibly small.
2) The main part of two pion exchange potential is attractive in the second

region and the outer part of the third region and, moreover, independent of the iso-spin and spin states.

3) $Z_1^{-1}Z_2 > 1$.

Of course, these conclusions are based on $PS(ps)$ π-nucleon coupling theory, and if not, the potential might not be so strong in the second region.

§ 4. Detection of ρ-meson

In this section, we shall analyse 5 Bev π-nucleon collision using our model.

Evidently, the simplest method of ρ-meson detection is to measure the momentum of the recoiled proton because this is a definite quantity in the process a) $\pi^- + p \to \rho^- + p$. Since ρ^- decays into π^- and π^0 directly, we observe this process as one pion production two prong event and the recoiled nucleon state is proton state. But one pion production events are only $\sim 1/3$ of total in two prong events. Therefore, it may be difficult to separate this process from total two prong proton events by the analysis of the momentum distribution only. However, with respect to the collision in the outer region, it is inferred that diagram Fig. 20 makes the main contribution on account of its long-range character. So, in the following, we shall focus our attention only on the small momentum transfer (forward) scattering. Fig. 21 and Fig. 22 show the momentum-angle distribution of the recoiled protons in the two and four, six prong events respectively.[15] In these data, most of the two prong events are $(p\pi^-\pi^0\pi^0)$ and $(p\pi^-\pi^0\pi^0\pi^0)$, and most of the four, six events are $(p\pi^-\pi^-\pi^+)$ and $(p\pi^-\pi^-\pi^+\pi^0)$. If we simply assume that there is not any essential difference between these processes as in statistical theory, the situation of the

Fig. 20.

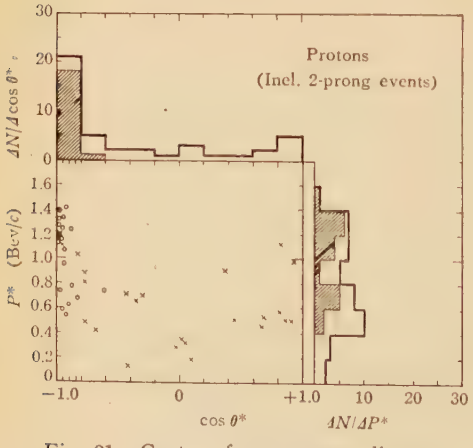

Fig. 21. Center-of-mass scatter diagram of protons from inelastic 2-prong events. For details, see reference 15).

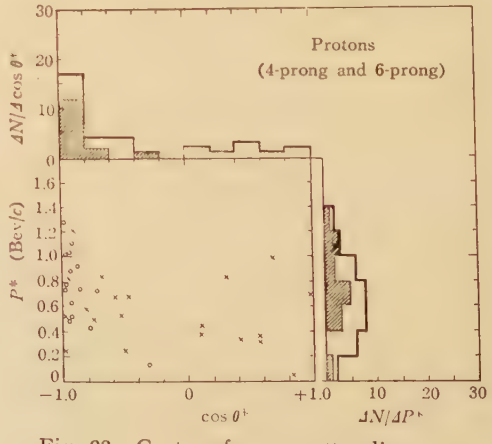

Fig. 22. Center-of-mass scatter diagram of protons from inelastic 4-prong and 6-prong events.

back ground of process a) in Fig. 21 may not be so different from Fig. 22. By this consideration, we can presume that process a) corresponds to an anomalous group A) in Fig. 21, where the range of A) is

$1.2 \leq p^* \leq 1.4$ Bev/c, (p^*: momentum at barycentric system)
$-1.0 \leq \cos\theta^* \leq -0.9$ (θ^*: scattering angle at barycentric system)

and contains 8 events.

It is reported that about 75% of the data were obtained in the 4.99 ± 0.33 Bev/c beam and the rest were obtained in the 4.49 ± 0.30 Bev/c beam. Table II

Table II. Momentum of recoiled proton p^* (Bev/c)
($\bar{p}$: momentum of the incident pion beam)

$\bar{p}$ \ $m(\mu)$	4	5	6	7
4.99	1.41±0.05	1.38±0.05	1.34±0.06	1.29±0.06
4.49	1.32±0.06	1.29±0.06	1.24±0.06	1.19±0.06

shows the relation between p^* and ρ-meson mass m. From this, we can conclude $m=5\sim6\mu$. Using π-ρ interaction (2·8), we get the following differential cross section

$$\sigma(\theta^*) = 4\frac{F^2}{4\pi}\frac{f^2}{4\pi}\frac{M^2}{T^2}\frac{|\boldsymbol{p}'|}{|\boldsymbol{p}|}\frac{1}{\mu^2}\frac{h^2}{(h^2+\mu^2)^2}\left(h^2+\frac{(h\cdot k')^2}{m^2}\right)|F_\rho(h^2)F_N(h^2)|^2,$$
$$h = k'-k = p-p', \qquad (4\cdot1)$$

where k, k' and p, p' represent the four momenta of incoming pion, outgoing ρ and nucleon at barycentric system respectively, and T is the total energy. F_ρ and F_N represent the form factors of the vertex π-ρ and π-nucleon respectivly. We can easily show that this formula is nearly independent of θ^* except the form factors. Since region (A) corresponds to the momentum transfer $|\boldsymbol{h}| \leq 4.6\mu$, process a) must damp in the core region. Considering that the electromagnetic form factor of nucleon damps in the neighbourhood $|\boldsymbol{h}| \sim 4\mu$, we may regard that the form factors F_ρ, F_N cause this damping. One is tempted to assume that (1, 0) determines the behavior of these form factors just as ρ-meson in the vector part of the electromagnetic structure of nucleon (Fig. 23). But the situation is not so simple because direct π-nucleon coupling is regarded not to be weak contrary to the γ-nucleon case.

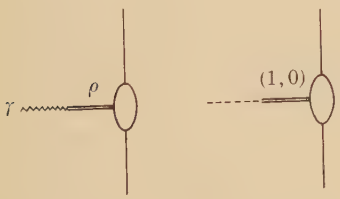

Fig. 23.

In order to estimate $F^2/4\pi$, we use square well form factor and write (4·1) as

$$\frac{F^2}{4\pi} = AT^2\frac{|\boldsymbol{p}|}{|\boldsymbol{p}'|}\frac{1}{a}\frac{d(a)}{\int_0^a f(x)\,dx},$$

$$f(x)=\frac{h^2}{(h^2+\mu^2)^2}\left(h^2+\frac{(h\cdot k')^2}{m^2}\right),$$

$$x=1+\cos\theta^*,$$

$$d(a)=(\text{number of events between } 0\leq x\leq a)/a,$$

$$A=0.925\times10^{-4}\mu^{-2},$$

a=radius of form factor. (4·2)

Using the value $a=0.07\sim0.10$, we get slightly larger value $F^2/4\pi=2.4\sim3.6$. But this may not contradict with our previous result considering that the number of the events is only 8 and the correct evaluation of $d(a)$ is very difficult.

The charge exchange process b) $\pi^-+p\rightarrow\rho^0+n$ has the same properties as a) but the cross section should be two times of case a). The experimental data (Fig. 24) do not contradict with this, taking into account the great uncertainty.

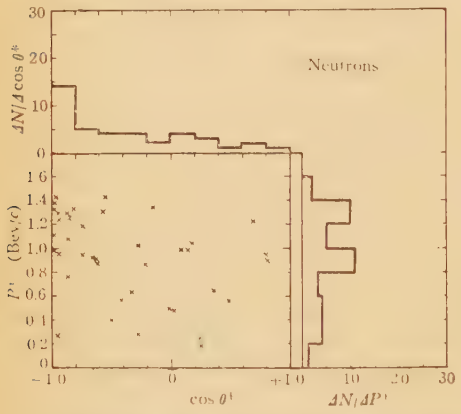

Fig. 24. Center-of-mass scatter diagram of neutrons from all inelastic events consistent with the emission of a single neutron.

Fig. 25.

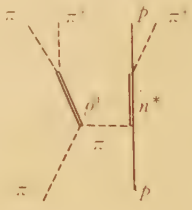

Fig. 26.

We can also investigate the rescattering effect arising from (3, 3) resonance (notation N^*) such as Fig. 25 and Fig. 26 from the data Fig. 21 and Fig. 22, because the recoiled protons resulted from these processes distribute uniformly on a broad region $0.67\lesssim p^*\lesssim1.28$ Bev/c of the forward scattering. This effect was studied by Drell[16] and Salzman.[17] But the estimated $R(N^*/N)$ (the ratio of the total excited nucleon state production to nucleon production) in our analysis is not so large as Salzman's result.

§ 5. Concluding remarks

The investigation results of ρ-meson mass and π-ρ coupling constant in § 4 are consistent with the results of the analysis of experiment in § 2. The analysis of the results of Anderson et al.[10] gave slightly smaller ρ-meson mass ($\sim4.7\mu$) than ours ($5\sim6\mu$), and this value does not contradict with the energy level $\sim5.1\mu$ of P-wave ρ-nucleon bound state (third resonance), if we take into

Fig. 27.

account the effect of the centrifugal barrier as suggested by Takeda. But in the case of such a small ρ-meson mass, one cannot predict the T and J independence of the P-wave bound states as the radii of the wave functions are rather small and the effective region of the potential may be the third region. And one may not disregard the rescattering effect and the three pion exchange tensor force (Fig. 27), in the calculation of $\Gamma^{in}/2$ and $\Gamma^{el}/2$ in such a short distance.

The smallness of ρ-π coupling constant was also a remarkable property of our model. This value is directly related sensitively to the problems of the electromagnetic structure of nucleon.

Our theoretical investigation of ρ-nucleon potential have given quite reasonable results, on the assumption of $PS(ps)$ π-nucleon coupling and that the main part of the potential is produced by the exchange of two S-wave pions. Usually, such an effect of S-wave pion is suppressed by the so-called pair suppression effect. In complete pair suppression case, V^2 becomes $\frac{1}{2}(1+\lambda)E_{ij}V_{4,1}$ and this is repulsive (see § 3). We can avoid this difficulty by putting suitable phenomenological attractive potential in the third region. But this situation involves some difficulties as in the above stated small ρ-meson mass case.

Though we have assumed that ρ-nucleon bound states correspond to the resonances of π-nucleon collision, one might also substitute the (0, 1) state for ρ-meson in this case. If we take the (0, 1) state, the possibility of the presence of $T=3/2$ bound states is excluded fortunately. But the analysis may not be so easy as in the ρ-meson case as (0, 1) interacts with three pions. So, in this paper we omitted this possibility tentatively. Of course, in our ρ-meson description (0, 1)-nucleon levels should be shielded, if they exist, by ρ-nucleon levels. As to the possible (1, 0)-nucleon or (0, 0)-nucleon levels, the effects may not be serious on account of the smaller angular momentum of the corresponding possible resonances.

Our calculation have essentially been based on Assumption B). This assumption enables us to reduce the systems of strongly interacting nucleons and antinucleons to the alternative systems of weakly interacting bosons (clusters) and nucleons. In the present paper, we have treated only the simplest cluster (i, j) but the possible existence of larger clusters might be an interesting problem. For example, we can explain the properties of the fourth resonance of π-nucleon scattering by the introduction of 2 nucleoneum cluster of the iso-spin 2. Our discussion have also limited in nucleoneum-nucleon system in this paper but one can expect that *nucleoneum-nucleus* system plays an important role in π-nucleus or nucleon-nucleus collision in high energy.

If a nucleoneum is produced in a nucleus by some collision, the nucleus may be rearranged around the nucleoneum by the attractive nucleoneum-nucleon force or the nucleoneum may be emitted from the nucleus accompanying some nucleons. The former is observed as the resonance of the elastic scattering by

nucleus and the latter is observed as the emission of excited nucleon N^* or excited nucleus d^*, He^*, etc. Since the main decay modes of these d^*, He^* etc., are inferred to be

$$d^* \to d+\pi+\cdots,$$
$$He^* \to He+\pi+\cdots,$$
$$\cdots\cdots,$$

one observes these processes as the emission of deuteron, He, etc., which accompany some pions. The detection of these phenomena is our future problem.

Recently, several authors[18] reported the presence of K^* particle. It may be an interesting fact that the estimated K^* mass $\sim 6\mu$ is very close to ρ-meson mass. If we assume that K^* is a composite particle of a Λ-particle and an antinucleon, K^* must have the same properties as those of ρ-meson, viz. a) the decay process $K^* \to K+\pi$ must be P-wave decay and b) there are some K^*-nucleon bound states which are observed as K^--p resonance in the energy region $4\sim 6\mu$.

We remark that our model could be translated into Sakata's model, if we identify the (1, 0) state with pion.

Acknowledgements

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Appendix

We summarize here the conventional method of the normalization of the Bethe-Salpeter equation.

In a nucleon and an antinucleon system, the Bethe-Salpeter amplitude is defined as

$$G^{ij}(x, y) = \langle 0 | (\psi(x)\bar{\psi}(y))_+ | {}^{ij}_{k} \rangle \sqrt{2k_0} N^{-1}, \qquad (A\cdot 1)$$

where k is the four momentum of (i, j) state and N is the normalization factor. As the function

$$\frac{N^2}{2k_0} \mathrm{spur} | G^{ij}(x, y) |^2_{x_0=y_0}$$

represents the density of the nucleon and antinucleon, we get

$$\int d^3x d^3y \, \frac{N^2}{2k_0} \mathrm{spur} | G^{ij}(x, y) |^2_{x_0=y_0} = 1. \qquad (A\cdot 2)$$

Using (i, j)-nucleon vertex $\Gamma^{ij}(p, p', k)$, we can write this G^{ij} as

$$G^{ij}(x,y)=\frac{-i}{(2\pi)^4}\int d^4p\,d^4p'\,e^{ipx-ip'y}\,\delta(p-p'-k)\frac{1}{i\gamma p+M}\Gamma^{ij}(p,p',k)\frac{1}{i\gamma p'+M}, \tag{A·3}$$

where p and p' represent the four vectors of the nucleon and antinucleon. From this, (A·2) becomes

$$\frac{1}{(2\pi)^3}\int d^3\boldsymbol{q}\ \mathrm{spur}|I^{ij}(\boldsymbol{q})|^2=2m_{ij},$$

$$I^{ij}(\boldsymbol{q})=\frac{1}{2\pi i}\int_{-\infty}^{\infty}dq_0\frac{1}{(i\gamma(k/2+q)+M)}\Gamma^{ij}\left(\frac{k}{2}+q,-\frac{k}{2}+q,k\right)\frac{1}{(i\gamma(-k/2+q)+M)},$$

$$k_0=m_{ij},\quad \boldsymbol{k}=0. \tag{A·4}$$

Neglecting the contribution from the pole of Γ^{ij} we get

$$\mathrm{spur}|I^{i0}|^2=8F_{i0}\frac{4E^2+k_0^2}{(4E^2-k_0^2)^2},$$

$$\mathrm{spur}|I^{i1}|^2=8F_{i1}\frac{4E^2+k_0^2}{(4E^2-k_0^2)^2}\left(1-\frac{q_r^2}{E^2}\right),$$

$$E=\sqrt{\boldsymbol{q}^2+M^2}, \tag{A·5}$$

where r represents the polarization of this vector states. If we disregard this small polarization term, we get

$$\frac{1}{(2\pi)^3}\int d^3\boldsymbol{q}\ \mathrm{spur}|I^{ij}|^2=\frac{F_{ij}^2}{\pi^2}M\left(\frac{\omega(2a_{ij}+\omega^2-1)}{a_{ij}^2+\omega^2}+(1-2a_{ij}^2)\frac{\tan^{-1}\omega/a_{ij}}{a_{ij}}\right). \tag{A·6}$$

From this and (A·4), we obtain (3·2).

Note added in proof : The recent experiments which were achieved by two groups in Brookhaven gave following results

ρ-meson mass$=\sim 5.5\,\mu$ $\quad F^2/4\pi\lesssim 4$, (by Walker et al.[19])

ρ-meson mass$=5\sim 5.5\,\mu$ $\quad F^2/4\pi=\sim 2$. (by Fowler et al.[20])

Both of them are quite consistent with our analysis.

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Renormalization of the Partition Function in the Relativistic Statistical Mechanics of Electron Gas

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A method of renormalization in calculating the grand partition function of the electron gas is presented. The renormalization is performed by relating the partition function to the finite time interval S-matrix. It is shown that the divergencies are eliminated by renormalizing the mass, the charge, and the wave function as in the S-matrix.

§ 1. Introduction

The quantum field theoretical method has been extensively applied in the statistical mechanics: In the nonrelativistic case T. Matsubara[1] has shown that the partition function can be calculated by means of the Feynmann diagram, by introducing the interaction representation. E. Fradkin[2] has shown that Schwinger's equation for the Green function of the statistical mechanics may also be used in this case.

If we want to treat the many-particle system, in which the particles are relativistic, and quantum effects have to be taken into account, we come across the ultra-violet divergencies which are familiar in the quantum field theory.

The renormalization prescription has been given by H. Ezawa, Y. Tomozawa and H. Umezawa[3] in connection with the theory of multiple particle production. In the perturbation expansion, the energy integrals are first carried out, then the Green function can be splitted into two parts, the external and the internal parts. The internal part of the Green function is nothing but the Feynman function which is familiar in the field theory. Correspondingly, the lines of the Feynman diagrams are splitted into external and internal ones. When the external lines are cut open, the diagrams become analogous to those in the scattering theory. With reference to these diagrams, the renormalization is performed as to the internal lines.

We wish to reformulate their prescription in two respects: First, not to rely upon the energy integration to split the lines, and second, to recover the parallelism with the usual renormalization theory. The first point is achieved by splitting the Green function from the outset (cf. Eqs. (AI·7) and (AI·8) in Appendix I). The Lorentz invariance of the internal Green functions, thus introduced, makes the second point possible.

In this paper, we take up an example of the relativistic electron gas. We introduce the interaction representation, and investigate the relation between the grand partition function and the S-matrix. We use the imaginary temperature here, so that the metric becomes Lorentz type. In two points, our problem is different from that of scattering theory: We have to deal with the finite time interval S-matrix on the one hand, and on the other we have to take the vacuum type diagram.

In the case of finite time interval also, E.C.G. Stueckelberg et al.[4] have shown that the divergencies can be managed in the same way as in the case of infinite time interval. Their proof, however, depends upon the assumption that the times, at which the interaction is switched on and off, are not so sharply defined. Corresponding to this assumption, we have to use a diffused boundary for the temperature interval $(0, i\beta)$ which restricts our argument to the case of very low temperature.

§ 2. General Formulation

The grand partition function Ξ is given by the following formula:

$$\begin{aligned}\Xi &= \mathrm{Tr}\exp\{-i\beta(H_0+H_I-\mu N)\},\\ &\equiv \mathrm{Tr}\{\rho\},\end{aligned} \tag{2·1}$$

where $i\beta=(1/kT)$, k is the Boltzmann constant and T is the temperature, μ is the chemical potential, H_0 and H_I are, respectively the free and the interaction Hamiltonians and N is the total number of electrons. The grand partition function of the ideal electron gas is denoted by

$$\begin{aligned}\Xi_0 &= \mathrm{Tr}\exp\{-i\beta(H_I-\mu N)\},\\ &\equiv \mathrm{Tr}\{\rho_0\}.\end{aligned} \tag{2·2}$$

Now we define the following quantity

$$S(\beta)=\rho_0^{-1}\rho, \tag{2·3}$$

which corresponds to the S-matrix.

Then the differential equation for $S(\beta)$ is

$$\frac{\partial S(\beta)}{\partial i\beta}=-\boldsymbol{H}_I(\beta)\,S(\beta), \tag{2·4}$$

where

$$\boldsymbol{H}(\beta)=\rho_0^{-1}H\rho_0. \tag{2·5}$$

Now, we introduce an imaginary temperature in order to recover the analogy with the S-matrix. In order to see whether $i\beta$ can be replaced by the imaginary temperature, let us examine the analytic property of Ξ. In terms of the density function $\rho(N, E)$, we can express Ξ as follows:

$$\Xi = \sum_N \lambda^N \int_0^\infty \rho(N,E)\, e^{-\beta E}\, dE, \tag{2·6}$$

where $\lambda = e^{\beta\mu}$. According to the theory of Laplace transform, if $\int_0^\infty \rho(N,E) \times e^{-\beta E} dE$ is convergent at a real point $\beta = \beta_0$, then it is convergent for any complex values of β which satisfies the condition $\mathrm{Re}(\beta) \geq \beta_0$. We may reasonably assume that $\Xi(\beta)$ is defined if β is real and positive. Then, Eq. (2·6) is regular as a function of β which satisfies the condition $\mathrm{Re}(\beta) > 0$. Similar arguments have been given by P. Martin and J. Schwinger.[5]

Now, we can replace β by the imaginary temperature

$$\beta = i\tau \tag{2·7}$$

$$\mathrm{Im}\,\tau < 0.$$

Then the equation for $S(\beta)$ turns out to be

$$i \frac{\partial S(\tau)}{\partial \tau} = \boldsymbol{H}_I(\tau)\, S(\tau), \tag{2·8}$$

where a convention is adopted to write $\boldsymbol{H}_I(\tau)$, say, for $\boldsymbol{H}_I(i\tau)$. This convention will not cause any confusion. Eq. (2·4) can be solved by iteration:

$$S(\tau) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^\tau dt_n \int d\boldsymbol{x}_n \cdots \int_0^\tau dt_1 \int d\boldsymbol{x}_1\, P(\boldsymbol{H}_I(x_n) \cdots \boldsymbol{H}_I(x_1)). \tag{2·9}$$

Since we are considering the problem of electron gas, the interaction Hamiltonian is

$$\boldsymbol{H}_I(x) = ie : \bar{\boldsymbol{\psi}}(x) \gamma_\mu \boldsymbol{\psi}(x) : \boldsymbol{A}_\mu(x), \tag{2·10}$$

where : : denotes the normal product in the field theory. The field operators used in Eq. (2·10) are

$$\boldsymbol{\psi}(x) = \rho_0^{-1}(t)\, \psi(\boldsymbol{x})\, \rho_0(t) = e^{i\mu t} \frac{1}{(2\pi)^{3/2}} \int d^3p \times \sum_r \{a_r(\boldsymbol{p})\, u_r(\boldsymbol{p})\, e^{ipx} + b_r^*(\boldsymbol{p})\, v_r(\boldsymbol{p})\, e^{-ipx}\} \tag{2·11}$$

and

$$\boldsymbol{A}_\mu(x) = \rho_0^{-1}(t)\, A_\mu(\boldsymbol{x})\, \rho_0(t) = \frac{1}{(2\pi)^{3/2}} \int d^3k \sqrt{\frac{1}{2\omega_k}} \{c_\mu(\boldsymbol{k})\, e^{ikx} + c_\mu^*(\boldsymbol{k})\, e^{-ikx}\}, \tag{2·12}$$

where $\epsilon_p = \sqrt{\boldsymbol{p}^2 + m^2}$, $\omega_k = |\boldsymbol{k}|$, and a_r, b_r are the electron and the positron annihilation operators, and c_μ is the photon annihilation operator.

The S-matrix given by Eq. (2·9) is that of finite time interval. In order to simplify the notation we introduce a boundary function $V(x)$

$$V(x)=\begin{cases}1\,;\ \delta\leqq \mathrm{Re}(t)\leqq \mathrm{Re}(\tau)-\delta \\ 0\,;\ \mathrm{Re}(\tau)\leqq \mathrm{Re}(t).\end{cases} \tag{2·13}$$

where δ is an arbitrary small positive number. This function is schematically shown in Fig. 1. The function $V(x)$ is assumed to be continuous at $t=0$, and τ. An example of V is given in Appendix II. We rewrite Eq. (2·19) by using Eqs. (2·10) and (2·13):

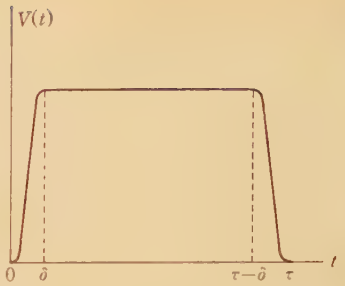

Fig. 1.

$$\begin{aligned}S(\tau)=\sum_{n=0}^{\infty} e^{n}\ \frac{1}{n!}\int\cdots\int dx_n\cdots dx_1 \\ \times P(:\bar{\psi}(x_n)\gamma_{\mu_n}\psi(x_n):\cdots:\bar{\psi}(x_1)\gamma_{\mu_1}\psi(x_1):) \\ \times P(A_{\mu_n}(x_n)\cdots A_{\mu_1}(x_1))\,V(x_n)\cdots V(x_1).\end{aligned} \tag{2·14}$$

Our aim is to calculate the following quantity:

$$\frac{\Xi}{\Xi_0}=\langle S(\beta)\rangle, \tag{2·15}$$

where $\langle A\rangle=\mathrm{Tr}(\rho A)/\mathrm{Tr}(\rho_0 A)$. To achieve this, we take two steps. In the first step, we decompose the time ordered products in Eq. (2·14) into the normal products which are defined in the field theory. The divergencies, which appear in the process of rearrangement, are eliminated by renormalizing the mass, the charge, and the wave function. We show details of the calculation in the next section. In the next step, we calculate the statistical expectation value of the S-matrix, that is to say, the expectation values of the normal products. For this purpose, we use the functions $D_\epsilon(x)$ and $S_\epsilon(x)$ which are the expectation values of the normal products which are defined in field theory

$$\langle:A_\mu(x)A_\nu(x'):\rangle=-i\delta_{\mu\nu}D_\epsilon(x-x') \tag{2·16}$$

$$\langle:\psi(x)\bar{\psi}(x'):\rangle=ie^{i\mu(t-t')}S_\epsilon(x-x'). \tag{2·17}$$

As is calculated in Appendix I, the Fourier representations of $D_\epsilon(x)$ and $S_\epsilon(x)$ are as follows:

$$D_\epsilon(x)=\frac{1}{(2\pi)^3}\int d^4k\, f_k\delta(k^2)\,e^{ikx} \tag{2·18}$$

$$\begin{aligned}S_\epsilon(x)=-\frac{1}{(2\pi)^3}(\gamma_\mu\partial_\mu-m)\int d^4p[f_p^{(-)}\theta(p_0) \\ +f_p^{(+)}\theta(-p_0)]\delta(p^2+m^2)\,e^{ipx},\end{aligned} \tag{2·19}$$

where

$$f_k=\frac{1}{\exp\{i\tau|k_0|\}-1}, \tag{2·20}$$

$$f_p^{(\pm)}=\frac{1}{\exp\{i\tau(|p_0|\pm\mu)\}+1}. \tag{2·21}$$

In the equation above, τ and t should satisfy the equation $|\mathrm{Im}\, t|<-\mathrm{Im}\,\tau$ so that the energy integration should converge. In this paper we assume that $\mathrm{Im}\, t=0$, and $\mathrm{Im}\,\tau=-\delta$, with a small positive number δ. Then the Fourier transforms of $D_e(x)$ and $S_e(x)$ tend to zero exponentially when the energy and the momentum become very large. We let δ tend to zero after the calculation is performed.

The formulae analogous to Wick's theorem hold for the expectation values of the normal products of field operators. These are proved in Appendix I.

$$\begin{aligned}&\langle : \boldsymbol{A}_\mu(x)\,\boldsymbol{A}_\nu(x')\,\boldsymbol{A}_\lambda(x'')\,\boldsymbol{A}_\sigma(x''') : \rangle\\ &\quad=-\delta_{\mu\nu}\delta_{\lambda\sigma}D_e(x-x')\,D_e(x''-x''')-\delta_{\mu\lambda}\delta_{\nu\sigma}D_e(x-x'')\,D_e(x'-x''')\\ &\quad-\delta_{\mu\sigma}\delta_{\nu\lambda}D_e(x-x''')\,D_e(x'-x'')\end{aligned} \tag{2·20}$$

$$\begin{aligned}&\langle : \boldsymbol{\psi}_\alpha(x)\,\bar{\boldsymbol{\psi}}_\beta(x')\,\boldsymbol{\psi}_\gamma(x'')\,\bar{\boldsymbol{\psi}}_\delta(x''') : \rangle\\ &\quad=-(S_e(x-x'))_{\alpha\beta}(S_e(x''-x'''))_{\gamma\delta}+(S_e(x''-x'))_{\gamma\beta}(S_e(x-x'''))_{\alpha\delta}\\ &\quad\times\exp\{i\mu(t-t'+t''-t''')\}.\end{aligned} \tag{2·21}$$

The same relations might hold for the expectation values of the arbitrary number of field operators.

Our process of calculation can be shown diagramatically: First we draw all diagrams of the S-matrix (Fig. 2) and then connect the external lines by D_e and S_e functions (Fig. 3).

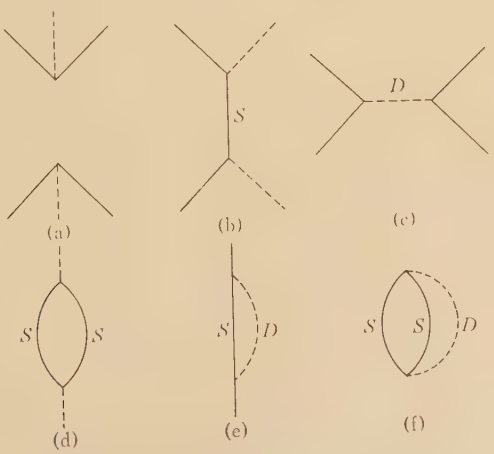

Fig. 2. The 2nd order S-matrix element

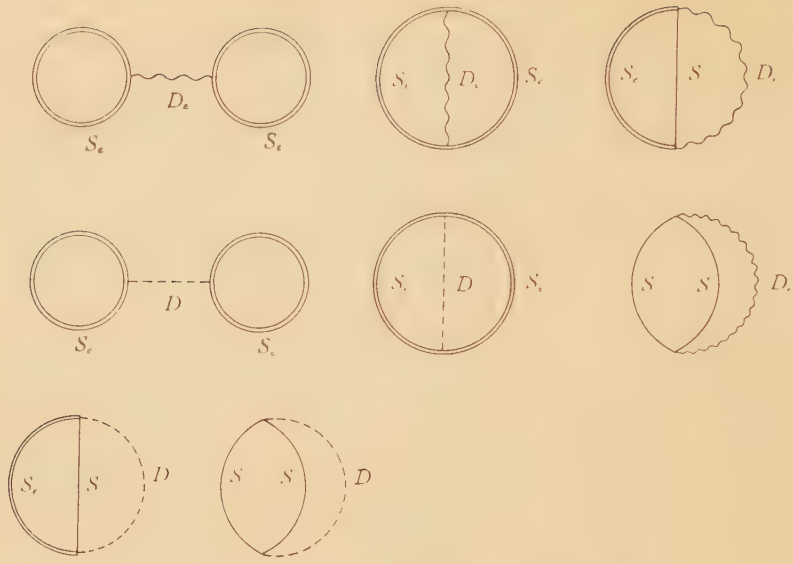

Fig. 3. The expectation value of S-matrix: $\langle S \rangle$.

§ 3. Calculation of S-matrix

In this section, we discuss the divergent terms which appear in calculating the finite time interval S-matrix.

Applying Wick's theorem, we obtain the 2nd order S-matrix:

$$
\begin{aligned}
S = {} & \frac{e^2}{2!}\iint P\{:\bar{\psi}(x)\gamma_\mu\psi(x):\cdot:\bar{\psi}(x')\gamma_\nu\psi(x'):\}\,P\{A_\mu(x)A_\nu(x')\}\,V(x)V(x')\,dx\,dx' \\
= {} & \frac{e^2}{2}\iint:[:\bar{\psi}(x)\gamma_\mu\psi(x):\cdot:\bar{\psi}(x')\gamma_\nu\psi(x'):+:A_\mu(x)A_\nu(x'):V(x)V(x')\,dx\,dx' \\
& + \frac{e^2}{2}\,i\iint:\bar{\psi}(x)\gamma_\mu S(x-x')\gamma_\nu\psi(x'):A_\mu(x)A_\nu(x'):e^{i\mu(t-t')}V(x)V(x')\,dx\,dx' \\
& + \frac{e^2}{2}\,i\iint:\bar{\psi}(x')\gamma_\nu S(x'-x)\gamma_\mu\psi(x):A_\mu(x)A_\nu(x'):e^{i\mu(t'-t)}V(x)V(x')\,dx\,dx' \\
& - \frac{e^2}{2}\,i\iint:[:\bar{\psi}(x)\gamma_\mu\psi(x):\bar{\psi}(x')\gamma_\nu\psi(x'):]:\partial_{\mu\nu}D(x-x')V(x)V(x')\,dx\,dx' \\
& + \frac{e^2}{2}\iint\mathrm{Tr}\{\gamma_\mu S(x-x')\gamma_\nu S(x'-x)\}:A_\mu(x)A_\nu(x'):V(x)V(x')\,dx\,dx' \\
& + e^2\iint:\bar{\psi}(x)\gamma_\mu S(x-x')\gamma_\nu\psi(x'):\partial_{\mu\nu}D(x-x')\,e^{i\mu(t-t')}V(x)V(x')\,dx\,dx' \\
& + \frac{e^2}{2}\,i\iint\mathrm{Tr}\{\gamma_\mu S(x-x')\gamma_\nu S(x'-x)\}\partial_{\mu\nu}D(x-x')V(x)V(x')\,dx\,dx', \qquad (3\cdot1)
\end{aligned}
$$

where $S(x)$ and $D(x)$ are Feynmann's causal functions of the electron and the photon respectively.

We consider only the fifth, sixth, and seventh terms, which correspond to the photon self energy part (Fig. 2 d), the electron self energy part (Fig. 2e), and the vacuum fluctuation part respectively (Fig. 2f). Other terms have nothing to do with ultra-violet divergencies. As for the vacuum fluctuation part, we shall discuss it in the next section.

The S-matrix corresponding to the electron self energy part is:

$$S_e=-i\iint:\bar{\psi}(x)\,\Sigma(x-x')\,\psi(x'):V(x)\,V(x')\,dx\,dx', \tag{3·2}$$

where

$$\Sigma(x-x')=ie^2\gamma_\mu S(x-x')\,\gamma_\mu D(x-x'). \tag{3·3}$$

$\Sigma(x)$ is the same function that appears in the scattering problem. The Fourier transform of Eq. (3·2) is given by,

$$S_e=-ie^2\iint:\bar{\psi}(p)\,\Sigma(p')\,\psi(p''):V(p-p')\,V(p'-p'')\,dp\,dp'\,dp'', \tag{3·4}$$

where

$$\Sigma(p)=\frac{ie^2}{(2\pi)^4}\int\gamma_\mu S(p-k)\,\gamma_\mu D(k)\,dk. \tag{3·5}$$

Now the divergent terms are separated in the following way:

$$\Sigma(p)=\delta m+B(i\not{p}+m)+(i\not{p}+m)^2\Sigma_f(p), \tag{3·6}$$

where δm and B are the mass and the wave function renormalization constants, and $\Sigma_f(p)$ is the finite part of $\Sigma(p)$. Thus, the finite term of the S-matrix is

$$(S_e)_f=-ie^2\iint:\bar{\psi}(p)\,(i\not{p}'+m)^2\Sigma_f(p')\,\psi(p''):V(p-p')\,V(p'-p'')\,dp\,dp'\,dp''. \tag{3·7}$$

The S-matrix which corresponds to the photon self energy part is

$$S_a=-\frac{i}{2}\iint\Pi_{\mu\nu}(x-x'):\boldsymbol{A}_\mu(x)\,\boldsymbol{A}_\nu(x'):V(x)\,V(x')\,dx\,dx', \tag{3·8}$$

where

$$\Pi_{\mu\nu}(x-x')=ie^2\,\mathrm{Tr}\{\gamma_\mu S(x-x')\,\gamma_\nu S(x'-x)\}. \tag{3·9}$$

The Fourier transform of Eq. (3·8) turns out to be

$$S_a=-\frac{i}{2}\iiint:\boldsymbol{A}_\mu(k)\,\Pi_{\mu\nu}(k')\,\boldsymbol{A}_\nu(k''):V(-k-k')\,V(k'-k'')\,dk\,dk'\,dk'', \tag{3·10}$$

where

$$\Pi_{\mu\nu}(k)=\frac{ie^2}{(2\pi)^4}\int\mathrm{Tr}\{\gamma_\mu S(p)\,\gamma_\nu S(p-k)\}\,dp. \tag{3·11}$$

We separate the divergencies from $\Pi_{\mu\nu}$, by taking account of the gauge invariance condition.

As shown in Appendix III, this condition is written as

$$k_\mu \Pi_{\mu\nu}(k) = k_\nu \Pi_{\mu\nu}(k) = 0. \tag{3·12}$$

Then, $\Pi_{\mu\nu}$ must have the following form:

$$\Pi_{\mu\nu}(k) = C\cdot(k_\mu k_\nu - k^2\delta_{\mu\nu}) + (k_\mu k_\nu - k^2\delta_{\mu\nu})\,\Pi_f(k), \tag{3·13}$$

where C is the charge renormalization constant, and $\Pi_f(k)$ is the finite part of $\Pi_{\mu\nu}(k)$. Then, the finite part of the S-matrix is

$$(S_d)_f = -\frac{1}{2}\int : \boldsymbol{A}_\mu(k)\,\boldsymbol{A}_\nu(k'') : (k_\mu' k_\nu' - k'^2\delta_{\mu\nu})\,\Pi_f(k') \times V(-k-k')\,V(k'-k'')\,dk\,dk'\,dk''. \tag{3·14}$$

The separation of divergencies from the finite time interval S-matrix can be performed also in the higher order perturbation series. These are eliminated by adding the counter term to the Lagrangean (we have ascertained this up to the fourth order perturbation series).

$$\begin{aligned}
\mathcal{L}^c = &-\{\delta m^{(2)} V^2(x) + \delta m^{(4)} V^4(x) + \cdots\}\cdot : \bar{\psi}(x)\,\psi(x) : \\
&-\frac{1}{2}\{B^{(2)} V^2(x) + B^{(4)} V^4(x) + \cdots\}\cdot : \left\{-\frac{\partial\bar{\psi}}{\partial x_\mu}\cdot\gamma_\mu\psi + \bar{\psi}\gamma_\mu\frac{\partial\psi}{\partial x_\mu} + 2m\bar{\psi}\psi\right\} : \\
&-\frac{1}{2}\{C^{(2)} + C^{(4)} V^2(x) + \cdots\}\cdot : \left\{\frac{\partial}{\partial x_\nu}(V\boldsymbol{A}_\mu)\frac{\partial}{\partial x_\mu}(V\boldsymbol{A}_\nu(x)) - \frac{\partial}{\partial x_\mu}(V\boldsymbol{A}_\nu)\frac{\partial}{\partial x_\mu}(V\boldsymbol{A}_\nu)\right\} : \\
&-ie\{L^{(3)} V^3(x) + \cdots\} : \bar{\psi}(x)\,\gamma_\mu\psi(x) : \boldsymbol{A}_\mu(x),
\end{aligned} \tag{3·15}$$

where $\delta m^{(2)}$, $\delta m^{(4)}$, $\cdots$, $B^{(2)}$, $B^{(4)}$, $\cdots$, $C^{(2)}$, $C^{(4)}$, $\cdots$, $L^{(3)}$ $\cdots$ are the mass, the electron wave function, the charge and the photon wave function renormalization constants respectively. These agree with those which appear in the scattering problem. The form of the counter term is the same as Bogoliugov and Shirkov have shown.[6]

§ 4. Vacuum type diagram

The divergencies of diagram (f) in Fig. 1 cannot be eliminated by renormalization. This diagram is analogous to the vacuum fluctuation diagram of the S-matrix. If energy is conserved at each vertex, it is nothing but the vacuum diagram. The S-matrix corresponding to this is

$$S_f = \frac{e^2}{2} i\iint \mathrm{Tr}\{\gamma_\mu S(x-x')\,\gamma_\mu S(x'-x)\}\,D(x-x')\,V(x)\,V(x')\,dx\,dx'. \tag{4·1}$$

By using the photon self energy $\Pi_{\mu\nu}$, we obtain

$$S_f=\frac{1}{2}\iint \Pi_{\mu\mu}(x-x')\,D(x-x')\,V(x)\,V(x')\,dx\,dx'. \qquad (4\cdot2)$$

The Fourier transform of Eq. (4·2) becomes

$$S_f=\frac{1}{2}\iint \Pi_{\mu\mu}(k)\,D(k')\,V(k+k')\,V(-k-k')\,dk\,dk'. \qquad (4\cdot3)$$

The divergent terms, which are separated from $\Pi_{\mu\nu}(k)$, cancel out with the statistical expectation value of the corresponding counter terms. Therefore, it is sufficient that we calculate the following quantity:

$$S_f=\frac{1}{2}\iint (-3k^2)\cdot\Pi_f(k)\,D(k')\,V(k+k')\,V(-k-k')\,dk\,dk'. \qquad (4\cdot4)$$

By using a property of the boundary function, $V(k)=\delta(\boldsymbol{k})\,V(k_0)$, we obtain the following formula:

$$S_f=\frac{V}{2(2\pi)^3}\int (-3k^2)\,\Pi_f(k)\cdot D(-\boldsymbol{k},\,q_0-k_0)\,V(q_0)\,V(-q_0)\,dq_0\,d^4k. \qquad (4\cdot5)$$

The integration over q_0 is performed by means of the cubic curve-boundary* which is given in Appendix II. After some calculations, the vacuum part is obtained to be

$$S_f=\frac{V\tau}{2(2\pi)^4}\int (-3k^2)\,\Pi_f(k)\,D(k)\,d^4k+\frac{V}{2(2\pi)^3}\int (-3k^2)\,\Pi_f(k)J_1(k,\tau)\,d^4k$$
$$+\frac{V}{2(2\pi)^3}\int (-3k^2)\,\Pi_f(k)J_2(k)\,d^4k. \qquad (4\cdot6)$$

The explicit forms of $J_1(k,\tau)$ and $J_2(k)$ are given in Appendix II. From Eq. (AII·5) we see that $J_1(k,\tau)$ tends to zero when the momentum becomes very large. Thus, the second term of Eq. (4·6) is the finite integral, but it diverges when the sharp boundary limit $\delta\to0$ is taken. The third term is a divergent integral, but it does not depend on the temperature τ.

Thus the contribution from the vacuum type diagram can be written in the form

$$\frac{\Xi_f}{\Xi_0}=\frac{V\tau}{2(2\pi)^4}\int (-3k^2)\,\Pi_f(k)\,D(k)\,d^4k+V\cdot f(\tau)+a\cdot V, \qquad (4\cdot7)$$

where $f(\tau)$ is convergent if $\delta\neq0$, and a is a divergent constant independent of τ.

---

* We calculated the integral (4·5) by using a linear boundary:

$$V(t)=\begin{cases} t/\delta & 0\leq t\leq\delta \\ 1 & \delta\leq t\leq\tau-\delta \\ (\tau-t)/\delta, & \tau-\delta\leq t\leq\tau. \end{cases}$$

In this case the second term of Eq. (4·6) is a divergent integral. This suggests that we have to use a boundary function whose first order derivative is continuous.

The contribution from the vacuum type diagrams can be factored out, then,

$$\Xi=\exp\left\{\frac{V\tau}{2(2\pi)^4}\int(-3k^2)\,\Pi_f(k)\,D(k)\,d^4k+a\cdot V+V\cdot f(\tau)\right\}\cdot\Xi_1. \quad (4\cdot8)$$

where Ξ_1 denotes the contribution of all diagrams but the vacuum type ones to the grand partition function, $f(\tau)$ is a surface dependent term.*

The energy of the system is given by

$$E=\frac{\partial}{\partial_i\beta}\log\Xi-E_1+\frac{V}{2(2\pi)^4}i\int(-3k^2)\,\Pi_f(k)\,D(k)\,d^4k+i\frac{\partial f}{\partial\tau}, \quad (4\cdot9)$$

where $\beta=i\tau$ and $E_1=\frac{\partial}{\partial_i\beta}\log\Xi_1$. The second term is divergent but is independent of the temperature. It only shifts the origin of the total energy of the system, which is not observed. What we are interested in is the specific heat. The third term, which is the surface effect, can be neglected if δ is large enough.

Now, we calculate the pressure of the system:

$$p=\frac{1}{\beta}\frac{1}{V}\log\Xi=p_1+\frac{i}{2(2\pi)^4}\int(-3k^2)\,\Pi_f(k)\,D(k)\,d^4k+\frac{1}{\beta}f(\tau), \quad (4\cdot10)$$

where $p_1=\frac{1}{\beta V}\log\Xi_1$. The second term gives the divergent constant pressure. But we can observe only pressure differences. The vacuum fluctuation part, therefore, has nothing to do with the observable effects.

In general, macroscopic energy-momentum tensor is defined as follows:

$$T_{\mu\nu}=T_n\{\rho\theta_{\mu\nu}\}-T_v\{\rho\theta_{\mu\nu}\}_{T=0},$$

where $\theta_{\mu\nu}$ is field theoretical energy-momentum tensor operator. This corresponds to subtract from energy and pressure obtained in Eqs. (4·9) and (4·10) the temperature independent terms, E_1 and p_1.

§ 5. Infrared divergencies

When we calculate the contribution of diagrams corresponding to Fig. 2, (a), (b), we have to deal with infrared divergencies. (b), (c) and (e) do not contain the divergency. To see the order of divergencies, we introduce the photon mass κ.

In the first place we calculate the diagram (a).

$$\langle S_a\rangle=\frac{e^2}{2}\iint\langle:\bar{\psi}(x)\gamma_\mu\psi(x):\rangle\langle:\bar{\psi}(x')\gamma_\nu\psi(x'):\rangle$$
$$\times\langle:A_\mu(x)\,A_\nu(x'):\rangle\,V(x)\,V(x')\,dx\,dx'$$

* This means that $f(\tau)$ depends on the shape of the boundary and diverges when $\delta\to0$, but can be neglected if τ and δ are sufficiently large, satisfying the condition $\tau\gg\delta$.

$$=i\frac{e^2}{2}\mathrm{Tr}\{\gamma_\mu S_e(0)\}\,\mathrm{Tr}\{\gamma_\mu S_e(0)\}\iint D_e(x-x')\,V(x)\,V(x')\,dx\,dx' \qquad (5\cdot1)$$

Now, the total number of electrons is given by

$$\boldsymbol{N}=i\int d\sigma_\mu :\bar{\psi}\gamma_\mu\psi: = i\int d^3x :\bar{\psi}\gamma_0\psi:. \qquad (5\cdot2)$$

The statistical expectation value of $\boldsymbol{N}$ is obtained to be

$$N=\langle\boldsymbol{N}\rangle = i\int d^3x\,\mathrm{Tr}\{\gamma_0 S_e(x)\}_{x=0} = iV\mathrm{Tr}\{\gamma_0 S_e(x)\}_{x=0}.$$

Therefore

$$\mathrm{Tr}\{\gamma_0 S_e(0)\} = -i\left(\frac{N}{V}\right). \qquad (5\cdot3)$$

Remembering $D_e(k)=f_k\delta(k^2+\kappa^2)$, we perform the integration of Eq. (5·1). Then we have

$$\langle S_a\rangle = \frac{e^2}{2}\left(\frac{N}{V}\right)^2 V\,\frac{1}{\kappa^3}\,(e^{-i\tau\kappa}-1). \qquad (5\cdot4)$$

Assuming that κ is very small, we expand Eq. (5·4) in the power series. Then,

$$\langle S_a\rangle = -\frac{e^2}{2}\left(\frac{N}{V}\right)^2 V\frac{i\tau}{\kappa^2}+\cdots. \qquad (5\cdot5)$$

In the same way, the diagram (d) is calculated, and we obtain

$$\langle S_d\rangle = -\frac{e^2}{2}\left(\frac{N}{V}\right)^2 V\left[\frac{1}{\kappa^3}(1-e^{-i\tau\cdot\kappa})+\frac{i\tau}{\kappa^2}\right] = -e^2\left(\frac{N}{V}\right)^2\frac{i\tau}{\kappa^2}+\cdots. \qquad (5\cdot6)$$

Therefore, the 2nd order perturbation series contains the infrared divergencies of the 2nd order.

Thus far, we have not assumed that the system is neutral. In the realistic problem, the system is neutral due to the positive ion sources. In order to take this into account, we subtract the term $\langle :\bar{\psi}\gamma_\mu\psi:\rangle\cdot A_\mu(x)$ from the interaction Hamiltonian:[7]

$$H_I' = -ie\int d^3x[:\bar{\psi}(x)\gamma_\mu\psi(x):-\langle :\bar{\psi}(x)\gamma_\mu\psi(x):\rangle]\boldsymbol{A}_\mu(x). \qquad (5\cdot7)$$

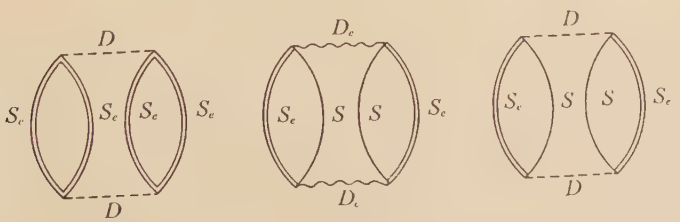

Fig. 4.

Replacing the Hamiltonian H_I by H_I', we can easily show that the diagrams (a) and (d) and the positive source terms cancel out.

As for the 4th order perturbation approximation infrared divergencies remain even if we assume the space charge to be zero. For example, the diagrams given in Fig. 4 are divergent for low frequencies.

We shall discuss the problem of infrared divergency in the near future,

§ 6. Discussion

As we have seen, the ultra-violet divergencies are eliminated by renormalizing the mass, the charge, and the wave function, and by shifting the origin of total energy of the system.

Now, we have to discuss about the diffused boundary which we have introduced in Appendix II. If we take the sharp boundary limit $\delta \to 0$, new divergencies arise from the vacuum and the diagrams of Fig. 2, (d), (e), which are not renormalizable. This is the situation that arises in the finite time interval S-matrix theory.[4] For example, we calculate the grand partition function corresponding to (e). The statistical expectation value of the S-matrix becomes

$$\langle S_d \rangle = -\frac{i}{2}\iint D_e(x-x')\,(\Pi_{\mu\mu}(x-x'))_f V(x)\,V(x')\,dx\,dx'. \qquad (6\cdot1)$$

The Fourier transform becomes

$$\langle S_d \rangle = -\frac{3}{2}\,iV\int D_e(k)k'^2\,\Pi_f(k')\,V(-k_0-k_0')\,V(k_0+k_0')\,\delta(\boldsymbol{k}+\boldsymbol{k}')\,dk_0'\,d\boldsymbol{k}. \qquad (6\cdot2)$$

From Eq. (AII·3) and the momentum dependence of $\Pi_f(k')$: $k'^2\Pi_f(k') \to k'^2 \log (k'^2/m^2)$ if $k'^2 \to \infty$, we find that Eq. (6·2) is the divergent integral in the sharp boundary limit $\delta \to 0$.

We take δ large enough so that the surface dependent terms, Eq. (6·1), can be neglected as compared with $\langle S_a \rangle$, $\langle S_b \rangle$ and $\langle S_c \rangle$: the statistical expectation values of (a), (b) and (c) in Fig. 2. Assuming that electron gas is degenerate: $\mu_f \beta \gg 1$, we obtain the condition for δ

$$\text{if} \quad \rho \ll m^3, \quad \beta \gg \delta \gg \beta^{-1}(m/\rho),$$

$$\text{if} \quad \rho \gg m^3, \quad \beta \gg \delta \gg \beta^{-1}\rho^{-2/3},$$

where ρ is the density of electron gas. Thus our theory is applicable in the case of low temperature or high temperature and high density.

We have to mention here that the surface dependent terms do not appear in the case of boson system. For example, if we consider the boson system interacting through $L' = g\phi^3$, the self energy part becomes, $k^2\Pi_f(k) \to \log (k^2/m^2)$ $(k^2 \to \infty)$. Therefore, Eq. (6·2) gives the finite result if the sharp boundary limit

is taken. In the same way, for the boson system interacting through $\lambda\phi^4$, which has been discussed by H. Ezawa et al.[3] no surface dependent terms appear.

Acknowledgements

The author would like to express his gratitude to Dr. H. Ezawa and Mr. O. Kamei, for helpful discussions.

Appendix I

Let us investigate the relation between the statistical expectation value of the T-product and the normal product of the field operators. The expectation values of the T-products are denoted by $\boldsymbol{D}(x-x')$, and $\boldsymbol{S}(x-x')e^{i\mu(t-t')}$:

$$\langle T(\boldsymbol{A}_\mu(x)\,\boldsymbol{A}_\nu(x'))\rangle = -i\delta_{\mu\nu}\boldsymbol{D}(x-x'), \tag{AI·1}$$

$$\langle T(\boldsymbol{\psi}(x)\,\bar{\boldsymbol{\psi}}(x'))\rangle = i\boldsymbol{S}(x-x')\,e^{i\mu(t-t')}, \tag{AI·2}$$

where $\boldsymbol{A}_\mu$ and $\boldsymbol{\psi}$ are defined in (2·11) and (2·12). In the interaction representation, $\boldsymbol{D}$ and $\boldsymbol{S}$ become

$$\boldsymbol{D}(x) = \frac{1}{(2\pi)^3} i\int \frac{d^3k}{2\omega_k}[f_k(e^{ikx}+e^{-ikx}) + e^{ikx}\theta(t) + e^{-ikx}\theta(-t)] \tag{AI·3}$$

where

$$f_k = \frac{1}{e^{i\omega_k\tau}-1},$$

$$\boldsymbol{S}(x) = \frac{1}{(2\pi)^3} i(\gamma_\mu\partial_\mu - m)\int \frac{d^3p}{2\epsilon_p}[-\{f_p^{(-)}e^{ipx} + f_p^{(+)}e^{-ipx}\} + e^{ipx}\theta(t) + e^{-ipx}\theta(-t)], \tag{AI·4}$$

where

$$f_p^{(\pm)} = \frac{1}{\exp\{i\tau(\sqrt{p^2+m^2}\pm\mu)\}+1}.$$

According to Wick's theorem, the T-product is written in terms of the normal product:

$$T(\boldsymbol{A}_\mu(x)\,\boldsymbol{A}_\nu(x')) = :\boldsymbol{A}_\mu(x)\,\boldsymbol{A}_\nu(x'): -i\delta_{\mu\nu}D(x-x'), \tag{AI·5}$$

$$T(\boldsymbol{\psi}(x)\,\bar{\boldsymbol{\psi}}(x')) = :\boldsymbol{\psi}(x)\,\bar{\boldsymbol{\psi}}(x'): +iS(x-x')\,e^{i\mu(t-t')}, \tag{AI·6}$$

where $D(x)$ and $S(x)$ are Feynmann's propagation function. Making use of the functions defined in Eqs. (2·16) and (2·17),

$$\langle :\boldsymbol{A}_\mu(x)\,\boldsymbol{A}_\nu(x):\rangle = -i\delta_{\mu\nu}D_e(x-x')$$

$$\langle :\boldsymbol{\psi}(x)\,\bar{\boldsymbol{\psi}}(x'):\rangle = ie^{i\mu(t-t')}S_e(x-x'),$$

we obtain the relations

$$\boldsymbol{D}(x)=D(x)+D_e(x) \tag{AI·7}$$

$$\boldsymbol{S}(x)=S(x)+S_e(x). \tag{AI·8}$$

In the statistical mechanics, the relation corresponding to Wick's theorem also holds.[8] This states

$$\begin{aligned}&\langle\!\langle T(\boldsymbol{A}_\mu(x)\,\boldsymbol{A}_\nu(x')\,\boldsymbol{A}_\lambda(x'')\,\boldsymbol{A}_\sigma(x'''))\rangle\!\rangle\\&\quad=-\delta_{\mu\tau}\delta_{\lambda\sigma}\boldsymbol{D}(x-x')\,\boldsymbol{D}(x''-x''')-\delta_{\mu\lambda}\delta_{\nu\sigma}\boldsymbol{D}(x-x'')\,\boldsymbol{D}(x'-x''')\\&\quad-\delta_{\mu\sigma}\delta_{\nu\lambda}\boldsymbol{D}(x-x''')\,\boldsymbol{D}(x'-x'').\end{aligned} \tag{AI·9}$$

$$\begin{aligned}&\langle\!\langle T\{\boldsymbol{\psi}_\alpha(x)\,\bar{\boldsymbol{\psi}}_\beta(x')\,\boldsymbol{\psi}_\gamma(x'')\,\bar{\boldsymbol{\psi}}_\delta(x''')\}\rangle\!\rangle\\&\quad=\{-\boldsymbol{S}_{\alpha\beta}(x-x')\,\boldsymbol{S}_{\gamma\delta}(x''-x''')+\boldsymbol{S}_{\gamma\beta}(x''-x')\,\boldsymbol{S}_{\alpha\delta}(x-x''')\}\\&\quad\times\exp\{i\mu(t-t'+t''-t''')\}.\end{aligned} \tag{AI·10}$$

The T-products are decomposed into the normal products:

$$\begin{aligned}&T(\boldsymbol{A}_\mu(x)\,\boldsymbol{A}_\nu(x')\,\boldsymbol{A}_\lambda(x'')\,\boldsymbol{A}_\sigma(x'''))\\&\quad=:\boldsymbol{A}_\mu(x)\,\boldsymbol{A}_\nu(x')\,\boldsymbol{A}_\lambda(x'')\,\boldsymbol{A}_\sigma(x'''):-i\delta_{\lambda\sigma}:\boldsymbol{A}_\mu(x)\,\boldsymbol{A}_\nu(x'):D(x''-x''')\\&\quad-i\delta_{\nu\sigma}:\boldsymbol{A}_\mu(x)\,\boldsymbol{A}_\lambda(x''):D(x'-x''')-i\delta_{\nu\lambda}:\boldsymbol{A}_\mu(x)\,\boldsymbol{A}_\nu(x'''):D(x'-x'')\\&\quad-i\delta_{\mu\sigma}:\boldsymbol{A}_\nu(x')\,\boldsymbol{A}_\lambda(x''):D(x-x''')-i\delta_{\mu\lambda}:\boldsymbol{A}_\nu(x')\,\boldsymbol{A}_\sigma(x'''):D(x-x'')\\&\quad-i\delta_{\mu\nu}:\boldsymbol{A}_\lambda(x'')\,\boldsymbol{A}_\sigma(x'''):D(x-x')-\delta_{\mu\nu}\delta_{\lambda\sigma}D(x-x')\,D(x''-x''')\\&\quad-\delta_{\mu\lambda}\delta_{\nu\sigma}D(x-x'')\,D(x'-x''')-\delta_{\mu\sigma}\delta_{\nu\lambda}D(x-x''')\,D(x'-x'').\end{aligned} \tag{AI·11}$$

$$\begin{aligned}&T(\boldsymbol{\psi}_\alpha(x)\,\bar{\boldsymbol{\psi}}_\beta(x')\,\boldsymbol{\psi}_\gamma(x'')\,\bar{\boldsymbol{\psi}}_\delta(x'''))=:\boldsymbol{\psi}_\alpha(x)\,\bar{\boldsymbol{\psi}}_\beta(x)\,\boldsymbol{\psi}_\gamma(x'')\,\bar{\boldsymbol{\psi}}_\delta(x'''):\\&\quad+:\boldsymbol{\psi}_\alpha(x)\,\bar{\boldsymbol{\psi}}_\beta(x'):ie^{i\mu(t''-t''')}S_{\gamma\delta}(x''-x''')\\&\quad-i:\boldsymbol{\psi}_\gamma(x'')\,\bar{\boldsymbol{\psi}}_\beta(x'):e^{i\mu(t-t''')}S_{\alpha\delta}(x-x''')\\&\quad+i:\boldsymbol{\psi}_\gamma(x'')\,\bar{\boldsymbol{\psi}}_\delta(x'''):e^{i\mu(t-t')}S_{\alpha\beta}(x-x')\\&\quad-i:\boldsymbol{\psi}_\alpha(x)\,\bar{\boldsymbol{\psi}}_\delta(x'''):e^{i\mu(t''-t''')}S_{\gamma\beta}(x''-x''')\\&\quad-e^{i\mu(t-t'+t''-t''')}S_{\alpha\beta}(x-x')\,S_{\gamma\delta}(x''-x''')\\&\quad+e^{i\mu(t''-t'+t-t''')}S_{\gamma\beta}(x''-x')\,S_{\alpha\delta}(x-x''').\end{aligned} \tag{AI·12}$$

Taking the statistical expectation values of Eqs. (AI·11) and (AI·12), and using Eqs. (AI·9) and (AI·10), we obtain the relations

$$\begin{aligned}&\langle\!\langle:\boldsymbol{A}_\mu(x)\,\boldsymbol{A}_\nu(x')\,\boldsymbol{A}_\lambda(x'')\,\boldsymbol{A}_\sigma(x''')\!:\rangle\!\rangle\\&\quad=-\delta_{\mu\lambda}\delta_{\nu\sigma}D_e(x-x')\,D_e(x''-x''')-\delta_{\mu\lambda}\delta_{\nu\sigma}D_e(x-x'')\,D_e(x'-x''')\\&\quad-\delta_{\mu\sigma}\delta_{\nu\lambda}D_e(x-x''')\,D_e(x'-x'')\end{aligned} \tag{AI·13}$$

$$\langle : \psi_\alpha(x)\,\bar{\psi}_\beta(x')\,\psi_\gamma(x'')\,\bar{\psi}_\delta(x''') : \rangle$$
$$= -(S_e(x-x'))_{\alpha\beta}(S_e(x''-x'''))_{\gamma\delta}\, e^{i\mu(t-t'+t''-t''')}$$
$$+(S_e(x''-x'))_{\gamma\beta}(S_e(x-x'''))_{\alpha\delta}\, e^{i\mu(t''-t'+t-t''')}. \qquad \text{(AI·14)}$$

Appendix II

By using a cubic curve, we examine the vacuum fluctuation diagram. The equation of the curve is

$$V(t)=\begin{cases} \dfrac{3}{\delta^2}t^2-\dfrac{2}{\delta^3}t^3, & 0\leq t\leq\delta \\ 1 & \delta\leq t\leq\tau-\delta \\ \dfrac{3}{\delta^2}(-t+\tau)^2-\dfrac{2}{\delta^3}(-t+\tau)^3, & \tau-\delta\leq t\leq\tau \\ 0 & \text{otherwise.} \end{cases} \qquad \text{(AII·1)}$$

This is shown in Fig. 1.

The Fourier transform of $V(t)$ is

$$V(p)=\int_{-\infty}^{\infty} V(t)\,e^{-ipt}\,dt=\frac{3}{\pi}\frac{i}{\delta^2 p^3}\{1-e^{-ip(\tau-\delta)}\}\left\{1+e^{-ip\delta}+\frac{2i}{\delta p}(1-e^{-ip\cdot\delta})\right\}. \qquad \text{(AII·2)}$$

If we take the sharp boundary limit $\delta\to 0$, then we have

$$V(p)=\frac{i}{2\pi}\frac{1}{p}(1-e^{ip\tau}). \qquad \text{(AII·3)}$$

The contribution of the vacuum type diagram to the S-matrix is given by Eq. (4·5):

$$S_{vac}=\frac{V}{2(2\pi)^3}\int(-3k^2)\,\Pi_f(k)\,D(-\boldsymbol{k},\,q_0-k_0)\,V(q_0)\,V(-q_0)\,d^4k\,dq_0. \qquad \text{(AII·4)}$$

To calculate S_{vac}, we first perform the q_0 integration:

$$J(k)=\int D(-\boldsymbol{k},\,q_0-k_0)\,V(q_0)\,V(-q_0)\,dq_0.$$

After somewhat tedius calculations, we obtain

$$J(k)=\frac{\beta}{2\pi}D(k)+\frac{9}{\pi}\frac{1}{\delta^4}i\frac{1}{|\boldsymbol{k}|}\left[\frac{1}{(k_0-|\boldsymbol{k}|)^6}\left\{2+2\exp(i\tau(k_0-|\boldsymbol{k}|))\right.\right.$$
$$\left.\left.-\exp(i(\tau-2\delta)(k_0-|\boldsymbol{k}|))-2\exp(i(\tau-\delta)(k_0-|\boldsymbol{k}|))-\exp(i\tau(k_0-|\boldsymbol{k}|))\right\}\right.$$

$$+\frac{4i}{\delta(k_0-|\boldsymbol{k}|)^7}\Big\{2\exp(i\delta(k_0-|\boldsymbol{k}|))+\exp(i(\tau-2\delta)(k_0-|\boldsymbol{k}|))$$
$$-\exp(i\tau(k_0-|\boldsymbol{k}|))\Big\}$$
$$+\frac{4}{\delta^2(k_0-|\boldsymbol{k}|)^8}\Big\{2-2\exp(i\delta(k_0-|\boldsymbol{k}|))+\exp(i(\tau-2\delta)(k_0-|\boldsymbol{k}|))$$
$$-2\exp(i(\tau-\delta)(k_0-|\boldsymbol{k}|))+\exp(i\tau(k_0-|\boldsymbol{k}|))\Big\}+(k_0\to-k_0)\Big]$$
$$+\frac{1}{2\pi}\frac{1}{\delta^4}\Big\{-\frac{2}{5}\frac{1}{\delta}\frac{\partial^6 D}{\partial k_0^6}+\delta\frac{\partial^4 D}{\partial k_0^4}+\frac{6}{5}\delta^3\frac{\partial^2 D}{\partial k_0^2}-\frac{44}{35}\delta^5 D\Big\}. \qquad \text{(AII·5)}$$

Thus we find out that $J(k)$ is expressed as a sum of three terms:

$$J(k)=\frac{\beta}{2\pi}D(k)+J_1(k,\tau)+J_2(k), \qquad \text{(AII·6)}$$

where $J_1(k,\tau)$ tends to zero like k^{-6} when $k\to\infty$, and $J_2(k)$ does not depend on the temperature. Thus we have

$$S_{vac}=-\frac{3V}{2(2\pi)^4}\beta\int k^2\,\Pi_f(k)\,D(k)\,d^4k$$
$$-\frac{3V}{2(2\pi)^3}\int k^2\,\Pi_f(k)\,J_1(k,\tau)\,d^4k-\frac{3V}{2(2\pi)^3}\int k^2\,\Pi_f(k)\,J_2(k)\,d^4k.$$

The second term is the surface effect, because it vanishes if the energy conserves. The third term is the divergent integral, but does not depend on the temperature.

Appendix III

It has been proved that the generalized Ward identity holds for the Green function of the statistical mechanics.[7] Here we give another proof of the Ward identity.

In our case, the Lagrangean is given by

$$\mathcal{L}=\bar{\psi}(\gamma_\mu\partial_\mu-i\gamma_0\mu-m)\psi-ie\,V(x)\,\bar{\psi}\gamma_\mu\psi\,\boldsymbol{A}_\mu+\frac{1}{4}\,(\partial_\mu\boldsymbol{A}_\nu-\partial_\mu\boldsymbol{A}_\nu)(\partial_\mu\boldsymbol{A}_\nu-\partial_\mu\boldsymbol{A}_\nu). \qquad \text{(AIII·1)}$$

This Lagrangean is invariant under the transformation

$$\psi(x)\to\psi(x)\,e^{ie\Lambda(x)V(x)}$$
$$\boldsymbol{A}_\mu(x)\to\boldsymbol{A}_\mu(x)+\partial_\mu\Lambda(x) \qquad \text{(AIII·2)}$$
$$\Lambda(x)=0 \quad \text{if} \quad t=0,\ \tau.$$

Therefore the S-matrix should be invariant under the transformation (AIII·2). The S-matrix corresponding to the photon self energy part is given by the following equation:

$$S=-\frac{1}{2}\int :\boldsymbol{A}_{\mu}(x)\,\Pi_{\mu\nu}(x,\,x')\,\boldsymbol{A}_{\nu}(x'):V(x)V(x')\,dxdx'. \qquad \text{(AIII·3)}$$

Eq. (AIII·2) requires that the following equations have to hold for $\Pi_{\mu\nu}(x,\,x')$:

$$\partial_{\mu}\Pi_{\mu\nu}(x,\,x')=\partial_{\nu}'\Pi_{\mu\nu}(x,\,x')=0. \qquad \text{(AIII·4)}$$

These relations are ascertained by direct computation, up to the fourth order in the coupling constant.

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On Vertical Representation of Bethe-Salpeter Amplitudes

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The Bethe-Salpeter equation for S-wave bound states of two scalar particles with equal mass interacting via scalar mesons is considered in the ladder approximation. It is shown that an integral representation previously attempted holds for all the solutions, normal and abnormal, of the B-S equation. Use is made of the fact that the B-S equation under consideration can be written in a form of an integral equation with a real symmetric and positive kernel.

Also given is a limitation on the lowest eigenvalue of such an integral kernel in terms of its first and second traces.

§ 1. Introduction

For the relativistic treatment of two-particle bound systems there are few examples in which we know the properties of wave functions in detail. Wick[1] and Cutkosky[2] were able to solve the problem of two scalar particles interacting through scalar photons in the ladder approximation. But the unexpected simplicity in this case is due to the extra degeneracy of eigenvalues characteristic of the Coulomb force. Although Wanders[3] obtained an integral equation, extending their approach to the case of mesons with finite mass, it is too complicate to be solved. Previously one of the present authors attempted to derive an integral representation of Bethe-Salpeter amplitudes from general principles,[4] which turned out to be essentially the same as what Wanders had assumed in his work. It has been pointed out that this so-called vertical representation does not give the most general functions satisfying the conditions stated.[5]* While Dyson's horizontal representation[6] is proved rigorously, it is not convenient for discussing the two-body problem. The vertical representation is now shown to be valid in every order of perturbation expansion.[7]

Of course perturbation theory as well as general principles gives us nothing about abnormal solutions (the solutions which appear in addition to normal ones when we consider a Bethe-Salpeter equation and have no corresponding solutions of a nonrelativistic Schrödinger equation), for they are believed to have no connection with eigenstates of a total Hamiltonian.[8] The purpose of this note is to show that the vertical representation holds for both the normal

* In this respect one of us (M. I.) is indebted to Dr. H. Araki and Dr. R. F. Streater for their kind communications.

and abnormal S-wave solutions of a B-S equation, at least in the ladder approximation. It seems impossible to separate normal solutions from abnormal ones without further study of their properties.

In § 2 a preliminary discussion is given. A model adopted here allows us to rewrite the B-S equation by an integral equation with a real symmetric kernel. Then the sum of inverses of eigenvalues is equal to the trace of the kernel. The kernel is also shown to be positive. Next in § 3, by assuming the above-mentioned integral representation for the B-S amplitudes we replace the B-S equation by an integral equation for spectral functions appearing in the representation. The question then arises whether all the original solutions are kept unmissed under this transformation. We answer to it in the affirmative by ascertaining the equality of the traces of a new integral kernel and the real symmetric and positive one studied in § 2, although we do not claim mathematical rigor for our proof. Finally in § 4 is given a limitation on the lowest eigenvalue of a real symmetric and positive kernel, which is the best possible in terms of its first and second traces. It is also applicable to an estimate of the lowest eigenvalue of the Schrödinger equation with a non-positive potential.

§ 2. Preliminary discussion

The Bethe-Salpeter amplitude for a compound system of two scalar particles is defined as

$$\tilde{\phi}(x, P)=\left\langle 0\left| T\left[\phi_1\left(\frac{x}{2}\right), \phi_2\left(-\frac{x}{2}\right)\right]\right| P\right\rangle, \tag{2·1}$$

where P denotes a four-momentum of the compound state $|P\rangle$ with rest mass M. Its Fourier transform $\phi(q, P)$ satisfies the B-S equation

$$\phi(q, P)=\Delta'_{F1}\left(\frac{P}{2}+q\right)\Delta'_{F2}\left(\frac{P}{2}-q\right)\int K(q, q', P; \lambda)\phi(q', P)d^4q', \tag{2·2}$$

where λ denotes a coupling constant and the relative energy has been continued to the imaginary axis, following Wick's procedure. q_4 and q'_4 are then real and $d^4q=d\boldsymbol{q}dq_4$. In the center-of-mass system $P_\mu=(\boldsymbol{0}, iM)$, which is regarded as an imaginary vector.

First we consider the limiting case $M\to 0$ ($P_\mu\to 0$), in which Eq. (2·2) is covariant under all rotations in the Euclidean four-dimensional q-space. Any solution can be written as a product of a function of q^2 and a four-dimensional spherical harmonics

$$H_{klm}(\alpha, \theta, \varphi)=N|\sin\alpha|^l C_{k-l}^{l+1}(\cos\alpha)\, Y_{lm}(\theta, \varphi), \tag{2·3}$$

where $k=l, l+1, \cdots$, $\cos\alpha=q_4/|q|$, N is a normalization constant and $C_\kappa^{l+1}(z)$ is a Gegenbauer function,[9] a polynomial of κ-th order. Eq. (2·2) now reduces

to an integral equation of one variable. If the latter has only discrete spectrum, then any eigenfunction has the form

$$\phi_{\kappa n l m}(q)=u_{n,l+\kappa}(q^2)H_{l+\kappa,l,m}(\alpha,\theta,\varphi), \tag{2·4}$$

with $\kappa=0, 1, 2, \cdots$ and $n=1, 2, \cdots$. The even-oddness of $\phi(q)$ with respect to q_4 is the same as that of κ. Eigenvalues λ are determined by n and $l+\kappa(=k)$, the degree of degeneracy being $(l+\kappa+1)^2$. For $M>0$ this degeneracy is destroyed partially and λ are determined by κ, n and l.

In order to discuss the case $M>0$, we consider a model of an S-wave system of two scalar particles interacting via scalar mesons with mass $\mu(>0)$. We confine ourselves to the ladder approximation and the case of two particles with equal mass. Eq. (2·2) can now be written as

$$[(q+i\eta)^2+m^2][(q-i\eta)^2+m^2]\phi(q)=\frac{\lambda m^2}{\pi^2}\int\frac{\phi(q')}{(q-q')^2+\mu^2}d^4q', \tag{2·5}$$

where $\eta=(\boldsymbol{0}, M/2)$ and λ is a dimensionless coupling constant. We assume the stability condition for meson, $\mu<2m$. Hereafter m will conveniently be taken to be unity, unless otherwise stated. Eq. (2·5) can easily be reduced to an integral equation with a real symmetric kernel,[1]

$$\Phi(q)=\lambda\int H(q, q')\Phi(q')d^4q', \tag{2·6}$$

where

$$\Phi(q)=\sqrt{f(q)}\,\phi(q),$$

$$f(q)=(q^2+\rho^2)^2+4\eta^2q_4^2, \quad \rho^2=1-\eta^2,$$

$$H(q, q')=\frac{1}{\pi^2}[f(q)f(q')]^{-1/2}\cdot[(q-q')^2+\mu^2]^{-1}.$$

Separating the S-wave part $\Phi_S(q)$ from $\Phi(q)$, we get an integral equation of two variables

$$u(|\boldsymbol{q}|, q_4)=\lambda\int_0^\infty d|\boldsymbol{q}'|\int_{-\infty}^{\infty}dq_4'K(|\boldsymbol{q}|, q_4; |\boldsymbol{q}'|, q_4')u(|\boldsymbol{q}'|, q_4'), \tag{2·7}$$

where

$$u(|\boldsymbol{q}|, q_4)=|\boldsymbol{q}|\Phi_S(q),$$

$$K(|\boldsymbol{q}|, q_4; |\boldsymbol{q}'|, q_4')=\frac{1}{\pi}[f(q)f(q')]^{-1/2}\log\frac{(|\boldsymbol{q}|+|\boldsymbol{q}'|)^2+(q_4-q_4')^2+\mu^2}{(|\boldsymbol{q}|-|\boldsymbol{q}'|)^2+(q_4-q_4')^2+\mu^2}.$$

This kernel has a real symmetric form, too. Since it is invariant for the replacement, $q_4\to-q_4$ and $q_4'\to-q_4'$, eigenfunctions of Eq. (2·7) can be classified by the even-oddness with respect to the relative energy. For symmetric or antisymmetric solutions K may be replaced by $K_+=(K+K')/2$ or $K_-=(K-K')/2$,

respectively, where $K'(|\boldsymbol{q}|, q_4; |\boldsymbol{q}'|, q_4') = K(|\boldsymbol{q}|, q_4; |\boldsymbol{q}'|, -q_4')$. We now use the equality between the trace of a symmetric kernel and the sum of inverses of its eigenvalues.[10] Then we get

$$\sigma_+ \equiv \sum_{even} \frac{1}{\lambda} = \mathrm{Tr}(K_+), \quad \sigma_- \equiv \sum_{odd} \frac{1}{\lambda} = \mathrm{Tr}(K_-). \tag{2·8}$$

By even or odd we mean that the sum is to be taken over the eigenvalues of symmetric or antisymmetric solutions. The traces of K and K' are given by

$$\mathrm{Tr}(K) = \frac{1}{\pi} \int_0^\infty d|\boldsymbol{q}| \int_{-\infty}^{\infty} dq_4 \{(|\boldsymbol{q}|^2 + q_4^2 + \mu^2)^2 + 4\eta^2 q_4^2\}^{-1} \log \frac{4|\boldsymbol{q}|^2 + \mu^2}{\mu^2}, \tag{2·9a}$$

$$\mathrm{Tr}(K') = \frac{1}{\pi} \int_0^\infty d|\boldsymbol{q}| \int_{-\infty}^{\infty} dq_4 \{(|\boldsymbol{q}|^2 + q_4^2 + \mu^2)^2 + 4\eta^2 q_4^2\}^{-1} \log \frac{4(|\boldsymbol{q}|^2 + q_4^2) + \mu^2}{4q_4^2 + \mu^2}. \tag{2·9b}$$

As $\log \frac{4|\boldsymbol{q}|^2 + \mu^2}{\mu^2} > \log \frac{4(|\boldsymbol{q}|^2 + q_4^2) + \mu^2}{4q_4^2 + \mu^2} > 0$ when $q_4 \neq 0$, it follows that $\sigma_+ > \sigma_- > 0$.

We first calculate $\sigma_\pm(\mu, \eta)$ for $\eta = 0$:

$$\sigma_+(\mu, 0) = \frac{1}{2(1-\mu^2/4)} \log \frac{2}{\mu},$$

$$\sigma_-(\mu, 0) = \sigma(\mu, 0) - \sigma_+(\mu, 0),$$

$$\sigma(\mu, 0) = \log \frac{1}{\mu} + \frac{\mu}{2} \frac{\cos^{-1} \mu/2}{\sqrt{1-\mu^2/4}}. \tag{2·10}$$

The last expression in (2·10) will be used in the Appendix. If we restrict $\phi(q)$ to functions of q^2 alone, we see by (2·3) and (2·2) that they correspond to solutions with $\kappa = 0$. We then obtain from Eq. (2·2) an integral equation with one variable of a real symmetric form,[1]

$$u(s) = \lambda \int_0^\infty K_0(s, s')\, u(s')\, ds', \tag{2·11}$$

where

$$u(s) = s^{1/2}(s+1)\phi(s), \quad s \equiv q^2,$$

$$K_0(s, s') = \{2(ss')^{1/2}(s+1)(s'+1)\}^{-1} \{s + s' + \mu^2 - [(s+s'+\mu^2)^2 - 4ss']^{1/2}\}.$$

If we denote by $\lambda_{\kappa n l}$ an eigenvalue with quantum numbers κ, n and l, then we get

$$\sigma_0(\mu, 0) \equiv \sum_n \lambda_{0n0}^{-1} = \mathrm{Tr}(K_0) = 1 - \mu \left\{ \frac{1-\mu^2/2}{\sqrt{1-\mu^2/4}} \cos^{-1} \frac{\mu}{2} - \mu \log \frac{1}{\mu} \right\}. \tag{2·12}$$

Considering the degeneracy discussed at the beginning of this section, we also obtain

$$\sigma_+(\mu, 0)-\sigma_0(\mu, 0)=\sum_{\substack{n\\ \kappa=2,4,\cdots}}\lambda_{\kappa n0}^{-1}=\sum_{\substack{n\\ l=2,4,\cdots}}\lambda_{0nl}^{-1},$$

$$\sigma_-(\mu, 0)=\sum_{\substack{n\\ \kappa=1,3,\cdots}}\lambda_{\kappa n0}^{-1}=\sum_{\substack{n\\ l=1,3,\cdots}}\lambda_{0nl}^{-1}. \tag{2·13}$$

We next calculate the traces of K and K' in the opposite case, $\eta=1$ (zero binding energy). After elementary integrations we have

$$\sigma(\mu, 1)=\log\frac{1}{\mu}+\frac{2}{\mu}\sqrt{1-\frac{\mu^2}{4}}\cos^{-1}\frac{\mu}{2},$$

$$\sigma'(\mu, 1)=\frac{1}{2}\sigma\left(\frac{\mu}{2}, 1\right)-\log\frac{1}{\mu}+1$$

$$-\frac{\mu}{2}\int_0^\infty\frac{t\,dt}{\cosh t}-\frac{\mu}{2}\int_{\mu/2}^1\frac{dz}{z}\frac{\cos^{-1}z}{\sqrt{1-z^2}}. \tag{2·14}$$

In order to make clear the connection with the non-relativistic limit, we replace μ by μ/m. For $\mu/m\ll 1$, $\sigma_\pm(\mu, 1)$ can be expressed as

$$\sigma_+(\mu, 1)=\pi\frac{m}{\mu}+\frac{1}{4}\log\frac{m}{\mu}-\frac{1}{4}(1-\log 2)-\frac{\pi}{8}\frac{\mu}{m}\log\frac{m}{\mu}+O\left(\frac{\mu}{m}\right), \tag{2·15a}$$

$$\sigma_-(\mu, 1)=\frac{3}{4}\log\frac{m}{\mu}-\frac{1}{4}(3+\log 2)+\frac{\pi}{8}\frac{\mu}{m}\log\frac{m}{\mu}+O\left(\frac{\mu}{m}\right). \tag{2·15b}$$

It is to be noted that logarithmic terms appear in $\sigma_\pm$, preventing their power series expansion in μ/m. As is known, in the nonrelativistic limit Eq. (2·2) reduces to a Schrödinger equation with the Yukawa potential $\lambda V(r)=-\lambda\pi e^{-\mu r}/r$. The sum of inverses of the nonrelativistic S-wave eigenvalues at zero binding energy is given by[11]

$$\sigma_{NR}(\mu, 1)\equiv\sum\lambda_{NR}^{-1}=m\int_0^\infty|V(r)|r\,dr, \tag{2·16}$$

and is equal to $\pi m/\mu$, which coincides with the first term in Eq. (2·15a). It is easy to see that

$$\lim_{\mu/m\to 0}\frac{\sigma_+(\mu, 1)}{\sigma_{NR}(\mu, 1)}=1,\quad \lim_{\mu/m\to 0}\frac{\sigma_-(\mu, 1)}{\sigma_{NR}(\mu, 1)}=0. \tag{2·17}$$

Finally we remark that the kernels $K(|\boldsymbol{q}|, q_4; |\boldsymbol{q}'|, q_4')$ and $K_0(s, s')$ are both positive, i.e. all the eigenvalues are positive. It is sufficient to show that $H(q, q')$ is such a kernel, because the spectrum of K or K_0 is a subset of that of H. As the kernel $H(q, q')$ is positive for $\mu=0$,[2] and all the eigenvalues of Eq. (2·5) are monotonously increasing functions of μ,[12] it is *a fortiori* positive for $\mu>0$.

By this property of the kernel K we have the inequalities

$$\underset{\kappa:\,even}{\mathrm{Min}}\,\lambda_{\kappa no}\geq\sigma_+(\mu,\eta)^{-1},\quad \underset{\kappa:\,odd}{\mathrm{Min}}\,\lambda_{\kappa no}\geq\sigma_-(\mu,\eta)^{-1}. \tag{2·18}$$

Particularly at $\eta=0$ we get

$$\mathrm{Min}\,\lambda_{ono}\geq\sigma_0(\mu,0)^{-1},\quad \underset{l:\,1,3,\cdots}{\mathrm{Min}}\,\lambda_{onl}\geq\sigma_-(\mu,0)^{-1}, \tag{2·19}$$

$$\underset{l:\,2,4,\cdots}{\mathrm{Min}}\,\lambda_{onl}\geq\{\sigma_+(\mu,0)-\sigma_0(\mu,0)\}^{-1}.$$

§ 3. Integral representation

We first assume that there exist S-wave solutions of the B-S equation,

$$\left\{\left(q-\frac{P}{2}\right)^2+1\right\}\left\{\left(q+\frac{P}{2}\right)^2+1\right\}\phi(q,P)=\frac{\lambda}{\pi^2}\int d^4q'\frac{\phi(q',P)}{(q-q')^2+\mu^2}, \tag{3·1}$$

which can be represented uniquely as

$$\phi(q,P)=\int_{-1}^{1}dz\int_0^\infty dt\,\frac{g(z,t)}{[q^2-zq\cdot P+t+\rho^2]^2}. \tag{3·2}$$

(3·2) is simply related to the vertical representation introduced previously:[4]

$$\phi(q,P)=\int_{-1}^{1}dz\int_0^\infty ds\,\frac{f(z,s)}{[(q-z\cdot P/2)^2+s]^2}. \tag{3·3}$$

Eq. (3·2) can be transformed into an integral equation for spectral functions $g(z,t)$ by Feynman's method of parametric integration:[3]

$$g(z,t)=\lambda\int_{-1}^{1}dz'\int_0^\infty dt'\,R(z,t;z',t')\,g(z',t'), \tag{3·4}$$

where

$$R(z,t;z',t')=\frac{1}{2}\int_0^1\frac{dx}{xu(z',t';x)}\left\{\delta(t)-\delta[t-R(z,z')\,u(z',t';x)]\right\}, \tag{3·5a}$$

$$u(z,t;x)=\frac{\mu^2}{1-x}+\frac{t+\rho^2+\eta^2z^2}{x}-(\rho^2+\eta^2z^2), \tag{3·5b}$$

$$R(z,z')=\frac{1\mp z}{1\mp z'},\quad \text{for } z\gtrless z'. \tag{3·5c}$$

From (3·4) and (3·5a) we obtain that

$$\int_0^\infty g(z,t)\,dt=0, \tag{3·6}$$

which means that $\tilde{\phi}(x, P)$ is non-singular at $x=0$ in the Euclidean x-space.

Some S-wave solutions of Eq. (3·1) might be missed in Eq. (3·4), because it is not evident whether all the S-wave solutions of Eq. (3·1) can be represented in the form Eq.(3·2). Let all the solutions of Eq. (3·4) correspond to a subset of solutions of Eq. (2·7). If P denotes the projection operator on the subspace spanned by this subset, then R and PKP are to be connected with each other by a non-singular transformation.* The traces of the two kernels are then equal, since traces are kept invariant under similarity transformations. As K is a symmetric positive kernel,

$$\mathrm{Tr}(PKP)=\mathrm{Tr}(K) \tag{3·7}$$

if and only if $P=1$. Then, by comparison of the first traces of R and K, we can determine whether R is equivalent to K or not. (In general, when K is not a positive kernel, we have to replace the first traces by the second traces in the above argument.) It is shown in the Appendix that the trace of R expressed as

$$\mathrm{Tr}(R)=\frac{1}{2}\int_{-1}^{1}dz\int_{0}^{1}\frac{dx}{xu(z,0;x)}=\int_{0}^{1}dx\int_{0}^{1}dz\,\frac{1-x}{x\mu^2+(1-x)^2(\mu^2+\eta^2z^2)} \tag{3·8}$$

is just equal to $\mathrm{Tr}(K)$ given by Eq. (2·9a). Thus we conclude that every S-wave solution of the B-S equation (3·1) can be represented in the form Eq.(3·2).

It is now evident that $\mathrm{Tr}(R')$ given in the form

$$\mathrm{Tr}(R')=\int_{0}^{1}dx\int_{0}^{1}dz\,\frac{(1-x)\,\theta[R(z,-z)-x]}{x\mu^2+(1-x)^2(\mu^2+\eta^2z^2)} \tag{3·9}$$

is equal to $\mathrm{Tr}(K')$, where $R'(z,t,z',t')=R(z,t,-z',t')$. By the way, at $\eta=0$ Eq. (3·4) can be transformed into an integral equation for $g(t)=\int_{-1}^{1}dz\;g(z,t)$. It has only solutions with $\kappa=0$, because we get for the trace of its integral kernel

$$\int_{0}^{1}dx\frac{(1-x)^2}{x\mu^2+(1-x)^2}=1-\mu\left\{\frac{1-\mu^2/2}{\sqrt{1-\mu^2/4}}\cos^{-1}\frac{\mu}{2}-\mu\log\frac{1}{\mu}\right\}, \tag{3·10}$$

the right-hand side being just what is given in Eq. (2·12). It also follows from that $\phi(q)$ is a function of q^2 only, for the parameter z disappears eventually in the integral representation. In the nonrelativistic limit in which abnormal solutions disappear, $(1-|z|)g(z,t)$ is a function of $t/(1-|z|)$ only,[3] and we have, for the trace of an integral equation it satisfies, that

* Since $g(z,t)$ is a singular function, it will be safer to consider an integral equation[3] for $G(z,t)=\int_0^t g(z,t')dt'$, instead of Eq. (3·4) for $g(z,t)$. It produces, however, no change in our argument here.

$$\frac{\pi}{2}\int_0^1 \frac{m\,dx}{\{x\mu^2+(1-x)^2 mB\}^{1/2}} = \frac{\pi}{2}\left(\frac{m}{B}\right)^{1/2}\log\left(1+\frac{2(mB)^{1/2}}{\mu}\right)=\sigma_{NR}\left(\mu, 1-\frac{B}{2m}\right), \tag{3·11}$$

where B denotes the binding energy.

On the support of $g(z, t)$ we see from (3·4) and (3·5) that

$$g(z, t)=0 \quad \text{when } 0<\frac{t}{1-|z|}<t_1, \tag{3·12}$$

where

$$t_1=\frac{1}{1-|z|}\operatorname*{Min}_{-1\leq z'\leq 1}\left[R(z, z')\{\mu^2+2\mu\sqrt{\rho^2+\eta^2 z'^2}\}\right].$$

If we write m explicitly, t_1 is given by

$$\begin{aligned} t_1&=\left\{1-\left(\frac{\mu}{2m}\right)^2\right\}^{1/2}\frac{M}{m}\rho\mu+\left(\frac{M}{2m}\right)^2\mu^2 \quad \text{for } M^2\geq m(2m+\mu),\\ &=m\mu+\frac{\mu^2}{2} \qquad\qquad\qquad\qquad \text{for } M^2\leq m(2m+\mu). \end{aligned} \tag{3·13}$$

Some analogy might be noted to the so-called anomalous threshold appearing in a form factor of a composite system. However, the critical mass, $\{m(2m+\mu)\}^{1/2}$, in our case depends on the mass of intermediary meson, while in the latter the critical mass is $\sqrt{2}m$ and does not depend on the dynamics. It is to be added that as $g(z, t)$ of antisymmetric solutions have no $\delta(t)$ term, $g(z, t)=0$ for arbitrary M when $(1-|z|)^{-1}t<m\mu+\mu^2/2$.

Lastly we refer to the spatial extension of a bound system of two particles with masses m_1 and m_2. For this purpose it is more convenient to use the vertical representation (3·3). The B-S amplitude at equal relative time in the center-of-mass system is

$$\tilde{\phi}(\boldsymbol{x}, x_4=0)=\int d^4q\, e^{iq\cdot x}\phi(q, P)=\int_0^\infty ds K_0(\sqrt{s}\,|\boldsymbol{x}|)\int_{-1}^1 dz f(z, s), \tag{3·14}$$

K_0 being the modified Bessel function of the zeroth order. We define the spatial extension a of the system by the inverse of the minimal value of $\sqrt{s}$ belonging to the support. With slight modification of the arguments given in this section, we get

$$a^{-2}\equiv\operatorname*{Min}_{f(z,s)\neq 0} s=\operatorname*{Min}_{-1\leq z\leq 1}\left[(m_1^2+m_2^2)/2+z(m_1^2-m_2^2)/2-(1-z^2)\left(\frac{M}{2}\right)^2\right].$$

We can take $m_1\geq m_2$ without loss of generality. Then a is given by

$$a^{-1}=\frac{1}{2M}\{(m_1+m_2)^2-M^2\}^{1/2}\{M^2-(m_1-m_2)^2\}^{1/2}$$

$$\text{for}\quad (m_1^2-m_2^2)^{1/2}\leq M<m_1+m_2, \tag{3·15a}$$

$$a^{-1}=m_2 \qquad \text{for}\quad m_1-m_2<M\leq(m_1^2-m_2^2)^{1/2}. \tag{3·15b}$$

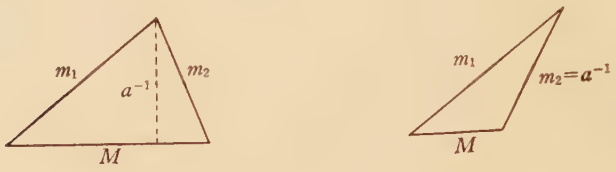

Fig. 1. The spatical extension a of a composite system of two particles with masses m_1 and m_2. Figs. (1·a) and (1·b) correspond to Eqs. (3·15a) and (3·15b), respectively.

The geometrical meaning of the above result is clear by Fig. 1. In the non-relativistic limit a^{-1} reduces to $\{2m_1m_2B/(m_1+m_2)\}^{1/2}$, where $B(=m_1+m_2-M)\ll m_1+m_2$. (3·15b) may be interpreted in such a way that any particle cannot be enclosed in a region smaller than its Compton wavelength, however strong its coupling with another particle may be.

§ 4. Limitation on the lowest eigenvalue

All the integral kernels discussed in § 2 were real symmetric and positive ones. In this section we present a limitation on the lowest eigenvalue of such a kernel, which might be useful for numerical estimate in realistic models. We consider an integral equation with a kernel of this kind,

$$\phi_i(x)=\frac{1}{s_i}\int K(x,y)\,\phi_i(y)\,dy, \quad s_1\geq s_2\geq\cdots(>0), \tag{4·1}$$

where some variables are represented by a single letter x or y. As is known, we have the equalities[10]

$$\begin{aligned}\sigma_1&\equiv\sum_i s_i=\int dx K(x,x),\\ \sigma_2&\equiv\sum_i s_i^2=\int dx\int dy\,K(x,y)^2,\end{aligned} \tag{4·2}$$

the existence of the integrals being assumed. It is easy to get a limitation on s_1 in terms of σ_1 and σ_2,

$$\sigma_1\geq\sqrt{\sigma_2}\geq s_1\geq\sigma_2/\sigma_1. \tag{4·3}*$$

* In general we get the inequalities

$$\sigma_1\geq\sqrt{\sigma_2}\geq\cdots\geq\sqrt[n]{\sigma_n}\geq\cdots\geq s_1\geq\cdots\geq\sigma_n/\sigma_{n-1}\geq\cdots\geq\sigma_2/\sigma_1,$$

σ_n being the n-th trace of a symmetric positive kernel.

The lower bound of s_1 given above can be made larger. If we note that

$$\sigma_2 = s_1^2 + \sum_{i\geq 2} s_i^2 \leq s_1^2 + (\sigma_1 - s_1)^2,$$

we have

$$s_1 \geq \{\sigma_1 + (2\sigma_2 - \sigma_1^2)^{1/2}\}/2 \quad \text{or} \quad s_1 \leq \{\sigma_1 - (2\sigma_2 - \sigma_1^2)^{1/2}\}/2,$$

when $\sigma_2 \geq \sigma_1^2/2$. But we must discard the second inequality because $s_1 \geq \sigma_2/\sigma_1$. In order to extend our argument to the case, $\sigma_2 < \sigma_1^2/2$, we assume first that $K(x, y)$ has at most m non-vanishing eigenvalues. We need to find the minimum and maximum values of $\sigma_2 = \sum_{i=1}^{m} s_i^2$ when σ_1 and $s_1 (< \sigma_1)$ are given and $s_2, s_3, \cdots, s_m$ are a set of numbers such that $\sum_{i=1}^{m} s_i = \sigma_1$ and $s_1 \geq s_2 \geq \cdots \geq s_m \geq 0$. To make σ_2 minimum we have to take $s_2 = \cdots = s_m = (\sigma_1 - s_1)/(m-1)$. It follows that $\sigma_2 \geq s_1^2 + (\sigma_1 - s_1)^2/(m-1)$. Thus we get $\sigma_2 \geq \sigma_1^2/m$ and

$$s_1 \leq \{\sigma_1 + [(m-1)(m\sigma_2 - \sigma_1^2)]^{1/2}\}/m.$$

To make σ_2 maximum we must set $s_2 = \cdots = s_{n-1} = s_1$, $s_n = \sigma_1 - (n-1)s_1$ and $s_{n+1} = \cdots = s_m = 0$, when $\sigma_1/(n-1) > s_1 \geq \sigma_1/n$ with $n = 2, 3, \cdots, m$. In this case $\sigma_2 \leq (n-1)s_1^2 + \{\sigma_1 - (n-1)s_1\}^2$ and we obtain

$$s_1 \geq \{\sigma_1 + [(n\sigma_2 - \sigma_1^2)/(n-1)]^{1/2}\}/n \quad \text{for } \sigma_1^2/(n-1) > \sigma_2 \geq \sigma_1^2/n,$$

with $n = 2, 3, \cdots m$. Letting $m \to \infty$ we finally get the limitation

$$\sqrt{\tau} \geq s_1/\sigma_1 \geq \frac{1}{n}\left\{1 + \left(\frac{n\tau - 1}{n-1}\right)^{1/2}\right\} \tag{4·4}$$

for $1/(n-1) > \tau > 1/n$, with $n = 2, 3, \cdots$ and $\tau = \sigma_2/\sigma_1^2$. The latter equality in (4·4) holds only when there exist just n non-vanishing eigenvalues. From this derivation we easily see that (4·4) is the best limitation in terms of σ_1 and σ_2. σ_1 and σ_2 also give an upper bound of s_2. From the inequalities $s_2 \leq \sigma_1 - s_1$ and $s_2^2 \leq 1/2(s_1^2 + s_2^2) \leq \sigma_2/2$ it follows that

$$\begin{aligned} s_2/\sigma_1 &\leq 1/2[1 - (2\tau - 1)^{1/2}] \quad \text{for } 1/2 \leq \tau < 1, \\ &\leq (\tau/2)^{1/2} \qquad\qquad\quad \text{for } 0 < \tau \leq 1/2. \end{aligned} \tag{4·5}$$

In the same way as before we can ascertain that this limitation is the best possible in terms of σ_1 and σ_2.

Some interesting examples are given by Schrödinger equations with negative semidefinite potentials ($V(\boldsymbol{x}) \leq 0$). For simplicity we limit ourselves to zero binding energy, although the present method is applicable without difficulty to the case of arbitrary energy. The eigenvalues s are nothing but the so-called well-depth parameters.[13] For a central potential $V(r)/s$ and l-wave states the Schrödinger equation given by

$$\frac{\hbar^2}{2m}\left(-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}\right)u_l(r)=\frac{1}{s}|V(r)|u_l(r)$$

and proper boundary conditions are expressed by the following integral equation :[11]

$$\phi_l(r)=\frac{1}{s}\int_0^\infty dr' K_l(r, r')\phi_l(r'), \tag{4·6}$$

where

$$\phi_l(r)=|V(r)|^{1/2}u_l(r),$$

$$K_l(r, r')=\frac{2m}{\hbar^2}|V(r)|^{1/2}g_l(r, r')|V(r')|^{1/2},$$

$$g_l(r,r')=\frac{1}{2l+1}r_<^{l+1}r_>^{-l},$$

and $r_>(r_<)$ denotes the larger (smaller) of r and r'. σ_1 and σ_2 are then given by

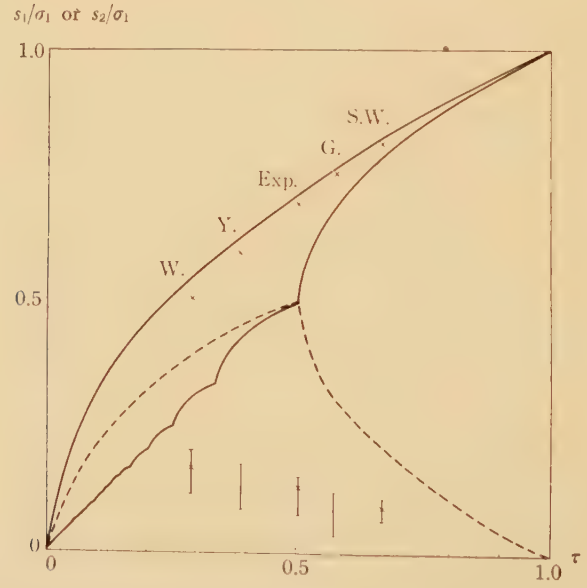

Fig. 2. Limitation on s_1/σ_1 and s_2/σ_1 in terms of $\tau\equiv\sigma_2/\sigma_1^2$. s_1/σ_1 is allowed in the domain enclosed by the two solid curves, and s_2/σ_1 under the dotted curve. Exact values of s_1/σ_1 and s_2/σ_1 are marked by crosses for several known cases (see text). When s_1/σ_1 is exactly known, we have a severer limitation on s_2/σ_1, which is indicated by vertical lines for the cases mentioned in the text.

$$\sigma_1=\frac{2m}{\hbar^2}\frac{1}{2l+1}\int_0^\infty dr\, r|V(r)|,$$
$$\sigma_2=\left(\frac{2m}{\hbar^2}\frac{1}{2l+1}\right)^2\int_0^\infty dr\int_0^\infty dr'\, r_<^{2l+2}r_>^{-2l}|V(r)|\cdot|V(r')|. \tag{4·7}$$

When $l=0$ the τ-values for square well, Gaussian, exponential and Yukawa potentials are $2/3$, $\pi/2-1$, $1/2$ and $2\log 2-1$, respectively (see Fig. 2). The longer the potential tail is, the smaller is the τ-value and so is the spacing of s_1/σ_1 and s_2/σ_1. In this figure is added the case (denoted by W.) of the integral equation (2·11) with $\mu=0$, in which $\tau=\pi^2/3-3$.

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Appendix

Equality of Tr(R) *and* Tr(K)

We first ascertain the equality for the case $\eta=0$ by a direct calculation. For $\eta=0$ we have

$$\mathrm{Tr}\,(R)=\int_0^1 dx\frac{1-x}{x\mu^2+(1-x)^2}=\log\frac{1}{\mu}+\frac{\mu}{2}\frac{\cos^{-1}\mu/2}{(1-\mu^2/4)^{1/2}}$$
$$=\frac{1}{\pi}\int_0^\infty d|\boldsymbol{q}|\int_{-\infty}^\infty dq_4(\boldsymbol{q}^2+q_4^2+1)^{-2}\log\frac{4\boldsymbol{q}^2+\mu^2}{\mu^2}=\mathrm{Tr}(K), \tag{A·1}$$

with the use of (2·10). Then it follows that

$$\int_0^1 dx\frac{1-x}{x\mu^2+(1-x)^2\alpha(z)}=\frac{1}{\pi}\int_0^\infty d|\boldsymbol{q}|\int_{-\infty}^\infty dq_4(\boldsymbol{q}^2+q_4^2+\alpha(z))^{-2}\log\frac{4\boldsymbol{q}^2+\mu^2}{\mu^2}, \tag{A·2}$$

where $\alpha(z)=\rho^2+\eta^2z^2$. We note that

$$\int_0^1 dz\int_{-\infty}^\infty dq_4(\boldsymbol{q}^2+q_4^2+\alpha(z))^{-2}=\frac{1}{2}\int_{-\infty}^\infty dq_4\int_{-1}^1 dz(\boldsymbol{q}^2+q_4^2+\rho^2+2iz\eta q_4)^{-2}$$
$$=\int_{-\infty}^\infty dq_4\{(\boldsymbol{q}^2+q_4^2+\rho^2)^2+4\eta^2q_4^2\}^{-1}. \tag{A·3}$$

Thus we can prove that

$$\mathrm{Tr}\,(R)=\int_0^1 dz\int_0^1 dx\frac{1-x}{x\mu^2+(1-x)^2\alpha(z)}$$

$$=\frac{1}{\pi}\int_0^\infty d|\boldsymbol{q}|\int_{-\infty}^{\infty} dq_4\{(\boldsymbol{q}^2+q_4^2+\rho^2)^2+4\eta^2 q_4^2\}^{-1}\log\frac{4\boldsymbol{q}^2+\mu^2}{\mu^2}=\mathrm{Tr}(K). \qquad (\mathrm{A}\cdot 4)$$

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Macroscopic Causality and Lower Limit for the Energy Derivative of the Scattering Phase Shift

——Relativistic Case——

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A relation between the macroscopic causality and lower bound of the energy derivative of the scattering phase shift is studied in the case of relativistic quantum field theory. From the requirement of the macroscopic causality the energy derivative of the real part of the phase shift must be non-negative in such an energy region that the imaginary part of phase shift does not vary rapidly with the energy, whereas in another region such inequality is not generally valid.

§ 1. Introduction

In the case of one-channel and potential scattering problems it is well known that the existence of lower bound of the energy derivative of the scattering phase shift is, fundamentally, a consequence of the macroscopic causality condition.[1),2)] The aim of this paper is to examine whether such a relation holds also in the case of relativistic quantum field theory. In this case the scattering phase shift must be dealt generally as a complex quantity owing to the existence of multi-channel scattering and this is a main feature different from the case of one-channel scattering.

At first one may suppose that our purpose can be achieved by making use of the analiticity properties of the scattering amplitudes. However, it is impossible to do so, because the analiticity properties of the scattering amplitudes cannot generally be derived from the requirement of macroscopic causality (see the last paragraph in § 4 of reference 3)). Therefore we shall adopt another course of study.

In order to require the macroscopic causality the scattering problem will be studied by means of the wave-packet-formalism.[3)] Here the "macroscopic causality" is formulated in such a way that the scattered wave cannot emerge before the incident wave arrives at the scatterer provided that the extensions of wave packets are neglected.

§ 2. Transition matrix

We consider the elastic scattering of the particles with mass M and mass μ. Denote the initial average momentum and average position of the particle

with mass M by $\boldsymbol{p}_1$ and $\boldsymbol{x}_1$, and those of the particle with mass μ by $\boldsymbol{p}_2$ and $\boldsymbol{x}_2$, then the initial state of the scattering can be represented in terms of the outgoing solution of the Schrödinger equation, $|\boldsymbol{k}_1, M; \boldsymbol{k}_2, \mu\rangle^+$, as follows:[3]

$$|\boldsymbol{p}_1, \boldsymbol{x}_1, M; \boldsymbol{p}_2, \boldsymbol{x}_2, \mu\rangle^+ = \int d\boldsymbol{k}_1 d\boldsymbol{k}_2 F_1(\boldsymbol{p}_1, \boldsymbol{x}_1, \boldsymbol{k}_1) F_2(\boldsymbol{p}_2, \boldsymbol{x}_2, \boldsymbol{k}_2) |\boldsymbol{k}_1, M; \boldsymbol{k}_2, \mu\rangle^+ \tag{1}$$

with

$$F(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{k}) = \exp(-i\boldsymbol{k}\cdot\boldsymbol{x}) f(\boldsymbol{p}, \boldsymbol{k}), \tag{2}$$

where $F(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{k})$ is the wave function describing the wave packet and $f(\boldsymbol{p}, \boldsymbol{k})$ is assumed to be nearly equal to zero except for $|\boldsymbol{k}-\boldsymbol{p}| \lesssim |\Delta\boldsymbol{p}|$, $\Delta\boldsymbol{p}$ being the uncertainty of the momentum of the wave packet. Here the spatial separation $|\boldsymbol{x}_1-\boldsymbol{x}_2|$ has also been assumed to be sufficiently larger than the extension of the wave packet, i.e. $|\boldsymbol{x}_1-\boldsymbol{x}_2| \gg |\Delta\boldsymbol{r}|$, $\Delta\boldsymbol{r}$ being the uncertainty of the position of the packet. Of course, $\Delta\boldsymbol{p}\cdot\Delta\boldsymbol{r} \approx 1$ holds by the uncertainty principle.

In a similar way the expression

$$|\boldsymbol{p}_3, \boldsymbol{x}_3, M; \boldsymbol{p}_4, \boldsymbol{x}_4, \mu\rangle^- = \int d\boldsymbol{k}_3 d\boldsymbol{k}_4 F_3(\boldsymbol{p}_3, \boldsymbol{x}_3, \boldsymbol{k}_3) F_4(\boldsymbol{p}_4, \boldsymbol{x}_4, \boldsymbol{k}_4) |\boldsymbol{k}_3, M; \boldsymbol{k}_4, \mu\rangle^- \tag{3}$$

is of the final state of the scattering, where $|\boldsymbol{k}_3, M; \boldsymbol{k}_4, \mu\rangle^-$ denotes the incoming solution of the Schrödinger equation.

Using the definitions (1) and (3) for the initial and final state of the scattering, we can now write the transition matrix for the collision:[3]

$$\begin{aligned} &{}^-\langle \boldsymbol{p}_3, \boldsymbol{x}_3, M; \boldsymbol{p}_4, \boldsymbol{x}_4, \mu | \exp(-iHt) | \boldsymbol{p}_1, \boldsymbol{x}_1, M; \boldsymbol{p}_2, \boldsymbol{x}_2, \mu\rangle^+ \\ &\quad = \int d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 \exp[-i\{E_3(\boldsymbol{k}_3)+E_4(\boldsymbol{k}_4)\}t] \\ &\quad \times F_3^\dagger(\boldsymbol{p}_3, \boldsymbol{x}_3, \boldsymbol{k}_3) F_4^\dagger(\boldsymbol{p}_4, \boldsymbol{x}_4, \boldsymbol{k}_4) {}^-\langle \boldsymbol{k}_3, M; \boldsymbol{k}_4, \mu | \boldsymbol{k}_1, M; \boldsymbol{k}_2, \mu\rangle^+ \\ &\quad \times F_1(\boldsymbol{p}_1, \boldsymbol{x}_1, \boldsymbol{k}_1) F_2(\boldsymbol{p}_2, \boldsymbol{x}_2, \boldsymbol{k}_2), \end{aligned} \tag{4}$$

where ${}^-\langle \boldsymbol{k}_3, M; \boldsymbol{k}_4, \mu | \boldsymbol{k}_1, M; \boldsymbol{k}_2, \mu\rangle^+$ is the so-called S-matrix and can be expanded in terms of the partial wave phase shift $\delta_l(W^2)$ in the following way:

$$\begin{aligned} &{}^-\langle \boldsymbol{k}_3, M; \boldsymbol{k}_4, \mu | \boldsymbol{k}_1, M; \boldsymbol{k}_2, \mu\rangle^+ \\ &\quad = \delta^3(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\, \delta[E_1(\boldsymbol{k}_1)+E_2(\boldsymbol{k}_2)-E_3(\boldsymbol{k}_3)-E_4(\boldsymbol{k}_4)] \\ &\quad \times W/(\pi^2 K) \cdot \sum_l (2l+1) \exp[2i\delta_l(W^2)] \cdot P_l(\cos\theta), \end{aligned} \tag{5}$$

where

$$\begin{aligned} &W^2 = [E_3(\boldsymbol{k}_3)+E_4(\boldsymbol{k}_4)]^2 - (\boldsymbol{k}_3+\boldsymbol{k}_4)^2, \\ &K^2 = [W^2-(M+\mu)^2][W^2-(M-\mu)^2]/4W^2, \\ &\cos\theta = 1-2\Delta^2/K^2 \end{aligned}$$

and

$$4\Delta^2=(\boldsymbol{k}_3-\boldsymbol{k}_1)^2-[E_3(\boldsymbol{k}_3)-E_1(\boldsymbol{k}_1)]^2.$$

§ 3. Condition of the scattering

Let us examine the conditions for $\boldsymbol{p}_1$, $\boldsymbol{x}_1$, $\boldsymbol{p}_2$, $\boldsymbol{x}_2$ and $\boldsymbol{p}_3$, $\boldsymbol{x}_3$, $\boldsymbol{p}_4$, $\boldsymbol{x}_4$ for getting non-vanishing transition matrix (4). We set $\delta_l(W^2)=\delta_{lR}(W^2)+i\delta_{lI}(W^2)$ and introduce the Fourier transform of $\exp[-2\delta_{lI}(W^2)]$ by

$$\exp[-2\delta_{lI}(W^2)]=\int_{-\infty}^{\infty}d\alpha\exp(2i\alpha W^2)\,E_l(\alpha) \tag{6}$$

with

$$E_l^{+}(\alpha)=E_l(-\alpha), \tag{6}'$$

in order to take into account the effect of the variation of $\exp[-2\delta_{lI}(W^2)]$. It should be noted that the Fourier transform $E_l(\alpha)$ is not unique, since $\exp[-2\delta_{lI}(W^2)]$ is defined only in the physical region $W^2\geqq(M+\mu)^2$.

In order that the transition matrix (4) does not vanish, the variation of the phase of the integrand (4) with respect to $\boldsymbol{k}_i$'s must be equal to zero apart from the errors of the extensions of the packets, when $\boldsymbol{k}_i$'s vary around $\boldsymbol{p}_i$'s:

$$\begin{aligned}&\Delta\boldsymbol{k}_3\cdot\boldsymbol{x}_3+\Delta\boldsymbol{k}_4\cdot\boldsymbol{x}_4-\Delta\boldsymbol{k}_1\cdot\boldsymbol{x}_1-\Delta\boldsymbol{k}_2\cdot\boldsymbol{x}_2-\Delta\boldsymbol{k}_3\cdot\boldsymbol{V}_3(\boldsymbol{p}_3)t-\Delta\boldsymbol{k}_4\cdot\boldsymbol{V}_4(\boldsymbol{p}_4)t\\&\quad+4W_p[\delta'_{lR}(W_p{}^2)+\alpha][\Delta\boldsymbol{k}_3\cdot\boldsymbol{V}_3(\boldsymbol{p}_3)+\Delta\boldsymbol{k}_4\cdot\boldsymbol{V}_4(\boldsymbol{p}_4)]\approx 0,\end{aligned} \tag{7}$$

where $\boldsymbol{V}(\boldsymbol{p})\equiv\boldsymbol{p}/E(\boldsymbol{p})$, $\delta'_{lR}(W_p{}^2)\equiv[d\delta_{lR}(W^2)/dW^2]_{W^2=W_p{}^2}$ and $W_p{}^2\equiv[E_3(\boldsymbol{p}_3)+E_4(\boldsymbol{p}_4)]^2$ in the centre-of-mass system ($\boldsymbol{p}_3+\boldsymbol{p}_4=0$). Here we have neglected the effects of the variations of $\cos\theta$ and W/K in (5). As is easily seen, this approximation becomes better at high energy $W^2\gg(M+\mu)^2$. $\Delta\boldsymbol{k}_i$'s in (7) are restricted by virtue of the conservation law of the total energy momentum as follows:

$$\Delta\boldsymbol{k}_1+\Delta\boldsymbol{k}_2-\Delta\boldsymbol{k}_3-\Delta\boldsymbol{k}_4=0 \tag{8}$$

and

$$\Delta\boldsymbol{k}_1\cdot\boldsymbol{V}_1(\boldsymbol{p}_1)+\Delta\boldsymbol{k}_2\cdot\boldsymbol{V}_2(\boldsymbol{p}_2)-\Delta\boldsymbol{k}_3\cdot\boldsymbol{V}_3(\boldsymbol{p}_3)-\Delta\boldsymbol{k}_4\cdot\boldsymbol{V}_4(\boldsymbol{p}_4)=0. \tag{9}$$

Using the undeterminate multiplier T, we get from (7), (8) and (9)

$$\boldsymbol{x}_2-\boldsymbol{x}_1\approx-[\boldsymbol{V}_2(\boldsymbol{p}_2)-\boldsymbol{V}_1(\boldsymbol{p}_1)]T, \tag{10}$$

$$\boldsymbol{x}_4-\boldsymbol{x}_3\approx[\boldsymbol{V}_4(\boldsymbol{p}_4)-\boldsymbol{V}_3(\boldsymbol{p}_3)][t-T-\beta W_p] \tag{11}$$

with

$$\beta\equiv 4[\delta'_{lR}(W_p{}^2)+\alpha],$$

where $\approx$ means that the relations (10) and (11) hold if the extension of the wave packet is neglected which is of the order $\boldsymbol{\Delta p}$ or $\boldsymbol{\Delta r}$ and if the effects of $\cos\theta$ and W/K in (5) is omitted.

The relations (10) and (11) represent the following fact that the incident particles which are initially localized at the positions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ collide with each other at the time T, then the scattered particles emerge after the time βW_p and finally arrive at the positions $\boldsymbol{x}_3$ and $\boldsymbol{x}_4$. (See Fig. 1.) The time interval $(0, T)$ stands for the initial state of the scattering and the time interval $(T+\beta W_p, t)$ stands for the final state of the scattering. The four-vector $\overline{PQ}$ is time-like.

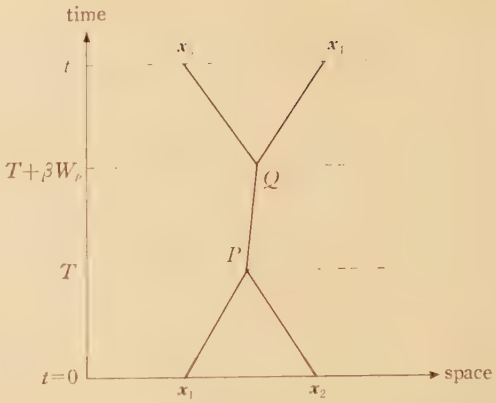

Fig. 1. Schematic representation of the relations (10) and (11).

§ 4. Condition of macroscopic causality and concluding remarks

The scattered waves cannot emerge before the incident waves collide with each other, so that βW_p must be non-negative,* i.e.

$$\delta'_{lR}(W_p^2)+\alpha \gtrsim 0, \tag{12}$$

provided that the extensions of the wave packets and the effects of $\cos\theta$ and W/K in (5) are neglected. It is well valid at high energy $W_p^2 \gg (M+\mu)^2$.

The condition (12) implies that the transition matrix (4) cannot contain the contribution from α smaller than $-\delta'_{lR}(W_p^2)$. This requirement is satisfied if we are able to make $E_l(\alpha)$ zero for $\alpha < -\delta'_{lR}(W_p^2)$ by making use of non-uniqueness of $E_l(\alpha)$. This possibility, however, is not always assured. The above requirement is also satisfied if the contribution from $\alpha < -\delta'_{lR}(W_p^2)$ vanishes after integration over α from $-\infty$ to $+\infty$, even though $E_l(\alpha)$ does not identically vanish for $\alpha < -\delta'_{lR}(W_p^2)$.

(1) If the first possibility for $E_l(\alpha)$ is true, we have the following relation from (6), (6)′ and (12):

$$\exp[-2\delta_{ll}(W_p^2)] \approx \int_{-\varphi}^{\varphi} d\alpha \exp(2i\alpha W_p^2) E_l(\alpha) \tag{13}$$

* Suppose βW_p is not smaller than $-t_0$ apart from the errors of the extensions of wave packets, where the extension of the interaction is given by $t_0[\boldsymbol{V}_2(\boldsymbol{p}_2)-\boldsymbol{V}_1(\boldsymbol{p}_1)]$. Then in another Lorentz frame the extension of the interaction becomes of the order $(t_0/\sqrt{1-\beta^2})\Delta\boldsymbol{V}$, $\Delta\boldsymbol{V}$ being the relative velocity in the new Lorentz frame. If β takes a sufficiently large value, this extension can be made measurable by a macroscopic observation whatever small t_0 may be. Therefore, in order that $(t_0/\sqrt{1-\beta^2})\Delta\boldsymbol{V}$ be always of microscopic order, t_0 must be zero. (See the last paragraph in § 6 in reference 3).)

with

$$E_l(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dW^2\exp(-2i\alpha W^2)\exp[-2\delta_{lI}(W^2)], \tag{14}$$

where $\varphi\equiv\delta'_{lR}(W_p^2)$ must always be non-negative, as is easily seen. Inserting (14) into (13), we get a new type of relation between the real part and imaginary part of l-wave phase shift:

$$\exp[-2\delta_{lI}(W_p^2)]\approx\frac{1}{2\pi}\int_{-\infty}^{\infty}dW^2\frac{\sin[2(W^2-W_p^2)\,\delta'_{lR}(W_p^2)]}{W^2-W_p^2}\exp[-2\delta_{lI}(W^2)], \tag{15}$$

which holds well at high energy $W_p^2\gg(M+\mu)^2$. As is easily seen, the unphysical region $-(M+\mu)^2\lesssim W^2\lesssim(M+\mu)^2$ hardly contributes to the integration in (15), if we are concerned with the high energy $W_p^2\gg(M+\mu)^2$. The region $W^2\leqq-(M+\mu)^2$ relates to the region $W^2\geqq(M+\mu)^2$ by virtue of the crossing symmetries. Note that $\delta'_{lR}(W_p^2)$ must always be non-negative.

(2) The second possibility for $E_l(\alpha)$, however, may be a general case in which we cannot obtain any new condition about the complex phase shift from the requirement of the macroscopic causality.

(3) In a particular case in which the imaginary part of phase shift is constant within some energy region, that is, we need not take into account the effect of the variation of $\exp[-2\delta_{lI}(W^2)]$, $(\alpha=0)$, we have the following condition in place of (12):

$$\delta'_{lR}(W_p^2)\gtrsim 0 \tag{16}$$

within such energy region. This inequality is well valid at high energy $W_p^2\gg(M+\mu)^2$.

In the case of one-channel and potential scatterings Wigner showed the following inequality:[1]

$$\delta_l'(k)\gtrsim -a, \tag{17}$$

where a is the radius of the potential. In the theory of potential scatterings the causality may be violated by the range $2a$ of the potential at most. However, the violation by more than the range $2a$ is unacceptable. The inequality (17) is an immediate consequence of this fact.[2]

In the covariant field theory, however, such an extension of the interaction, whatever small it may be, always leads to the measurable violation of the macroscopic causality owing to the Lorentz covariance (see the footnote). The inequality (16) is an immediate consequence of this requirement in some energy region.

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The Contribution of Multiple Scattering within the Nucleus to the Elastic Scattering of Nucleons by Nuclei

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The impulse approximation expansion of the nucleon-nucleus scattering operator in terms of nucleon-nucleon scattering operators is rearranged so that it is given by the sum of a term representing repeated scatterings from the same nuclear nucleon and a term representing repeated scatterings from two different nuclear nucleons. When the expansion is limited to double scattering, the result explains some well-known experimental phenomena, namely, the parallelism between the elastic differential scattering cross section and the Born approximation cross section, and the sharp dip in the polarization at the Born approximation diffraction zero.

§ 1. Introduction and summary

A great deal of work has been done in the attempt to express the experimental parameters describing nucleon-nucleus elastic scattering at high energies in terms of the nucleon-nucleon scattering parameters at the same energy.[1] The usual procedure is to neglect the binding energy of the nuclear nucleons compared to the kinetic energy of the incident particle (impulse approximation), to assume that the nucleon-nucleus potential is a sum over the individual nucleon-nucleon potentials, and then to express these two-body potentials in terms of the two-body scattering matrices. The resultant "optical model potential", linear in the two-body scattering amplitude, can then be used to calculate the nucleon-nucleus scattering cross section and polarization. The nucleon-nucleus scattering amplitude calculated in the first Born approximation from the optical model potential is then linear in the nucleon-nucleon scattering amplitude and can be interpreted as the scattering of the incident particle by a single nucleon in the nucleus, the nucleus then recoiling as a whole. We shall often refer to the "first Born approximation amplitude" and the "single scattering amplitude" interchangeably.

It has been shown that the polarization predicted by the optical model potential can be computed quite adequately at very small scattering angles, using the Born approximation scattering amplitude,[2] and several attempts have been made to use this result to distinguish among the various proposed sets of nucleon-nucleon phase shifts,[3] usually without success.[4] The polarization cal-

culated with the single scattering amplitudes should be independent of nuclear structure, depending only on the nucleon-nucleon scattering amplitude. In practice, this has been found to be true only at very small scattering angles, the measured polarization showing a marked dip (or a zero in the lighter nuclei such as carbon) at that angle for which there is a diffraction minimum in the scattering cross section; the diffraction pattern depends, of course, on the specific nucleus involved. As far as the scattering cross section is concerned, there has been no reason to believe that the single scattering amplitude should give any representation of the data at all. It is therefore remarkable to note that the Born approximation differential cross section is almost exactly four times the measured cross section at scattering angles between the pure Coulomb region and the first diffraction minima and for energies between 150 Mev and 300 Mev.[5]

In this paper an attempt will be made to show that double scattering of the incident nucleon by the nucleons in the nucleus can account for the above phenomena. That is, if the complete scattering amplitude is expanded to include scattering by a single nuclear nucleon, propagation in the nucleus and then another scattering, either by the same or a different nucleon; the resulting amplitude predicts a cross section proportional to the Born cross section and a minimum in the polarization at the diffraction zero of the Born amplitude.

§ 2. Formalism

Let the subscript 0 refer to the incident particle while the subscripts i, j, or $k(1\leq i, j, k\leq A)$ represent the nuclear nucleons. We treat the incident particle as distinguishable since it has been shown[6] that this leads to correct results for elastic scattering provided that the nucleon-nucleon scattering amplitudes which are used are properly antisymmetrized and include recoil effects. Let h_α represent the kinetic energy of the α-th nucleon, $V_{\alpha\beta}$ the interaction potential between any two nucleons $\alpha\neq\beta(0\leq\alpha, \beta\leq A)$.

Define $H_0\equiv\sum_{i=0}^{A}h_i$; $H_N\equiv\sum_{i=1}^{A}h_i+\sum_{i<j=1}^{A}V_{ij}$ is the Hamiltonian of the free nucleus; the interaction between the incident nucleon and the nucleus is given by $H_{int}\equiv\sum_{i=1}^{A}V_{0i}$, and the total Hamiltonian is $H=H_N+h_0+H_{int}$.

The free particle wave functions are defined by

$$h_i\varphi_i(\epsilon_i)=\epsilon_i\varphi_i(\epsilon_i). \tag{1}$$

The nuclear wave functions, including center of mass motion of the nucleus, are given by

$$H_N\chi_N=E_N\chi_N. \tag{2}$$

We also require the solutions for the case where there is no interaction between

the incident particle and the nucleus, i.e.

$$(H_N+h_0-E_a)\psi_a=0; \tag{3}$$

here E_a includes the kinetic energy of the incident particle (ϵ_0) and the nucleus as well as the binding energy of the nucleus. The actual scattering problem is then represented by the solutions of

$$(H-E_a)\Psi_a^{\pm}=0, \tag{4}$$

where

$$\Psi_a^{+}\to\psi_a \quad \text{as } t\to-\infty,$$
$$\Psi_b^{-}\to\psi_b \quad \text{as } t\to+\infty$$

represent the ingoing and outgoing scattering wave solutions given by

$$\Psi_a^{\pm}=\psi_a-\frac{1}{H-E_a\mp i\epsilon}H_{int}\psi_a. \tag{5}$$

The scattering matrix can be represented as

$$S_{ba}=(\Psi_b^{-},\ \Psi_a^{+})=(\psi_b,\ \psi_a)+\delta S_{ba}$$

where $(\psi_b,\ \psi_a)$ is the unit operator in the $A+1$ particle space and vanishes except for forward elastic scattering. Here[7]

$$\delta S_{ba}=-2\pi i\delta(E_b-E_a)\ (\psi_b|H_{int}-H_{int}\frac{1}{H-E_a-i\epsilon}H_{int}|\psi_a). \tag{6}$$

Defining the transition matrix by $S_{ba}=1_{ba}-2\pi i\delta(E_b-E_a)T_{ba}$, we get

$$T_{ba}=\sum_{i=1}^{A}[(\psi_b|\{D_i\}|\psi_a)-\sum_{j\neq i}^{A}(\psi_b|\{ND_{ij}\}|\psi_a)], \tag{7}$$

where

$$\{D_i\}\equiv V_{0i}-V_{0i}\frac{1}{H-E_a-i\epsilon}V_{0i}, \tag{8}$$

$$\{ND_{ij}\}\equiv V_{0i}\frac{1}{H-E_a-i\epsilon}V_{0j}.$$

We define a two-body energy ε_{0i}; $E_a-\varepsilon_{0i}$ is then the remaining energy of the $A-1$ nucleons. Writing

$$\kappa_i\equiv H_N+\sum_{k\neq i}^{A}V_{0k}-h_i=H-h_0-h_i-V_{0i},$$

we can make the formal expansion

$$\frac{1}{H-E_a-i\epsilon}=\frac{1}{a(\varepsilon_{0i})}\sum_{n=0}^{\infty}\left[(E_a-\varepsilon_{0i}-\kappa_i)\frac{1}{a(\varepsilon_{0i})}\right]^n, \tag{9}$$

where the denominators are now the two-body operators

$$a(\varepsilon_{0i}) = h_0 + h_i + V_{0i} - \varepsilon_{0i} - i\epsilon. \tag{10}$$

Letting $b(\varepsilon_{0i}) \equiv a(\varepsilon_{0i}) - V_{0i}$, we can now define the two-nucleon transition operator,

$$t(\varepsilon_{0i}) = V_{0i} - V_{0i}\frac{1}{a(\varepsilon_{0i})}V_{0i} = V_{0i} - V_{0i}\frac{1}{b(\varepsilon_{0i})}t(\varepsilon_{0i}). \tag{11}$$

This is an integral equation for the nucleon-nucleon scattering operator and can be inverted to give an integral equation for the nucleon-nucleon potential, viz.

$$V_{0i} = t(\varepsilon_{0i}) + t(\varepsilon_{0i})\frac{1}{b(\varepsilon_{0i})}V_{0i} = t(\varepsilon_{0i}) + t(\varepsilon_{0i})\frac{1}{b(\varepsilon_{0i}) - t(\varepsilon_{0i})}t(\varepsilon_{0i}). \tag{12}$$

Using (8), (9), (10), (11) and (12), we find that the diagonal term can be written as

$$\{D_i\} = t(\varepsilon_{0i})\left\{1 - \frac{1}{b(\varepsilon_{0i})}\sum_{n=0}^{\infty}\left[(E_a - \varepsilon_{0i} - \kappa_i)\frac{1}{b(\varepsilon_{0i})}\left(1 - t(\varepsilon_{0i})\frac{1}{b(\varepsilon_{0i})}\right)\right]^n \times (E_a - \varepsilon_{0i} - \kappa_i)\frac{1}{b(\varepsilon_{0i})}t(\varepsilon_{0i})\right\}, \tag{13}$$

where

$$\kappa_i = H_N - h_i + \sum_{k\neq i}^{A}\left[t(\varepsilon_{0k}) + t(\varepsilon_{0k})\frac{1}{b(\varepsilon_{0k}) - t(\varepsilon_{0k})}t(\varepsilon_{0k})\right], \tag{14}$$

so that the "diagonal part" of the scattering is completely represented by free two-body scattering operators with the nuclear Hamiltonian appearing in between successive scatterings of the incident particle. Similarly, we can express the "non-diagonal" part of the scattering in terms of two-body operators, i.e.

$$\{ND_{ij}\} = t(\varepsilon_{0i})\frac{1}{b(\varepsilon_{0i})}\sum_{n=0}^{\infty}\left[(E_a - \varepsilon_{0i} - \kappa_i)\frac{1}{b(\varepsilon_{0i})}\left(1 - t(\varepsilon_{0i})\frac{1}{b(\varepsilon_{0i})}\right)\right]^n \times t(\varepsilon_{0j})\left[1 + \frac{1}{b(\varepsilon_{0j}) - t(\varepsilon_{0j})}t(\varepsilon_{0j})\right]. \tag{15}$$

If $\{D_i\} \equiv t(\varepsilon_{0i})\{d_i\}$, and $\{ND_{ij}\} \equiv t(\varepsilon_{0i})\{Nd_{ij}\}$, it follows that

$$T_{ba} = \sum_{i=1}^{A}(\psi_b | t(\varepsilon_{0i})[\{d_i\} - \sum_{j\neq i}^{A}\{Nd_{ij}\}]|\psi_a). \tag{16}$$

We can therefore picture the total scattering as the sum of a diagonal term which represents repeated scatterings from the same target nucleon (i) and the non-diagonal term which represents repeated scatterings from two target nucleons (i) and (j); in both cases there is "propagation" and "interaction" with the nucleus between successive scatterings from the (i) or (j) nucleon. The nuclear "interaction" and "propagation" may be said to refract the scattered

wave back so that repeated scatterings from the same target nucleon are possible.

We now limit ourselves to double scattering of the incident particle, i.e. we consider only those parts of the nucleon-nucleus scattering operator which are linear or bilinear in the nucleon-nucleon scattering operator involving the incident particle. This means keeping only those parts of $\{d_i\}$ and $\{Nd_{ij}\}$ which give rise to a single scattering of the incident nucleon. We take $\kappa_{is} \equiv \sum_{k \neq i}^{A} V_{0k}$ as the part of κ_i which involves a scattering of the incident particle; $\kappa_{iN} = \kappa_i - \kappa_{is}$. The single scattering terms are then

$$\{d_i\}_1 = 1 - \frac{1}{b(\varepsilon_{0i})} \sum_{n=1}^{\infty} \left[(E_a - \varepsilon_{0i} - \kappa_{iN}) \frac{1}{b(\varepsilon_{0i})} \right]^n t(\varepsilon_{0i}), \tag{17}$$

and

$$\{Nd_{ij}\}_1 = \frac{1}{b(\varepsilon_{0i})} \sum_{n=0}^{\infty} \left[(E_a - \varepsilon_{0i} - \kappa_{iN}) \frac{1}{b(\varepsilon_{0i})} \right]^n t(\varepsilon_{0j}). \tag{18}$$

Eqs. (17) and (18) include not only a single scattering of the incident particle but an infinite number of scatterings of the nuclear nucleons with each other, each accompanied by the free propagation of the incident particle. For incident energies greater than 100 Mev, the average velocity of the nuclear nucleons is much less than that of the incident particle and hence, during the time the incident particle traverses the nucleus, we would expect the nuclear particles to make fewer collisions with each other than with the incident particle. Therefore, to be consistent, we put

$$\{d_i\}_1 = 1 - \frac{1}{b(\varepsilon_{0i})} (E_a - \varepsilon_{0i} - \kappa_{iN}) \frac{1}{b(\varepsilon_{0i})} t(\varepsilon_{0i}) \tag{19}$$

and

$$\{Nd_{ij}\}_1 = \frac{1}{b(\varepsilon_{0i})} \left[1 + (E_a - \varepsilon_{0i} - \kappa_{iN}) \frac{1}{b(\varepsilon_{0i})} \right] t(\varepsilon_{0j}). \tag{20}$$

To this approximation, the scattering matrix can be written:

$$\begin{aligned} T_{ba} = \sum_{i=1}^{A} \Bigg\{ & (\psi_b | t(\varepsilon_{0i}) | \psi_a) - (\psi_b | (E_a - \varepsilon_{0i} - \kappa_{iN}) \frac{1}{b(\varepsilon_{0i})} t(\varepsilon_{0i}) \frac{1}{b(\varepsilon_{0i})} t(\varepsilon_{0i}) | \psi_a) \\ & - (\psi_b | \left[t(\varepsilon_{0i}) \frac{1}{b(\varepsilon_{0i})}, \ (E_a - \varepsilon_{0i} - \kappa_{iN}) \frac{1}{b(\varepsilon_{0i})} \right] t(\varepsilon_{0i}) | \psi_a) \Bigg\} \\ & - \sum_{\substack{i=1 \\ j \neq i}}^{A} \Bigg\{ (\psi_b | t(\varepsilon_{0i}) \frac{1}{b(\varepsilon_{0i})} t(\varepsilon_{0j}) | \psi_a) \\ & + (\psi_b | (E_a - \varepsilon_{0i} - \kappa_{iN}) \frac{1}{b(\varepsilon_{0i})} t(\varepsilon_{0i}) \frac{1}{b(\varepsilon_{0i})} t(\varepsilon_{0j}) | \psi_a) \\ & + (\psi_b | \left[t(\varepsilon_{0i}) \frac{1}{b(\varepsilon_{0i})}, \ (E_a - \varepsilon_{0i} - \kappa_{iN}) \frac{1}{b(\varepsilon_{0i})} \right] t(\varepsilon_{0j}) | \psi_a) \Bigg\}. \end{aligned} \tag{21}$$

The commutator terms have been shown to be of the order of the ratio of nuclear binding energies to incident particle energy;[5] since we are concerned with elastic scattering at incident kinetic energies greater than 100 Mev, we neglect the commutator terms. By definition, $\kappa_{iN}=H_N-h_i$; furthermore ψ_b is an eigenstate of H_N with energy $E_b-\epsilon_{0b}$ (from Eqs. (1) and (3)) so that

$$(\psi_b|(E_a-\mathcal{E}_{0i}-\kappa_{iN})\frac{1}{b(\mathcal{E}_{0i})}=(\psi_b|\frac{E_a-\mathcal{E}_{0i}-E_b+\epsilon_{0b}+h_i}{h_0+h_i-\mathcal{E}_{0i}-i\epsilon}=(\psi_b|, \tag{22}$$

where we have used conservation of energy ($E_a=E_b$) and $h_0\psi_b=\epsilon_{0b}\psi_b$ (Eqs. (1) and (3)). The result is the simple form,

$$T_{ba}=\sum_{i=1}^{A}\Big\{(\psi_b|t(\mathcal{E}_{0i})|\psi_a)-(\psi_b|t(\mathcal{E}_{0i})\frac{1}{b(\mathcal{E}_{0i})}t(\mathcal{E}_{0i})|\psi_a) \tag{23}$$

$$-2\sum_{f\neq i}^{A}(\psi_b|t(\mathcal{E}_{0i})\frac{1}{b(\mathcal{E}_{0i})}t(\mathcal{E}_{0f})|\psi_a)\Big\}.$$

The $t(\mathcal{E}_{0i})$ is an operator in the space of the two nucleons (i) and (0). Taking into account the translational invariance of the individual scattering events, we can write

$$t(\mathcal{E}_{12})=(\boldsymbol{p}_1',\boldsymbol{p}_2'|t(\mathcal{E}_{12})|\boldsymbol{p}_1,\boldsymbol{p}_2)=\delta(\boldsymbol{p}_1'+\boldsymbol{p}_2'-\boldsymbol{p}_1-\boldsymbol{p}_2)t(\mathcal{E}_{12};\boldsymbol{p}_1'-\boldsymbol{p}_2';\boldsymbol{p}_1-\boldsymbol{p}_2). \tag{24}$$

Let $\bar{\chi}$ represent the nuclear wave function in the nuclear center of mass system and let $\Delta\boldsymbol{p}_0=\boldsymbol{p}_{0b}-\boldsymbol{p}_{0a}$ be the momentum transfer of the incident particle. Then, for elastic scattering, the first term of Eq. (23) becomes, to order $1/A$:

$$(\bar{\chi}_a|\boldsymbol{q}_i-\Delta\boldsymbol{p}_0)t(\mathcal{E}_{0i};\boldsymbol{p}_{0b}-\boldsymbol{q}_i+\Delta\boldsymbol{p}_0;\boldsymbol{p}_{0a}-\boldsymbol{q}_i)(\boldsymbol{q}_i|\bar{\chi}_a), \tag{25}$$

where we are summing over the nuclear nucleon momentum $\boldsymbol{q}_i$. Assuming the range of momenta of the nucleons bound in the nucleus to be small compared to the incident nucleon momenta, t in (25) does not depend upon $\boldsymbol{q}_i$ and can be taken outside the sum. Then, we may write

$$(\bar{\chi}_a|\boldsymbol{q}_i-\Delta\boldsymbol{p}_0)(\boldsymbol{q}_i|\bar{\chi}_a)=\int d^3x_i\rho_a(\boldsymbol{x}_i)e^{-i\boldsymbol{p}_0\boldsymbol{x}_i}\equiv F_a(\Delta\boldsymbol{p}_0) \tag{26}$$

where $\rho_a(\boldsymbol{x}_i)$ is the single particle density in the nucleus of state a (ground state for our purposes), and F_a is the nuclear form factor. To order $1/A$, $\boldsymbol{p}_{0b}=\boldsymbol{p}_{0a}=(2m_0\mathcal{E}_{0i})^{1/2}$; since t must be a scaler, it depends only on $\mathcal{E}_{0i}$ and $\Delta\boldsymbol{p}_0$ for momentum transfers small compared to the incident momenta. Assuming identical particles in the nucleus and letting $\bar{t}$ represent t after the appropriate spin and isotopic spin averaging has been carried out, the single scattering part of the scattering operator is

$$AF_a(\Delta\boldsymbol{p}_0)\bar{t}(\mathcal{E}_{0i};2\Delta\boldsymbol{p}_0). \tag{27}$$

Note that the scattering angle appearing in t is twice that appearing in F_a, i.e. t is a function of the scattering angle in the two-nucleon center of mass system

which is one half the laboratory scattering angle. The two-body amplitude in (27) is "off the energy shell" since the momentum is transferred to the entire nucleus rather than to a single nucleon. Eq. (27) is the usual first Born approximation and represents the polarization quite well at very small scattering angles; it gives a cross section four times the measured quantity over a very large angular range - out to the first diffraction zero of F.[5]

In a similar manner, the diagonal double scattering part of (23) can be written as

$$AF_a(\Delta\boldsymbol{p}_0)\int d^3p\frac{\bar{t}(\varepsilon_{0i};\,-\Delta\boldsymbol{p}_0-\boldsymbol{p}_{0b};\,\boldsymbol{p}_{0a}-2\boldsymbol{p})\,\bar{t}(\varepsilon_{0i};\,\boldsymbol{p}_{0a}-2\boldsymbol{p};\,-\boldsymbol{p}_{0a})}{p^2/2m_0+(\boldsymbol{p}_{0a}-\boldsymbol{p})^2/2m_i-\varepsilon_{0i}-i\epsilon}. \tag{28}$$

We define the two-nucleon correlation function $g(\boldsymbol{x}_i,\,\boldsymbol{x}_j)$ by

$$\rho(\boldsymbol{x}_i,\,\boldsymbol{x}_j)=\rho(\boldsymbol{x}_i)\rho(\boldsymbol{x}_j)[1+g(\boldsymbol{x}_i,\,\boldsymbol{x}_j)] \tag{29}$$

where $\rho(\boldsymbol{x}_i,\,\boldsymbol{x}_j)$ is the two-nucleon density in the ground state of the nucleus. Then

$$\begin{aligned}(\bar{\chi}_a|\boldsymbol{q}_i+\boldsymbol{p}_{0a}-\boldsymbol{p},\,\boldsymbol{q}_j+\boldsymbol{p}-\boldsymbol{p}_{0b})\,(\boldsymbol{q}_i,\,\boldsymbol{q}_j|\bar{\chi}_a)&=\int d^3x_i d^3x_j\rho(\boldsymbol{x}_i,\,\boldsymbol{x}_j)\exp i[(\boldsymbol{p}_{0a}-\boldsymbol{p})\boldsymbol{x}_i\\&\quad+(\boldsymbol{p}-\boldsymbol{p}_{0a})\boldsymbol{x}_j]\\&=F_a(\boldsymbol{p}-\boldsymbol{p}_{0a})F_a(\boldsymbol{p}_{0b}-\boldsymbol{p})+G_a(\boldsymbol{p}-\boldsymbol{p}_{0a};\,\boldsymbol{p}_{0b}-\boldsymbol{p}),\end{aligned} \tag{30}$$

and carrying out the proper averaging, the non-diagonal double scattering part of T_{ab} becomes

$$\begin{aligned}A(A-1)\int d^3p[F_a(\boldsymbol{p}_{0a}-\boldsymbol{p})F_a(\boldsymbol{p}-\boldsymbol{p}_{0b})+G_a(\boldsymbol{p}_{0a}-\boldsymbol{p};\,\boldsymbol{p}-\boldsymbol{p}_{0b})]\bar{t}(\varepsilon_{0i};\,\boldsymbol{p}-2\boldsymbol{p}_{0b};\,-\boldsymbol{p})\\\times\bar{t}(\varepsilon_{0j};\,\boldsymbol{p}_{0a}-2\boldsymbol{p};\,-\boldsymbol{p}_{0a})/[p^2/2m_0-\varepsilon_{0i}-i\epsilon].\end{aligned} \tag{31}$$

Taking ε_{0i} to be the kinetic energy of two particles where the struck particle (i) is initially at rest, we see that the pole of the non-diagonal term corresponds to scattering from an infinite mass target particle, i.e. the recoil energy equals the initial energy. On the other hand, the diagonal pole, Eq. (28), corresponds to the elastic scattering of two bodies of masses m_0 and m_i. Thus the diagonal double scattering lies on the two-body "energy shell" whereas the non-diagonal term takes the two-body scattering amplitudes way off the two-body "energy shell." In the diagonal case, the incident particle scatters once from a nuclear nucleon, leaving the remaining nucleus unmoved ($\sim 1/A$) but in an excited state. The second scattering then restores the nucleus to the ground state resulting in elastic scattering. In between the two scatterings, the struck nucleon is "free" and hence has the two-body relation between momentum and energy. In the non-diagonal case, if the first scattering from nucleon j leaves the nucleus in an excited state, it is very improbable that a scattering from a different nucleon (i) will restore the resulting nucleus to the ground state.

Hence, here the intermediate state consists of the nucleus recoiling as a unit; the second scattering changes the nuclear recoil but keeps the nucleus in the ground state. The entire nucleus is the " struck particle " and hence the relation between energy and momentum in the intermediate state is characteristic of the scattering from an infinite mass target particle.

§ 3. Small scattering angles

We assume the momentum transferred to the nucleus is small compared to the incident momentum; in Eq. (28) this implies $\Delta p_0 \ll p_{0b}$, $p_{0b} \approx p_{0a}$. Assuming parity invariance of the t operators, and defining

$$\mathcal{A}(\varepsilon_{0i}) \equiv \int d^3p \frac{\bar{t}(\varepsilon_{0i}; \boldsymbol{p}_{0a}; 2\boldsymbol{p}-\boldsymbol{p}_{0a})\bar{t}(\varepsilon_{0i}; 2\boldsymbol{p}-\boldsymbol{p}_{0a}; \boldsymbol{p}_{0a})}{p^2/2m_0 + (\boldsymbol{p}_{0a}-\boldsymbol{p})^2/2m_i - \varepsilon_{0i} - i\epsilon} \tag{32}$$

which is independent of scattering angle, the diagonal part of the nucleon-nucleus scattering operator (Eq. (23)) can be written as follows:

$$T_{ba}(\text{diagonal}) = AF_a(\Delta\boldsymbol{p}_0)[\bar{t}(\varepsilon_{0i}; 2\Delta\boldsymbol{p}_0) - \mathcal{A}(\varepsilon_{0i})]. \tag{33}$$

Assuming $g(\boldsymbol{x}_i, \boldsymbol{x}_j) = g(\boldsymbol{x}_i - \boldsymbol{x}_j)$, and defining

$$\frac{1}{V}G(\boldsymbol{q}) = \frac{1}{V}\int d^3y g(\boldsymbol{y}) e^{i\boldsymbol{q}\boldsymbol{y}} \tag{34}$$

where V is the nuclear volume, the non-diagonal term (31) becomes an angle independent constant $B(\varepsilon_{0i})$ where

$$B(\varepsilon_{0i}) = A(A-1)\int d^3p \frac{[|F_a(\boldsymbol{p}_{0a}-\boldsymbol{p})|^2 + G_a(\boldsymbol{p}_{0a}-\boldsymbol{p})/V]}{p^2/2m_0 - \varepsilon_{0i} - i\epsilon}$$
$$\times \bar{t}(\varepsilon_{0i}; 2\boldsymbol{p}_{0a}-\boldsymbol{p}; \boldsymbol{p})\bar{t}(\varepsilon_{0j}; 2\boldsymbol{p}-\boldsymbol{p}_{0a}; \boldsymbol{p}_{0a}). \tag{35}$$

The form factor F_a decreases very rapidly from one as the momentum transfer increases whereas the correlation term vanishes for very small or very large arguments. Hence, the constant $B(\varepsilon_{0i})$ is expected to be much less than $\mathcal{A}(\varepsilon_{0i})$ and therefore not important in determining the cross section at small angles. It will, of course, be important at the first diffraction zero of the form factor where the diagonal part of the scattering operator vanishes and the entire cross section is given by the non-diagonal term.

The nucleon-nucleon scattering operator varies slowly with scattering angle when compared to F_a. Thus, the major angular dependence of the Born approximation cross section comes from the angular variation of the form factor. Eq. (33) then shows that, neglecting non-diagonal scattering, the differential cross section, calculated at small angles including double scattering, will be proportional to the cross section calculated in the first Born approximation. Considering the experimental data, this is a very desirable result.[5]

§ 4. Polarization

Let $\boldsymbol{n}$ represent a unit vector normal to the scattering plane; $F_a G(\theta)$ and $F_a H(\theta)$ are, respectively, the scalar and spin amplitudes as calculated in the Born approximation,[5] $\Delta G(\theta)$ and $\Delta H(\theta)$ are the corrections due to diagonal double scattering, $K(\theta)$ and $L(\theta)$ are the non-diagonal double scattering amplitudes. The nucleon-nucleus scattering amplitudes can now be written as follows:

$$g(\theta)+\boldsymbol{\sigma}_0\cdot\boldsymbol{n}h(\theta)=AF_a(\theta)[G(\theta)+\Delta G(\theta)+\boldsymbol{\sigma}_0\cdot\boldsymbol{n}(H(\theta)+\Delta H(\theta))] \\ +A(A-1)[K(\theta)+\boldsymbol{\sigma}_0\cdot\boldsymbol{n}L(\theta)]. \qquad (36)$$

The polarization is then given by

$$P(\theta)=2\frac{\mathrm{Im}\,Pg^*(\theta)h(\theta)}{|g(\theta)|^2+|h(\theta)|^2}, \qquad (37)$$

where Im P is the imaginary part operator and the asterisk represents the complex conjugate. It is obvious that if the non-diagonal terms are neglected, the nuclear form factor cancels from Eq. (37) leaving the polarization independent of nuclear structure. This is observed to be true at small angles where the cross section is determined by diagonal scattering.

$L(\theta)$ is the term which gives rise to spin-flip (and hence polarization in an unpolarized incident beam) via non-diagonal double scattering, and is expected to be zero or very small. For, consider an incident unpolarized beam; let the first scattering take place in the median plane of the nucleus, the plane of the first scattering being the plane of the paper. The resultant polarization must be perpendicular to the plane of scattering—hence assume the particle scattered up has its spin directed into the paper, that scattered down has its spin out of the paper. Assume the second scatterings to be perpendicular to and directed out of the paper (the resultant plane of the total scattering is therefore perpendicular to the paper). In the second scattering, the incident spin is in the scattering plane and hence unaffected; the resultant beam scattered from the nucleus has equal probability of having spin parallel or anti-parallel to its direction and hence is unpolarized, i.e. $L(\theta)\approx 0$.

At the diffraction zero of $F(\theta)$, only $K(\theta)$ will be non-zero. The cross section will not vanish and hence, from Eqs. (36) and (37), the polarization will have a node at the first minimum in the cross section. This is exactly the phenomenon observed in light nuclei like carbon; it is not so obvious in heavier nuclei like lead where, presumably, higher order scattering terms must be important.

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Observables of a Dirac Particle

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The most general forms, consistent with reasonable physical requirements, for the position and other observables of a Dirac particles are obtained.

§ 1. Introduction

In a recent paper[1] the authors investigated the extreme relativistic representation of the dynamical variables of a Dirac particle. The choice of representatives was restricted by requiring that they satisfy certain conditions considered physically reasonable. In particular, the position operator $\boldsymbol{X}$ was required to have the following properties:

(i) Any component X_i of $\boldsymbol{X}$ must obey the usual commutation rules with any component p_j of the momentum operator $\boldsymbol{p}$:

$$[X_i, p_j]=i\delta_{ij}; \tag{1}$$

(ii) $\boldsymbol{X}$ must be defined separately on positive and negative energy states, and consequently must commute with the sign of the energy, given by (H/E), where H is the Hamiltonian and $E=+(p^2+m^2)^{1/2}$, m being the mass of the particle;

(iii) $\boldsymbol{X}$ must be such that the velocity operator $\dot{\boldsymbol{X}}$ is given by

$$\dot{\boldsymbol{X}}=\frac{\boldsymbol{p}}{E}\cdot\frac{H}{E}; \tag{2}$$

(iv) The components of $\boldsymbol{X}$ must commute among themselves.

The operator considered there was such that it satisfied the further condition

(v) that $\boldsymbol{X}$ should be a polar-vector under space reflections and invariant with respect to time inversions.*

Some of the conditions given above have been considered before in the literature,[2),3)] but no attempt has hitherto been made to deduce the most general class of position operators which enjoy these properties. We direct our atten-

* The authors are grateful to the referee for pointing out the fact, overlooked by them, that some undesirable terms in the general position operator would naturally be eliminated by the requirement of time reversal invariance.

tion to this problem in the present paper. We show that there is a two-parameter family of operators having all the above properties, except (iv), and that even if the fourth condition of commutability of components is imposed, there still remain *four* different possibilities* for $\boldsymbol{X}$, of which one corresponds to the Foldy-Wouthuysen[4] 'mean position' operator. This situation remains unchanged even if a further specification is made that all the other dynamical variables, whose operator representatives are determined once the representative of position is chosen, should also satisfy the usual commutation relations.

§ 2. The position operator

To carry out the determination of the position operator one may work in the usual (D) representation in which the Hamiltonian has the form

$$H^D=\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta m, \tag{3}$$

or in the C representation in which the Hamiltonian is

$$H^C=\beta E, \tag{4}$$

or in any other suitable representation.** However, owing to the extreme simplicity of H^C, it is most convenient to work in the C representation. The position operator in this representation will be denoted by $\boldsymbol{X}^C$.

$\boldsymbol{X}^C$ must have commutation relations of the type (1) with the momentum operator, which, on account of the nature of the transformation between the C and D representations,[1] is given by $\boldsymbol{p}$ in both the representations. Thus we require

$$[X_i^C, p_j]=i\delta_{ij}. \tag{5}$$

We know, on the other hand, that

$$[x_i, p_j]=i\delta_{ij}, \tag{6}$$

$\boldsymbol{x}$ being the variable labelling points in the ordinary three-dimensional space, in terms of which $\boldsymbol{p}=-i\boldsymbol{\nabla}$. Hence it is reasonable to look for an expression for $\boldsymbol{X}^C$ in the form

$$\boldsymbol{X}^C=\boldsymbol{x}+\boldsymbol{\xi}^C, \tag{7}$$

where $\boldsymbol{\xi}^C$ is a vector which commutes with $\boldsymbol{p}$. This means that $\boldsymbol{\xi}^C$ can have no dependence on $\boldsymbol{x}$, but may contain $\boldsymbol{\alpha}$, β and $\boldsymbol{p}$ which are the only other independent variables entering the description of the Dirac particle. We may

* In a preprint received while this manuscript was in preparation, Bardakci and Acharya[5] conclude, erroneously, that conditions no more restrictive than the above determine a unique position operator.

** For definitions of the various representations, see Bose, Gamba and Sudarshan[6] or Mathews and Sankaranarayanan.[1]

now impose on $\boldsymbol{X}^C$ the conditions (ii) to (v) successively, remembering that for all the variables occurring in the defining equations we must take expressions appropriate to the C representation in which we are working. Thus the condition (ii) becomes, in view of (4),

$$[\boldsymbol{X}^C, \beta]=0,$$

which immediately reduces to

$$[\boldsymbol{\xi}^C, \beta]=0\,; \tag{8}$$

and (2), written in terms of a typical component, has the form

$$\frac{1}{i}[X_i^C, \beta E]=\frac{p_i}{E}\beta.$$

But since $[x_i, \beta E]=[x_i, E]\beta=ip_i/E\cdot\beta$ and $[\xi_i^C, E]=0$, this again reduces to (8) which therefore expresses both the conditions (ii) and (iii). The most general $\boldsymbol{\xi}^C$ consistent with (8) can now be constructed as a linear combination (with coefficients which may depend on p) of *polar* vectors that can be formed from $\boldsymbol{\alpha}$, β and $\boldsymbol{p}$. It is easily verified that the only such vectors which enjoy the property of commutation with β are

$$(\boldsymbol{\sigma}\times\boldsymbol{p}),\ (\boldsymbol{\sigma}\times\boldsymbol{p})\beta,\ \boldsymbol{p}\beta \text{ and } \boldsymbol{p}.$$

The last two of these, however, change sign under time inversion and are therefore, by (v), not admissible as terms in $\boldsymbol{\xi}^C$. Hence we obtain

$$\boldsymbol{\xi}^C=A(\boldsymbol{\sigma}\times\boldsymbol{p})+B(\boldsymbol{\sigma}\times\boldsymbol{p})\beta, \tag{9}$$

where A, B are functions of p and m which must be homogeneous and of degree -2 in these two quantities in order that $\boldsymbol{\xi}^C$ may have the same dimensions as $\boldsymbol{x}$.

It is clear from the above form that the freedom in the choice of $\boldsymbol{X}^C$ allowed by the conditions other than commutability of components is enormous. Certain specific choices of the functions A, B lead to some of the operators suggested by earlier workers. This is most easily shown by passing over to the D representation in which the position operator is represented by*

$$\boldsymbol{X}^D=e^{-iS}\boldsymbol{X}^C e^{iS}=\boldsymbol{x}+\frac{i\beta\boldsymbol{\alpha}}{2E}-\frac{i\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{p}+(\boldsymbol{\sigma}\times\boldsymbol{p})E}{2E^2(E+m)}$$
$$+\frac{A}{E}\{m(\boldsymbol{\sigma}\times\boldsymbol{p})+i\beta\boldsymbol{\alpha}p^2-i\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{p}\}+B(\boldsymbol{\sigma}\times\boldsymbol{p})\beta. \tag{10}$$

We can now identify the following special cases:

(a) If $A=(E/m-1)/2p^2$, $B=0$, we get Pryce's[2] covariant position operator which he denotes by $\boldsymbol{X}$.

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* For the explicit form of the transformation matrix e^{iS} see ref. 1.

(b) When $A=(m/E-1)/2p^2$, $B=0$, we obtain Pryce's operator $\boldsymbol{q}$.

(c) $A=-1/2p^2$, $B=0$ leads to Pac's[7] position variable.

All three of these have non-commuting components.

(d) If we set $A=B=0$, we recover the position operator obtained by Pryce,[2] Newton and Wigner[3] and Foldy and Wouthuysen.[4] This will appear below as one of the four possibilities for a position operator with commuting components, to the consideration of which we now turn.

§ 3. Commutability of the components of the position operator

The requirement that the components of $\boldsymbol{X}^C=\boldsymbol{x}+\boldsymbol{\xi}^C$ (where $\boldsymbol{\xi}^C$ is given by (9)) should commute in pairs imposes severe restrictions on the functions A and B, but by no means determines a unique solution. In fact, as we have already mentioned, there are four different solutions. This result emerges from a straightforward evaluation of the commutator of any two components of $\boldsymbol{X}^C$. The commutator vanishes in the following cases:

$$\begin{aligned} &A=B=0; && A=-\frac{1}{p^2},\ B=0; \\ &A=-\frac{1}{2p^2},\ B=-\frac{1}{2p^2}; && A=-\frac{1}{2p^2},\ B=+\frac{1}{2p^2}. \end{aligned} \tag{11}$$

The first of these evidently leads to the Foldy-Wouthuysen mean position operator. The other three give new possibilities; for $\boldsymbol{X}^C$ we have, respectively,

$$\begin{aligned} \boldsymbol{X}^C&=\boldsymbol{x}-\frac{1}{p^2}(\boldsymbol{\sigma}\times\boldsymbol{p}), \\ \boldsymbol{X}^C&=\boldsymbol{x}-\frac{1}{2p^2}(\boldsymbol{\sigma}\times\boldsymbol{p})(1+\beta), \\ \boldsymbol{X}^C&=\boldsymbol{x}-\frac{1}{2p^2}(\boldsymbol{\sigma}\times\boldsymbol{p})(1-\beta). \end{aligned} \tag{12}$$

§ 4. The spin operator

Let us now consider the operator for spin which goes together with any particular choice for the position operator. We *define* the spin operator $S=\Sigma/2$ as the difference between the operators for the total and the orbital angular momenta, the latter being, by definition, the vector product of the position and the linear momentum operators:

$$\boldsymbol{S}=\frac{1}{2}\boldsymbol{\Sigma}=\boldsymbol{J}-\boldsymbol{X}\times\boldsymbol{p}. \tag{13}$$

If, in particular, we work in the C representation, we get the corresponding C representation expression

$$\boldsymbol{S}^C = \frac{1}{2}\boldsymbol{\Sigma}^C = \boldsymbol{J}^C - \boldsymbol{X}^C \times \boldsymbol{p}. \tag{14}$$

From the relation of the total angular momentum $\boldsymbol{J}$ to the generator of infinitesimal rotations, we know that $\boldsymbol{J}$ is given (in the D representation) by $\boldsymbol{J}^D = \boldsymbol{x}\times\boldsymbol{p} + 1/2\cdot\boldsymbol{\sigma}$, and unitary transformation with e^{iS} shows* that $\boldsymbol{J}^C$ also is given by exactly the same expression. Thus we have, finally,

$$\frac{1}{2}\boldsymbol{\Sigma}^C = \boldsymbol{x}\times\boldsymbol{p} + \frac{1}{2}\boldsymbol{\sigma} - \boldsymbol{X}^C\times\boldsymbol{p} = \frac{1}{2}\boldsymbol{\sigma} - \boldsymbol{\xi}^C\times\boldsymbol{p}. \tag{15}$$

It is natural to require that the components of the spin operator so defined should obey the usual commutation relations among themselves. If for $\boldsymbol{\xi}^C$ in (15) the general form (9) is taken, it is found that the imposition of the above condition restricts the possible values of the coefficients in (9) to just the four sets given in (11), which were obtained there from the condition of commutability of the components of the position operator. Conversely, if a position operator corresponding to any of the four sets of values given in (11) is chosen, the components of the spin operator which results from the choice will automatically satisfy the usual commutation relations.

Exactly the same situation obtains also in regard to commutabilities of the type

$$[X_i, \Sigma_j] = 0, \quad \text{etc.}$$

§ 5. Discussion

We thus have, finally, four position operators (and corresponding sets of other observables) that are satisfactory in every respect. Of these, the set corresponding to $A=B=0$ is well known from the work of Foldy and Wouthuysen,[4] but the others are believed to be new. It must be emphasized that the latter sets have all the desirable properties associated with the F-W operators, including the fact that the spin and orbital angular momenta are separately constants of the motion. There appears to be no reason, therefore, to prefer any one of the four sets above the others as long as one is dealing with free particles. The interaction with electromagnetic fields and its bearing on a possible choice of one among the above four sets of operators to represent observables of a Dirac particle will be examined in a later publication.

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* See, for instance, Foldy and Wouthuysen, ref. 4).

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On a Radiative Decay of Σ^+

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The radiative decay of Σ^+ is examined by using a dispersion theoretic method. The matrix element is related to those of the pion-photoproduction from proton and the ordinary Σ^+-decay in the lowest mass approximation. In order to obtain the real part of the matrix element, according to Fujii-Iwata's rule we introduce the parameter which reflects the damping behavior of the decay matrix element corresponding to the increase of Q-values. Concerning the effective interaction of $\Sigma^+ \to n^0 + \pi^+$, we assumed the conventional type, i.e. $a^+ + b^+\gamma_5$. Examining our results in the light of the observed decay rate, we conclude that b^+ may be excluded in some accuracy as seen in Table II.

§ 1. Introduction

Recently the studies on the weak interation have been concentrated on the 4-fermion interaction currents.[1] Particularly the ordinary pionic decays of hyperons are being investigated in this connection. The rare modes of the decay then became quite significant and much attention has been directed to the beta decays of Σ and Λ.[1] In the present paper we shall consider another rare mode of decay of Σ^+, i.e.

$$\Sigma^+ \to p^+ + \gamma. \tag{1}$$

Although the above mode is not connected directly with the 4-fermion interaction, it will play an important role when one considers the decay as a whole.

Since one has not assumed any primary interaction which leads directly to the decay (1), the studies of this mode are necessarily related to the ordinary pionic decay modes of which we refer to (0) and (+) as follows:

$$(0): \quad \Sigma^+ \to p^+ + \pi^0 \tag{2}$$

and

$$(+): \quad \Sigma^+ \to n^0 + \pi^+.$$

In early days before the parity non-conservation was discovered Kawaguchi and Nishijima had first studied this process in exploiting the unitary property of the S-matrix.[2] Recently, however, several works have been done with respect to the analysis of (1), much attention being paid to the special interactions V-A or S-P, and calculations have been proceeded in the perturbation method. The results are then obtained in giving the ratio of probabilities of two decay modes:

$$\rho_\gamma = \frac{P(\Sigma^+ \to p^+ + \gamma)}{P(\Sigma^+ \to p^+ + \pi^0)}. \tag{3}$$

In these approaches there are usually many unknown parameters involved in the final results such as various strong and weak interaction coupling constants. Since any statement derived from these results depends necessarily on the values of these parameters, the usual theoretical predictions spread over very wide range of magnitude for ρ_γ. Behrends,[3] with the $(S\text{-}P)$ type and a provisional $(V\text{-}A)$ type[4] weak interactions, obtained

$$\rho_\gamma = \frac{1}{66} \sim \frac{1}{2500},$$

while Prakash and Zimerman gave[4]

$$\rho_\gamma = \frac{7}{100} \sim \frac{100}{100}$$

with the weak interactions of $(V\text{-}A)$ type. They further showed that, with an intermediate boson theory of weak interaction,[5]

$$\rho_\gamma = 1.4 \times 10^{-4}.$$

Experimentally on the other hand observations of (1) are being established to give

$$\rho_\gamma^{\text{obs}} = (0.5 \sim 1)\%, \tag{4}$$[6]

so that the agreement between the theoretical prediction and the observed values is quite poor. We here study this radiative decay of Σ^+ from a dispersion theoretic approach by using as few arbitrary parameters as possible.

§ 2. General expressions of the decay*

Subject to the requirement of invariance of the theory under the proper Lorentz transformation and the gauge transformation for the photon, the general matrix element for the decay (1) can be written as

$$(f|S|i) = -i(2\pi)^4 \delta(q-p-k) \frac{1}{(2\pi)^{9/2}} \sqrt{\frac{mM}{2k_0 p_0 q_0}} \times \bar{u}(p)[A + B\gamma_5] i\gamma k \gamma a u(q), \tag{5}$$

where only two invariant scalar functions are included as parameters characteristic of (1). The 4-momenta, masses and spin angular momenta assignments are shown in Fig. 1. Ad-

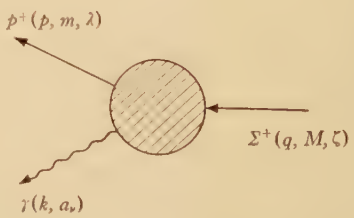

Fig. 1.

* Contents of this section are seen also in references 3) and 4).

ditional requirements for invariances such as T-, C- and P-invariances restrict the range of parameters as

(i) T-invariance
Im $A^*B=0$, the phase difference between A and B is $\pi(\mathrm{mod}\ \pi)$;
(ii) C-invariance
Re $A^*B=0$, A and B have a phase difference of $\pi/2$ $(\mathrm{mod}\ \pi)$;
(iii) P-invariance
A or B is zero. (6)

The partial lifetime or the transition probability of (1) is given by

$$P(\Sigma^+\to p^+ +\gamma)=\frac{1}{\tau_\gamma}=\frac{1}{8\pi}\frac{(M^2-m^2)^3}{M^3}(|A|^2+|B|^2). \tag{7}$$

The differential transition probability reads, in the rest system of the decaying particle, as

$$\begin{aligned} &W(\boldsymbol{\zeta};\boldsymbol{\lambda},\hat{\boldsymbol{k}},\hat{\boldsymbol{a}})\\ &\quad\propto(|A|^2+|B|^2)(\hat{\boldsymbol{a}}\hat{\boldsymbol{a}})[1-\operatorname{sech}\omega(\hat{\boldsymbol{k}}\boldsymbol{\lambda})(\hat{\boldsymbol{k}}\boldsymbol{\zeta})]\\ &\quad+(|A|^2-|B|^2)\{(\hat{\boldsymbol{a}}\hat{\boldsymbol{a}})[(\boldsymbol{\zeta}\boldsymbol{\lambda})-(\hat{\boldsymbol{k}}\boldsymbol{\lambda})(\hat{\boldsymbol{k}}\boldsymbol{\zeta})]-2(\boldsymbol{\zeta}\hat{\boldsymbol{a}})(\boldsymbol{\lambda}\hat{\boldsymbol{a}})\}\\ &\quad+(A^*B+B^*A)(\hat{\boldsymbol{a}}\hat{\boldsymbol{a}})[(\hat{\boldsymbol{k}}\boldsymbol{\zeta})-\operatorname{sech}\omega(\hat{\boldsymbol{k}}\boldsymbol{\lambda})]\\ &\quad-i(A^*B-B^*A)\{(\hat{\boldsymbol{a}}\hat{\boldsymbol{a}})(\hat{\boldsymbol{k}}\cdot[\boldsymbol{\zeta}\times\boldsymbol{\lambda}])+2(\boldsymbol{\zeta}\hat{\boldsymbol{a}})(\hat{\boldsymbol{a}}\cdot[\hat{\boldsymbol{k}}\times\boldsymbol{\lambda}])\}\\ &\qquad\left(\operatorname{sech}\omega=\frac{m}{p_0}=\frac{2mM}{m^2+M^2},\ \ln\frac{M}{m}\equiv\omega\right) \end{aligned} \tag{8}*$$

in the case of a linearly polarized photon. $\hat{\boldsymbol{k}}$ is a unit 3-vector in the direction of the photon propagation, $\boldsymbol{\zeta}$ and $\boldsymbol{\lambda}$ are unit 3-vectors in the directions of the spin angular momenta of Σ^+ and p^+, respectively, and $\hat{\boldsymbol{a}}$ is the photon polarisation vector.

The corresponding expression for the case of a circularly polarized photon is

$$\begin{aligned} &W(\boldsymbol{\zeta};\boldsymbol{\lambda},\hat{\boldsymbol{k}},\epsilon)\\ &\quad\propto\{|A|^2+|B|^2+\epsilon(A^*B+B^*A)\}\\ &\quad\times\{1-\operatorname{sech}\omega(\hat{\boldsymbol{k}}\boldsymbol{\lambda})(\hat{\boldsymbol{k}}\boldsymbol{\zeta})+\epsilon[(\hat{\boldsymbol{k}}\boldsymbol{\zeta})-\operatorname{sech}\omega(\hat{\boldsymbol{k}}\boldsymbol{\lambda})]\}, \end{aligned} \tag{9}$$

where

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* In deriving (8) and (9) use has been made of the projection operator of the fermion spin:[7]

$$S(\boldsymbol{\lambda})=1/2(1-i\lambda\gamma\gamma_5);\ \{\lambda\}=\{\boldsymbol{\lambda},\ i(\boldsymbol{p}\boldsymbol{\lambda})/p_0\},$$

and

$$\boldsymbol{\lambda}=\hat{\boldsymbol{\lambda}}+\boldsymbol{p}(\hat{\boldsymbol{\lambda}}\boldsymbol{p})/m(p_0+m),\ \lambda^2=1,\ \lambda p=0.$$

The vector $\boldsymbol{\lambda}$ is the direction of the spin of the particle with momentum $\boldsymbol{p}$ having $\hat{\boldsymbol{\lambda}}$ as spin direction in the coordinate system where it is at rest.

$$\epsilon = \begin{cases} +1 \text{ for a right-handed photon,} \\ -1 \text{ for a left-handed photon.} \end{cases} \quad (\epsilon^2=1)$$

Now it is interesting to notice that the whole information about the decay (1) is derived from only the two functions, A and B. We shall try to evaluate them in the following sections.

§ 3. Dispersion theoretic approach to A and B

The S-matrix element of the process (1) is written as[8),9)]

$$(f|S|i) = (p, k, a_\mu^{out}|^{in}q) = (-i)(2\pi)^4\delta(q-p-k) \times \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2k_0}} a_\mu^+(k)\langle p|J_\mu(0)|q\rangle, \tag{10}$$

where $J_\mu(x)$ is the source of the photon field:

$$-J_\mu(x) = \Box A_\mu(x). \tag{11}$$

As was suggested before, we shall not assume any primary interaction that leads directly to the decay (1) throughout. The matrix element in (10) is further rewritten as

$$\langle p|J_\mu(0)|q\rangle = \frac{i}{(2\pi)^{3/2}}\sqrt{\frac{M}{q_0}}\int dx e^{iqx}\langle p|T(J_\mu(0), F(x))|0\rangle u(q)$$

$$= \frac{i}{(2\pi)^{3/2}}\sqrt{\frac{M}{q_0}}\int dx\, e^{iqx}\{\langle p|\theta(-x)[J_\mu(0), F(x)]|0\rangle + \langle p|F(x)J_\mu(0)|0\rangle\}u(q), \tag{12}$$

and

$$F(x) \equiv (\gamma\partial + M)\psi_2(x). \tag{13}$$

Since we note that

$$\sqrt{\frac{M}{q_0}}\int dx\langle p|F(x)J_\mu(0)|0\rangle u(q)\, e^{iqx}$$

$$= (2\pi)^4\sqrt{\frac{M}{q_0}}\sum_s \delta(q-p+s)\langle p|F|s\rangle\langle s|J_\mu|0\rangle u(q),$$

which makes no contribution to the matrix element for $\Sigma^+ \to p^+ + \gamma$, we obtain

$$\langle p|J_\mu(0)|q\rangle \equiv \mathcal{H}_\mu u(q)$$

$$= \frac{i}{(2\pi)^{3/2}}\sqrt{\frac{M}{q_0}}\int dx e^{iqx}\langle p|\theta(-x)[J_\mu(0), F(x)]|0\rangle u(q) = (2\pi)^{3/2}\sqrt{\frac{M}{q_0}} \times \sum_s \left\{\frac{\langle p|J_\mu|s\rangle\langle s|F|0\rangle}{s_0-q_0-i\epsilon}\delta(s-q) + \frac{\langle p|F|s\rangle\langle s|J_\mu|0\rangle}{s_0+q_0-p_0+i\epsilon}\delta(s+q-p)\right\}u(q)$$

$$\equiv [\mathcal{D}_\mu(q_0) + i\mathcal{C}_\mu(q_0)]u(q). \tag{14}$$

That $\mathcal{H}_\mu$ satisfies a standard type of the dispersion relation as a function of complex q_0, and therefore of $-q^2$, will be shown in a similar way to that Kawarabayashi and Machida did in their study of the electromagnetic structure of the nucleon.[10]* Taking the rest system of p and considering $\mathcal{H}_\mu$ as a function of the complex q_0, it can be analytically continuated into the complex plane; namely, no analytic continuation in terms of "τ" is not required as in the attempts of Chew et al.[11] After changing the variable from q_0 to an invariant $-q^2$, we have

$$\mathcal{H}_\mu(-q^2)=\frac{1}{\pi}\int\frac{\mathrm{Im}\,\mathcal{H}_\mu(\tilde{\xi})}{\tilde{\xi}+q^2}\,d\tilde{\xi}. \tag{15}$$

By inserting in (14) a sum over intermediate states, we obtain the absorptive part:

$$\mathcal{A}_\mu(q_0)\,u(q)=\sqrt{\frac{M}{(2\pi)^3 q_0}}\,(2\pi)^3\pi\sum_s\delta(s-q)\langle p|J_\mu|s\rangle\langle s|F|0\rangle u(q). \tag{16}$$

Terms with J_μ and F interchanged are dropped in (16), because they make no contributions to the present problem.

In order to proceed with calculations it may be noticed that the transition we consider is the one between two states with the different strangeness quantum numbers, while the other quantities are conserved, i.e. the electric charge, the nucleon number and the angular momentum remain unchanged from the initial to the final states, +1, +1, and 1/2, respectively. Taking this into account, one finds that, among the states that contribute to (16), the lowest mass states are the $(p^+\gamma's)$- and $(N\pi)$-states. The former are shown to give so much smaller contributions that they may be neglected in the present calculations. Turning to $(N\pi)$-states which have strangeness 0, the F factor should manage the only weak interaction. The matrix element is expressed as products of those for the ordinary Σ^+ decays (2) and for the inverse processes of the photoproduction of π. In order to estimate contributions from states of higher energies, the detailed knowledge about the related matrices should be required. Since, however, they have been seldom observed, we shall not enter into details of these things, but we shall make their estimates through an extrapolation from the lowest energy contributions, by making use of an "empirical" cutoff parameter.

§ 4. $(N\pi)$-states contributions

The two $(N\pi)$-states, $(p^+\pi^0)$- and $(n^0\pi^+)$-states, contribute to the absorptive part:

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* See also references 9).

$$a_\mu{}^+(k)\,\mathcal{O}_\mu(q_0)\,u(q)=(2\pi)^{3/2}\pi\sqrt{\frac{M}{q_0}}\int d\Omega_l\begin{pmatrix}Lp_0'l_0\\ q_0\end{pmatrix}\langle p|Ja^+|p'l\rangle\langle p'l|F|0\rangle u(q), \tag{17}$$

where the momenta and masses in the intermediate states are denoted as p', m and l, μ for the nucleon and the pion, respectively.

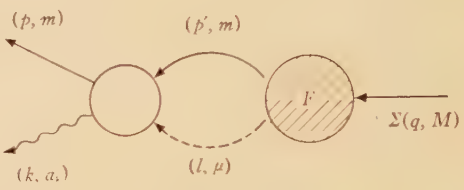

Fig. 2.

The second factor in (17) is related to the matrix elements of processes (2), (0) : $\Sigma^+\to p^+\pi^0$ and (+) : $\Sigma^+\to n^0+\pi^+$, of which the most general expression has a form

$$\langle p'l|F|0\rangle u(q)=\frac{1}{(2\pi)^3}\sqrt{\frac{m}{2l_0p_0'}}\,\varphi_\alpha{}^*(l)\,\bar{u}(p')[a^\alpha+b^\alpha\gamma_5]u(q), \tag{18}$$

and the parameters therein are experimentally fairly well known.

The first factor, on the other hand, is the matrix element of the inverse process of the photo-pion production which is well investigated by Chew et al., and others.[12]* According to them, this can be decomposed into 4 invariant forms. Correspondingly, the contribution from each $(N\pi)$-state is written as a sum of 4 terms :

$$a_\nu{}^+(k)\,\mathcal{O}_\nu u(q)=\frac{i}{2(2\pi)^5}\sum_\alpha\sqrt{\frac{mM'}{p_0q_0}}[\varphi_\alpha(l)\,\varphi_\alpha{}^+(l)]\left(\frac{m}{2q_0}\right)L\int d\Omega_l[\sum_{j=A}^{D}X_j{}^\alpha|,$$

and

$$X_A{}^\alpha\equiv\bar{u}(p)\left(-\frac{i}{2m}A^\alpha\right)(\gamma a^+)(\gamma k)\,\mathcal{O}^\alpha u(q), \tag{19}$$

$$X_B{}^\alpha\equiv\bar{u}(p)\left(\frac{i}{m}B^\alpha\right)\{(Pa^+)(kl)-(la^+)(Pk)\}\,\mathcal{O}^\alpha u(q),$$

$$X_C{}^\alpha\equiv\bar{u}(p)\left(\frac{1}{2m}C^\alpha\right)\{(\gamma a^+)(kl)-(la^+)(\gamma k)\}\,\mathcal{O}^\alpha u(q),$$

$$X_D{}^\alpha\equiv\bar{u}(p)\left(\frac{1}{m}D^\alpha\right)\{(\gamma a^+)(Pk)-(Pa^+)(\gamma k)-im(\gamma k)(\gamma a^+)\}\,\mathcal{O}^\alpha u(q),$$

where

$$\mathcal{O}^\alpha\equiv b^\alpha(m-M'-i\gamma l)+a^\alpha\gamma_5(m+M'-i\gamma l)$$

$$P\equiv\frac{1}{2}(p+p')$$

* Hereafter their works will be referred to as CGLN.

$$\alpha = \begin{cases} 0 & \text{for the intermediate } (p^+\pi^0)\text{-state,} \\ + & \text{for the intermediate } (n^0\pi^+)\text{-state.} \end{cases} \tag{19}$$

$$M'^2 = -q^2.$$

A^α, B^α, etc., are certain functions of invariants:

$$\nu = -Pk/m = Pl/m,$$

$$\nu_B = -kl/2m$$

and

$$q^2, \tag{20}$$

as well as the nucleon isotopic spin. In CGLN each of them in its turn consists of three parts according to its isotopic spin character, so that there are totally 12 quantities which should be denoted by H_j ($j=1, 2, \cdots, 12$). CGLN then gave a set of dispersion relations for H_j:

$$H_j(\nu, \nu_B) = R_j\left(\frac{1}{\nu_B - \nu} \pm \frac{1}{\nu_B + \nu}\right) + \frac{1}{\pi}\int_{\nu_0} d\nu' \operatorname{Im} H_j(\nu', \nu_B)\left(\frac{1}{\nu' - \nu} \pm \frac{1}{\nu' + \nu}\right), \tag{21}$$

where

$$\nu_0 \equiv \nu_B + \mu + \mu^2/2m.$$

($\pm$) signs correspond to the crossing symmetry thereof.

Since we are concerned with the matrix elements between states of $J=1/2$, the usual resonant states in the $(N\pi)$-system will not affect the result so far as our interest is confined in the lowest mass configurations. Then the rescattering corrections of the $(N\pi)$-scattering in the $J=1/2$ state may be so small that one may neglect them without touching on the characteristic structure of the matrix element. With this expectation we shall drop the second terms in (21). The explicit expressions of A's, etc., are given in this approximation as follows.

For the intermediate $(p^+\pi^0)$-state, $(\alpha=0)$,

$$A^0 = -f_r e_r\left(\frac{1}{\nu_B - \nu} + \frac{1}{\nu_B + \nu}\right)$$

$$B^0 = +f_r e_r \frac{1}{2m\nu_B}\left(\frac{1}{\nu_B - \nu} + \frac{1}{\nu_B + \nu}\right)$$

$$C^0 = f_r \mu_p\left(\frac{1}{\nu_B - \nu} - \frac{1}{\nu_B + \nu}\right)$$

$$D^0 = f_r \mu_p\left(\frac{1}{\nu_B - \nu} + \frac{1}{\nu_B + \nu}\right),$$

and for the intermediate $(n^0\pi^+)$-state, $(\alpha=+)$,

$$A^+=-f_\tau e_\tau \frac{1}{\nu_B-\nu}$$

$$B^+=+f_\tau e_\tau \frac{1}{2m\nu_B}\frac{1}{\nu_B-\nu}$$

$$C^+=+f_\tau \mu_p \frac{1}{\nu_B-\nu}-f_\tau \mu_n \frac{1}{\nu_B+\nu}$$

$$D^+=f_\tau \mu_p \frac{1}{\nu_B-\nu}+f_\tau \mu_n \frac{1}{\nu_B+\nu}, \tag{22}$$

where

$$f_\tau^2/4\pi=0.08,$$
$$e_\tau^2/4\pi=1/137.0,$$
$$\mu_p=\mu^S+\mu^V=1.793 \ (e_\tau/2m),$$
$$\mu_n=\mu^S-\mu^V=-1.913 \ (e_\tau/2m). \tag{23}$$

Substituting these into (19), the contributions to the imaginary parts from two $(N\pi)$-states are calculated straightforwardly. Those for the initial lowest mass state, $-q^2=M^2$, run as

$$a_\nu^+ \mathcal{A}_\nu u(q)\Big|_{p^+\pi^0}=i\frac{1}{(2\pi)^3}\sqrt{\frac{mM}{p_0 q_0}}\frac{1}{\text{Mev}}$$
$$\times\bar{u}(p)\left\{-3.71\times10^{-6}b^0-7.26\times10^{-5}a^0\gamma_5\right\}i\gamma k\gamma a^+ u(q)$$

and

$$a_\nu^+ \mathcal{A}_\nu u(q)\Big|_{n^0\pi^+}=i\frac{1}{(2\pi)^3}\sqrt{\frac{mM}{p_0 q_0}}\frac{1}{\text{Mev}}$$
$$\times\bar{u}(p)\left\{+1.55\times10^{-6}b^+ +9.14\times10^{-5}a^+\gamma_5\right\}i\gamma k\gamma a^+ u(q). \tag{24}$$

Now we return to the dispersion relations (15). Since what we want to know is the value of the function at $-q^2=M^2$, it may conveniently be rewritten as

$$\mathcal{H}_\nu(M^2)=i\,\text{Im}\,\mathcal{H}_\nu(M^2)+\frac{1}{\pi}\int_{(1+\mu)^2}\frac{\text{Im}\,\mathcal{H}_\nu(\xi)}{\xi-M^2}d\xi. \tag{25}$$

Even with the approximations applied above, only the first term of (25) has been obtained. Our next task is to evaluate the second term or the real part of the matrix element. Though detailed information about it has not yet been obtained, its gross behavior may be guessed in some plausible ways as follows.

First we try to get an extention of Fujii-Iwata's rule.[13] According to them,

there would be some regularities among magnitudes of matrix elements of the various decay modes of the elementary particle. One of them will read as follows: when a common elementary particle, say, the K-meson, decays through a variety of channels, the magnitudes of the individual decay matrix elements are related as

$$\text{(I)} \quad M=\text{constant}\times e^{\frac{b-Q}{2a}}, \quad (Q:\ \text{Q-values}) \tag{26}$$

where

$$a \approx (36 \sim 38)\ \text{Mev}$$

and

$$b \approx (950 \sim 1000)\ \text{Mev}.$$

One of the most surprising things in this formula is that, while the matrix elements themselves there vary over 10^2, any deviation from it does not exceed more than factor 2. Although the formula, as it stands, is not suitable for the present materials, we can examine the same method to the cases with common daughters. Some examples are studied in the Appendix by taking 3 sets of decay daughters. We might then put

$$2a \approx (200 \sim 1000)\ \text{Mev} \tag{27}$$

for the present problem. We also tried another type of the damping factor,

$$\text{(II)} \quad M=\text{constant}\times Qe^{-\frac{Q}{2a}} \tag{28}$$

with

$$2a \approx (80 \sim 100)\ \text{Mev}.$$

After these assumptions the dispersion relation converges to lead to a finite real part as the things would be. Secondly, the damping property of the matrix element is also suggestive from a field theoretical point of view. As Goldberger and Treiman showed in their studies on the β-decay, μ-capture, etc., by using their dispersion relation, states of the higher mass configuration would contribute in an exponential to make the amplitude damping.[9]

Under the presumptions (26) to (28), we are ready to obtain the real part of the transition matrix element at $-q^2=M^2$, the square of mass of Σ^+.

$$\text{Re}\,\mathcal{H}_\nu(M^2)=\text{Im}\,\mathcal{H}_\nu(M^2)\,\frac{1}{\pi}\int_{(\lambda+\mu)^2}\frac{[\text{Damping factor}]}{\xi-M^2}\,d\xi$$

Because of their non-uniqueness of the cutoff or the damping parameters, results are shown for cases with extended sets of $2a$ values.

§ 5. Numerical results and discussions

In the preceding sections the transition matrix element for (1) was related to those of the pion photoproduction and the ordinary decays (2), and to another

parameter which we might call the damping factor. The lowest mass state contributions are given in (24), and the other higher order contribution was suggested to be obtained by an extrapolation method. Now in order to get final estimates of A and B, it is necessary to fix the parameters of the ordinary decay (2). The experiments have shown that[14]

$$\tau(\Sigma^+ \to p^+ + \pi^0) = 1.65 \times 10^{-10} \text{ sec.}$$
$$\frac{\tau(\Sigma^+ \to p^+ + \pi^0)}{\tau(\Sigma^+ \to n^0 + \pi^+)} = 1.00 \pm 0.04, \tag{29a}$$

and for the asymmetry parameters[15]

$$\alpha^0 \bar{P} = 0.75 \pm 0.17,$$
$$\alpha^+ \bar{P} = 0.03 \pm 0.08. \tag{29b}*$$

Assuming that the $|\Delta I| = 1/2$ is valid in the decay:

$$A_+ + 2A_0 = A_-,$$

Cork et al. also gave an arranged data that

$$|\alpha^0| = 0.99 \begin{matrix} +0.01 \\ -0.05 \end{matrix}$$

and

$$|\alpha^+| = 0.04 \pm 0.11. \tag{29c}$$

The pionic decay is completely described in giving the two complex quantities a^α and b^α. If we require the T-invariance in this decay we are led to 4 sets of parameters which can equally well reproduce (29) (Table I). With these sets of parameters, from Case I to Case IV, corresponding ρ_γ values are listed in Table II.

Table I. Parameters of the Σ^+-ordinary decay (with $|\Delta I| = 1/2$ and T-invariance).

Decay modes	$\Sigma^+ \to p^+ + \pi^0$		$\Sigma^+ \to n^0 + \pi^+$	
Cases	a^0 (in 10^{-7})	b^0 (in 10^{-7})	a^+ (in 10^{-7})	b^+ (in 10^{-7})
I	+2.9	+29	−4.1	0
II	+2.9	+29	0	−42
III	+2.9	−29	−4.1	0
IV	+2.9	−29	0	+42

* α^0 and α^+ denote the characteristic asymmetry parameters in the decays (0) and (+) of (2), respectively; $\bar{P}$ is the average polarization of Σ^+ samples which decay through channels (0) and (+), too.

Table II. Calculated ρ_γ for several *extended* damping parameters, to be compared with $\rho_\gamma^{ob}=(0.5\sim1.0)$ % (4).

Damping factors			I					II
$(2a)$ in Bev			0.2	1	2	4	10	0.1
Real part/Imag. part		0	0.29	0.79	1.1	1.4	~2	0.3
ρ_γ in %	Cases I, III	0.32	0.35	0.52	0.70	0.94	~1.2	0.35
	Cases II, IV	0.066	0.072	0.11	0.15	0.2	~0.33	0.072

Referring to the experimental values (4), $\rho_\gamma^{ob}=(0.5\sim1)\%$, it will be seen from Table II that for the purpose of reconciling (4) with the ordinary pionic decays it is necessary to consider that Case I or III is realized. Namely, as far as the present calculations are concerned the transition (+), $\Sigma^+\to n^0+\pi^+$, may be caused mainly through an interaction of parity reserving character and, if any, much less through the (γ_5)-interaction. This can also be a reflection of the fact that the photoproduction of positive pions from the proton occurs through the $E1$ interaction at low energies.

In the course of calculations use has been made of the $|\Delta I|=1/2$ rule, or (29c). At first sight this would be helpful to make the transition structure much simplified. Even without this assumption, however, the same conclusion will be attained. $|\Delta I|=1/2$ here manifests itself in giving predictions of almost the largest ρ_γ values among other possibilities. It will then be remarked that ρ_γ(I, III) and ρ_γ(II, IV) will never overlap with each other and ρ_γ(I, III) $>\rho_\gamma$(II, IV). Referring to the fact that the observed ratio ρ_γ^{ob} is high and examining the results in Table II with respect to this, one can conclude that Case I or III may be realized in the decay (2). Further selection between Cases I and III, however, is not possible as long as only ρ_γ is concerned.

For this purpose there should be further examination of relations between the two decay modes, (2 ; 0) and (1), in their up-down asymmetries and polarizations of the ejected protons (Table III). As will be seen from the table, the proton polarization will follow its sign from (2 ; 0) to (1), so that the helicity of the emitted protons is + in Cases I and II, and is − in Cases III and IV. The up-down asymmetry in the proton distributions on the other hand will be opposite in the two modes, i.e. protons in (1) will dart forth more in downward direction with respect to the Σ^+-spin, while those from $\Sigma^+\to p^+ +\pi^0$ will come preferentially upwards for Cases I and II, or *vice versa* in Cases III and IV.

The present calculation has many restrictions on its validity. First, in order to obtain the real part of the matrix element, it is required to know the true imaginary part over the whole region of the argument. In our calculations, however, a certain damping behavior was assumed. Second, approximate matrix

Table III. A and B with the vanishing real parts, and qualitative relations in the decay parameters.

	A(in 10^{-12}/Mev)	B(in 10^{-12}/Mev)	$\Sigma^+ \to p^+ + \gamma$		$\Sigma^+ \to p^+ + \pi^0$	
			proton helicity	proton emitted preferentially	proton helicity	proton emitted preferentially
I	+11	+59	+	downwards	+	upwards
II	+17	+21	+	downwards	+	upwards
III	−11	+59	−	upwards	−	downwards
IV	−17	+21	−	upwards	−	downwards

elements of the photoproduction processes are used even when the lowest mass contributions are considered. Further, one would be able to calculate each factor in (17), with recourse to the dispersion relations, for all values of the argument $-q^2$. This would however bring many undeterminable parameters into the final results. At any rate, the minute structures at higher energies were not considered in the present calculations. When we take account of the higher mass configurations, for example, $(N2\pi)$-states, the well-known resonant states will become to play an important role. Further, some new energy levels or the so-called elementary particles may also be expected to exist which can make certain contributions to the present problem. In this paper, however, these things were not treated individually. One may hope that in the near future the qualitative relations in Table III will be investigated experimentally, and ρ_γ will be established. All these things then put the present method of approach to a test and an idea of Fujii-Iwata's damping factor may be examined. Finally it is remarked that there has not been assumed any primary interaction which will directly cause the decay (1). Whether the primary interaction large enough to alter the present conclusions exists is still an open question.

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Appendix

The damping of the decay matrix elements with increasing Q-values will be investigated, by taking as examples 3 pairs of particles where both particles in each pair decay into a common set of daughters. After Fujii and Iwata,[13] we first assume that the decay matrix elements depend on the Q-value as

$$\text{(I)} \quad e^{-\frac{Q}{2\sigma}}. \qquad \text{(A·I)}$$

Our problem is to apply this to decays with somecommon daughters. For $K\to\mu+\nu$ and $\pi\to\mu+\nu$, the ratio of magnitudes of the corresponding (observed) matrix elements and that of Q-values are respectively

$$1 : 6.25, \qquad (A\cdot 2)$$

and

$$388 : 33. \qquad (A\cdot 2)'$$

This requires to take

$$2a \approx 200 \text{ Mev}. \qquad (A\cdot 3)$$

Similar values, (150~200) Mev, are also derived for $2a$ by arguing the second pair, $n\to p+e+\nu$ and $\Lambda\to p+e+\nu$. For $\Sigma\to N+\pi$ and $\Lambda\to N+\pi$, on the other hand, those ratios are

$$1.3 : 1 \quad \text{(for matrix elements)},$$

and

$$110 : 37 \quad \text{(for Q-values)},$$

so that there is no satisfactory solution for $2a$ ($2a$ becomes negative!). Taking into account that the rule (I) is not strict but has allowances of factor 2, we may take in this case

$$2a \sim 1000 \text{ Mev}. \qquad (A\cdot 4)$$

Complying with these situations we calculated in Table II the ratios ρ_γ for several extended values of $2a$ from 200 Mev to 10000 Mev.

If we assume, instead of (I), for the Q-value dependence of the matrix elements, that

$$\text{(II)} \quad Qe^{-\frac{Q}{2a}}, \qquad (A\cdot 5)$$

we can fix the parameter as

$$2a = (50 \sim 100) \text{ Mev} \qquad (A\cdot 6)$$

for all the 3 examples. This may indicate that the ratio of the real part to the imaginary part is less than 0.3, and accordingly ρ_γ is rather small.

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Effects of the γ-3π Interaction on the Electromagnetic Properties of Deuteron*

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Various effects of the γ-3π interaction on the electromagnetic properties of the deuteron are examined by making use of the effective interaction Hamiltonian suggested from the isoscalar part of the electromagnetic form factors of the nucleon. The contribution of the γ-3π interaction to the magnetic moment of the deuteron is unreasonably large, namely comparable with the observed value, although this result depends strongly on the inner part of the deuteron wave function and the pion dissociation probability of the nucleon. The electric quadrupole moment due to the γ-3π interaction is about 20% of the observed value. The γ-3π interaction also contributes to the photodisintegration of the deuteron through the $M1$ and $E2$ transitions with the ${}^3S+{}^3D$ final state. The interference term between the usual $E1$ term and ours contributes appreciably to an asymmetry in the angular distribution.

§ 1. Introduction

The electromagnetic structure of the nucleon has been extensively investigated for several years from both experimental and theoretical points of view. After much efforts were devoted, an important question is still open concerning the isoscalar part of the nucleon form factors. It has been pointed out that the three-pion state would be responsible for the large isoscalar radius of the charge.[2),3)] The contribution of the γ-3π interaction to the isoscalar part of the form factors seems to be promising for the interpretation of experimental results, though quantitative agreement has not yet been obtained.[4),5)] Because of difficulties in calculating the form factors, it would be worth while to investigate various kinds of photoeffects in order to get further information on the γ-3π interactions.

In the present paper we would like to point out that the γ-3π interaction gives rise to considerable effects on the electromagnetic properties of the deuteron.[1)]

First of all, we postulate the phenomenological local Hamiltonian as[5)]

$$H=ie\lambda\varepsilon_{\lambda\mu\nu\rho}A_\lambda(x)\frac{\partial\varphi_1}{\partial x_\mu}\frac{\partial\varphi_2}{\partial x_\nu}\frac{\partial\varphi_3}{\partial x_\rho}, \tag{1·1}$$

* The preliminary report has been published in the Nuovo Cimento. See reference 1).

where λ is the effective coupling constant, $A_\lambda(x)$ the electromagnetic potential, and φ_i the real pion field operator. We take the unit $\hbar=c=\mu$ (pion mass)$=1$ throughout this paper. In order to give the idea on sign and rough magnitude of λ, the diagram of the γ-3π process through a nucleon closed loop is calculated in the lowest order perturbation, and one gets

$$\lambda_0=\frac{16}{\sqrt{\pi}}f^3=0.2 \quad \text{for } f^2=0.08. \tag{1·2}$$

This is of the right sign, but too small to interpret the isoscalar charge radius.

Fig. 1. Diagram of pion photoproduction due to the γ-3π interaction. Nucleon is expressed by a solid line, pion by a broken line, and photon by a wavy line. The shaded part is replaced by the isovector part of the nucleon electromagnetic form factors.

One of the evidences that λ in (1·1) cannot be much larger than λ_0 of (1·2) is found in the plus-minus ratio of the pion photoproduction at threshold.[1),6)] Taking the diagram in Fig. 1 into account, the ratio is modified by the γ-3π interaction as

$$\frac{\sigma(-)}{\sigma(+)}=\frac{\left[1+\frac{1}{2M}+\frac{1}{2M}(\mu_p+\mu_n)+\frac{\lambda}{g}\left\{\frac{1}{4}(\mu_p-\mu_n)+\frac{1}{2}\zeta\right\}\right]^2}{\left[1-\frac{1}{2M}-\frac{1}{2M}(\mu_p+\mu_n)-\frac{\lambda}{g}\left\{\frac{1}{4}(\mu_p-\mu_n)+\frac{1}{2}\zeta\right\}\right]^2}, \tag{1·3}$$

where M is the nucleon mass, μ_p and μ_n the anomalous magnetic moments of proton and neutron in the nuclear magneton, respectively, ζ the pion dissociation probability of a nucleon, and the pion-nucleon coupling constant g is chosen as $g^2/4\pi=15$. Comparing with the experimental value $\sigma(-)/\sigma(+)\sim1.35$, we find λ to lie between

$$\lambda=0.17 \quad \text{for } \zeta=1,$$

and

$$\lambda=0.26 \quad \text{for } \zeta=0.* \tag{1·4}$$

The electromagnetic properties of the deuteron is affected by the γ-3π interaction given by (1·1) with the coupling constant $\lambda=0.2$. Since this interaction does not change the isotopic spin, it does contribute to the magnetic moment and the electric quadrupole moment of the deuteron as will be shown in § 3 and § 4, respectively.

The results calculated using the conventional deuteron wave function are surprisingly large, although they depend too strongly on the behavior of the wave function in the inner region to draw any definite conclusion. It would

* The isovector part of the charge form factor $G_1(q^2)$ is about $0.2\times e/2$ for large momentum transfer q^2. This suggests $\zeta\gtrsim0.8$.[7)]

still be worth while to note that even a very moderate estimation, taking the contribution only from the outer region into account, gives the extra magnetic moment of a magnitude comparable with the difference between the observed value and the sum of the intrinsic ones of the constituents.

§ 5 is devoted to the photodisintegration. In the high energy region around $E_\gamma \approx 150$ Mev, this process is usually described in terms of the $E1$ transition into the ^3P and ^3F final states and of the $M1$ transition into the ^1D state through the (33) pion-nucleon resonance.[8] Other transitions are considered to be relatively small. Theoretical angular distribution is, therefore, almost symmetric with respect to 90°, while experiment shows a big forward peak. Since the γ-3π interaction contributes to the $M1$ and $E2$ transitions into the ^3S and ^3D states, the $\cos\theta$ term and $\sin^2\theta\cos\theta$ term are obtained from the interference between the usual $E1$ transition and the $M1$ and $E2$ transitions due to the γ-3π interaction.

In the Appendix, it is shown, on the basis of the chain approximation, that the coupling constant λ is common for all the processes associated with the γ-3π interaction, even if the pion-pion rescattering corrections are taken into account.

§ 2. Effective Hamiltonian for the deuteron photoeffects

In order to obtain the effective Hamiltonian for the electromagnetic interaction of the deuteron from (1·1), we calculate the transition matrix element of the Hamiltonian (1·1) between the initial state with two nucleons and a photon and the final state with two nucleons. Corresponding to the diagram (a) in Fig. 2, one sees[9]

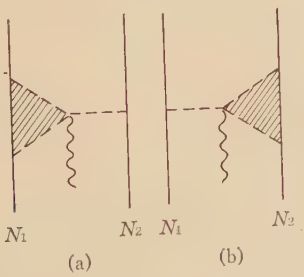

Fig. 2. Diagrams for the electromagnetic processes of the deuteron due to the γ-3π interaction. N_1 and N_2 denote the first and the second nucleon, respectively. The meaning of the lines are the same with Fig. 1.

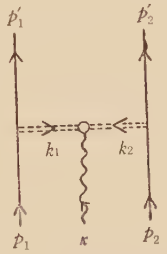

Fig. 3. The assignment of the momenta of the lines in Fig. 2. k_1, for instance, means the sum of momenta of two pions in Fig. 2 (a), while it means the momentum of single pion in Fig. 2 (b).

$$\langle p_1', p_2'|H|p_1, p_2; \kappa\rangle_a = i\frac{1}{4}\lambda\varepsilon_{\mu\nu\lambda\rho}\langle p_2'|\partial_\rho\varphi_k|p_2\rangle F_{\lambda\mu}(\kappa)\langle p_1'|\varphi_i\varepsilon_{ijk}\partial_\nu\varphi_j|p_1\rangle, \tag{2·1}$$

where

$$F_{\lambda\mu}(\kappa) = i[\kappa_\lambda A_\mu(\kappa) - \kappa_\mu A_\lambda(\kappa)].$$

The momenta of the particles are specified as shown in Fig. 3. The bold face letters represent three-dimensional momenta while the light face ones denote four-momenta.

(2·1) can be calculated by using

$$\langle p_2'|\partial_\rho\varphi_k|p_2\rangle = g\frac{(p_2-p_2')_\rho}{(p_2-p_2')^2+1}\bar{u}(p_2')\gamma_5^{(2)}\tau_k^{(2)}u(p_2), \tag{2·2}$$

and

$$\langle p_1'|\varphi_i\varepsilon_{ijk}\partial_\nu\varphi_j|p_1\rangle = -\frac{1}{e}\bar{u}(p_1')\tau_k^{(1)} \times\left\{i\gamma_\nu^{(1)}G_1((p_1-p_1')^2)+\sigma_{\nu\sigma}^{(1)}(p_1-p_1')_\sigma G_2((p_1-p_1')^2)\right\}u(p_1), \tag{2·3}$$

where G_1 and G_2 are the two-pion part of the isovector electromagnetic form factors of the nucleon,[2),3)] corresponding to the charge density and magnetization density, respectively.

Introducing T_a as

$$\langle p_1', p_2'|H|p_1, p_2; \kappa\rangle_a = \frac{1}{\sqrt{2\kappa}}\bar{u}(p_2')\bar{u}(p_1')T_a u(p_2)u(p_1),$$

we have

$$T_a = -i\frac{1}{4}\lambda g\boldsymbol{\tau}^{(1)}\boldsymbol{\tau}^{(2)}\varepsilon_{\mu\nu\lambda\rho}F_{\lambda\mu}k_{2\rho}\frac{1}{k_2^2+1}[i\gamma_\nu^{(1)}G_1(k_1^2)+\sigma_{\nu\sigma}^{(1)}k_{1\sigma}G_2(k_1^2)]\gamma_5^{(2)}, \tag{2·4}$$

where k_1 and k_2 are introduced in Fig. 3 as

$$k_1 = p_1 - p_1', \quad k_2 = p_2 - p_2'.$$

Similarly, we have

$$T_b = -i\frac{1}{4}\lambda g\boldsymbol{\tau}^{(1)}\boldsymbol{\tau}^{(2)}\varepsilon_{\mu\nu\lambda\rho}F_{\lambda\mu}k_{1\rho}\frac{1}{k_1^2+1}\gamma_5^{(1)}\left[i\gamma_\nu^{(2)}G_1(k_2^2)+\sigma_{\nu\sigma}^{(2)}k_{2\sigma}G_2(k_2^2)\right]. \tag{2·5}$$

In the nonrelativistic approximation for Dirac spinors T_a and T_b can be written in the form

$$\begin{aligned} T_a &= -\boldsymbol{\tau}^{(1)}\boldsymbol{\tau}^{(2)}\frac{1}{4M}g\lambda\frac{1}{k_2^2+1}(\boldsymbol{e}_a\boldsymbol{E}+\boldsymbol{h}_a\boldsymbol{H}), \\ T_b &= -\boldsymbol{\tau}^{(1)}\boldsymbol{\tau}^{(2)}\frac{1}{4M}g\lambda\frac{1}{k_1^2+1}(\boldsymbol{e}_b\boldsymbol{E}+\boldsymbol{h}_b\boldsymbol{H}), \end{aligned} \tag{2·6}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the Fourier components of the electric and magnetic field strengths, respectively. $\boldsymbol{e}_a$ and $\boldsymbol{h}_a$ are given by

$$\boldsymbol{e}_a = \frac{1}{M}(\boldsymbol{k}_1\times\boldsymbol{q}_1)(\boldsymbol{\sigma}_2\boldsymbol{k}_2)\,G_1(k_1^2)+i[(\boldsymbol{\sigma}_1\boldsymbol{k}_1)\boldsymbol{k}_1-\boldsymbol{k}_1^2\boldsymbol{\sigma}_1](\boldsymbol{\sigma}_2\boldsymbol{k}_2)\,G(k_1^2), \tag{2·7}$$

$$\boldsymbol{h}_a = -\boldsymbol{k}_1(\boldsymbol{\sigma}_2\boldsymbol{k}_2)\,G_1(k_1^2)+k_{10}\left\{\left[\boldsymbol{q}_1+\frac{1}{2}i(\boldsymbol{k}_1\times\boldsymbol{\sigma}_1)\right]\frac{1}{M}G_1(k_1^2)\right.$$
$$\left.+i(\boldsymbol{k}_1\times\boldsymbol{\sigma}_1)\,G_2(k_1^2)\right\}(\boldsymbol{\sigma}_2\boldsymbol{k}_2), \tag{2·8}$$

where

$$\boldsymbol{q}_1=\frac{1}{2}(\boldsymbol{p}_1+\boldsymbol{p}_1'),$$
$$\boldsymbol{q}_2=\frac{1}{2}(\boldsymbol{p}_2+\boldsymbol{p}_2'), \tag{2·9}$$

and

$$G(k^2)=\frac{1}{2M}G_1(k^2)+G_2(k^2). \tag{2·10}$$

$\boldsymbol{e}_b$ and $\boldsymbol{h}_b$ are obtained by exchanging the subscripts 1 and 2 of momenta and spins in (2·7) and (2·8).

$G_1(k^2)$ and $G_2(k^2)$ can be given in a spectral representation as follows:

$$G_1(k^2)=\frac{1}{\pi}\int dm^2\frac{g_1(m^2)}{k^2+m^2},$$
$$G_2(k^2)=\frac{1}{\pi}\int dm^2\frac{g_2(m^2)}{k^2+m^2}. \tag{2·11}$$

No subtraction is assumed both for G_1 and G_2, because we are calculating only the contribution from the two-pion state.[9] In view of the tentative nature of the present calculation we use the following approximate forms for G in the numerical estimations,

$$G_1(k^2)=\frac{e}{2}\zeta\frac{b^2}{k^2+b^2},$$
$$G_2(k^2)=\frac{e}{4M}(\mu_p-\mu_n)\frac{b^2}{k^2+b^2}. \tag{2·12}$$

These are normalized as

$$G_1(0)=\frac{e}{2}\zeta,$$
$$G_2(0)=\frac{e}{4M}(\mu_p-\mu_n).$$

b^2 is related to the root-mean-square radius r of the form factor as

$$\frac{1}{6}r^2=\frac{1}{b^2}.$$

In the following we assume

$$b^2=18 \quad \text{or} \quad 24,$$

corresponding to

$$r=0.81\times10^{-13}\,\text{cm},$$

and

$$r=0.70\times10^{-13}\,\text{cm},$$

respectively.

§ 3. Magnetic moment

Let us calculate the magnetic moment of the deuteron. For a constant magnetic field $\boldsymbol{H}$, the biggest contribution comes from the first term of (2·8). Then the magnetic interaction Hamiltonian is found to be

$$-\boldsymbol{\mu}\boldsymbol{H}=\frac{3}{4}\frac{1}{M}\lambda g\frac{1}{(2\pi)^3}\frac{1}{\pi}\int dm^2 g_1(m^2)$$
$$\times\int d\boldsymbol{q}\int d\boldsymbol{r}\,\Phi^*(r)\frac{1}{(\boldsymbol{q}^2+1)(\boldsymbol{q}^2+m^2)}(\boldsymbol{q}\cdot\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)(\boldsymbol{q}\boldsymbol{H})\,e^{-i\boldsymbol{q}\boldsymbol{r}}\Phi(r). \qquad (3\cdot1)$$

$\Phi(r)$ is the deuteron wave function given by

$$\Phi(r)=\frac{1}{\sqrt{4\pi}}\frac{1}{r}\left[u(r)+\frac{\sqrt{2}}{4}S_{12}w(r)\right]\chi, \qquad (3\cdot2)$$

where r is the relative coordinate of the two nucleons, S_{12} the usual tensor operator, and χ the spin function. Here we put $\boldsymbol{k}_1=-\boldsymbol{k}_2=\boldsymbol{q}$ since $\boldsymbol{\kappa}=0$. Furthermore, q_0 in G_1 and in the meson propagator is neglected, because it is proportional to the recoil energy of the nucleons.

First of all, the S-S expectation value of the magnetic moment is calculated, using the S-state wave function $u(r)$ given by[10]

$$u(r)=\begin{cases}Ae^{-\alpha r}-Be^{-\beta r} & \text{for } r>r_c=0.2,\\ 0 & \text{for } r<r_c,\end{cases} \qquad (3\cdot3)$$

where

$$A=1.039,\quad B=1.392,$$
$$\alpha=0.328,\quad \beta=2.360.$$

Introducing the total spin $\boldsymbol{S}=(1/2)(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)$, and taking into account the integrations over angular variables, we obtain

$$-\boldsymbol{\mu}\boldsymbol{H}=\frac{1}{2M}\lambda g\frac{1}{(2\pi)^3}\frac{1}{4\pi}\frac{1}{\pi}\int dm^2 g_1(m^2)\int d\boldsymbol{q}\int d\boldsymbol{r}$$

$$\times\frac{1}{r^2}u^2(r)\frac{\boldsymbol{q}^2}{(\boldsymbol{q}^2+1)(\boldsymbol{q}^2+m^2)}(\boldsymbol{SH})\,e^{-i\boldsymbol{q}\boldsymbol{r}}. \tag{3·4}$$

The magnetic moment in the unit of the nuclear magneton is found to be

$$\mu_{SS}=\frac{\lambda g}{e}\frac{1}{\pi}\int dm^2\frac{g_1(m^2)}{m^2-1}\int_{r_c}^{\infty}dr u^2(r)[U_1(r)-m^2U_m(r)], \tag{3·5}$$

where $U_m(r)$ is defined as

$$U_m(r)=\frac{e^{-mr}}{4\pi r}. \tag{3·6}$$

Hereafter we choose the approximate form for G_1 given by (2·12). Finally we have

$$\mu_{SS}=\frac{1}{2}\lambda g\zeta\frac{b^2}{b^2-1}\int_{r_c}^{\infty}dr u^2(r)[U_1(r)-b^2U_b(r)]. \tag{3·7}$$

The integration in (3·7) can be easily done and the numerical results are tabulated in Table I.

Table I. The magnetic moment of the deuteron due to the γ-3π interaction in the unit of nuclear magneton.

		$r_c=0.2$	$r_c=0.3$
μ_{SS}	$b^2=18$	$-0.033\,\zeta$	$-0.025\,\zeta$
	$b^2=24$	$-0.030\,\zeta$	$-0.021\,\zeta$

		$r_c=0.2$	$r_c=0.3$	Matsumoto's $w(r)$ with $r_c=0.2$
μ_{SD}	$b^2=18$	$-0.576\,\zeta$	$-0.173\,\zeta$	$+0.108\,\zeta$
	$b^2=24$	$-0.739\,\zeta$	$-0.218\,\zeta$	$+0.125\,\zeta$

Next we calculate the *S-D* cross term of the magnetic moment. In the spin state $S_z=+1$ where the z-axis is chosen parallel to the magnetic field, one gets

$$-\boldsymbol{\mu}_{SD}\boldsymbol{H}=\frac{3}{8}\frac{g\lambda e\zeta}{M}\frac{b^2}{\sqrt{2}}\frac{1}{\pi}\frac{1}{(2\pi)^3}\int d\boldsymbol{r}\frac{1}{r^2}u(r)\,w(r)$$

$$\times\int d\boldsymbol{q}\frac{1}{(\boldsymbol{q}^2+1)(\boldsymbol{q}^2+b^2)}e^{-i\boldsymbol{q}\boldsymbol{r}}\left[-\sqrt{\frac{12\pi}{5}}Y_2^1(\boldsymbol{r})\,q_-+\sqrt{\frac{4\pi}{5}}Y_2^0(\boldsymbol{r})\,q_z\right]q_z H, \tag{3·8}$$

where $Y_l^m(\boldsymbol{r})$ is the spherical harmonics with respect to the direction of $\boldsymbol{r}$, and

$$q_\pm = \mp \frac{q_x \pm i q_y}{\sqrt{2}}.$$

By partial integration with respect to $\boldsymbol{q}$, (3·8) can be put into the form

$$\mu_{SD} = -\frac{3}{8}\frac{g\lambda}{M} e\zeta b^2 \frac{1}{\pi(2\pi)^3}\frac{1}{\sqrt{2}}\int d\boldsymbol{r}\frac{1}{r^4}u(r)\,w(r)$$
$$\times\int d\boldsymbol{q}\, e^{-i\boldsymbol{q}\boldsymbol{r}}\left[-\sqrt{\frac{12\pi}{5}}\,Y_2^1(-i\nabla_q)q_- + \sqrt{\frac{4\pi}{5}}\,Y_2^0(-i\nabla_q)\,q_z\right]\frac{q_z}{(\boldsymbol{q}^2+1)(\boldsymbol{q}^2+b^2)}, \tag{3·9}$$

where $Y_l^m(-i\nabla_q)$ is a polynomial of l-th order in $-i\nabla_q$ obtained by replacing $\boldsymbol{r}/r$ in $Y_l^m(\boldsymbol{r})$ by $-i\nabla_q$. Performing the differentiations and the integrations with respect to $\boldsymbol{q}$ we obtain the magnetic moment as

$$\mu_{SD} = \frac{3}{\sqrt{2}} g\lambda\zeta \frac{b^2}{b^2-1}\int_{r_c}^{\infty} dr\frac{1}{r^2}u(r)\,w(r)$$
$$\times\left[\left(1+r+\frac{1}{3}r^2\right)U_1(r) - \left(1+br+\frac{1}{3}b^2r^2\right)U_b(r)\right]. \tag{3·10}$$

The integration in (3·10) can be done analytically using the "pion theoretical" wave function $w(r)$ given by[10]

$$w(r) = \begin{cases} \left(1+\dfrac{3}{\alpha r}+\dfrac{3}{\alpha^2 r^2}\right)Ce^{-\alpha r} - \dfrac{D}{r^2}e^{-\delta r} & \text{for } r > r_c \\ 0 & \text{for } r < r_c \end{cases} \tag{3·11}$$

where

$$C = 0.02624, \quad D = 1.298, \quad \delta = 0.962.$$

Numerical results are given in Table I.

It is found that μ_{SD} is much larger than μ_{SS} in magnitude and even amounts to the comparable order with the observed value itself. These figures, however, should not be accepted literally. In order to see the situation more in detail, we investigate the integrand in (3·10), which is given, apart from $u(r)\,w(r)$, by

$$\mathcal{M}(r) = \mathcal{M}_1(r) + \mathcal{M}_b(r),$$
$$\mathcal{M}_1(r) = \frac{1}{r^2}\left(1+r+\frac{1}{3}r^2\right)U_1(r), \tag{3·12}$$
$$\mathcal{M}_b(r) = -\frac{1}{r^2}\left(1+br+\frac{1}{3}b^2r^2\right)U_b(r).$$

It is readily shown that

$$\mathcal{M}(r) \geq 0, \tag{3·13}$$

$$\mathcal{M}(r) \approx \frac{1}{6}\frac{1}{4\pi}\frac{b^2-1}{r}, \quad \text{for } r \approx 0. \tag{3·14}$$

The functions $\mathcal{M}_1(r)$, $\mathcal{M}_b(r)$ and $\mathcal{M}(r)$ are plotted in Fig. 4. We find that $\mathcal{M}(r)$ behaves roughly as r^{-2} for $r \geq r_c = 0.2$. This leads to the undesired conclusion that the μ_{SD} depends strongly on the behavior of the deuteron wave function in the inner region.

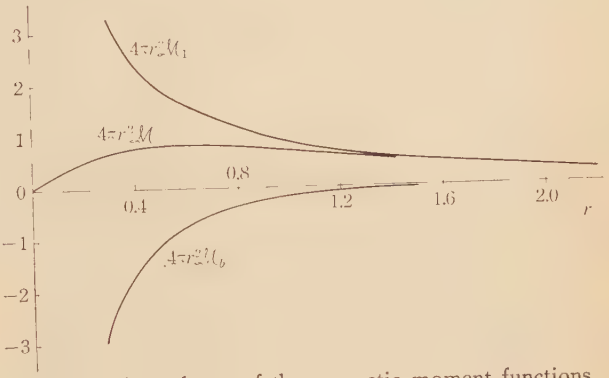

Fig. 4. r dependence of the magnetic moment functions $\mathcal{M}_1(r)$, $\mathcal{M}_b(r)$ and $\mathcal{M}(r)$.

The inequality (3·13) shows that

$$\mu_{SD} > 0, \tag{3·15}$$

provided $w(r)$ is positive in the inner region as well as in the outer region. The results tabulated in Table I contradicts (3·15). This comes from the fact that $w(r)$ given by (3·11) is negative for $r \lesssim 0.6$, due to the negative contribution from the second term of (3·11) in the inner region. For the sake of reference we calculate μ_{SD} using $w(r)$ proposed by Matsumoto,[11]

$$w(r) = C_1 e^{-\alpha_1 r} + C_2 e^{-\alpha_2 r} + C_3 e^{-\alpha_3 r}, \tag{3·16}$$

where

$$C_1 = 0.111, \quad C_2 = 0.656, \quad C_3 = -0.767,$$

$$\alpha_1 = 0.40, \quad \alpha_2 = 1.00, \quad \alpha_3 = 2.00.$$

This function agrees with the pion-theoretical one (3·11) in the outer region

$$r \geq 1.0,$$

and is positive definite in the whole region. The calculated μ_{SD} is also tabulated in Table I.* The values obtained are positive in agreement with (3·15) and smaller than the values calculated using (3·11). The difference may be considered to come from the considerable difference of two $w(r)$'s in the region $r \lesssim 0.8$.

Furthermore we can conjecture that μ_{SD} would be further reduced if the inner part of $w(r)$ is assumed to be appreciably smaller without changing the

* We cut off the integration at $r_c = 0.2$, though the parameters in (3·16) are chosen without assuming hard core.

outer part. We may, however, impose a "lower limit" to μ_{SD} which is obtained by cutting off the integration, say, at $r=1.0$, using $w(r)$ (3·11) or (3·16),

$$\mu_{SD} \geq 0.015\,\zeta, \quad \text{for } b^2=18. \tag{3·17}$$

Even this lower limit is still surprisingly large, namely comparable with the difference between the magnetic moment of the deuteron and its constituents,

$$\mu_d-(1+\mu_p+\mu_n)=-0.022.$$

§ 4. Electric quadrupole moment

In order to obtain the electric quadrupole moment of the deuteron, we have to find the charge density, $\rho(\boldsymbol{r})$. Here the electric field is described by a scalar potential $\phi(\boldsymbol{r})$ as

$$\boldsymbol{E}(\boldsymbol{r})=-\mathrm{grad}\,\phi(\boldsymbol{r}). \tag{4·1}$$

The interaction Hamiltonian density is

$$\rho(\boldsymbol{r})\,\phi(\boldsymbol{r})=(2\pi)^{-6}\int d\boldsymbol{s}\int d\boldsymbol{q}\int d\boldsymbol{\kappa}\,\Phi^*(\boldsymbol{s})\,e^{-i\boldsymbol{\kappa}\boldsymbol{r}-i\boldsymbol{q}\boldsymbol{s}}\,T(\boldsymbol{q},\boldsymbol{\kappa})\,\Phi(\boldsymbol{s}), \tag{4·2}$$

where

$$\boldsymbol{q}=\frac{1}{2}(\boldsymbol{k}_1-\boldsymbol{k}_2)=\frac{1}{2}(\boldsymbol{p}_1-\boldsymbol{p}_2)-\frac{1}{2}(\boldsymbol{p}_1'-\boldsymbol{p}_2').$$

$\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are expressed by $\boldsymbol{q}$, $\boldsymbol{\kappa}$ and $\boldsymbol{k}$, where

$$\boldsymbol{k}_1=\boldsymbol{q}-\frac{1}{2}\boldsymbol{\kappa}, \quad \boldsymbol{k}_2=-\boldsymbol{q}-\frac{1}{2}\boldsymbol{\kappa},$$

$$\boldsymbol{q}_1=\boldsymbol{k}-\frac{1}{4}\boldsymbol{\kappa}, \quad \boldsymbol{q}_2=-\boldsymbol{k}-\frac{1}{4}\boldsymbol{\kappa}.$$

It is to be noted that $\boldsymbol{k}$ is related only to the velocity dependent term, so that it does not appear here. From (4·1) and (4·2), the charge density reads

$$\rho(\boldsymbol{r})=(2\pi)^{-6}\int d\boldsymbol{s}\int d\boldsymbol{q}\int d\boldsymbol{\kappa}\,\Phi^*(\boldsymbol{s})\,e^{-i\boldsymbol{\kappa}\boldsymbol{r}-i\boldsymbol{q}\boldsymbol{s}}\,\mathcal{J}(\boldsymbol{q},\boldsymbol{\kappa})\,\Phi(\boldsymbol{s}), \tag{4·3}$$

where $\mathcal{J}(\boldsymbol{q},\boldsymbol{\kappa})$ is defined by substituting $-i\boldsymbol{\kappa}$ for $\boldsymbol{E}$ in (2·6).

The quadrupole moment is, then written as

$$eQ=\int d\boldsymbol{r}\,(3z^2-r^2)\,\rho(\boldsymbol{r}), \tag{4·4}$$

for the $S_z=1$ state. Inserting (4·3) in (4·4), the quadrupole moment is readily expressed as

$$eQ=-(2\pi)^{-6}\int d\boldsymbol{r}\int d\boldsymbol{s}\int d\boldsymbol{q}\int d\boldsymbol{\kappa}\,\Phi^*(\boldsymbol{s})\,e^{-i\boldsymbol{\kappa}\boldsymbol{r}-i\boldsymbol{q}\boldsymbol{s}}\left(3\frac{\partial^2}{\partial\kappa_z^2}-\Gamma_\kappa^2\right)\mathcal{J}(\boldsymbol{q},\boldsymbol{\kappa})\,\Phi(\boldsymbol{s})$$

$$=-(2\pi)^{-3}\int ds\int dq\,\Phi^*(s)\,e^{-iq\cdot s}\left(3\frac{\partial^2}{\partial\kappa_z^2}-\Gamma_\kappa^2\right)\mathcal{J}(q,\kappa)\Big|_{\kappa=0}\Phi(s). \quad (4\cdot5)$$

It should especially be remarked that the expectation value of the electric quadrupole moment for the initial and final states both in the ^3S does not vanish. It is well known that the ^3S-^3S expectation value always vanishes for the usual nucleon current contribution. In the present case, however, the operator $(3z^2-r^2)$ does not work on the proton coordinate, but on the meson cloud. Hence, we do not have any *a priori* reason that this expectation value has to vanish.

It is, therefore, necessary to examine the ^3S-^3S expectation value of the quadrupole moment, even if it is expected to be small any way. In (4·5) we calculate the quadrupole moment operator using (2·7),

$$\left\langle S_z=1\left|\left(3\frac{\partial^2}{\partial\kappa_z^2}-\Gamma_\kappa^2\right)\mathcal{J}(q,\kappa)\right|S_z=1\right\rangle_{\kappa=0}$$
$$=\frac{3g\lambda}{M}\left[(2q_z^2-q^2)FG+3q_z^2(q^2-q_z^2)(G'F-GF')\right], \quad (4\cdot6)$$

where we introduce

$$F(q^2)=\frac{1}{q^2+1}. \quad (4\cdot7)$$

The contribution from the ^3S-^3S term in question is

$$Q_{SS}=\frac{\lambda}{5}\frac{g}{4M^2}\frac{b^2}{b^2-1}(\mu_p-\mu_n+\zeta)$$
$$\times\int_0^\infty u^2(r)\,dr\left\{-\frac{7b^2+5}{b^2-1}U_1(r)+\frac{-5b^2+17}{b^2-1}b^3U_b(r)+3r[U_1(r)+b^3U_b(r)]\right\}. \quad (4\cdot8)$$

Using the explicit form of the S-wave function (3·3), we obtain

$$Q_{SS}\approx-1.1\times10^{-4} \quad \text{for } \zeta=1 \text{ and } b^2=18. \quad (4\cdot9)$$

This value is negligible compared with the observed one, 0.14, as was expected.

Next, the ^3S-^3D cross term is calculated. By applying the same technique used in the calculation of μ_{SD}, the quadrupole moment operator is evaluated in parallel with (4·6) as

$$\left\langle S_z=1\left|\left(3\frac{\partial^2}{\partial\kappa_z^2}-\Gamma_\kappa^2\right)\mathcal{J}(q,\kappa)S_{12}\right|S_z=1\right\rangle_{\kappa=0}$$
$$=-6\frac{g\lambda}{M}\frac{1}{r^2}\left[5GF+\frac{41}{3}q^2G'F-\frac{1}{3}q^2GF'\right.$$

$$+q^4\left(\frac{104}{15}G''F+\frac{8}{3}G'F'-\frac{64}{15}GF''\right)+\frac{4}{5}q^6(G'''F+G''F'-G'F''-GF''')\Big].$$

(4·10)

Here the integration over the angular variables is taken into account. The quadrupole moment itself is now readily written down as

$$2Q_{SD}=-\frac{\sqrt{2}}{2}(2\pi)^{-3}\frac{1}{4\pi}\int d\boldsymbol{r}\frac{1}{r^2}u(r)w(r)\int d\boldsymbol{q}\,e^{-i\boldsymbol{q}\boldsymbol{r}}$$

$$\times\left\langle S_z=1\left|\left(3\frac{\partial^2}{\partial\kappa_z^2}-\nabla_\kappa^2\right)\mathcal{J}(\boldsymbol{q},\boldsymbol{\kappa})S_{12}\right|S_z=1\right\rangle_{\kappa=0}$$

$$=\frac{1}{\sqrt{2}}\frac{3}{10}\frac{\lambda g}{M^2}\frac{b^2}{(b^2-1)^2}(\mu_p-\mu_n+\zeta)\int_{r_c}^{\infty}dr\frac{1}{r^2}u(r)w(r)$$

$$\times\Big\{2(4b^2-1)[(1+r)U_1(r)-(1+br)U_b(r)]$$

$$+\frac{1}{6}[(13b^2-1)r^2U_1(r)-(19b^2-7)b^2r^2U_b(r)]$$

$$-\frac{1}{2}(b^2-1)[r^3U_1(r)+b^3r^3U_b(r)]\Big\}. \qquad (4\cdot11)$$

Numerical results are displayed in Table II.

Table II. The quadrupole moment of the deuteron due to the γ-3π interaction. Here the pion dissociation probability, ζ, of the nucleon is taken to be unity.

		$r_c=0.2$	$r_c=0.3$
Q_{SS}/Q_{exp} (%)	$b^2=18$	−0.078	+0.079
	$b^2=24$	+0.090	+1.25

		$r_c=0.2$	$r_c=0.3$	Matsumoto's $w(r)$ with $r_c=0.2$
Q_{SD}/Q_{exp} (%)	$b^2=18$	−22.9	−6.86	+5.04
	$b^2=24$	−29.7	−9.00	+5.96

We encounter the same situation as that in μ_{SD}. By analogy with the case of μ_{SD} the integrand (apart from $u(r)w(r)$) in (4·11) is written as

$$\mathcal{Q}(r)=\mathcal{Q}_1(r)+\mathcal{Q}_b(r),$$

$$\mathcal{Q}_1(r)=\frac{1}{r^2}\left[2(4b^2-1)(1+br)+\frac{1}{6}(13b^2-1)r^2-\frac{1}{2}(b^2-1)r^3\right]U_1(r),$$

$$\mathcal{Q}_b(r)=-\frac{1}{r^2}\left[2(4b^2-1)(1+br)+\frac{1}{6}(19b^2-7)b^2r^2+\frac{1}{2}(b^2-1)b^3r^3\right]U_b(r). \tag{4·12}$$

We can show that

$$\begin{aligned}&\mathcal{Q}(r)\geq 0,\\ &\mathcal{Q}(r)\approx\frac{5}{24\pi}(b^2-1)^2\frac{1}{r}, \quad \text{for } r\approx 0.\end{aligned} \tag{4·13}$$

The functions (4·12) are plotted in Fig. 5. We see that the curves are very similar to those in Fig. 4. Therefore Q_{SD} also depends strongly on the behavior of the deuteron wave function in the inner region.

From (4·13) we have

$$Q_{SD}>0, \tag{4·14}$$

provided $w(r)$ is positive in the inner region. The values calculated by using $w(r)$ given by (3·11) are again negative and contradict (4·14). The values with $w(r)$ given by (3·16) are also listed in Table II, which are positive as they should be. Corresponding to (3·17) we have a "lower limit" to Q_{SD} with $r_c=1.0$ as

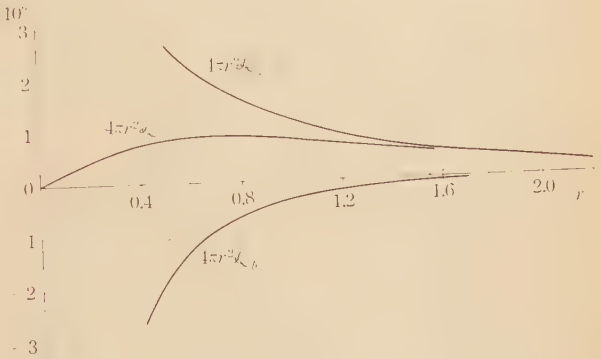

Fig. 5. r dependence of the quadrupole moment functions $\mathcal{Q}_1(r)$, $\mathcal{Q}_b(r)$ and $\mathcal{Q}(r)$.

$$Q_{SD}\geq +0.78\%, \quad \text{for } b^2=18. \tag{4·15}$$

§ 5. Photodisintegration

The main term of the photodisintegration of the deuteron is usually described by the $E1$ transition for 100 Mev~350 Mev gamma rays, namely by the transition

$$E1: {}^3S+{}^3D\rightarrow {}^3P+{}^3F. \tag{5·1}$$

Another important contribution comes from the $M1$ transition into the ${}^1D$ state through the (33) pion-nucleon resonance. Here we would like to see only the order of magnitude of the contribution from the γ-3π interaction, especially the interference term with the usual direct $E1$ transitions.

The matrix element of the direct $E1$ transition is given by[8]

$$
\begin{aligned}
&M^d = M_1^d + M_2^d + M_3^d, \\
&M_1^d = (\boldsymbol{\varepsilon}\boldsymbol{n})\, U(\kappa), \\
&M_2^d = [(\boldsymbol{\sigma}_1\boldsymbol{\varepsilon})(\boldsymbol{\sigma}_2\boldsymbol{n}) + (\boldsymbol{\sigma}_1\boldsymbol{n})(\boldsymbol{\sigma}_2\boldsymbol{\varepsilon})]V(\kappa), \\
&M_3^d = (\boldsymbol{\varepsilon}\boldsymbol{n})(\boldsymbol{\sigma}_1\boldsymbol{n})(\boldsymbol{\sigma}_2\boldsymbol{n})\, W(\kappa),
\end{aligned}
\tag{5·2}
$$

$\boldsymbol{\varepsilon}$ being the polarization vector of the photon, and $\boldsymbol{n}$ the unit vector of the final proton momentum. $U(\kappa)$, $V(\kappa)$ and $W(\kappa)$ are given by[8]

$$
\begin{aligned}
U(\kappa) &= -\pi \frac{e}{\sqrt{4\pi}} \frac{\kappa}{t} \frac{1}{3} \left(L_0 + \frac{3}{2} L_1 + \frac{7}{2} L_2 + \frac{3}{2} L_F \right), \\
V(\kappa) &= \pi \frac{e}{\sqrt{4\pi}} \frac{\kappa}{t} \frac{1}{6} \left(L_0 - \frac{3}{2} L_1 + \frac{1}{2} L_2 - 3 L_F \right), \\
W(\kappa) &= \pi \frac{e}{\sqrt{4\pi}} \frac{\kappa}{t} \frac{5}{2} L_F .
\end{aligned}
\tag{5·3}
$$

$\boldsymbol{\kappa}$ and $\boldsymbol{t}$ are the momenta of the initial photon and of the final proton, respectively. L's are defined as

$$
\begin{aligned}
L_0 &= J_s - \sqrt{2}\, J_d, \\
L_1 &= J_s + \frac{1}{\sqrt{2}} J_d, \\
L_2 &= J_s - \frac{\sqrt{2}}{10} J_d,
\end{aligned}
\tag{5·4}
$$

where

$$
J_s = t \int_0^\infty dr\, r^2 u(r) j_1(tr),
$$

$$
J_d = t \int_0^\infty dr\, r^2 w(r) j_1(tr),
$$

and

$$
L_F = \frac{3}{5} \sqrt{2}\, t \int_0^\infty dr\, r^2 w(r) j_3(tr).
\tag{5·5}
$$

It is obvious that $U(\kappa)$ represents the transition from the ${}^3S$ and ${}^3D$ initial states, $V(\kappa)$ and $W(\kappa)$ from the ${}^3D$ initial state. We are going to pick up the following transition due to the γ-3π interaction,

$$
{}^3S + {}^3D \rightarrow {}^3S + {}^3D.
\tag{5·6}
$$

This is mostly described by the $M1$ and $E2$ transitions. As far as the interference term is concerned, the biggest contribution is expected from

$$M_{S\to S}^{\gamma-3\pi} M_1^{d*}, \tag{5·7}$$

because only M_1^d involves the transition from the ${}^3S$ state of the deuteron. The next ones are

$$M_{S\to S}^{\gamma-3\pi}(M_2^d+M_3^d)^*, \tag{5·8}$$

$$M_{S\to D}^{\gamma-3\pi} M_1^{d*}, \tag{5·9}$$

and

$$M_{D\to S}^{\gamma-3\pi} M_1^{d*}. \tag{5·10}$$

The terms $(M_{S\to D}^{\gamma-3\pi}+M_{D\to S}^{\gamma-3\pi})\times(M_2^d+M_3^d)^*$ and $M_{D\to D}^{\gamma-3\pi}$ are neglected. The matrix element due to the γ-3π interaction is

$$M^{\gamma-3\pi}=(2\pi)^{-3}\int d\boldsymbol{r}\int d\boldsymbol{q}\, e^{-i\boldsymbol{q}\boldsymbol{r}-i\boldsymbol{k}\boldsymbol{r}} T(\boldsymbol{q},\boldsymbol{\kappa})\,\Phi(r), \tag{5·11}$$

where $T(\boldsymbol{q},\boldsymbol{\kappa})$ is given by (2·6).

In order to obtain the cross section for unpolarized initial state, the summation over all spin states for the final state and the average for the initial state is taken. For the nucleon spin this procedure is done as

$$\sum_{spin} M^{d*} M^{\gamma-3\pi}=\frac{1}{3} Sp(M^{d*}\Lambda_t M^{\gamma-3\pi}\Lambda_t), \tag{5·12}$$

for (5·7), (5·8), (5·9) and (5·10). Λ_t is the projection operator of the spin triplet state defined by

$$\Lambda_t=\frac{1}{4}(3+\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2). \tag{5·13}$$

In (5·12), the trace is taken separately for $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$.

As far as the interference with the direct $E1$ transition is concerned, the first terms of (2·7) and (2·8) vanish after the summation (5·12). The second term of (2·8) involves the factor k_{10}, which is of order $1/M$, so that we may only take the second term of (2·7). In the following, we approximate (2·7) and (2·8) as

$$\begin{aligned}&\boldsymbol{e}_a\approx i\{(\boldsymbol{\sigma}_1\boldsymbol{k}_1)\boldsymbol{k}_1-\boldsymbol{k}_1^2\boldsymbol{\sigma}_1\}(\boldsymbol{\sigma}_2\boldsymbol{k}_2)\,G(\boldsymbol{k}_1^2),\\ &\boldsymbol{h}_a\approx 0.\end{aligned} \tag{5·14}$$

Writing the matrix element $M^{\gamma-3\pi}$ in a form $(\boldsymbol{\sigma}_1\boldsymbol{a})(\boldsymbol{\sigma}_2\boldsymbol{b})$, we give the formulas for the spin sum,

$$\sum_{spin}(\boldsymbol{\varepsilon}\boldsymbol{n})U(\boldsymbol{\sigma}_1\boldsymbol{a})(\boldsymbol{\sigma}_2\boldsymbol{b})=\frac{1}{3}U(\boldsymbol{\varepsilon}\boldsymbol{n})(\boldsymbol{a}\boldsymbol{b}), \tag{5·15}$$

$$\begin{aligned}&\sum_{spin}[(\boldsymbol{\sigma}_1\boldsymbol{\varepsilon})(\boldsymbol{\sigma}_2\boldsymbol{n})+(\boldsymbol{\sigma}_1\boldsymbol{n})(\boldsymbol{\sigma}_2\boldsymbol{\varepsilon})]V(\boldsymbol{\sigma}_1\boldsymbol{a})(\boldsymbol{\sigma}_2\boldsymbol{b})\\ &\quad=\frac{2}{3}V[-(\boldsymbol{\varepsilon}\boldsymbol{n})(\boldsymbol{a}\boldsymbol{b})+2(\boldsymbol{\varepsilon}\boldsymbol{a})(\boldsymbol{n}\boldsymbol{b})+2(\boldsymbol{\varepsilon}\boldsymbol{b})(\boldsymbol{n}\boldsymbol{a})],\end{aligned} \tag{5·16}$$

$$\sum_{spin}(\boldsymbol{\varepsilon n})(\boldsymbol{\sigma}_1\boldsymbol{n})(\boldsymbol{\sigma}_2\boldsymbol{n})W(\boldsymbol{\sigma}_1\boldsymbol{a})(\boldsymbol{\sigma}_2\boldsymbol{b})=\frac{1}{3}W(\boldsymbol{\varepsilon n})[4(\boldsymbol{na})(\boldsymbol{nb})-(\boldsymbol{ab})]. \quad (5\cdot17)$$

The average over the polarization of the photon is taken in the coordinate system in Fig. 6. First, the matrix element M_1^d is proportional to $(\boldsymbol{\varepsilon n})$. In the present coordinate system this term is rewritten as

$$M_1^d=U(\kappa)\sin\theta\cos\varphi. \quad (5\cdot18)$$

On the other hand, $M_{S\to S}^{\gamma-3\pi}$ in (5·7) is apparently independent of φ, so that the interference between these terms vanishes when we take average over φ. In the same reason (5·10) also does not contribute. For the unpolarized gamma ray, we have, accordingly, to calculate only (5·8) and (5·9).

The matrix element $M^{\gamma-3\pi}$ is now calculated from (2·6) inserting (5·14) for $\boldsymbol{e}_a$ and $\boldsymbol{h}_a$. $\boldsymbol{E}$ is substituted by

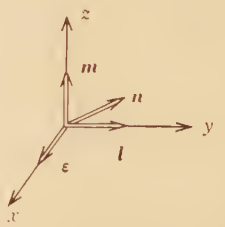

Fig. 6. The coordinate system used for the photodisintegration of the deuteron. $\boldsymbol{\varepsilon}$ is the unit vector of the polarization of photon, $\boldsymbol{m}$ is the unit vector parallel to the photon momentum, and $\boldsymbol{l}$ is the unit vector perpendicular to $\boldsymbol{\varepsilon}$ and $\boldsymbol{m}$. $\boldsymbol{n}$ is the unit vector parallel to the proton momentum $\boldsymbol{t}$.

$$\boldsymbol{E}\to i\kappa\boldsymbol{\varepsilon}, \quad (5\cdot19)$$

from the relation $\boldsymbol{E}=-\dot{\boldsymbol{A}}$. Then T reads

$$\begin{aligned}T(\boldsymbol{q},\boldsymbol{\kappa})=&\frac{3}{4}\frac{g}{M}\lambda\kappa\Big[L(-)(\boldsymbol{\varepsilon q})\Big\{(\boldsymbol{\sigma}_1\boldsymbol{q})(\boldsymbol{\sigma}_2\boldsymbol{q})\\&+\frac{1}{2}[(\boldsymbol{\sigma}_1\boldsymbol{q})(\boldsymbol{\sigma}_2\boldsymbol{\kappa})-(\boldsymbol{\sigma}_1\boldsymbol{\kappa})(\boldsymbol{\sigma}_2\boldsymbol{q})]-\frac{1}{4}(\boldsymbol{\sigma}_1\boldsymbol{\kappa})(\boldsymbol{\sigma}_2\boldsymbol{\kappa})\Big\}\\&-L(12)(Q^2-q\kappa w)(\boldsymbol{\sigma}_1\boldsymbol{\varepsilon})\Big[(\boldsymbol{\sigma}_2\boldsymbol{q})+\frac{1}{2}(\boldsymbol{\sigma}_2\boldsymbol{\kappa})\Big]\\&-L(21)(Q^2+q\kappa w)\Big[-(\boldsymbol{\sigma}_1\boldsymbol{q})+\frac{1}{2}(\boldsymbol{\sigma}_1\boldsymbol{\kappa})\Big](\boldsymbol{\sigma}_2\boldsymbol{\varepsilon})\Big],\end{aligned} \quad (5\cdot20)$$

where

$$\begin{aligned}&L(12)=G(k_1^2)F(k_2^2),\\&L(21)=G(k_2^2)F(k_1^2),\\&L(\pm)=L(12)\pm L(21).\end{aligned} \quad (5\cdot21)$$

Other notations are given in terms of the polar angle (θ,Φ) of $\boldsymbol{q}$ as

$$\begin{aligned}&Q^2=q^2+\frac{1}{4}\kappa^2,\\&w=\cos\theta,\\&k_1^2=Q^2-q\kappa w,\quad k_2^2=Q^2+q\kappa w.\end{aligned} \quad (5\cdot22)$$

In order to obtain the matrix element $M_{S\to S}^{\gamma-3\pi}$, the following substitution is made in (5·11),

$$e^{-itv}\to\frac{\sin tr}{tr},$$

and

$$\Phi(r)\to\frac{1}{\sqrt{4\pi}}\frac{u(r)}{r}. \tag{5·23}$$

Thus we have

$$M_{S\to S}^{\gamma-3\pi}=(2\pi)^{-3}\frac{1}{\sqrt{4\pi}}\int d\boldsymbol{r}\int d\boldsymbol{q}\,e^{-iqr}\frac{\sin tr}{tr}\frac{1}{r}u(r)\,T(\boldsymbol{q},\boldsymbol{\kappa})$$

$$=(2\pi)^{-3}\sqrt{4\pi}\frac{1}{t}\int_0^\infty dr\sin tr\frac{u(r)}{r}\int d\boldsymbol{q}\frac{\sin qr}{q}T(\boldsymbol{q},\boldsymbol{\kappa}). \tag{5·24}$$

The cross term (5·8) with the direct $E1$ transition is given after taking the summation over nucleon spins using (5·16) and (5·17) and over photon polarization,

$$\sum_{\substack{spin\\pol}}(M_2^{d*}+M_3^{d*})\,M_{S\to S}^{\gamma-3\pi}=\cos\theta\frac{1}{\pi}\left[V(\kappa)+\frac{1}{2}W(\kappa)\sin^2\theta\right]\frac{g\lambda}{\sqrt{4\pi}}\frac{\kappa}{Mt}$$

$$\times\int_0^\infty dr\int_0^\infty dq\int_{-1}^1 dw\frac{1}{r}u(r)\,q\sin tr\sin qr$$

$$\times\left[\kappa L(+)\left(q^2w^2-\frac{1}{2}Q^2\right)+qwL(-)\left(\frac{1}{4}\kappa^2-q^2w^2\right)\right]$$

$$=\cos\theta\frac{1}{\pi}\left[V(\kappa)+\frac{1}{2}W(\kappa)\sin^2\theta\right]\frac{eg\lambda}{\sqrt{4\pi}}\frac{\kappa}{t}\frac{b^2}{2M^2}(\mu_p-\mu_n+\zeta)I_1, \tag{5·25}$$

where

$$I_1=\int_0^\infty dr\int_0^M dq\int_{-1}^1 dw\frac{1}{r}u(r)\sin tr\sin qr\frac{q}{(Q^2+b^2-q\kappa w)(Q^2+1+q\kappa w)}$$

$$\times\left[\kappa\left(q^2w^2-\frac{1}{2}Q^2\right)+qw\left(\frac{1}{4}\kappa^2-q^2w^2\right)\right]. \tag{5·26}$$

For easiness of computation $u(r)$ is used with $r_c=0$. On the other hand, the integration with respect to q is cut off at the nucleon mass. The main contribution comes from the region of $q=2\sim3$, and rather insensitive to the cutoff momentum.

The resulting cross section is

$$\frac{d\sigma_1}{d\omega}=\frac{e^2}{4\pi}\frac{g\lambda}{24(2\pi)^2}\frac{b^2}{M}(\mu_p-\mu_n+\zeta)\frac{\kappa}{t}I_1$$
$$\times\left[\left(L_0-\frac{3}{2}L_1+\frac{1}{2}L_2-3L_F\right)+\frac{15}{2}L_F\sin^2\theta\right]\cos\theta. \tag{5·27}$$

The other term $M_{S\to D}^{\gamma-3\pi}$ is obtained by putting

$$e^{-i\boldsymbol{u}\boldsymbol{r}}\to-4\pi j_2(tr)\sum_{m=-2}^{2}Y_2^m(\boldsymbol{r})Y_2^{m*}(\boldsymbol{t}),$$

and

$$\Phi(r)\to\frac{1}{\sqrt{4\pi}}\frac{u(r)}{r}, \tag{5·28}$$

in (5·11). In a way similar to (5·24) $M_{S\to D}^{\gamma-3\pi}$ is calculated as

$$M_{S\to D}^{\gamma-3\pi}=-4\pi(2\pi)^{-3}\int d\boldsymbol{r}\,j_2(tr)\int d\boldsymbol{q}\,e^{-i\boldsymbol{q}\boldsymbol{r}}\sum_m Y_2^m(\boldsymbol{r})\,Y_2^{m*}(\boldsymbol{t})\,T(\boldsymbol{q},\boldsymbol{\kappa})\frac{1}{\sqrt{4\pi}}\frac{1}{r}u(r)$$
$$=\frac{2}{\pi}\frac{1}{\sqrt{4\pi}}\int_0^\infty dr\int d\boldsymbol{q}\,ru(r)j_2(tr)j_2(qr)\sum_m Y_2^{m*}(\boldsymbol{t})\,Y_2^m(\boldsymbol{q})\,T(\boldsymbol{q},\boldsymbol{\kappa}). \tag{5·29}$$

Using the formula (5·15), the interference term with the direct $E1$ transition (5·9) is finally found to be

$$\sum_{\substack{spin\\pol}}M_1^{d*}M_{S\to D}^{\gamma-3\pi}=\sin^2\theta\cos\theta\frac{15}{16\pi}U(\kappa)\frac{g\lambda}{\sqrt{4\pi}}\frac{\kappa}{M}$$
$$\times\int_0^\infty dr\int_0^M dq\int_{-1}^1 dw\,ru(r)j_2(tr)j_2(qr)q^3(1-w^2)w\left[L(+)\kappa qw-\frac{1}{2}\kappa^2L(-)\right]$$
$$=\sin^2\theta\cos\theta\frac{15}{16\pi}U(\kappa)\frac{eg\lambda}{\sqrt{4\pi}}\kappa\frac{b^2}{2M^2}(\mu_p-\mu_n+\zeta)I_2, \tag{5·30}$$

where

$$I_2=\int_0^\infty dr\int_0^M dq\int_{-1}^1 dw\,ru(r)j_2(tr)j_2(qr)\,q^3w(1-w^2)$$
$$\times\frac{1}{(Q^2+b^2-q\kappa w)(Q^2+1+q\kappa w)}\left(q\kappa w-\frac{1}{2}\kappa^2\right). \tag{5·31}$$

The cross section reads

$$\frac{d\sigma_2}{d\omega}=-\frac{e^2}{4\pi}\frac{5g\lambda}{64(2\pi)^2}\frac{b^2}{M}(\mu_p-\mu_n+\zeta)\kappa I_2$$
$$\times\left(L_0+\frac{3}{2}L_1+\frac{7}{2}L_2+\frac{3}{2}L_F\right)\sin^2\theta\cos\theta. \tag{5·32}$$

The resulting differential cross section is displayed in Table III, by adding (5·27) and (5·32). Comparison with experiment[12),13)] is done only for the $\cos\theta$ and $\sin^2\theta\cos\theta$ terms, in order only to know whether the contribution from the γ-3π term is appreciable or not. The effect on the $\cos\theta$ term is negligible, but that on the $\sin^2\theta\cos\theta$ term is of appreciable order of magnitude with the opposite sign.

Table III. The coefficient of $\cos\theta$ and $\sin^2\theta\cos\theta$ in the differential cross section of the photodisintegration of the deuteron. Theoretical values are the interference terms between the direct $E1$ transition and the γ-3π term.

E_γ (Mev)	Calculated coefficient of $\cos\theta$ (μb)	Observed coefficient of $\cos\theta$ (μb)	Calculated coefficient of $\sin^2\theta\cos\theta$ (μb)	Observed coefficient of $\sin^2\theta\cos\theta$ (μb)
80	0.01	1.6	−0.14	1.4
190	0.02	0.92	−0.16	0.64
290	0.05	0.61	−0.24	—

§ 6. Discussions

It has been found in § 3 that the contribution of the γ-3π interaction to the magnetic moment of the deuteron is surprisingly large. The S-S term is negative, so that this works so as to reduce the D-state probability. The order of magnitude of this term is comparable with the difference between the magnetic moment of the deuteron and of its constituents. Furthermore, the S-D term is much larger than the S-S term. It is difficult to find the magnitude and even the sign as is seen in Table I, because the inner part of the D-wave function plays an important role. From Tables I and II, one can find that the contribution from $0.2\leq r\leq 0.3$ is larger than that from $r\geq 0.3$ for both the magnetic moment and quadrupole moment. It is, however, remarkable that the contribution from the outer part, $r\geq 1$, to μ_{SD} is as large as the S-S term.

It seems to be difficult to draw any conclusion from our analysis, because we have to take the following points into account:

1) μ_{SS} and μ_{SD} are both proportional to the pion dissociation probability of the nucleon, ζ, so that the magnetic moment is sensitive to ζ. It is plausible to take $\zeta\geq 0.8$, which is suggested from the isovector part of the charge form factor for large momentum transfer. If ζ is found to be smaller, the effect on the magnetic moment is reduced.

2) The effective coupling constant λ of the γ-3π interaction has been taken as 0.2. Since this value is obtained by the perturbation theory through the nucleon closed loop, λ may be reduced by the pair suppression, as other phenomena related to the nucleon closed loop. In this case, we have to give up to inter-

pret the isoscalar form factors.[4] They might be explained in terms of, for example, the isoscalar ω meson.[14]

3) Other contributions to the magnetic moment should also be added, for example, the additional magnetic moment from the L-S potential,[15] and the relativistic effects.[16] They are both considerable order of magnitude.

The results on the electric quadrupole moment is summarized as follows: Q_{SS} is negligibly small, though the sign and magnitude again critically depend on the cutoff radius. Q_{SD} is about 20 to 30% of the observed value. This strongly depends on the inner part of the D-wave function. The contribution from the outer region, $r \geq 1$, is 0.78% of the observed value. Contrary to the magnetic moment, the quadrupole moment does not depend on ζ so much, because ζ appears as $(\mu_p - \mu_n + \zeta)$. The effect of the γ-3π term on the quadrupole moment is in any way less embarrassing than that on the magnetic moment.

The effect of the γ-3π interaction on the photodisintegration is less important than the above two problems. The γ-3π term certainly contributes to $\cos\theta$ and $\sin^2\theta \cos\theta$ terms in the angular distribution, but the magnitude is not so large. In this calculation, we have introduced, on account of the technical reason, the cutoff momentum at the nucleon mass, instead of the core radius in the integration with regard to r. This is not an essential point. It is found that the inner part of the wave function is not so important in the photodisintegration, furthermore the D-wave function does not appear as far as the γ-3π effect is concerned.

In conclusion we are tempted to emphasize that the γ-3π interaction cannot be disregarded in any photonuclear reaction, especially in pion photoproduction and electromagnetic properties of the deuteron.

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Appendix

We shall show that there appears a modified coupling constant λ commonly in various processes.[1] The pion-pion rescattering effect is taken into account, for definiteness, by the chain approximation method,[17] with the interaction Hamiltonian given by

$$H = -4\pi a_0(\varphi_i \varepsilon_{ijk} \partial_\mu \varphi_j)^2. \qquad (A\cdot 1)$$

If we neglect the pion-pion rescattering correction completely in the diagrams, in Fig. 1, Fig. 2 and Fig. 7 (the electromagnetic structure of the nucleon), there appears a common factor,

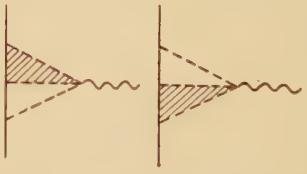

Fig. 7. The three-pion diagrams of the isoscalar part of the nucleon form factors with the pion-pion interaction included in the shaded part.

$$\mathcal{O}_0=\lambda_0 G_\alpha^{\,0}(k^2), \quad (\alpha=1, 2) \tag{A·2}$$

where $G_\alpha^{\,0}(k^2)$ is the two-pion part of the isovector electromagnetic form factor of the nucleon with the pion-pion interaction neglected. If we substitute the chain diagram, illustrated in Fig. 8, for the pion-pion scattering part in the shaded area of Fig. 1, Fig. 2 and Fig. 7, $\mathcal{O}_0$ is altered in the following way,

Fig. 8. Pion-pion interaction in the chain approximation.

$$\mathcal{O}_0\to\mathcal{O}=\frac{\mathcal{O}_0}{1-X(k^2)}, \tag{A·3}$$

where $X(k^2)$ is a contribution from a single bubble in Fig. 8, given by

$$X(k^2)=-4\pi a_0 i(2\pi)^{-4}\int\frac{dp\,dp'}{(p^2+1)(p'^2+1)}(p-p')^2\delta(p+p'-k)$$

$$=\frac{a_0}{3\pi^3}\int_4^{(2\Lambda)^2}d\xi\frac{\xi}{\xi+k^2}\left(1-\frac{4}{\xi}\right)^{3/2}.$$

According to the renormalization procedure given by Miyamoto,[17] $\mathcal{O}$ in (A·3) can be put into the form

$$\mathcal{O}=\frac{1}{1-X(-4)}\frac{\mathcal{O}_0}{D(k^2)}, \tag{A·4}$$

where

$$D(k^2)=1-\frac{1}{1-X(-4)}[X(k^2)-X(-4)]$$

$$=1+\frac{2a}{\pi}\left(1+\frac{k^2}{4}\right)\left[\log 2\Lambda-\sqrt{1+\frac{4}{k^2}}\log\left(\sqrt{1+\frac{k^2}{4}}+\sqrt{\frac{k^2}{4}}\right)\right],$$

with the renormalized coupling constant (the observed scattering length) a, given by

$$a=\frac{a_0}{1-X(-4)}. \tag{A·5}$$

On the other hand, $G_\alpha^{\,0}(k^2)$ is also modified to the "observed" form factor $G_\alpha(k^2)$ due to the pion-pion interaction,

$$G_\alpha^{\,0}(k^2)\to G_\alpha(k^2)=G_\alpha^{\,0}(k^2)\,F_\pi(k^2), \tag{A·6}$$

where $F_\pi(k^2)$ is the electromagnetic form factor of a pion, given by

$$F_\pi(k^2)=\frac{D(0)}{D(k^2)}. \tag{A·7}$$

Substituting (A·2), (A·5), (A·6) and (A·7) in (A·4), we obtain

$$\mathcal{O}=\frac{a}{a_0}\lambda_0\frac{1}{D(0)}G_a(k^2), \tag{A·8}$$

which can be written in the form

$$\mathcal{O}=\lambda G_a(k^2), \tag{A·9}$$

in terms of the observed form factor $G_a(k^2)$ and the modified coupling constant λ, given by

$$\lambda=\lambda_0\frac{a}{a_0}\frac{1}{D(0)}. \tag{A·10}$$

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Methods for Detecting High Energy Photons in Primary Cosmic Rays

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Detection of the high energy photons in primary cosmic rays provides us with useful data for the study of the origin of cosmic rays, because they move in straight lines and bring information on the directions of their sources. Two methods are described to detect such photons through the secondary particles produced by them in air.

i) A part of photons can produce mesons capable of reaching the sea level through photo-nuclear interactions with air nuclei, although the majority of them are absorbed in the atmosphere by producing electron pairs when their energies are 10^{10}–10^{11} ev. These mesons do not suffer considerable scattering from the original direction of photons, and higher intensity of mesons in a small solid angle containing the direction of the source can be detected in comparison with the average intensity. The lower limit of the photon flux detectable by this method is estimated to be about $4 \cdot 10^{-6}$ cm^{-2} sec^{-1} for photons of 200 Gev.

ii) For photons with energies near 10^{12} ev, it is possible to detect cascade showers initiated from them when observations are made at mountain altitude. Discussions are made on the detection efficiency of these showers by the special arrangement of shower detectors. Minimum detectable flux of photons of 10^{12} ev by this arrangement is estimated to be about $2 \cdot 10^{-7}$ cm^{-2} sec^{-1}.

§ 1. Introduction

It is well known that primary cosmic rays are nearly isotropic in the vicinity of the earth. Besides the slight anisotropy coming from the solar activity, one hardly finds any anisotropy exceeding 0.1 per cent. This extremely small degree of anisotropy can be attributed to the effect of galactic magnetic fields; the magnetic field strength of the order of 10^{-5} Gauss is strong enough to stir cosmic-ray particles which would otherwise bring some information from their sources.

In view of this situation, attention should be paid to the neutral components of primary cosmic rays. Among various neutral particles known at present, only γ-rays are of our interest. Other particles (neutrons, neutral mesons, neutrinos, etc.) are inadequate for our purpose either because they are unstable or have too weak interactions to be observed efficiently.

Several attempts were made[1] to catch primary γ-rays in the upper atmosphere. In the present stage, however, the results are preliminary and no information has been obtained so far about the source direction.

Most of the γ-rays falling on the earth produce electron-positron pairs in

the upper atmosphere and are absorbed in the air without causing any effects to the cosmic-ray intensity at sea level when their energies are 10^{10}–10^{11} ev. It should be noted, however, that a small fraction of the γ-rays can produce mesons through nuclear interactions with air nuclei. If mesons produced by the photons from a certain source are observed at sea level without suffering appreciable deflections from the original direction of photons, we shall detect these mesons whose intensity is above the average level within a small solid angle containing the direction of the source. Due to the small probability of the nuclear interactions of γ-rays, a very large detector with good angular resolution has to be employed in order to detect this effect. This effect was proposed as a possible mechanism for the interpretation of the point source of cosmic rays in Orion found by Sekido et al.[2] The present analysis is, in a more general viewpoint, to discuss the validity of the experiment making use of the above-mentioned effect.

When the energy of a primary photon is sufficiently high, an electron-positron pair produced by the photon develops itself to the so-called "cascade shower" and its tail will reach the ground level being observed as an extensive air shower. According to the calculation on cascade showers, energy of the order of 10^{14}ev is required for incoming photons in order to be detected efficiently with ordinary air shower detectors at sea level. This threshold energy can be reduced to about 10^{13}ev when observations are made at mountain altitude. These pure cascade showers can be distinguished from the showers produced by the nuclear component of primary cosmic rays by reference to the μ-meson contents in them. A plan of the experiment of this type is presented by Maze et al.,[3] and the analysis from this viewpoint is carried out also by the Tokyo Air Shower Group.[4]

To detect showers of 10^{12}ev, a special arrangement of air shower detectors is necessary to increase the detection efficiency of low density showers as well as the S-N ratio to nucleonic air showers. Quantitative analysis on the detection efficiency of such showers are given in the second part (§ 8–§ 9) of the present paper.

§ 2. Observation of meson produced by photons

Let I_p be the intensity of mesons at sea level produced by primary γ-rays with energies greater than E_0. I_p is related to the intensity $J_p(E_0)$ of the primary photons at the top of the atmosphere in the following way:

$$I_p = M_p P J_p(E_0) \quad (\mathrm{cm}^{-2}\mathrm{sec}^{-1}), \tag{2·1}$$

where P is the probability of the nuclear interaction of a photon and M_p is the mean multiplicity of mesons which are produced in the nuclear collision and are capable of reaching the sea level. On the other hand, the intensity I_n of mesons produced by the nuclear component of primary cosmic rays is

$$I_n = M_n J_n(E_0) \quad (\mathrm{cm}^{-2}\,\mathrm{sec}^{-1}\,\mathrm{str}^{-1}), \tag{2·2}$$

where $J_n(E_0)$ is the intensity of the nuclear component of primary cosmic rays, and M_n is a quantity corresponding to M_p in (2·2). If photons and nucleons with the same energy are assumed to behave similarly in the atmosphere, one may put $M_p = M_n$.*

At the top of the atmosphere, the dispersion of the direction of each photon from a source will be extremely small. As a result of the deflection of various causes of secondary mesons in the air, however, the image of the source viewed with mesons will have some extension. Experimentally, I_p in (2·1) is estimated from the ratio of the counting rates of particles when the source direction is in and out of the field of vision of the detector. Assuming that mesons from photons are distributed uniformly in the finite solid angle Ω_0 and that these particles are observed by a detector having a solid angle Ω_1, we obtain the above ratio for $\Omega_1 \geq \Omega_0$ as

$$(I_p + I_n \Omega_1)/I_n \Omega_1 . \tag{2·3}$$

It is clear that this ratio is determined most precisely when $\Omega_1 \simeq \Omega_0$. In this case, the degree of anisotropy is given from (2·1) and (2·2) as

$$R = \frac{I_p + I_n \Omega_0}{I_n \Omega_0} - 1 = \frac{M_p P J_p(E_0)}{M_n J_n(E_0) \Omega_0} = \frac{P J_p(E_0)}{J_n(E_0) \Omega_0} . \tag{2·4}$$

§ 3. Nuclear and electromagnetic interaction cross sections of photons

The meson production in photo-nuclear collisions is experimentally studied with γ-ray beams from an electron-synchrotron. The knowledge on these interactions is restricted, therefore, to about 1 Gev for γ-ray energy, and no direct information is available for energies exceeding this limit, while field theory suggests that the cross section of the photo-nuclear interaction roughly remains constant for higher energies. We now assume that the cross section σ_π is constant and is equal to that determined experimentally for $E \simeq 1$ Gev, i.e.

$$\sigma_\pi = 10^{-28}\ \mathrm{cm}^2 \text{ per nucleon}, \tag{3·1}$$

for photon energies larger than 1 Gev.

As for the electromagnetic interaction, theory is considered to be correct up to considerably high energy. The pair production cross section in air is expressed as follows:

$$\sigma_{pair} = \frac{A}{N} \frac{7}{9 X_0} \simeq 3.4 \cdot 10^{-26} A \quad \mathrm{cm}^2, \tag{3·2}$$

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* This assumption may not be so bad, as one would suppose, in view of the recently emphasized mechanism of the meson production by photon-nucleon and nucleon-nucleon collisions; see reference 5).

where A is the mean atomic weight of air nuclei, N represents the Loschmidt number and X_0 is the radiation length in air. (3·2) holds only in the case where $E \gg 137 m_e c^2 Z^{-1/3}$, which is well satisfied in the energy of our interest. Combination of (3·1) and (3·2) leads to the value of P in (2·4),

$$P = \frac{\sigma_\pi A}{\sigma_{pair}} \simeq 3 \cdot 10^{-3}, \tag{3·3}$$

where we tentatively assume the A dependence of the photo-nuclear cross section.

§ 4. Scattering of mesons

As mentioned before, the observed direction of μ-mesons deviate from that of primary photons. The following processes are considered as those resulting in the deflection of mesons in the atmosphere.

a) Angular distribution of π-mesons produced in the photo-nuclear interaction.
b) Deflection at π-μ decay.
c) Coulomb scattering of μ-mesons on traversing air.

To compare the magnitude of the angular deflections through the above-mentioned processes, it is convenient to compare the transverse momenta given to the mesons in each process.

a) It is currently accepted that the transverse momentum of mesons produced in the nuclear interactions is approximately 500 Mev/c essentially irrespective of their energies.
b) The maximum transverse momentum given to μ-mesons through π-μ decay is easily estimated to be about 30 Mev/c from the mass difference between the π-meson and the μ-meson.
c) Mean transverse momentum acquired by mesons when they pass X radiation length of matter is given as $P_c \simeq E_s X^{1/2}/c$, where $E_s \simeq 21$ Mev. For mesons coming vertically, $X \simeq 25$ and $P_c \simeq 100$ Mev/c.

It is clear that deflections of mesons due to the processes (b) and (c) are sufficiently smaller than that due to the process (a) to be neglected with a good approximation except for mesons with zenith distances near $\pi/2$. Then, the mean amount of the scattering of mesons is simply expressed as follows as a function of their energy E_m,

$$\theta_s = 500/E_m \text{ Mev}. \tag{4·1}$$

Table I.

E_m	E_0	θ_s
2 Gev	10 Gev	14°
5	30	5.7
10	80	2.9
20	200	1.4

In Table I, values of θ_s are given for various energies of mesons E_m with a rough estimate of E_0 corresponding to each value of E_m. One may regard that θ_s is related to Ω_0 in the following way,

$$\Omega_0=2\pi(1-\cos\theta_s). \tag{4·2}$$

In addition to the effects considered above, the geomagnetic deflection must be taken into account especially for particles with large zenith distances.* To estimate the order of magnitude of this effect, we shall treat a special case in which particles come from the north. Let H be the absolute strength of geomagnetic field and α its inclination. H can be considered as constant through the paths of mesons. For particles with a zenith distance ζ, the component of geomagnetic force effective for the motion of the particle is horizontal with a magnitude

$$H_{eff}=H\sin(\alpha+\pi/2-\zeta). \tag{4·3}$$

The angle of deflection becomes for $\theta_m \ll 1$

$$\theta_m=300\,HL\sin(\alpha+\pi/2-\zeta)/E_m, \tag{4·4}$$

where L is the path length of mesons. L can be expressed as follows:

$$L=h(x_0\sec\zeta)\sec\zeta. \tag{4·5}$$

Here $h(x)$ represents the height of the atmospheric level with the depth x g·cm^{-2} from the top of the atmosphere and x_0 is the mean depth of nuclear interactions of primary photons along their paths. The value of x_0 can be equated to the mean free path for the pair production of photons and $9X_0/7$ g·cm^{-2} from (3·2). For $\zeta=0$, $E_m\simeq 3$ Gev and $L\simeq 2.1\cdot 10^6$ cm from (4·5), and $\theta_m\simeq 6\cdot 10^{-2}$ or 3 degrees for $H=0.47$ Gauss and $\alpha=49°$ from (4·4). On the other hand, for $\zeta=80°$, $E_m\simeq 20$ Gev, $L\simeq 1.8\cdot 10^7$ cm and $\theta_m\simeq 0.11$ or 6 degrees. Here it is noted that mesons with positive and negative charges are deflected in the opposite directions. Then, for $\zeta=80°$, the separation of images of a source due to mesons of different signs of charge is about 12 degrees which is sufficiently large compared with the extension of each image defined by θ_s in Table I.

In this special case, the geomagnetic force does not alter the zenith distance of the incoming particles. This condition no more holds in general. As an example, the difference in the paths between positive and negative particles is shown in Fig. 1 when the detector is directed to the east. As shown in this figure, the path length of a positive particle is longer than that of a negative one, which results in the difference in deflection angles and in the intensities of two images of the source because the survival probability of positive mesons is smaller than that of negative ones.

* Numerical computations of this effect was carried out by Murayama and Kamiya,[6] and Kamiya[7] pointed out the east-west asymmetry in the cosmic-ray intensity due to this effect.

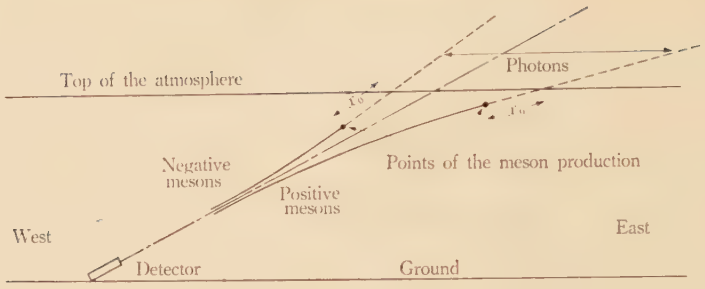

Fig. 1. Difference in paths of positive and negative mesons observed in the same direction at sea level.

§ 5. Minimum observable flux of photons

The lower limit of the flux of γ-rays detectable by the method suggested above depends on the accuracy of the determination of R in (2·4). For instance, when R is determined within an error of 1%, the value of J_{min} deduced from the following equation obtained by combining (2·4) and (4·2) is regarded as the minimum detectable flux of γ-rays:

$$J_{min}=\frac{10^{-2}\,\Omega_0}{3\cdot 10^{-3}}\,J_n(E_0)=6.7\,\pi(1-\cos\theta_s)\,J_n(E_0). \qquad (5\cdot 1)$$

Table II shows the values of J_{min} for different values of E_0.

Table II.

E_0	$J_n(E_0)$	Ω_0	J_{min}
10 Gev	$3.0\ 10^{-2}$ cm^{-2} sec^{-1} str^{-1}	0.20 str	$1.9\ 10^{-2}$ cm^{-2} sec^{-1}
30	$8.0\ 10^{-3}$	$3.1\ 10^{-2}$	$8.3\ 10^{-4}$
80	$2.3\ 10^{-3}$	$7.9\ 10^{-3}$	$6.0\ 10^{-5}$
200	$6.0\ 10^{-4}$	$2.0\ 10^{-3}$	$4.0\ 10^{-6}$

In the above discussion, R is assumed to be determined within an error of 1%. This error depends on the total number of observed cosmic rays and, therefore, depends on the area of the detector and the time of observation. (Here we neglect errors other than the statistical one.)

Let t_s be the time in which the detector is directed towards the source, t_b be the time measuring the background intensity and N_s and N_b be the total numbers of particles counted in the respective time intervals. Then, (2·4) is written in another form:

$$R=\frac{N_s/t_s}{N_b/t_b}-1. \qquad (5\cdot 2)$$

The error in R is given by the expression

$$\left(\frac{\Delta R}{R}\right)^2=(1+R)^2\left\{\left(\frac{\Delta N_s}{N_s}\right)^2+\left(\frac{\Delta N_b}{N_b}\right)^2\right\}=(1+R)^2\left(\frac{1}{N_s}+\frac{1}{N_b}\right), \quad (5\cdot3)$$

where ΔN_s and ΔN_b are the errors of N_s and N_b, which can be put $N_s^{1/2}$ and $N_b^{1/2}$ respectively, by considering the Poisson fluctuation only.

When the direction of the detector is fixed, the time of observation of one source is limited in the time of its passing through the field of vision of the detector. This time depends on the position of the source in the celestial sphere. For the given declination δ of the source, it has a value

$$720\theta_t \sec\delta/\pi, \quad \text{(min.)} \quad (5\cdot4)$$

where θ_t is a diameter of the field of vision of the detector.

When $E_m=10$ Gev, θ_t should be about 6 degrees according to the value of θ_s in Table I, and the time of observation is about 24 minutes for a source near the celestial equator ($\delta=0°$). In general, we can catch a source twice a day with the detector of a fixed zenith distance. The total time, therefore, becomes about 50 minutes per day. Remaining time can be used to measure the background intensity, so that we can put $N_s \ll N_b$. Then, (5·3) becomes, with a good approximation,

$$\Delta R/R=(1+R)N_s^{-1/2}. \quad (5\cdot5)$$

To make ΔR smaller than 1%, N_s must be larger than 10^4 for $R \ll 1$. Since the intensity of mesons with $E_m \geq 10$ Gev is several $m^{-2}\,sec^{-1}\,str^{-1}$, an observation for about one month is necessary with a detector having an effective area of 10 m^2 in order to detect a γ-ray source by means of the above-mentioned method with an accuracy within 1%. A cosmic-ray gas Čerenkov telescope constructed by Sekido et al.[8] has an area of 20 m^2 and has an angular resolution of about 3.5 degree. This will be adequate for the present purpose.

§ 6. Possible sources of high energy photons

Estimates of the intensity of γ-rays through various processes have been performed by Morrison[9] and by Savedoff[10] for energies up to 10^8 ev. For energies above 10^9 ev, γ-rays will exclusively be produced in nuclear interactions through the decay of π_0-mesons. Therefore, the amount of γ-rays produced in a source depends on the mechanism of acceleration of nuclei, the density of matter in the region in which the acceleration takes place, the time of storage of particles in this region, etc. Uncertainties in these factors make it difficult to estimate the γ-ray flux quantitatively. We shall refer to two possible sources which permit some discussions on the γ-ray flux.

i) *The sun.* The sun emits cosmic rays associated with intense solar flares. Such an event is observed by cosmic-ray detectors distributed over the world as an unusual increase of cosmic-ray intensity. Morrison[9] has pointed

out the possible existence of photons produced through the bremsstrahlung of electrons accelerated at the same stage as protons. It is further noted that a part of accelerated protons may undergo nuclear interactions in the solar atmosphere which result in γ-rays as decay products of π_0-mesons. On account of the very steep energy spectrum of solar cosmic rays, however, secondary particles of these γ-rays produced in the earth atmosphere will be more easily detected by neutron detectors at high latitudes rather than by the meson detectors described in the preceding sections. It is noted that this effect may be observed only by neutron detectors placed at the point where the sun is directly overhead of it at the time of the flare.

At the most significant increase on Feb. 23, 1956, over ten times above the normal intensity was observed at high latitudes. Let η_1 be the average ratio of the number of photons to that of protons of the same energy near 1 Gev produced at the time of the flare, and η_2 the probability of escape of protons from the accelerated region where the magnetic field is supposedly strong. The expected percentage of increase of the neutron intensity will be

$$\frac{10^3 P I_h \eta_1}{I_l \eta_2} \%, \tag{6·1}$$

where P is given in (3·3), I_h and I_l are the intensities of neutrons at high and low latitudes, respectively. If η_1/η_2 is 0.1, about one per cent increase would be expected for $P=3\cdot10^{-3}$ and $I_h/I_l=2$.

ii) *Crab nebula.* It has become evident from the observation of the strong polarization of light from the Crab nebula that electrons with energies 10^{11}–10^{12} ev exist in the nebula with an appreciably high density. Hayakawa[11] pointed out that about $3\cdot10^{-5}\,\mathrm{cm^{-2}\,sec^{-1}}$ would be expected for the flux of photons of 10^{11} ev at the top of our atmosphere according to Burbidge's model[12] on the formation of these high energy electrons in the nebula. This value is much larger than those estimated from the models by Oort et al.[13] or by Hayakawa et al.[14] which predict the photon flux of about $10^{-9}\,\mathrm{cm^{-2}\,sec^{-1}}$. Our method will provide the useful data to test whether Burbidge's model is acceptable or not.

§ 7. Comparison with an emulsion experiment

The minimum detectable flux of γ-rays shown in Table II should be compared with that in an experiment to catch γ-rays with emulsion near the top of the atmosphere, being about $5\cdot10^{-5}$ photons $\mathrm{cm^{-2}\ sec^{-1}}$ [10] irrespective of their energy above 10 Mev. Our value (10^{-4}–$10^{-5}\,\mathrm{cm^{-2}\,sec^{-1}}$) is comparable to this value for photons of energies near 10^{11} ev.

Although the present method is not so direct as the observations with emulsions, it has advantages over the latter in the following respects. Our method allows to obtain day-to-day data especially for sources with time vari-

ations of intensity and quickly to follow sudden events such as solar flare effects and nova explosion.

§ 8. Detection of the showers of photons

Cocconi[15] has pointed out the possibility of detecting γ-rays with energies around 10^{12} ev by observing the cascade showers initiated from them. As mentioned in § 1, however, it must be taken into account that these showers have very low particle densities even in the region near their axes, and the detection probability of these showers depends strongly on the number and the arrangement of the shower trays.

Let us confine ourselves to the case in which observations are performed at 600 $g \cdot cm^{-2}$ in atmospheric depth (4,500 m above sea level) with nine particle detectors, each surface being one meter square. At this height, photon showers of 10^{12}ev have a mean size of about 400 particles and have a mean value of the age parameter s of about 1.4. On the other hand, the mean age of the ordinary air showers initiated from the nuclear component of primary cosmic rays is about 1.2 and their size spectrum is represented approximately by the following expression,

$$F_n(\geq N) = 5.3 \cdot 10^{-7} (N/10^6)^{-[1.50 + 0.025 \ln (N/10^6)]}. \qquad (8 \cdot 1)$$

This equation is identical with that given by Greisen[16] except the two times larger value of the first numerical constant taking the difference in the altitude into account. (His equation is derived from the analysis of the density spectrum of extensive air showers at 3,200 m.) (8·1) is also in good agreement with the size spectrum of extensive air showers measured by Kulikov et al.[17] at 3,860 m above sea level. Fig. 2 shows the lateral distribution of particles in the showers with the size of 400 and with $s=1.2$ and $s=1.4$. As shown in Fig. 2, the density of particles in the shower with

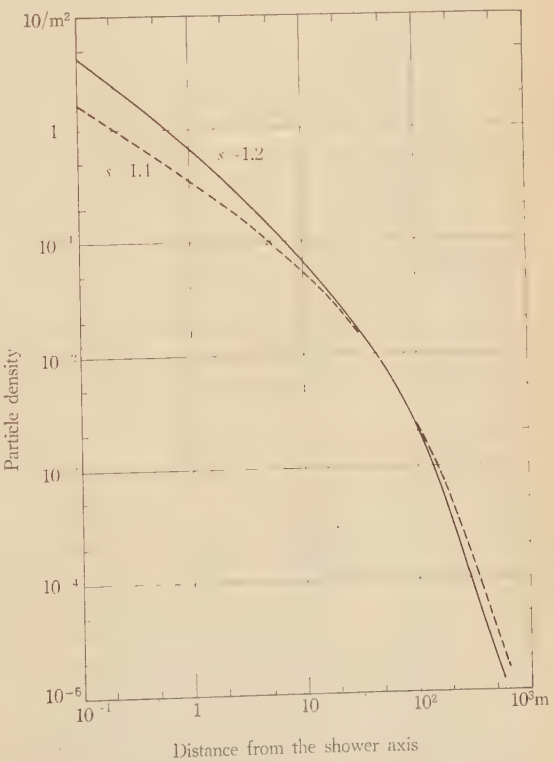

Fig. 2. Lateral distribution of particles in the showers with the size of 400. Solid line : nucleonic air showers ($s=1.2$). Dotted line : photon showers ($s=1.4$).

the size of 10^3 or less is of the order of one per meter square even within one meter from the axis and therefore it is hardly expected to determine the size and the arrival direction of the individual shower with a good accuracy.

§ 9. Estimates on the frequency of the showers

To make discussions quantitative, we assume that the nine trays are arranged as shown in Fig. 3. For the distance between the neighboring trays, it is desirable to make it as small as possible in order to obtain a high detection efficiency for small showers. We assume it, however, to be five meters to avoid the possible contamination of local showers.

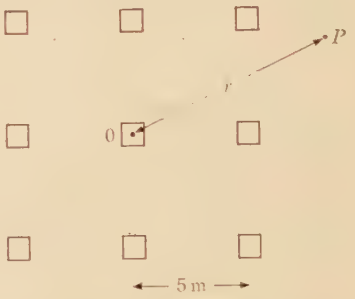

Fig. 3. An example of the arrangement of the shower trays to detect photon showers.

Let the point P in Fig. 3 be the position of the axis of a shower, r_i be the distance between the axis and the i-th tray and Δ_i be the particle density of the shower at the position of that tray. We further assume that showers are recorded whenever

i) particles are detected in any three of the nine trays (showers having particles in four or more trays are rejected), and

ii) only one particle is detected in each of these three trays (showers having two or more particles in any of these three trays are rejected).

These limitation to the showers must be necessary to discriminate the low density photon showers from the nucleonic air showers. Then, for instance, the probability of detecting one particle in each of the first, second and third tray and none in the trays from fourth to the ninth is

$$\prod_{i=1}^{3} \Delta_i e^{-\Delta_i} \prod_{j=4}^{9} e^{-\Delta_j} = \prod_{i=1}^{3} \Delta_i \exp\left[-\sum_{j=1}^{9} \Delta_j\right]. \tag{9·1}$$

Therefore, the probability of detecting one particle in each of any three (and not four or more) trays is

$$\eta(N, s, r) = \left\{\sum \left(\prod^{3} \Delta_i\right)\right\} \exp\left[-\sum_{j=1}^{9} \Delta_j\right], \tag{9·2}$$

where r represents the distance of the axis of the shower from the center of the detector. Here η is regarded as constant in the different positions with the same value of r. This approximation causes little error on the final result. The first summation in this equation must be performed for all combinations of any three trays out of the nine, the number of such combinations being ${}_9C_3=84$. Fig. 4 shows η as a function of r for showers of different sizes.

An effective area of the detector for showers with the size N and with the age s is defined by

$$S(N, s)=2\pi\int_0^\infty r\eta(N, s, r)dr. \tag{9·3}$$

Fig. 5 shows the size dependence of the effective area of the detector.

Counting rates of photon showers and nucleonic air showers with our detector are given respectively as follows:

Nucleonic air showers

$$I_n=\int_0^\infty \frac{F_n(\geq N)}{dN} S(N, s=1.2)dN,$$

Photon showers (9·4)

$$I_p=\int_0^\infty f_p(N)\, S(N, s=1.4)dN,$$

where $f_p(N)dN$ represents the differential size spectrum of the photon showers. Table III shows the counting rate of nucleonic air showers for different ranges of their size.

Table III.

N	I_n
$3\cdot10^2-10^3$	3.5 10^3 hr^{-1} str^{-1}
10^3-10^4	12.5 10^3
10^4-10^5	7.2 10^3
10^5-10^6	1.9 10^3
total	2.5 10^4

Since the effective solid angle for accepting the air showers is about one steradian, the total number

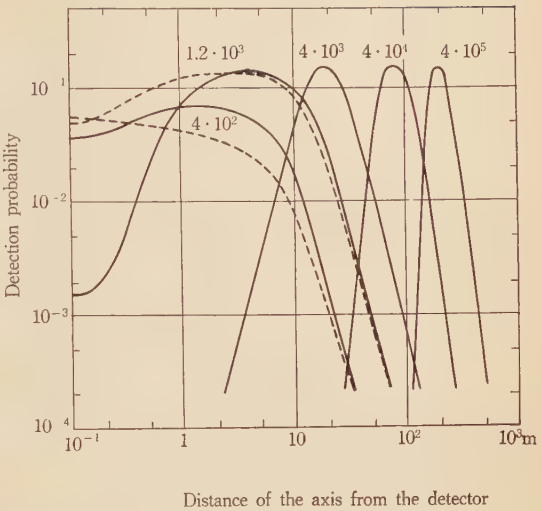

Fig. 4. Detection efficiency *versus* distance of the shower axis from the center of the detector. Solid line: nucleonic air showers (s=1.2). Dotted line: photon showers (s=1.4). The number attached to each curve represents the size of the shower.

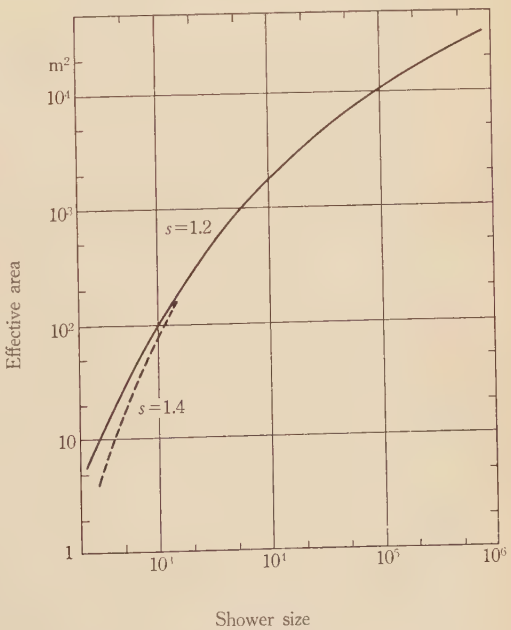

Fig. 5. Effective area of the detector *versus* shower size. Solid line: nucleonic air showers. Dotted line: photon showers.

of showers observed by our arrangement of the trays is approximately $2.5 \cdot 10^4$ per hour.

As one of the possible sources of high energy γ-rays, we again refer to the Crab nebula. According to the estimate by Cocconi,[15] $6.4 \cdot 10^{-4}$ photons m^{-2} sec^{-1} may be expected for energies in $10^{12} \pm 10^{10.8}$ev. These showers correspond to those with the size of 400 ± 25 at our altitude. If we assume that the size spectrum of photon showers has the form of $N^{-2.4}dN$ in the range of the size from $3 \cdot 10^2$ to 10^3 ($7.5 \cdot 10^{11}$ev$-2.5 \cdot 10^{12}$ ev in energy), the frequency of showers in this energy range normalized to the value given by Cocconi is about 30 m^{-2} hr^{-1}. Since the average effective area of the detector is about 30 m^2 in this range as shown in Fig. 5, the expected counting rate of the showers by photons from the Crab nebula is $9 \cdot 10^2$ hr^{-1}. This value may be compared with that of background air showers ($2.5 \cdot 10^4$ hr^{-1}), resulting in the signal to background ratio of about 4%. If the size spectrum of the photon showers extends beyond 10^3, higher S-N ratio may be obtained.

These photon showers have very steep zenithal dependence owing to the strong attenuation in size with atmospheric depth. In fact, the size of showers by photons of 10^{12} ev is about 150 when they fall with a zenith distance of 30° and will hardly be detected by our trays. Thus we shall detect a few per cent higher intensity of showers compared with the average value when the Crab nebula is within about twenty degrees from the zenith of the detector. It also means that the experiment must be carried out in the latitude band where we can observe the source near the zenith. For example this should be $7°N-39°N$ in order to aim at the Crab nebula (declination : $22°N$).

In general we define the minimum flux of photons observable by this method as the value giving the S-N ratio of 1%. For γ-rays of 10^{12} ev, the minimum detectable flux is $2.5 \cdot 10^4 \cdot 10^{-2} (30\ m^2)^{-1}$ $hr^{-1} \simeq 2.3 \cdot 10^{-7}$ cm^{-2} sec^{-1}.

To facilitate the experiment, it is desirable to make the above-mentioned S/N as large as possible. The following ways are effective for this purpose.

i) Observation at higher altitude

At 500 g cm^{-2}(5,800 m above sea level), the sizes of photon showers are about two times larger than those at 600 g cm^{-2}. Then S/N becomes about three times larger taking the increase of the frequency of background showers into account.

ii) Use of the shower trays of larger area

By making the area of each tray as 2 m^2, S/N can be increased by a factor of six.

iii) Increase of the number of trays

By adding one more tray, S/N can be about 50% higher.

iv) Anti-coincidence method

To reduce the effect of background showers, it is effective to place several trays about 20~30 meters far from the center of the detector in

anti-coincidence with the shower signal. It will be effective for rejecting showers with the size near 10^4, which are the main part of the background showers.

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$K\pi$ Resonance Effect to the Form Factors in the K_{l3} Decays*

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The form factors in the leptonic three-body decays of K-meson are studied in terms of the dispersion theoretic techniques developed by Goldberger and Treiman. Assuming that the $K\pi$-resonance has a dominant contribution to the dispersion integral, we can show that the consequences of the partially conserved current hypothesis studied by Bernstein and Weinberg follow from the theory under the weaker conditions on the high energy behavior of the divergence of the current.

§ 1. Introduction

The extension of the conserved vector current hypothesis[1] to the other weak currents has been proposed by a number of authors.[2] For the strangeness non-conserving current, Bernstein and Weinberg[3] have discussed a possible relationship between two form factors appearing in the leptonic three-body decay of K-meson. In their treatment, B.W. have made two basic assumptions on the high-energy behavior of the form factors, i.e. (a) form factors have, at most, definite finite limits when the momentum transfer q^2 becomes infinite; (b) the matrix elements of the divergence of the current J_a between the K-meson and the pion state becomes zero in the limit $q^2 \to \infty$.

In view of the fact that we do not have much knowledge concerning the high energy behavior of form factors in the general case, assumption (a) is usually adopted. Assumption (b) is the so-called partially conserved current hypothesis. In the nucleon β-decay, this hypothesis seems to be well justified for the axial vector part of the process.[2] This is partly because we know the structure of each term of that part (a pure axial vector term and an induced pseudo-scalar term) and partly because of the equal mass approximation for the neutron and the proton masses. The complications due to a mass difference between a parent particle and decay products have been remarked by Goldberger and Treiman[4] in connection with the hyperon β-decays. In fact, a number of authors[5] have already pointed out an inevitably approximate nature of the above hypothesis in the hyperon leptonic decays.

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In the K_{l3} processes, this point would be much more serious because of apparent large mass differences between the K-meson and the pion.

Thus, in the K_{l3} decays, it is desirable to seek a weaker assumption than (b) of B.W., yet one under which we may hope to obtain a certain relation between two form factors.

In this paper we shall consider this point and shall show that the assumption (b) of B.W. can be weakened as follows: (b′) the matrix element mentioned in (b) has, at most, a definite finite limit in the limit $q^2 \to \infty$. Namely, even under these two assumptions, (a) and (b′), only one parameter is needed to describe the K_{l3} decays.

In the following we shall give a brief review of the problem in § 2 and the form factors are studied in § 3 by means of the dispersion theoretic techniques developed by Goldberger and Treiman.[4] In particular we shall show that the result of B.W., as well as of the phenomenological intermediate vector boson theory can be obtained in the pole approximation for the s-wave or the p-wave $K\pi$-resonant state. In § 4, we shall briefly discuss the $K\pi$ scattering correction to the pole approximation and the final section is devoted to the discussion of our results.

§ 2. Effective Hamiltonian[6]

We study the following reactions,

$$K^{\pm} \to \pi^0 + l^{\pm} + \left\{\begin{matrix}\nu\\ \bar{\nu}\end{matrix}\right\}, \tag{1}$$

where $l^{\pm}$ means $\mu^{\pm}$ or $e^{\pm}$.

The relevant Hamiltonian may have a form:

$$\begin{aligned} H_I &= \frac{G^0}{\sqrt{2}} j_\alpha \cdot J_\alpha + \text{hermitian conjugate},\\ j_\alpha &= (\bar{\psi}_\mu \gamma_\alpha (1+\gamma_5)\psi_\nu) + (\bar{\psi}_e \gamma_\alpha (1+\gamma_5)\psi_\nu) + \text{h. c.},\\ J_\alpha &= (\bar{\psi}_P \gamma_\alpha (1+\gamma_5)\psi_A) + \cdots, \end{aligned} \tag{2}$$

where ψ_α is the field operator of the particle α and we assume that the $|\Delta I| = 1/2$ rule holds for the strangeness non-conserving current; i.e. J_α is an isospinor in isotopic-spin space. We further assume H_I to be time-reversal invariant.

The matrix element of the process, $K^+ \to \pi^0 + l^+ + \nu$, is given by the following form:

$$\begin{aligned} M_{fi} = (2\pi)^4 i \delta^4 (P_l + P_\nu + P_\pi - P_K)\, \bar{v}_l(P_l)\, \gamma_\alpha (1+\gamma_5)\, u_\nu(P_\nu)\\ \times \langle \pi^0 | \frac{G^0}{\sqrt{2}} J_\alpha(0) | K^+ \rangle, \end{aligned} \tag{3}$$

where p_i is the momentum of the particle i.

We shall study the strong interaction part in Eq. (3) by means of dispersion relations. By invariance arguments, we have

$$\langle \pi^0 | \frac{G^0}{\sqrt{2}} J_\alpha(0) | K^+ \rangle = \frac{1}{\sqrt{4P_{\pi 0} P_{K0}}} \{(P_K + P_\pi)_\alpha F_1(q^2) + (P_K - P_\pi)_\alpha F_2(q^2)\},$$
$$q^2 = (P_K - P_\pi)^2, \tag{4}$$

where $F_i(q^2)$ is a real function of q^2. $f_v(q^2)$ and $g_v(q^2)$ of B.W. are simply related with $F_i(q^2)$ as follows:

$$f_v(q^2) = 2F_1(q^2),$$
$$g_v(q^2) = F_2(q^2) - F_1(q^2). \tag{5}$$

§ 3. Dispersion relations

Instead of studying Eq. (4), we shall study the matrix element given below:

$$H_\alpha(P_\pi, P_K) = \sqrt{4P_{\pi 0} P_{K0}} \langle 0 | \frac{G^0}{\sqrt{2}} J_\alpha(0) | \pi^0 K^+ \text{ (in)} \rangle,$$
$$= (P_K - P_\pi)_\alpha F_1(\xi) + (P_K + P_\pi)_\alpha F_2(\xi),$$
$$\xi = (P_K + P_\pi)^2. \tag{6}$$

We assume that $F_i(\xi)$ is analytic and regular on the ξ-plane except on the cut $[-(m_K + m_\pi)^2, -\infty]$. In order to write down the dispersion relations for $F_i(\xi)$, we need a knowledge of the high-energy behavior of $F_i(\xi)$. As was mentioned in the Introduction, we make the following two assumptions concerning this point[7]:

(a) $F_i(\xi)$ has, at most, a definite limit (zero or constant) for $\xi \to \infty$ along the real axis.

(b′) $(P_K + P_\pi)_\alpha H_\alpha = f(\xi) = \xi F_2(\xi) - (m_k^2 - m_\pi^2) F_1(\xi)$ has, at most, a definite limit (zero or constant) for $\xi \to \infty$ along the real axis. From these two assumptions, we see that $F_1(\xi)$ may remain finite in the limit $\xi \to \infty$, while $F_2(\xi)$ must go to zero, at least, like $1/\xi$. Sugawara and Kanazawa[8] have recently proved that in this case $F_i(\xi)$ can be expressed in the following dispersion relations,

$$F_1(\xi) = F_1(\infty) + \frac{1}{\pi} \int_{(m_K + m_\pi)^2}^{\infty} d\xi' \frac{\operatorname{Im} F_1(-\xi')}{\xi' + \xi}, \tag{7a}$$

$$F_2(\xi) = \frac{1}{\pi} \int_{(m_K + m_\pi)^2}^{\infty} d\xi' \frac{\operatorname{Im} F_2(-\xi')}{\xi' + \xi}, \tag{7b}$$

and since $f(\xi)$ has the same analytic properties as $F_i(\xi)$, we have

$$f(\xi)=f(\infty)+\frac{1}{\pi}\int_{(m_K+m_\pi)^2}^{\infty}d\xi'\frac{\mathrm{Im}f(-\xi')}{\xi'+\xi}. \tag{8}$$

In the above expressions $F_1(\infty)$ and $f(\infty)$ are unknown real values of the functions at $\xi=\infty$.

Actually this representation of $F_i(\xi)$ is equivalent to that of B.W. The difference is that they have set $f(\infty)=0$ (the partially conserved current hypothesis) and from this condition they have obtained the relation between $F_1(\infty)$ and $\int_{(m_K+m_\pi)^2}^{\infty}d\xi'\ \mathrm{Im}\,F_2(-\xi')$, i.e.

$$F_1(\infty)=\frac{1}{m_K{}^2-m_\pi{}^2}\frac{1}{\pi}\int_{(m_K+m_\pi)^2}^{\infty}d\xi'\,\mathrm{Im}\,F_2(-\xi').$$

We have already argued in the Introduction the reason why we do not take $f(\infty)=0$ in our studies (see also § 5).

At this stage we should like to emphasize that our following discussions are valid if the one subtracted forms of Eqs. (7a) and (8) are meaningful, even if Eqs. (7a) and (8) are not. Thus our conclusions[7] obtained in this section may be independent of the validity of the theorem due to Sugawra and Kanazawa.

Now we see from the above considerations and Eq. (7) that only one parameter has appeared in our theory; i.e. $F_1(\infty)$ or the subtracted constant in the case of the subtracted form of Eq. (7a) (see later discussion). This is the consequence of assumptions (a) and (b′), and if we make further weaker assumptions, two or more parameters will appear. Thus we may say that our set of assumptions[7] seems to be the weakest one in the general circumstance, in which a dynamical structure of the external source J_α is not known, to obtain a possible relation between $F_i(\xi)$'s.

Let us turn to the study of Eqs. (7a) and (8). In order to make the unknown high-energy contributions less important in the following calculations, it is appropriate to make one subtraction for Eqs. (7a) and (8) at $\xi=0$.

$$\begin{aligned}F_1(\xi)&=F_1(0)-\frac{\xi}{\pi}\int_{(m_K+m_\pi)^2}^{\infty}d\xi'\frac{\mathrm{Im}\,F_1(-\xi')}{\xi'(\xi'+\xi)},\\ f(\xi)&=f(0)-\frac{\xi}{\pi}\int_{(m_K+m_\pi)^2}^{\infty}d\xi'\frac{\mathrm{Im}f(-\xi')}{\xi'(\xi'+\xi)},\end{aligned} \tag{9}$$

where

$$f(0)=-(m_K{}^2-m_\pi{}^2)F_1(0).$$

In the following, we suppose $F_1(0)$ to be a parameter[9] which is determined

from the experimental lifetime of K-meson. The absorptive parts in Eqs. (7b) and (9) are obtained from the following equations:

$$\mathrm{Im}\, H_\alpha(P_\pi P_K) = -\frac{1}{2}\int d^4x\, e^{iP_\pi\cdot x}\,\langle 0|\,\frac{G^0}{\sqrt{2}}\,J_\alpha(0)\,j_{\pi^0}(x)|K^+\rangle\sqrt{2P_{K0}},$$

$$= -\pi\sum_n{}'\,\delta^4(P_\pi+P_K-P_n)\langle 0|\,\frac{G^0}{\sqrt{2}}\,J_\alpha(0)|n\rangle\langle n|j_{\pi^0}(0)|K^+\rangle\sqrt{2P_{K0}},$$

$$(\Box-m_\pi^2)\phi_{\pi^0}(x)=j_{\pi^0}(x), \tag{10}$$

where we have put a complete set of states $|n\rangle$ between $J_\alpha(0)$ and $j_{\pi^0}(0)$. Following Goldberger and Treiman,[4] we suppose in Eq. (10) that the sum of a complete set of states is taken as half the sum over "in" and "out" states to make sure of the reality of $\mathrm{Im}\, F_i$ at all stages of approximation. The lowest mass intermediate states contributing to Eq. (10) are the $K\pi$ states. As we have no sound knowledge concerning the higher mass states in Eq. (10), we must be satisfied to treat the problem in the lowest mass approximation. Then we follow the familiar line in the dispersion theory treatment. In the c.m.s. of the initial $K\pi$ system, if we consider the spatial part of H_α in Eq. (10), we easily obtain evidence that $\mathrm{Im}\, F_1$ has contributions from the p-wave states of the $K\pi$ system. In the same way, $(P_\pi+P_K)_\alpha\cdot\mathrm{Im}\, H_\alpha(P_\pi P_K)$ gives us $\mathrm{Im}\, f$, and the s-wave states contribute to it. We need the total isospin 1/2 states for the $K\pi$ system, because of our assumption of the $|\Delta I|=1/2$ rule for the strangeness non-conserving current.

Thus we have, in the lowest mass approximation,

$$\begin{aligned}&\mathrm{Im}\, F_1(-\xi')=\tan\varphi_P(\xi')\,\mathrm{Re}\, F_1(-\xi'),\\ &\mathrm{Im} f(-\xi')=\tan\varphi_S(\xi')\,\mathrm{Re} f(-\xi'),\\ &\xi'\geq(m_K+m_\pi)^2.\end{aligned} \tag{11}$$

$\varphi_i(\xi')$ is related with the phase shifts of $K\pi$ scattering, i. e.

$$\tan\varphi_S(\xi')=\frac{\mathrm{Re}\,(e^{i\delta_{S1/2}}\sin\delta_{S1/2})}{1-\mathrm{Im}(e^{i\delta_{S1/2}}\sin\delta_{S1/2})},$$

$$\tan\varphi_P(\xi')=\frac{\mathrm{Re}\,(e^{i\delta_{P1/2}}\sin\delta_{P1/2})}{1-\mathrm{Im}\,(e^{i\delta_{P1/2}}\sin\delta_{P1/2})}, \tag{12}$$

where $\delta_{r1/2}$ is, in general, a complex r-wave phase shift of the $K\pi$ scattering with the total isospin 1/2.

From Eqs. (9) and (11), we obtain the approximate solutions of the form factors

$$F_1(\xi)=F_1(0)\exp\Big[-\frac{\xi}{\pi}\int_{(m_K+m_\pi)^2}^{\infty}d\xi'\,\frac{\varphi_P(\xi')}{\xi'(\xi'+\xi)}\Big],$$

$$f(\xi)=f(0)\exp\left[-\frac{\xi}{\pi}\int_{(m_K+m_\pi)^2}^{\infty}d\xi'\frac{\varphi_S(\xi')}{\xi'(\xi'+\xi)}\right]. \tag{13}$$

Here we note that these solutions are meaningful for finite ξ, and since we know from the experiment[10] that the $K\pi$ system has a resonance, Eq. (13) would be a good approximation for $-\xi$ near that resonance energy (the pole approximation).

From the definition of $f(\xi)$ and Eq. (13), we have

$$F_1(\xi)=F_1(0)\exp\left[-\frac{\xi}{\pi}\int_{(m_K+m_\pi)^2}^{\infty}d\xi'\frac{\varphi_P(\xi')}{\xi'(\xi'+\xi)}\right],$$

$$F_2(\xi)=(m_K^2-m_\pi^2)F_1(0)\frac{1}{\xi}\left\{\exp\left[-\frac{\xi}{\pi}\int_{(m_K+m_\pi)^2}^{\infty}d\xi'\frac{\varphi_P(\xi')}{\xi'(\xi'+\xi)}\right]\right.$$

$$\left.-\exp\left[-\frac{\xi}{\pi}\int_{(m_K+m_\pi)^2}^{\infty}d\xi'\frac{\varphi_S(\xi')}{\xi'(\xi'+\xi)}\right]\right\}, \tag{14}*$$

and from this, we have

$$F_2(0)=\frac{1}{\pi}(m_K^2-m_\pi^2)F_1(0)\int_{(m_K+m_\pi)^2}^{\infty}d\xi'\frac{\varphi_S(\xi')-\varphi_P(\xi')}{\xi'^2}. \tag{15}$$

Now let us derive the result of B.W. in the pole approximation for the $K\pi$ resonant state. If the elastic scattering of $K\pi$ dominates the integral in Eq. (14) in the physical region of K_{l3}, i.e. for $\xi:[-(m_K-m_\pi)^2, -m_l^2]$,[11] we may approximate $\varphi_i(\xi')$ by

$$\varphi_S(\xi')\cong\delta_{S1/2}(\xi'),\quad \varphi_P(\xi')\cong\delta_{P1/2}(\xi'). \tag{16}$$

(A) The s-wave resonance case

If the $K\pi$ resonance is s-wave, the limiting case of the pole approximation for this state can be expressed as follows:

$$\begin{aligned}&\delta_{S1/2}(\xi')\cong\pi\theta(\xi'-m_r^2),\\ &\delta_{P1/2}(\xi')\cong 0,\end{aligned} \tag{17}$$

where $\theta(\xi'-m_r^2)$ is a step function and m_r denotes the resonance energy of the $K\pi$ system.

Then we have

$$\begin{aligned}&F_1(\xi)\cong F_1(0),\\ &F_2(\xi)\cong\frac{m_K^2-m_\pi^2}{\xi+m_r^2}F_1(0).\end{aligned} \tag{18}$$

* See Note added in proof.

This is the result of B.W., as is easily seen from Eq. (5).

(*B*) *The p-wave resonance case*

In the same way, for the p-wave resonance, we have

$$F_1(\xi) \simeq \frac{m_r^2}{\xi + m_r^2} F_1(0),$$
$$F_2(\xi) \simeq -\frac{m_K^2 - m_\pi^2}{\xi + m_r^2} F_1(0). \tag{19}$$

This is the result[12] of the intermediate vector boson theory with the phenomenological interactions of the following form:

$$H_I' = g_I' B_\alpha (\phi_{\pi^0} \partial_\alpha \phi_K - \phi_K \partial_\alpha \phi_{\pi^0}) + \text{h. c.},$$

together with

$$H_I = g_I B_\alpha [(\bar{\psi}_\mu \gamma_\alpha (1+\gamma_5)\psi_\nu) + (\bar{\psi}_e \gamma_\alpha (1+\gamma_5)\psi_\nu)] + \text{h. c.},$$

where g_I, g_I' are the coupling constants and B_α is the vector boson field with the mass m_r. The only difference between our calculations and those of B.W. or the intermediate boson case is just the normalization convention. But this does not make any difference, for, as we have mentioned before, we must fix $F_1(0)$ so as to give the correct lifetime of the K_{l3}.

Thus we see that the practically interesting result of B.W. can be reproduced, without the partially conserved current hypothesis, in the lowest order approximation of dispersion theory treatment. Of course, our calculations are meaningful only if $F_1(\xi)$ has no zero-point at $\xi = 0$. In order to study this question, it will be necessary to know a dynamical structure of J_α as well as that of strong interactions. From the experimental viewpoint, it will be possible, in principle, to know $F_1(0)$ by looking at the high energy tail of the π^0-energy spectrum. In particular, this is promising for the K_{e3} decay[13] because of the negligible electron mass.

In connection with the subtraction point, the real trouble compared to the β-decays[14] is the fact that we cannot define or find the unique point of ξ, where the renormalized coupling constant of the process can be defined. Thus we shall await future experimental information concerning this zero-point problem of $F_1(\xi)$.

Finally we should like to make two obvious remarks: i) From our dispersion theoretic viewpoint, the exactly conserved current hypothesis[15] is not acceptable if $F_1(0)$ is not zero; ii) if $F_1(0)$ turns out to be zero, the results of B.W. in the pole approximation are not meaningful. This is because, from Eq. (9), $F_1(\xi)$ can be expressed in the power series expansion of $\xi/(m_K + m_\pi)^2$ in the physical K_{l3} processes and the first term starts with ξ^1. Only a first few terms in the above expansion may be important. This fact gives a quite dif-

ferent effect compared to B.W.'s formula, when the K_{e3} decay is considered (see also reference 13)).

§ 4. Correction due to $K\pi$ scattering[16]

In the preceding section we have shown the equivalence between our treatment and that of B.W. in the pole approximation for the $K\pi$ s-wave resonant state. To consider the effect of $K\pi$ scattering in detail seems to be very difficult and may not be meaningful due to the present situation of experimental data of both $K\pi$ and K_{l3} processes. For the $K\pi$ resonant state, however, we could use the experimental data. For a non-resonant amplitude, there is no reliable calculation, but it is expected to be small. Assuming $K\pi$ s-wave resonance,[17] we may tentatively write amplitudes for these states in the following form:

$$\frac{\sqrt{\xi'}}{k'}\sin\delta_{S1/2}e^{i\delta_{S1/2}}=\frac{\sqrt{\xi'}\,\Gamma}{m_r^2-\xi'-ik'\Gamma}, \tag{20}$$

$$\frac{\sqrt{\xi'}}{k'}\sin\delta_{P1/2}e^{i\delta_{P1/2}}\simeq\frac{h(\xi')}{1-\frac{1}{\pi}\int_{(m_K+m_\pi)^2}^{\infty}dt\,\frac{k_t}{\sqrt{t}}\,\frac{h(t)}{t-\xi'-i\epsilon}}, \tag{21}$$

$$h(\xi')=\frac{\lambda^2}{2k'^2}Q_1(s),$$

$$s=1+\frac{2(m_K^2+m_\pi^2)-m_r^2-\xi'}{2k'^2},$$

where m_r, Γ and k' are the mass of K^*, the reduced width of the resonance and the c.m.s. momentum of the $K\pi$ system, respectively. λ^2 is defined as the residue of the resonance amplitude in the pole approximation (the effective coupling constant of $K^*K\pi$) and is related to m_r and Γ as follows:

$$\lambda^2\simeq\frac{1}{3}m_r\Gamma. \tag{22}$$

The factor 1/3 comes from the isospin consideration. $Q_1(s)$ is the Legendre function of the second kind.

Eq. (21) can be obtained from the dispersion relations for the partial wave amplitudes for the $K\pi$ scattering by assuming that the contributions from the unphysical cuts are well approximated by the K^*-resonance and by taking this contribution to be a pole. Although this approximation neglects the low energy $\pi\pi$ scattering effect completely, recent experiments[18] seem to show a resonant feature in rather high energy $\pi\pi$ systems. Therefore as an approximation to the nearest singularity, our treatment would make some sense.

Eqs. (20) and (21) are too complicated to enable us to draw a quantitative

conclusion. As our approximation Eq. (17) depends solely on the value of the reduced width Γ, we shall postpone the detailed study to a future communication.

§ 5. Concluding remarks

In this paper we have studied a possible relationship between two form factors in the K_{l3} decays by means of dispersion theoretic techniques. By choosing the subtracting point at $\tilde{\xi}=0$ in the dispersion relations for $F_1(\tilde{\xi})$ and $f(\tilde{\xi})$, the one parameter $F_1(0)$ was introduced into the theory. Supposing $F_1(\tilde{\xi})$ has no zero point at $\tilde{\xi}=0$, we have found the approximate solution, Eq. (14), for $F_i(\tilde{\xi})$ in the lowest mass approximation. Then we have shown that we can obtain the result of B.W. as well as that of the intermediate boson theory depending on the assumed resonant state of the $K\pi$ scattering. In § 4, a possible correction to Eq. (17) due to $K\pi$ scattering effect has been briefly suggested.

Turning to the zero-point problem of $F_1(\tilde{\xi})$ at $\tilde{\xi}=0$, it is suggested that the experimental information on the π^0-energy spectrum in the K_{e3} decay could provide us an unambiguous answer to this question. Thus, if $F_1(0)$ turns out to be non-zero, we can obtain the following conclusions from our standpoint: i) The exactly conserved current hypothesis is objectionable; ii) even if the results of B.W. are in agreement with experimental data, this fact does not necessarily mean the validity of the partially conserved current hypothesis; iii) in the case of the p-wave $K\pi$ resonance, while the B.W. hypothesis is meaningless, our solution Eq. (14) may be applicable to the analysis of the processes. On the other hand, if $F_1(0)$ is proved to be zero, we can conclude: 1′) Our work has no meaning at all; ii′) the exactly conserved current hypothesis is not inconsistent with the analyticity requirements in the dispersion theory framework; iii′) the practically interesting results of B.W. are not meaningful. (This is clear if we look at the π^0-energy spectrum in K_{e3} decay.)

Now let us discuss the hypothesis of B.W. in detail. These authors have speculated that $\partial_\alpha J_\alpha$ transforms like the K^* field. This is based on an analogy with those of nuclear β-decay,[2] where the divergence of the axial vector current $(\partial_\alpha J_\alpha{}')$ transforms like the pion field. As is well known, this is approximately true, only when that operator is taken into the matrix form between nucleons (the mass degenerate fields). If we go to the hyperon decays, these considerations may become less conclusive. Although the global symmetry could eliminate this difficulty in good approximation, so far there seems no positive evidence that this symmetry does work well in the hyperon β-decays.[19] In the K_{l3} processes, the mass difference between the K-meson and the pion could cause much serious difficulties, $(\delta=(m_K-m_\pi)/m_\pi\cong 2.5)$. Although Tiomno, Behrends and others[20] have proposed the attractive symmetry in which the

pion and the K-meson are treated in the unified way, these authors[21] have argued at the same time another possibility such that the proposed symmetry is valid only in the absence of the K-coupling so as to give the various mass splitting among baryons. These considerations have also been proposed by a number of authors.[22] One of the common features in these schemes is that the pion and the K-meson fields are distinctively treated. In these situations we should ask if it makes sense to introduce the hypothesis of the partially conserved current in the K_{l3} decays.

Finally we must add that we do not necessarily argue that our assumptions (a) and (b′) are suitable and weak enough to analyze the processes. Merely we should like to point out that there will be no relation between two form factors beyond these assumptions.

Acknowledgments

The author is indebted to Dr. D. Lurié for instructive discussions and comments. In particular, the author would like to thank Professor Y. Nambu for his suggestive discussions and valuable comments.

Note added in proof :

This result has been obtained by S. W. MacDowell in a slightly different way. See S. W. MacDowell, Phys. Rev. **116** (1959), 1047. The author is indebted to Dr. S. Furuichi to call his attention to this work. The above author has discussed the subtraction problem by examining the high energy behaviors of spectral functions, Im F_1 and Im $F_2-\{(m_K{}^2-m_\pi{}^2)/\xi\}$ Im F_1 in our notation, and has concluded that F_1 need at least one subtraction, while $F_2-\{(m_K{}^2-m_\pi{}^2)/\xi\}F_1$ does not. As we have studied F_1 and $\xi F_2-(m_K{}^2-m_\pi{}^2)F_1$ both with one subtraction, it seems to be no apparent contradiction between his work and ours.

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3) J. Bernstein and S. Weinberg, Phys. Rev. Letters **5** (1960), 481.
In the remainder of the text we shall refer to this work as B. W.
4) M. L. Goldberger and S. B. Treiman, Phys. Rev. **110** (1958), 1178; **111** (1958), 354.
M. L. Goldberger, Rev. Mod. Phys. **31** (1959), 797.
5) Y. Nambu's article in reference 2).
B. d'Espagnat, J. M. Jauch and Y. Yamaguchi, L. Lederman and C. Peyrou, CERN Report, "Strange Particle Physics" (1959). See especially Lectures given by Y. Yamaguchi.
S. Okubo, Nuovo Cimento **13** (1959), 292.
6) We follow the notation of Y. Yamaguchi, reference 5). The earlier work on the K_{l3} will be found in this lecture note.
7) Instead of (a) and (b′), we may make the following assumption : $F_1(\xi)$ and $F_2(\xi)$ can be expressed by the dispersion relations of one subtracted and non-subtracted form,

respectively. This may be slightly weaker than (a) and (b′). A possible connection between the partially conserved current hypothesis and the subtraction problem was fully discussed in the nucleon β-decay by Bernstein et al.; see last article in reference 2).

8) M. Sugawara and A. Kanazawa, preprint. The author should like to thank Professor Y. Nambu for informing him of this pre-publication work.

9) For the possibility that $F_1(\xi)$ may have a zero point at $\xi=0$; see later discussions.

10) M. Alston et al., Phys. Rev. Letters **6** (1961), 300.

11) Here we think of the K-decay at rest.

12) We mean the result in the second order Born approximation. S. Oneda and J. C. Pati, Phys. Rev. Letters **2** (1959), 125. See also Y. Yamaguchi, reference 5).

13) All the important observable quantities are summarized in the lecture note of reference 5). The π^0-energy spectrum in the K_{e3} decay may be given, in good approximation, as follows:

$$\frac{d\sigma}{d\omega_\pi} \propto F_1^2(\xi)\, m_K P_\pi^3 - \frac{3}{8m_K}\, m_e^2 P_\pi \xi\, [F_2(\xi)-F_1(\xi)]^2,$$

$$\xi = 2m_K \omega_\pi - m_K^2 - m_\pi^2.$$

14) See reference 6).

15) S. Okubo, R. E. Marshak, E. G. C. Sudarshan, W. B. Teutsch and S. Weinberg, Phys. Rev. Letters **1** (1958), 25; see also M. L. Goldberger and S. B. Treiman, Phys. Rev. **110** (1958), 1478 (L).

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These last two works argue in favor of p-wave resonance, but we need further experimental information.

18) J. A. Anderson et. al., Phys. Rev. Letters **6** (1961), 365.

19) Y. Yamaguchi's lecture in reference 5).
William E. Humphrey et al., Phys. Rev. Letters **6** (1961), 478.

20) J. Tiomno, Nuovo Cimento **6** (1957), 69.
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The other references will be found in these works.

21) See, for example, R. E. Behrends and A. Sirlin, Phys. Rev. **121** (1961), 324; but Tiomno and Peaslee argue different viewpoints on the mass-splitting.

22) See, for example, A. Pais, Phys. Rev. **122** (1961), 317 and the series of his works found in this article.

Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30 cm.), so that each letter may be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

Some Notes on the Effects of Vector Meson on the Electromagnetic Structure of Nucleon

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July 12, 1961

We achieved a detailed analysis[1] of the properties of ρ-meson,[2] using the data of second and third π-nucleon resonances and of 5 Bev π-nucleon inelastic scattering under the basic assumption: ρ-meson is a composite particle of a nucleon and an anti-nucleon. By this analysis, we got rather large ρ-meson mass $5\sim6\mu$ and small ρ-π coupling constant $F^2/4\pi = 1\sim2$ as compared with the results of Frazer and Fulco's theory.[3] It may be an interesting problem that whether these results are consistent with the theory of the electromagnetic structure of nucleon or not. In this short note, we shall give an outline of our theoretical investigation of this problem.

According to Frazer and Fulco, the imaginary part of electromagnetic form factor (vector part) is expressed as follows:

$$\mathrm{Im}\,F_i(t) = -(eF_\pi^*(t)\,\nu^{3/2}/\sqrt{t})\,\Gamma_i(t).^*$$

$$\nu = \frac{t}{4} - \mu^2. \quad (\mu: \pi\text{-meson mass}) \qquad (1)$$

By the direct calculation of diagrams Fig. 1, Fig. 2 and Fig. 3, we get

$$F_\pi(t) = \frac{m^2 - \Sigma(0)}{m_\pi^{\,2}} \frac{m_\pi^{\,2} - t}{m^2 - t - \Sigma(t)},$$

$$\Sigma(t) =$$

$$-\frac{F^2}{4\pi}\frac{2\nu}{3\pi}\sqrt{\frac{\nu}{\nu+1}}\log\left(\sqrt{|\nu|} + \sqrt{|\nu+1|}\right) + i\frac{F^2}{12\pi}\sqrt{\frac{\nu^3}{\nu+1}}\,\theta(\nu)$$

$$(\nu \geq 0 \text{ or } \nu \leq -1),$$

$$\frac{-F^2}{4\pi}\frac{\nu}{3}\sqrt{\frac{-\nu}{1+\nu}} \times \left(1 - \frac{2}{\pi}\cot^{-1}\sqrt{\frac{1+\nu}{-\nu}}\right)$$

$$(-1 < \nu < 0), \qquad (2)$$

$$(\mu = \text{unit})$$

where m and F represent renormalized ρ-meson mass and ρ-π coupling constant respectively. m_π is a constant related to the bare ρ-meson mass as mentioned by Gell-Mann.[4] By the calculation of Fig. 4, Fig. 5 and Fig.

* As to the meaning of the notation, see reference 3), though our notation is slightly different from theirs.

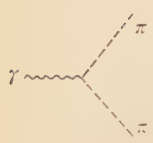

Fig. 1.

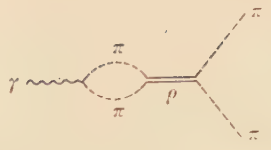

Fig. 2.

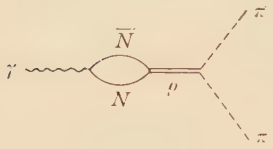

Fig. 3.

6, we get the following dispersion relation:

$$\Gamma_i(t)=\frac{F_{\rho_i}(t)}{\pi}\int_{-\infty}^{a}dt'\frac{\mathrm{Im}\,\Gamma_i(t')}{(t'-t)\,F_{\rho_i}(t')},$$

$$a=4\mu^2(1-\mu^2/4M^2),$$

$$F_{\rho_i}(t)=\frac{m_i^2-t}{m^2-t-\Sigma(t)}, \qquad (3)$$

Fig. 4.

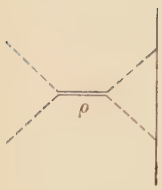

Fig. 5.

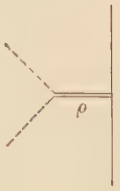

Fig. 6.

where m_1 and m_2 are the constants which are related closely to the coupling constants of the direct ρ-nucleon coupling (Fig. 6). We can prove the validity of this expression in any higher order diagrams. Using (1), (2) and (3), we get

$$\mathrm{Im}\,F_i(t)=-(e\nu^{3/2}/\sqrt{t}\,)Z\times\frac{(m_\pi^2-t)(m_i^2-t)}{|m^2-t-\Sigma(t)|^2}\frac{1}{\pi}\times\int_{-\infty}^{a}dt'\frac{\mathrm{Im}\,\Gamma_i(t')}{(t'-t)\,F_{\rho_i}(t')},$$

$$Z=\frac{m^2-\Sigma(0)}{m_\pi^2}. \qquad (4)$$

Since the special case $m_\pi=m_1=m_2=\infty$ corresponds to Frazer and Fulco's theory, our theory contains three additional parameters, viz. three right-hand zero points. If we use (0,1) state[1] (neutral vector meson) instead of ρ-meson, we get the same expression as (4) for the iso-scaler part by the same calculation.

In the case $m,\ m_i\gg 4\mu^2$ and $m_i^2/m^2\sim 1$, we may put $F_{\rho_i}=m_i^2/m^2-\Sigma(0)$ in the integral of this expression. In this approximation, (4) becomes

$$\mathrm{Im}\,F_i(t)=\frac{(m^2-\Sigma(0))^2}{|m^2-t-\Sigma(t)|^2}\left(1-\frac{t}{m_\pi^2}\right)\times\left(1-\frac{t}{m_i^2}\right)\mathrm{Im}\,F_i^0(t), \qquad (5)$$

where $\mathrm{Im}\,F_i^0(t)$ is the one calculated by C.K.G.Z.[5] and F.G.T.[6] The analysis of the recent data using this expression is in progress.

This method is also applicable to various problems related to π-π resonance such as two-pion exchange nucleon-nucleon potential and $KK\pi\pi$ interaction.

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Evidence for Quadratic Spin-Orbit Interaction*

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July 24, 1961

This note reports some results of an investigation into the static potential shapes which will account for the widely accepted proton-proton singlet even-parity data. The results support the hypothesis of a quadratic spin-orbit term, such as the one proposed by Hamada.[1]

A systematic search, using a high-speed electronic computor, has been performed in an attempt to fit a central potential to the pp S and D phase shifts[2] at 95 Mev and 310 Mev and the low-energy np parameters[3] a_0 and r_0. The potentials used were of a continuous, connected line-segment form, their values at arbitrary connecting points being adjustable.[4] An adjustable hard core was usually used. In addition, the potential was required to connect with an OPEP tail at a radius of $2\hbar/\mu c$ (μ=pion mass). The non-relativistic Schrödinger equation was solved numerically in an automatic program which varied the potential at each point of adjustment to minimize a chi-squared function of the deviations from the experimental parameters (a_0, r_0 and phase shifts). Many searches were made, using different *initial* potential shapes and core radii. It was found that the parameters were matched quite well by the final potentials,[5] with the sole exception of the D phase at 310 Mev. The latter was in every case 16^0 to 17^0, although the experimental value was supplied as 12.5^0. It was thought that perhaps some variation should be allowed in the OPEP tail, so the connecting point was moved out to $2.4\,\hbar/\mu c$. The search then gave a better overall fit, including to the D phase at 310 Mev. However, in order to produce this better fit, the potential became $+0.5$ Mev at $2\,\hbar/\mu c$. Since the OPEP is -0.75 Mev at $2\,\hbar/\mu c$, the change was not acceptable if one wished the potential to be near OPEP in this region. Since the D phase was too large, a potential was needed which was more repulsive in the D state at high energy. A short-range repulsive term proportional to $L(L+1)$ was added[6] and it produced the desired effect. We therefore conclude that if one requires a static potential with an OPEP tail, a quadratic spin-orbit term is necessary.

* Supported in part by the U. S. Atomic Energy Commission.

The computer work was carried out in the A. E. C. Computation Center at New York University and in the Com-

putation Center of the Pennsylvania State University.

1) T. Hamada, Prog. Theor. Phys. **24**, (1960), 1033.
2) The phases used were ${}^1\delta_0=22\pm1.5^0$, ${}^1\delta_2=4\pm0.4^0$ at 95 Mev and ${}^1\delta_0=-11\pm5^0$, ${}^1\delta_2=12.5\pm2^0$ at 310 Mev.
3) Used $a_0=-13.4\pm2.8\ \hbar/\mu c$ and $r_0=1.87\pm0.05\ \hbar/\mu c$.
4) This eliminated restricting the potential to a particular functional form.
5) The shape parameter predicted by these potentials was between +0.003 and +0.014.
6) This is the quadratic spin-orbit term; from invariance arguments no other static forms are available.

Motion of an Impurity Atom in One-Dimensional Lattice

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August 7, 1961

Recently, Rubin investigated the statistical dynamical properties of a heavy particle (of mass m') contained in a cubic lattice (of particles of mass m) and derived a Gaussian Markoff process for the motion of a heavy particle in one- and three-dimensional lattice.[1] His discussion, however, is based on an approximate solution of autocorrelation function for the case $m'\gg m$. Turner also obtained the autocorrelation function of the velocity of a heavy particle in a one-dimensional lattice.[2] This note represents the exact solution of an infinitely extended linear lattice containing an isotopic impurity particle with arbitrary mass.

Because of the linearity of equations of motion, the solution for the impurity particle (at site 0) can be expressed in the form

$$u_0(t)=\sum_{m=-\infty}^{\infty}U_m(t)\,u_m(0)+\sum_{m=-\infty}^{\infty}V_m(t)\,v_m(0), \qquad (1)$$

where $u_m(0)$ and $v_m(0)$ are the initial values of the displacement of the m-th particle from its equilibrium position and its velocity, respectively. It can be shown that the fundamental system of solutions of our linear differential equations of motion $\{U_m(t), V_m(t)\}$ is given by, when $m/m'-1=\varepsilon<0$, $(\varepsilon>-1)$

$$U_m(t)=U_{-m}(t)=[1+\varepsilon(1-\delta_{m,0})]\times\{J_{2m}(2\alpha t)+2\varepsilon\sum_{p=1}^{\infty}(-1)^p\times(1+2\varepsilon)^{p-1}J_{2(m+p)}(2\alpha t)\}, \qquad (2)$$

$$V_m(t)=V_{-m}(t)=\frac{1}{\alpha}[1+\varepsilon(1-\delta_{m,0})]\times\{\sum_{p=0}^{\infty}J_{2(m+p)+1}(2\alpha t)-\frac{\varepsilon}{1+\varepsilon}\times\sum_{p=1}^{\infty}[1-(-1)^p(1+2\varepsilon)^p]\times J_{2(m+p)+1}(2\alpha t)\}, \qquad (3)$$

and when $\varepsilon>0$

$$U_m(t)=U_{-m}(t)=[1+\varepsilon(1-\delta_{m,0})]\times\{(-1)^m 2\varepsilon(1+2\varepsilon)^{-m-1}\times\cos\{2\alpha(1+\varepsilon)(1+2\varepsilon)^{-1/2}t\}+J_{2m}(2\alpha t)+2\varepsilon\sum_{p=0}^{\infty}(-1)^{-p-1}\times(1+2\varepsilon)^{-p-1}J_{2(m-p)}(2\alpha t)\}, \qquad (2')$$

$$V_m(t)=V_{-m}(t)$$
$$=\frac{1}{\alpha}\{1+[(1+2\varepsilon)^{-1}-1]\delta_{m,0}\}$$
$$\times\{(-1)^m\varepsilon(1+2\varepsilon)^{-m-1/2}$$
$$\times\sin\{2\alpha(1+\varepsilon)(1+2\varepsilon)^{-1/2}t\}$$
$$+\sum_{p=0}^{\infty}J_{2(m+p)+1}(2\alpha t)$$
$$+\varepsilon\sum_{p=0}^{\infty}(-1)^{-p-1}(1+2\varepsilon)^{-p-1}$$
$$\times J_{2(p-m)+1}(2\alpha t)\}, \qquad (3')$$

where $\alpha=(K/m)^{1/2}$, K is the harmonic force constant and J_n is the Bessel function of order n. It is also readily seen that there is a relation between two functions $U_m(t)$ and $V_m(t)$, namely

$$U_m(t)=dV_m(t)/dt. \qquad (4)$$

It must be noted that all terms that appeared in Eqs. (2) and (2′) tend to zero in the limit $t\to\infty$, except the trigonometric terms which appear in the case $\varepsilon>0$. Apparently, these non-vanishing oscillating terms express the fact that the localized vibrational motion can exist when $m'<m$.[3]

The velocity autocorrelation function of the impurity particle and the conditional probability distribution function are given by

$$\langle v_0(t)\,v_0(0)\rangle/\langle v_0(0)^2\rangle=U_0(t), \qquad (5)$$

$$P(v,t|v^0,0)$$
$$=\mathfrak{N}\{v|v^0U_0(t),\,kTm'^{-1}[1-U_0(t)^2]\}, \qquad (6)$$

where the average means the ensemble average of an initially canonical distribution and $\mathfrak{N}\{v|M,\sigma\}$ represents a normal distribution with the mean M and the variance σ. The autocorrelation functions given by Eqs. (2) and (2′) show the following facts:

(i) When the impurity particle is lighter than other particles the autocorrelation function $U_0(t)$ given by Eq. (2′) does not vanish in the limit $t\to\infty$ and it can not be expected that the distribution (6) approaches to the equilibrium Maxwellian distribution independently of v^0. The equipartition of energy also does not establish.

(ii) When the impurity particle is heavier than other particles, the autocorrelation function $U_0(t)$ given by Eq. (2) gives just the same result with that obtained by Turner. In this case, the approach to the equilibrium distribution and the equipartition of energy are ensured in the limit $t\to\infty$.

Details of this problem will be discussed in a full paper in near future.

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Note on K^--p Interactions at 400 Mev/c

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August 2, 1961

According to the experimental results for K^--p collisions,[1] the

angular distributions for the three processes

$$K^-+p\rightarrow\pi^-+\Sigma^+,$$

$$K^-+p\rightarrow\pi^+ +\Sigma^-,$$

$$K^-+p\rightarrow K^-+p,$$

at 400 Mev/c seem to contain certain large $\cos^2\theta$ terms, although the processes up to about 250 Mev/c may be approximately described in terms of the strong s-wave interactions in K^--p system. In order to explain the remarkable property, Capps[2] assumed that the large $\cos^2\theta$ terms (which he referred as the 400 Mev/c anomaly) result from the strong interactions in the total angular momentum $J=3/2$ state of the K^--p system and discussed the parity of the $J=3/2$ state. His assumption may probably be natural since the main feature of the angular distributions for the three processes at 400 Mev/c may be approximately expressed by the form $(1+3\cos^2\theta)$. When we pay attention to the magnitude of the cross sections for K^--p collisions, however, it may be impossible to find out any anomaly in the neighborhood of 400 Mev/c from the experimental results[1] for the energy dependence of cross sections not only of absorption but also of elastic scattering.* Moreover the experimental results are inconsistent with an assumption such that only an isotopic spin state plays the most important role in K^--p interaction at 400 Mev/c or 300 Mev/c because of the following facts. i) The contributions from $I=1$ state to the absorption processes are comparable to those from $I=0$ state (see Table VI in reference 1)), where I indicates the isotopic spin of $\bar{K}$-N system. ii) $\sigma(K^-+p\rightarrow\bar{K}^0+n)\cong(1/10)\,\sigma(K^-+p\rightarrow K^-+p)$. This means the following circumstances. The contributions from $I=1$ state to the elastic K^--p scattering are comparable to those from $I=0$ state and the interference between the two isotopic spin states has a large effect on the scattering. Therefore it may be unreasonable to assume that the large $\cos^2\theta$ terms in the angular distributions result from the strong interaction only in a state belonging to $J=3/2$, $I=1$ (or $J=3/2$, $I=0$), although it is necessary to have much experimental data in order to draw the conclusion with confidence. In this note we should like to point out that the interference between $s_{1/2}$ and $d_{3/2}$ states (or the interference between $p_{1/2}$ and $p_{3/2}$ states) is mainly responsible for the angular distributions of K^--p scatterings at 400 Mev/c.

For simplicity let us examine the angular distributions for K^--p scattering omitting the suffix for isotopic spin state. We denote the partial wave amplitudes for $K^-+p\rightarrow K^-+p$ and $K^-+p\rightarrow\pi^{\mp}+\Sigma^{\pm}$ reactions as $R_J{}^l=|R_J{}^l|\exp(2i\delta_J{}^l)$ and $T_J{}^l=|T_J{}^l|\times\exp(2i\alpha_J{}^l)$ respectively, where J and l represent respectively the total angular momentum and the orbital angular momentum in the initial K^--p

* Because the experimental data are scanty, we think it necessary to examine in detail whether or not there is an anomaly with respect to the magnitude of cross sections.

state. When the states up to $J=3/2$ are taken into consideration, the experimental results for the angular distributions $1+(4\sim12)\cos^2\theta$ may not be explained unless the interference effects are taken into account, because the $s_{1/2}$ state which gives isotropic angular distributions may still play an important role in K^--p collision at 400 Mev/c. Among the interference effects between the $s_{1/2}$ and other states, it is only the interference between the $s_{1/2}$ and $d_{3/2}$ states that gives the large $\cos^2\theta$ term. If $\cos 2(\delta^0_{1/2}-\delta^2_{3/2})$ is nearly equal to 1, the experimental results may be explained even if the contributions not only from p-wave but also from d-wave are smaller than those from s-wave. In the case of $|R^2_{3/2}|\cong(1/3)|R^0_{1/2}|$, for example, the angular distribution for K^--p scattering may have the form $(1+5\cos^2\theta)$ if $\cos 2(\delta^0_{1/2}-\delta^2_{3/2})\cong 1$.

In the reactions $K^-+p\to\pi^{\mp}+\Sigma^{\pm}$, we can also develope our discussion along the same line as in K^--p scattering. If the interference between $T^0_{1/2}$ and $T^2_{3/2}$ is mainly responsible for the angular distribution with the large $\cos^2\theta$ term, then $|\sin 2(\alpha^0_{1/2}-\alpha^2_{3/2})|$ turns out to be very small. Capps[2] suggested a Σ^+ polarization measurement in $K^-+p\to\pi^-+\Sigma^+$ reaction in order to examine the parity of the $J=3/2$ state and showed that the interference between $T^0_{1/2}$ and the $J=3/2$ amplitudes in the polarization intensity is proportional to

$$2[\pm|T^0_{1/2}|\cdot|T^1_{3/2}|\sin 2(\alpha^0_{1/2}-\alpha^1_{3/2}) -3|T^0_{1/2}|\cdot|T^2_{3/2}|\sin 2(\alpha^0_{1/2}-\alpha^2_{3/2})\cos\theta]\times\sin\theta. \quad (1)$$

If $|\sin 2(\alpha^0_{1/2}-\alpha^2_{3/2})|$ is very small as was mentioned above, it may be difficult to expect a large $\cos\theta\sin\theta$ dependence of the polarization intensity at 400 Mev/c.

Finally we want to make some discussion about p and d wave interactions in K^--p collision.* It may be expected that d-wave K^--p scattering takes place through the mechanism via some kind of $KK\pi\pi$-interaction. Since the K^- meson momentum 400 Mev/c corresponds to the threshold energy of $K^-+p\to Y_1^*+\pi$ reaction, the 400 Mev/c anomaly may have something to do with the Y_1^* production. According to the experimental data,[3] the Y_1^* production angular distributions are isotropic in the K^--p collision up to about 850 Mev/c. If the spin of Y_1^* is equal to 3/2 and its parity is even as is expected from the assumption of Global symmetry, and if the parity of K^- is odd, then d-wave K^--p interaction may play the most important role in $K^-+p\to Y_1^*+\pi$ at low energy.

Since the K^- meson momentum in K^--p scattering at 400 Mev/c is $p^*\cong 1.25\times10^{13}$ cm^{-1}, the impact parameter r ($\cong 0.8\times10^{-13}$ cm) which is responsible for p-wave interaction is nearly equal to the impact parameter which is responsible for the strong (3/2, 3/2)-interaction in π-N scatter-

* We cannot neglect the contributions from p or d wave because the experimental value for the absorption cross section due to $I=0$ (or $I=1$) state seems to be larger than the limiting value of s-wave absorption ($\pi\lambda^2/2=10$ mb).

ing or nearly equal to the mean radius of the charge distribution for proton. Therefore we may expect the strong p-wave interaction in K^--p collision at 400 Mev/c. If it is the $p_{1/2}$ state instead of the $s_{1/2}$ state that plays the most important role in K^--p collision, the large $\cos^2\theta$ terms in the angular distributions may result mainly from the interference between $p_{1/2}$ and $p_{3/2}$ states in K^--p system.

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On the Generalized Nilsson Diagram

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Since the proposition of Davydov and Filippov,[1] some modified asymmetric rotor models have appeared,[2] and succeeded in explaining the properties of the low-lying levels of the even-even nuclei in the extended region of mass number. On the other hand, in the rare earth region, the actinide region and mass 25 region, rotational spectra of axial symmetric rotor is observed.[3] Nilsson[4] calculated the single particle level of the spheroidal potential, and coupling with the unified theory, succeeded in explaining the low-lying level structures of odd nuclei in these regions. At the same time, this simple model was used to determine the equilibrium shape of the nucleus in these region with plausible result.[5]

Using the Davydov-Filippov theory, several authors[6] found the two groups of nuclei, with asymmetry $\gamma\sim10°$ and $\gamma\sim26°$, respectively. The transition between them occurs at $N=88-90$, $N=100-116$, and $N=136-138$. This gross structure is of interest. To see whether this is explainable by the Nilsson model or not, his potential is generalized to asymmetric ellipsoidal potential, changing the expressions (A2a) and (A2b) in his paper, by the following expressions

$$\omega_\kappa=\omega_0(\varepsilon)\left\{1-\frac{2}{3}\varepsilon\cos\left(\gamma-\frac{2}{3}\pi\kappa\right)\right\},$$

$$\kappa=1,\ 2,\ 3.$$

Eigenvalues and eigenfunctions were computed for $N=1\sim7$, $\eta=6\sim-6$, $\gamma=10°$, $20°$ and $30°$, and μ values used are those which Mottelson and Nilsson have used in their analysis of odd nuclei.

Figs. 1 and 2 show the diagrams for $N>126$ and $Z>82$, respectively, the solid line giving the diagram for $\gamma=0$ (Nilsson) and broken line for $\gamma=30°$. Diagrams for intermediate γ values, omitted in order to avoid confusion, lie between these two extreme

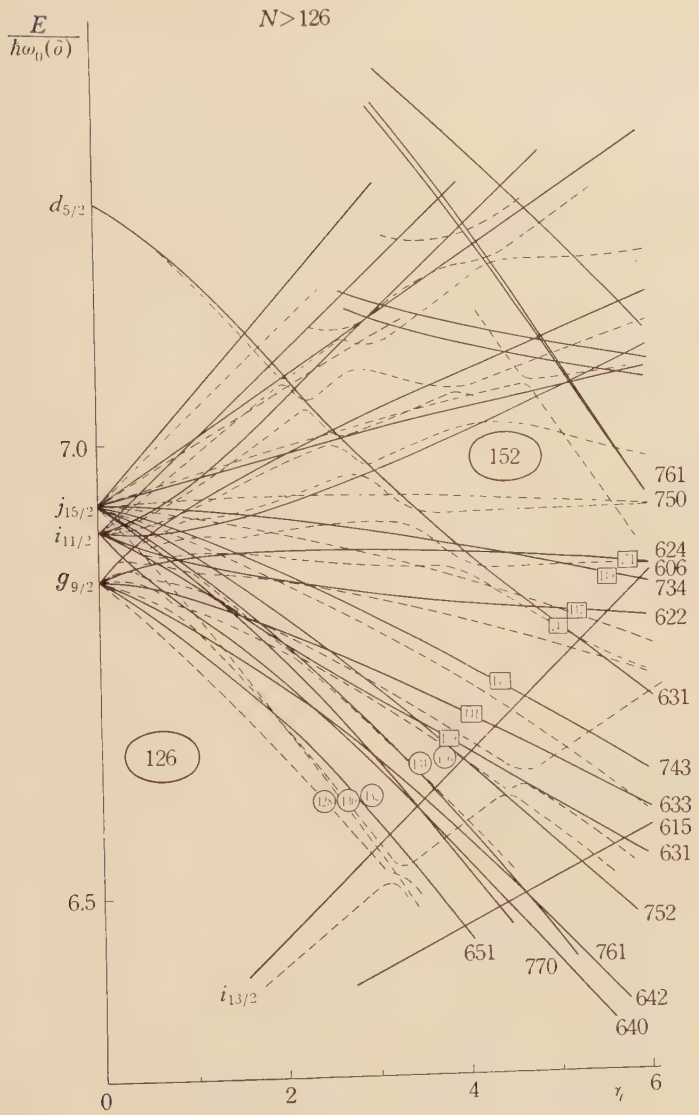

Fig. 1.

cases except for the points where the Nilsson diagrams cross.

Stephens et al.[7] analyzed rotational spectra of odd nuclei in this region and assigned Nilsson orbitals for the last odd nucleons. The number of these nucleons are given in the squares of the corresponding orbitals. In this region, according to the Davydov-Filippov theory, $Po^{212,214}$ and $Em^{218-222}$ are those which have equilibrium shape with $\gamma\sim26°$. The last nucleons of these nuclei may occupy the configurations specified with a circle in these

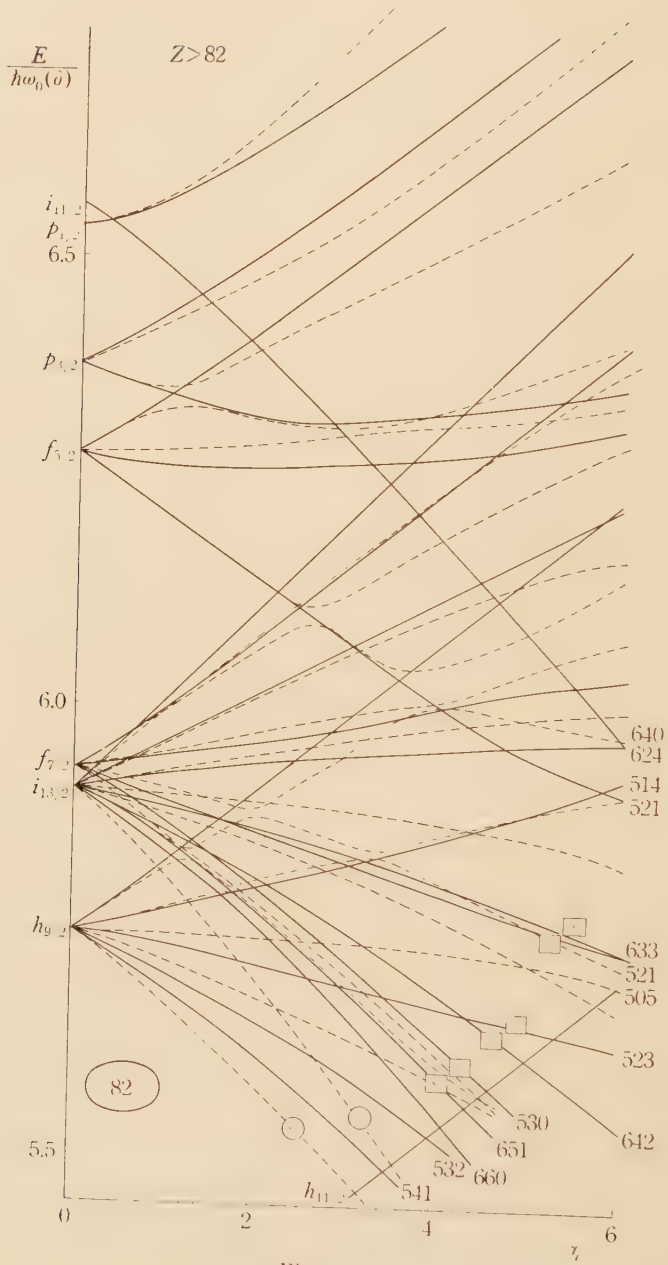

Fig. 2.

figures. It is interesting that most of these orbitals have, both in the proton diagram and the neutron diagram, broken lines far lower than corresponding solid lines, showing that, equilibrium shape of these nuclei may be $\gamma=30°$. It should be emphasized that high angular momentum states of

upper N-shell ($j_{15/2}$ and $i_{13/2}$ here) play an essential role in lowering large γ configurations. An analogous profile can be shown for the region $126 > N > 82$ and $82 > Z > 60$, and less decisively in the low mass number region.[8)]

Although the single particle model using the effective potential is too rough to predict anything decisively, it seems to the author plausible to say that these nuclei have asymmetric deformation.

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Approximate Constancy of Adiabatic Invariants

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In this paper we give general criteria to be satisfied by a slowly time-dependent Hamiltonian in order to possess adiabatic invariants of m-th order. We evaluate also the degree of approximate constancy of such adiabatic invariants. We end up the paper applying our methods to the motion of a charged particle in a magnetic field, i.e. to the harmonic oscillator.

§ 1. Introduction

In plasma physics the so-called guiding center approximation treats the motion of charged particles in a varying electromagnetic field as a gyration about the guiding center which in turn moves through space. The separation of these two motions can be valid only if the change of the field in space and time is slow.

To study those motions the concept of adiabatic invariant is important. Precise formulation of the notion of adiabatic invariance has been given by Chandrasekhar[1] and Lenard.[2] Kulsrud[3] believes that there are many quantities that are adiabatic invariants of higher order, even though we only know that they are invariants to a lower order.

If the guiding center approximation is valid, the adiabatic invariant of the gyration is assumed constant. It is interesting to know the error made when this is done. Therefore we should try to study the approximate constancy of adiabatic invariants. Adiabatic invariants are strict constants of the motion if the fields are constant.

Lenard proved the adiabatic invariance of the action integral of a one-dimensional non-linear oscillator to all orders. His paper is a generalization of the work of Kulsrud who studied the same concept for the harmonic oscillator. In Lenard's paper the time-dependent Hamiltonian corresponds, for fixed time, to a periodic motion.

In this paper we will generalize the results obtained by other authors in several points. First of all, we will not require that the time-dependent hamiltonian represent instantaneously a periodic motion; the unperturbed hamiltonian, though, must correspond to a periodic motion. Secondly, we give general criteria to be satisfied by the slowly time-dependent hamiltonian in order to possess adiabatic invariants of m-th order. Thirdly, our method allows us to

evaluate the degree of approximate constancy of such adiabatic invariants.

Adiabatic invariants are quantities that remain constants of the motion during an infinitely slow variation of the external parameters of the system. Their usefulness lies in the fact that they remain constant to a very good approximation even when the external parameters vary at a finite rate. In real plasma physics the external parameters always vary at a finite, though slow, rate. That is why our calculations can be applied to actual plasma problems. The method used in this paper is based on the introduction of the interaction picture[4] for classical mechanics.

We end up our paper applying the general techniques presented in the same to the motion of a charged particle in a magnetic field that varies in time slowly but at a finite rate. We evaluate the errors that are made in plasma physics when the adiabatic invariants are taken to be exact constants of the motion.

§ 2. Adiabatic invariants

We suppose that the Hamiltonian $H(t)$ of the system depends explicitly on time. As a matter of fact such a Hamiltonian, $H(t)$, varies continuously with time from an initial value, H_0, at the instant t_0 to a certain final value, H_1, at the instant t_1. We write

$$T=t_1-t_0, \qquad \tau=\frac{t-t_0}{T} \tag{1}$$

and design by $H(\tau)$ the value of the Hamiltonian $H(t)$ at the time instant $t=t_0+\tau T$. $H(\tau)$ is a continuous function of τ that we suppose given. The rate of evolution of the system from time t_0 till time t_1 depends only on the parameter T. We plan to study the case in which T is very large.

If we define the parameter $\varepsilon=1/T$, the limit $\varepsilon\to 0$ implies the indefinite decrease in the rates of change of external parameter. While the physical time t goes from t_0 to t_1, the fictitious time parameter τ changes from $\tau=0$ to $\tau=1$.

Therefore

$$H(0)=H_0, \qquad H(1)=H_1. \tag{2}$$

We have to study the evolution of the system under the action of the hamiltonian $H(\tau)$ when τ goes from $\tau=0$ to $\tau=1$. We plan to expand the corresponding evolution operator in powers of T^{-1}.

A quantity $\mathcal{L}^c$ is an adiabatic invariant to the m-th order, if a positive constant M can be found such that during the time interval $T(T\to\infty)$ the variation of $\mathcal{L}^c$ satisfies

$$|\Delta\mathcal{L}^c| < \frac{1}{T^m} M. \tag{3}$$

Even though we have studied elsewhere[4] the interaction picture for classical mechanics, we will do it again now for the case in which the Hamiltonian $H(t)$ depends explicitly on time in order to write the corresponding equations in terms of the fictitious time τ and of the fictitious Hamiltonian $H(\tau)$.

§ 3. Interaction picture

The dynamical time evolution of a function $A=A(q, p)$ of the sets of canonical conjugate variables, i.e. the total time evolution if that function A does not depend explicitly on time, is given by

$$\frac{dA}{dt}=\sum_i \frac{\partial H}{\partial p_i}\frac{\partial A}{\partial q_i}-\frac{\partial H}{\partial q_i}\frac{\partial A}{\partial p_i} \tag{4}$$

where $H=H(q, p, t)$ is the Hamiltonian of the system referred to the physical time t. The same equation written in terms of the new time variable τ and of the Hamiltonian $H(\tau)$ is

$$\frac{dA}{d\tau}=T[\Omega, A], \tag{5}$$

where the square parenthesis stands for the commutator between the quantities within. The operator Ω is Liouiville's differential operator given by

$$\Omega=\sum_i \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}-\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}. \tag{6}$$

The time evolution may be integrated introducing a differential operator $S(\tau)$ written as a function of Liouiville's operator Ω. Such operator $S(\tau)$ is defined by means of

$$A(q(\tau), p(\tau))=S(\tau)\, A(q_0, p_0)\, S^{-1}(\tau) \tag{7}$$

where q_{i0} and p_{i0} are the values of the sets of canonical variables at the time origin, i.e.

$$q_{i0}=q_i(0), \qquad p_{i0}=p_i(0). \tag{8}$$

The boundary condition satisfied by $S(\tau)$ is evidently

$$S(0)=1. \tag{9}$$

If we enter with (7) into (5) we obtain the differential equation satisfied by the evolution operator

$$\frac{dS(\tau)}{d\tau}=TS(\tau)\Omega(q_0, p_0, \tau). \tag{10}$$

The differential equation and the boudary condition can be both included in the following integral equation

$$S(\tau)=1+T\int_0^{\tau} d\tau_1 S(\tau_1)\Omega(q_0 p_0 \tau_1). \tag{11}$$

We are interested in the operator $S(1)$.

We would like to discuss now the time dependence of the operator Ω. The Hamiltonian never has dynamical time dependence since its total time derivative is equal to its partial time derivative. Therefore $H=H(q, p, \tau)$ depends on time τ only because of the explicit time dependence of the same. Besides Liouiville's operator may be written formally as

$$\Omega=H\, I \tag{12}$$

where I is the double operator defined as

$$I=\sum_i \frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q_i}-\frac{\overleftarrow{\partial}}{\partial q_i}\frac{\overrightarrow{\partial}}{\partial p_i}. \tag{13}$$

But I is invariant with respect to a canonical transformation. As a matter of fact, this is the actual content of the well-known invariance of the Poisson's brackets with respect to the above-mentioned transformations. Therefore I is invariant with respect to the time evolution of the system which is a canonical transformation. Consequently, Ω depends on τ only through the explicit time dependence of $H(\tau)$. Let us now pass to the interaction picture. We define $H_0=H(0)$ as the unperturbed Hamiltonian, and $S_0(\tau)$ the evolution operator generated by H_0. We define the interaction picture evolution operator $S_1(\tau)$ by

$$A(q(\tau), p(\tau))=S_1(\tau)\, S_0(\tau)\, A(q_0, p_0)\, S_0^{-1}(\tau)\, S_1^{-1}(\tau). \tag{14}$$

The perturbation Hamiltonian is defined by

$$H_1(\tau)=H(\tau)-H_0, \tag{15}$$

Hamiltonian that depends on time explicitly. From it we can construct another Liouiville's differential operator that we will call $\Omega_1(q_0 p_0 \tau)$.

Entering into (5) with expression (14), we find out the integral equation satisfied by $S_1(\tau)$.

$$S_1(\tau)=1+T\int_0^{\tau} d\tau' S_1(\tau')\,\Omega_1[\tau'] \tag{16}$$

where

$$\Omega_1[\tau]=S_0(\tau)\,\Omega_1(q_0 p_0 \tau)\, S_0^{-1}(\tau) \tag{17}$$

is the perturbing Liouiville's operator in the interaction picture. Such operator depends on time τ on two accounts: Firstly because $H_1(\tau)$ has an explicit time dependence, secondly because the unperturbed evolution operators $S_0(\tau)$ acting on the canonical set of conjugate variables that enter in the definition

of $\Omega_1(\tau)$ will induce the time dependence of the unperturbed motion in the same. Now, since the operator I of (13) is invariant under canonical transformations and remembering the symbolic expression (12), this second time dependence induced in $\Omega_1(\tau)$ corresponds to the time dependence induced in $H_1(\tau)=H_1(q, p, \tau)$ by $S_0(\tau)$; so, symbolically

$$\Omega_1[\tau]=H_1(q_0(\tau), p_0(\tau), \tau)\cdot I \tag{18}$$

where

$$q_{i0}(\tau)=S_0(\tau)\, q_{i0} S_0^{-1}(\tau)\ ;\ p_{i0}(\tau)=S_0(\tau)\, p_{i0} S_0^{-1}(\tau). \tag{19}$$

The slow time dependence of the perturbing Hamiltonian implies that $\Omega_1(q_0, p_0, \tau)$ does not contain the large parameter T, while $S_0(\tau)$ of (17) should more exactly be written, as

$$S_0(\tau)\to S_0(T\tau)=S_0(t) \tag{20}$$

when $t_0=0$ as we will suppose from now on.

§ 4. Periodic motion

We suppose that the unperturbed motion generated by H_0 is periodic. There are two kinds of periodic motions. Consider a system with a single degree of freedom. For such a system phase space is a two-dimensional plane. The first type, designated by the name "libration", occurs whenever both q and p are periodic functions of the time with the same frequency. Its orbit in phase space is closed. For the second type the coordinate q itself is not periodic, but is such that when q is increased by some value, the configuration of the system remains essentially unchanged. This motion will be referred to, simply, as rotation. The values of the position coordinate, that indeed in this type of periodicity is invariably an angle of rotation, are no longer bounded, but can increase indefinitely. In dealing with systems of more than one degree of freedom, the motion of the system is said to be periodic if the projection of the system point on each (q_i, p_i) plane is simply periodic in the sense defined for motion of only one degree of freedom. We suppose that the Hamilton-Jacobi equation is separable in at least one set of canonical variables. Then the projected motions are independent of each other, and their nature may be readily examined.

It is well known that, for periodic motions, when the value of the angle coordinate changes by unity the corresponding separation coordinate goes through a complete cycle. For the libration case this means return to its original value, while for the rotation type of periodicity the behaviour is more complicated. In general it is therefore possible to express a periodic separable coordinate as a Fourier series. The same may be said of a function of the separation coordinates such as, for instance, the perturbing Hamiltonian. We

will limit ourselves to the cases when this Fourier expansion is possible.

We have to assume now that the necessary conditions are met so that the unperturbed periodic motion introduces a time dependence in $H_1(q_0(\tau), p_0(\tau), \tau)$ such that this function may be represented as a sum of simple harmonic motions involving variable coefficients that account for the explicit time dependence of H_1. So we may write

$$H_1[q_0(\tau), p_0(\tau), \tau] = \sum_j h_j(\tau)\, e^{i\omega_j \tau T}. \tag{21}$$

The coefficients $h_j(\tau)$ would not depend on time τ if the perturbing Hamiltonian did not have explicit time dependence. The constants ω_j represent the fundamental frequencies and all its harmonics of the unperturbed periodic motion.

§ 5. Approximate constancy of adiabatic invariants

Let us suppose that the functions $A(q, p)$ represent constants of the motion of the unperturbed motion. For them it holds

$$S_0(\tau)\, A(q_0 p_0)\, S_0^{-1}(\tau) = A(q_0 p_0) \tag{22}$$

and, therefore, its time evolution will be given by $S_1(\tau) A(q_0 p_0) S_1^{-1}(\tau)$. As usually done, the integral equation for $S_1(\tau)$ is solved by iteration. Thus we get the following expansion for $S_1(1)$.

$$S_1(1) = 1 + T\int_0^1 d\tau'\, \Omega_1[\tau'] + T^2 \int_0^1 d\tau' \int_0^{\tau'} d\tau''\, \Omega_1[\tau'']\Omega_1[\tau']$$

$$+ \cdots T^n \int_0^1 d\tau' \int_0^{\tau'} d\tau'' \cdots \int_0^{\tau^{(n-1)}} d\tau^{(n)}\, \Omega_1[\tau^{(n)}]\Omega_1[\tau^{(n-1)}]\cdots\Omega_1[\tau'] + \cdots. \tag{23}$$

The general term in this series is equal to a multiple integral of a chronological ordered product of Liouiville's operators $\Omega_1[\tau]$. This fact is a well-known theorem in quantum mechanics, which has recently been extended to classical mechanics.[5]

Therefore the m-th term of series (23) satisfies

$$T^n \int_0^1 d\tau' \int_0^{\tau'} d\tau'' \cdots \int_0^{\tau^{(n-1)}} d\tau^{(n-1)}\, \Omega_1[\tau^{(n)}]\cdots\Omega_1[\tau']$$

$$= \frac{T^n}{n!} \int_0^1 d\tau' \int_0^1 d\tau'' \cdots \int_0^1 d\tau^{(n)} \{\Omega_1[\tau']\Omega_1[\tau'']\cdots\Omega_1[\tau^{(n)}]\}_- \tag{24}$$

where the symbol $\{\ \}_-$ means chronological ordering: It rearranges the product of time-labeled operators in the same order as the time sequence of their

label, the latest one in time being last in the product, i.e. operators appear in the order, reading from left to right, of growing time values.

Let us now study the value of an integral whose integrand is a rapidly varying periodic operator as it is $\Omega_j[\tau]$ since in (21) we are going to consider T large. For instance we will study the value of

$$R_j(T) = \int_0^1 \Omega_j(\tau)\, e^{i\omega_j \tau T}\, d\tau \tag{25}$$

where $\Omega_j(\tau)$ is Liouiville's operator corresponding to $h_j(\tau)$. Integrating $R_j(T)$ m-times by parts and provided that $\Omega_j(\tau)$ and its first m derivatives are zero for $\tau=0$ and for $\tau=1$, we find

$$R_j(T) = \frac{(-1)^{m+1}}{(i\omega_j T)^{m+1}} \int_0^1 e^{i\omega_j \tau T} \frac{d^{m+1}\Omega_j(\tau)}{d\tau^{m+1}}\, d\tau, \tag{26}$$

integral that is quite small for large T. The operator $d^{m+1}\Omega_j(\tau)/d\tau^{m+1}$ is Liouiville's operator for $d^{m+1}h_j(\tau)/d\tau^{m+1}$. Let us now try to evaluate the action of $R_j(T)$ on the adiabatic invariants if the derivatives of order higher than m of $\Omega_j(\tau)$ are not zero at the initial and final times. We integrate again by parts and obtain

$$R_j(T) = \frac{(-1)^{m+1}}{(i\omega_j T)^{m+1}} \left\{ \left[\frac{e^{i\omega_j \tau T}}{i\omega_j T} \frac{d^{m+1}\Omega_j(\tau)}{d\tau^{m+1}} \right]_{\tau=0}^{\tau=1} - \frac{1}{i\omega_j T} \int_0^1 e^{i\omega_j \tau T} \frac{d^{m+2}\Omega_j}{d\tau^{m+2}}\, d\tau \right\}. \tag{27}$$

To find an upper limit of this expression for $T\to\infty$ we neglect the term containing the integral since by the same procedure of integration by parts we can show that such a term contains powers of $1/T$ higher than the other terms. Since the operator $d^{m+1}\Omega_j(\tau)/d\tau^{m+1}$ does not contain the parameter T, we can in general find an upper bound to the action of the same when it acts on a certain constant of the motion of the unperturbed system. Let us call M_1 and M_0 the upper bounds of $d^{m+1}\Omega_j(\tau)/d\tau^{m+1}$ at the instants $\tau=1$ and $\tau=0$ respectively. Then the action of $R_j(T)$ on such adiabatic invariant is bounded as follows,

$$\|R_j(T)\| < \frac{M_1+M_0}{(\omega_j T)^{m+2}}. \tag{28}$$

The same can be said for any of the integrals whose sum is the right-hand side of the series (23). Applying identical procedure of partial integrations to each one of the n integrals of the general term (24) of (23), we will find that for large T such a term is of the order

$$\frac{1}{T^{(m+2).n}} \tag{29}$$

provided that $H_1(\tau)$ and its m first derivatives are zero at $\tau=0$ and $\tau=1$.

So we deduce that all the constants of the motion of the initial Hamiltonian H_0 are adiabatic invariants of order $m+1$ of the slowly time dependent Hamiltonian $H(\tau)$ provided that $H_1(\tau)$ and its m first derivatives are zero at the beginning and at the end of the interval $t_1-t_0=T$ which we suppose to be very large.

The precedent statement includes as a particular example the case of adiabatic invariance to all orders when $H_1(\tau)$ and all its derivatives are zero at the extremes of the long time interval. We may deduce for this case that the difference between $S_1(\tau)$ and 1 is smaller than any power of $1/T$. We know that in certain cases,[6] such a difference behaves as e^{-T}.

The above said conditions can be simplified if T is large but finite. Then the constants of the motion of the periodic motion are adiabatic invariants of order $m+1$ of the Hamiltonian $H(t)$ if this Hamiltonian and its m first derivatives are zero at the beginning and at the end of the time interval.

The present technique provides a method of evaluating the approximate constancy of the adiabatic invariants for large but finite T. Indeed for the constants of H_0 we have

$$A(q(1),p(1))-A(q(0),p(0))=\Delta A$$

$$=\left[\sum_j \frac{T}{(i\omega_j T)^{m+1}} \int_0^1 e^{i\omega_j \tau T} \frac{d^{m+1}\Omega_j}{d\tau^{m+1}}\, d\tau,\ A(q(0),p(0))\right] \qquad (30)$$

since all the other terms of $S_1(1)$ contribute much less to this difference.

We would like to emphasize that this proof of the adiabatic invariance of the constants of the motion of the unperturbed Hamiltonian is not based on the smallness of the perturbation Hamiltonian, but rather we have required that the perturbation Hamiltonian evolving under the action of the unperturbed one be a purely oscillating function of time without non-oscillating components.

We are going to examine presently the adiabatic invariance of the constants of the motion of the Harmonic oscillator. We will see that the deduction of many well-known results implies a small modification of techniques developed so far.

We would like to remark here that the procedure of introducing the interaction picture for classical mechanics allows us not only to show when we have adiabatic invariants of m-th order, but also to evaluate the variation of such adiabatic invariants during the evolution of the system when T is large but finite, this last result being of great interest.

§ 6. A differential equation

In the same way as, from the differential equation of motion (4) of analytical dynamics, we can deduce an operational equation (7), for the adiabatic

invariants whose time evolution is given by $S_1(\tau)$, we can deduce a differential equation

$$\frac{\delta A(\tau)}{\delta \tau} = T \sum_i \frac{\partial H_1[\tau]}{\partial p_i} \frac{\partial A(\tau)}{\partial q_i} - \frac{\partial H_1[\tau]}{\partial q_i} \frac{\partial A(\tau)}{\partial p_i} \tag{31}$$

corresponding to a Hamiltonian that is the perturbation Hamiltonian in interaction picture. With the help of this differential equation, stability conditions can be discussed.

§ 7. Application

We plan to study now the motion of a charged particle in a slowly time depending uniform magnetic field. In particular it is interesting to examine the approximate constancy of the magnetic moment of the charged particle.

The motion of such particle is essentially equivalent to that of a simple harmonic oscillator provided that we identify the Larmor frequency with twice the frequency of the oscillator.[7] The magnetic moment is proportional to the action variable of the oscillator.

Therefore in this paper we will limit ourselves to study the approximate constancy of the action variable for the simple harmonic oscillator.

The Hamiltonian for this problem is

$$H = \frac{p^2}{2m} + m\omega^2 \frac{q^2}{2}. \tag{32}$$

Here ω is the frequency that we will suppose a slow function of time

$$\omega = \omega(\tau) = \omega\left(\frac{t}{T}\right). \tag{33}$$

The action integral J for a simple harmonic oscillator is equal to the energy divided by the frequency of the oscillation. Making use of the method described above, we can easily determine the change in J for a variation of ω.

The usefulness of performing a canonical transformation to the action-angle variables to study this kind of problems is well known. Therefore we shall make a canonical transformation from the set (q, p) of canonical conjugate variables to the new variables (θ, J) defined in such a way that they reduce to the ordinary action-angle variables when the Hamiltonian (32) does not contain explicit time dependence, i.e. when the oscillator has a constant frequency. As is usually done, we shall choose the generating function to depend on q, θ, and τ, and in such a way that it reduces for constant ω, to the generating function of the well-known[8] canonical transformation to the action-angle variables for the simple harmonic oscillator of constant frequency. Therefore the generating function is

$$G(q, \theta, \tau) = \frac{1}{2} m\omega(\tau)\, q^2 \cot\theta. \tag{34}$$

which yields the following expressions for the momentum p and the action J,

$$p=m\omega q\cot\theta,$$
$$J=\frac{m\omega q^2}{2\sin^2\theta}, \tag{35}$$

and the new Hamiltonian written in terms of θ, J is

$$H'=H+\frac{d\tau}{dt}\frac{\partial G}{\partial\tau}=\omega J+\frac{1}{2T}\frac{\dot\omega}{\omega}J\sin 2\theta \tag{36}$$

where

$$\dot\omega\equiv\frac{d\omega}{d\tau}=T\frac{d\omega}{dt}. \tag{37}$$

The partition of H' into unperturbed term H_0' and perturbation in order to obtain the interaction picture is immediate,

$$H_0'=\omega J,$$
$$H_1'=\frac{1}{2T}\frac{\dot\omega}{\omega}J\sin 2\theta. \tag{38}$$

Let us examine if the conditions sufficient for the adiabatic invariance of J are fulfilled. The unperturbed equations of motion are

$$\frac{d\theta}{dt}=\omega(\tau)=\omega\left(\frac{t}{T}\right),$$
$$\frac{dJ}{dt}=0, \tag{39}$$

whose solutions are

$$\theta_0(t)=\theta_0+\int_0^t\omega\left(\frac{t'}{T}\right)dt'=\theta_0+T\int_0^\tau\omega(\tau')\,d\tau',$$
$$J_0(t)=J_0 \tag{40}$$

where $\theta_0=\theta_0(0)$, $J_0=J_0(0)$ are the initial values of these variables. Indeed J is a constant of the motion of H_0'. The perturbing Hamiltonian $H_1'=H_1'(\tau)$ in the interaction picture is

$$H_1'[\tau]=S_0(\tau)\,H_1'(\tau)\,S_0^{-1}(\tau)=\frac{1}{2T}\frac{\dot\omega}{\omega}J_0\sin 2\left(\theta_0+T\int_0^\tau\omega(\tau')\,d\tau'\right) \tag{41}$$

which for large T is a rapidly oscillating function of the general form (21) that fulfills all the sufficient conditions to assure the adiabatic invariance of J. In our case only one function $h_J(\tau)$ appears and it is $1/2T\cdot\dot\omega(\tau)/\omega(\tau)\cdot J_0$.

We would like to remark a peculiarity proper to the simple harmonic oscil-

lator, that is, that $H_1'[\tau]$ is not quite a periodic function without constant Fourier term, but it is rapidly oscillating and such that $H_1'(\tau)$ of (38) is a small perturbation compared with H_0' property that is much stronger than those needed to prove the adiabatic invariance of the constants of the motion of H_0'. Indeed we have

$$\frac{H_1'}{H_0'} \simeq \frac{1}{2T} \frac{\dot{\omega}}{\omega^2} \tag{42}$$

that goes to zero for large T provided that ω never becomes zero, as is the case for a charged particle in a magnetic field. As a matter of fact the proofs presented so far to show the adiabatic invariance of the action integral of the harmonic oscillator were based on this peculiar property, that is, why they could not be applied to other mechanical systems.

From formula (30) we obtain the following value for the variation of the action integral in the interval $(0, T)$

$$\delta J \equiv J(1) - J(0) = -J_0 \int_0^1 \frac{\dot{\omega}(\tau)}{\omega(\tau)} \cos 2\left(\theta_0 + T \int_0^\tau \omega(\tau')\, d\tau'\right) d\tau \tag{43}$$

for whose deduction we have used

$$I = \frac{\overleftarrow{\partial}}{\partial J} \frac{\overrightarrow{\partial}}{\partial \theta} - \frac{\overleftarrow{\partial}}{\partial \theta} \frac{\overrightarrow{\partial}}{\partial J} \tag{44}$$

and the fact that

$$\frac{\partial \theta_0(t)}{\partial \theta_0(t_1)} = 1. \tag{45}$$

If $\dot{\omega}$ and its m first derivatives are zero at $\tau=0$ and $\tau=1$, i.e. if the first $m+1$ derivatives of $\omega(\tau)$ are zero at $\tau=0$ and $\tau=1$, the integral in (43) is of the order $1/T^{m+2}$. In such a case, therefore, the action integral is adiabatic invariant of order $m+2$. Observe that this is so even if $\omega(0) \neq \omega(1)$, i.e. in the case when the magnetic field in which the charged particle moves changes from a constant initial value B_0 to another constant final value B_1, provided that this change takes place in a very long time compared with the Larmor period.

Integrating (43) by parts we will obtain the corresponding value of δJ. Finally we should remark that sometimes it is convenient to evaluate the average of the resulting δJ over equally weighted values of θ_0; and to evaluate this average $\langle \delta J \rangle_{\theta_0}$ we may need second terms of the perturbation series (23) since this average of (43) is identically zero.

§ 8. Conclusions

We should like to call the reader's attention to a paper by S. Tamor[9] in

which he extended the adiabatic invariance concept for a one-dimensional oscillator to the case where the frequency has one singularity for finite time. We have studied here only the case where the Hamiltonian does not possess such a singularity. However, the case of a singular Hamiltonian can be handled by the formalism here presented. The study of this problem and the new conclusions at which we arrive, will be done elsewhere.

The assumptions taken in this paper may be summarized stating that once the Hamiltonian is splitted into the unperturbed part and the perturbation, this last one considered as function of $q=q_0(t)$, $p=p_0(t)$ solutions of the unperturbed motion, should be a sum of rapidly oscillating terms when the time interval T becomes very large. Under these conditions the constants of the motion of the unperturbed Hamiltonian are adiabatic invariants of m-th order if the perturbing Hamiltonian and its $m-1$ time derivatives with respect to the explicit time dependence of the perturbation are zero at the initial and final time instant.

In order to compare this approach with the one by Kulsrud[3] we may say that ours is more general since it is based on an operational development of classical mechaics and on the introduction of the interaction picture. By our approach we can treat the adiabatic invariants of the harmonic oscillator as a particular example of our general theory as soon as we split the Hamiltonian into the unperturbed part and the perturbation as it is done in the application in which, precisely, we studied the harmonic oscillator. If this is done, the conclusions at which both approaches arrive coincide.

Finally we remark that, as it happened for the harmonic oscillator, the unperturbed Hamiltonian may be time depending also. Its constants of the motion are adiabatic invariants provided that the perturbation in the interaction picture fulfills the required conditions.

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Theory of Ultrasonic Attenuation in Metals*

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A quantum mechanical theory of the attenuation of the longitudinal ultrasonic wave in metals due to the conduction electrons is developed according to the general theory of linear response given by Kubo. When the mean free path of electrons is longer than the wavelength of the ultrasonic wave, the result is reduced to that given by the ordinary perturbational treatment and is not equivalent to the semiclassical one obtained by Pippard. The discrepancy between these two treatments appears mainly in the effect of the interaction between conduction electrons and moving impurities. It arises because in Pippard's theory the interaction is averaged out from the outset and appears as a fictitious field driving electrons, not as a random force. The treatment of scattering in Pippard's theory seems to be incorrect when the mean free path is longer than the wavelength. Though the correction is small in the absence of a static magnetic field, it seems to be important when a static magnetic field is applied in perpendicular to the wave propagation.

§ 1. Introduction

The experiments of Bömmel[1] and later workers[2] have revealed that in pure metals at low temperatures the cause of ultrasonic attenuation is a direct interaction between the ultrasonic vibration and the conduction electrons, and that the magnetic field dependence of the attenuation constant is oscillatory when a static magnetic field is applied perpendicular to the direction of the wave propagation. Many theoretical investigations of this phenomenon have been put forward. Among them, the first complete theory for the case in which a magnetic field is not present was the semiclassical one developed by Pippard.[3]

Pippard[4] also explained the oscillatory behaviour of the magnetic field dependence pointing that it may arise from matching of the wavelength of the sound wave and the diameter of the cyclotron motion of conduction electrons, and proposed the possibility to determine Fermi surface from these measurements. His explanation has been shown to be essentially correct by a more complete discussion of Cohen, Harrison and Harrison,[5] whose theory is a straightforward generalization of Pippard's one[3] to the case in which a magnetic field is present.

The feature of Pippard's theory[3] is semiclassical; where, using the Maxwell

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equation and the transport equation of the electronic current by which electric field and current produced by the sound wave are related to one another, field and current are determined in a self-consistent way. In the calculation of the electronic current, we must treat the scattering mechanism carefully. Electrons are scattered by impurities (or thermal phonons) which are fixed on the lattice. When the sound wave is present, impurities also move with the lattice and drag electrons. Therefore the electronic current produced by the interaction with moving impurities should be added to the ordinary one written by the form σE, σ being the conductivity and E the electric field. This effect may be called *collision drag effect*. Pippard made the assumption that the scattering can be described by a relaxation time τ, and that the distribution toward which relaxation takes place is a displaced Fermi distribution centred at the local velocity of the lattice. Because of the assumption, the collision drag effect appears in his theory as a fictitious field proportional to $\boldsymbol{u}/\tau$, where $\boldsymbol{u}$ is the particle velocity of the lattice.

The theory is in good agreement with experiments.[9] From the theoretical point of view, however, we shall not be satisfied only by such agreement. As long as the wavelength of the sound wave is longer than the mean free path of electrons, Pippard's treatment may perhaps be permissible. However, when the mean free path is longer, an electron experiences by no means the dragging force as such a field. Isn't it required to treat the scattering more carefully as a random force? It is our purpose in this article to answer this question.

In the absence of a magnetic field, however, the problem becomes only an academic one. For, in this case, the attenuation remains to be finite even at the limit $\tau\to\infty$, and so, when $ql\gg 1$, effects of the scattering give only small corrections to this main term which is independent of the treatment of the scattering mechanism. When a static magnetic field is applied perpendicular to the direction of the wave propagation, the circumstance is quite different. As the motion of electrons in the plane perpendicular to the magnetic field is quantized, there appears no attenuation at the limit $\tau\to\infty$, unless the frequency of the sound wave is equal to integral multiple of the electronic cyclotron frequency. Therefore, the attenuation essentially comes from the scattering, and we must answer the above question at first, before analysing experimental data of the magneto-acoustic effect.

The attenuation has been observed experimentally for the transverse waves as well as for the longitudinal ones, and there exists little difference between these two cases. However, the former is a little more complicated to treat theoretically, for we must take into account magnetic interactions there. As main features of the present problem are included in the case of the longitudinal wave, we treat, for simplicity, only this case. The results can be easily generalized to the case of the transverse wave with a minor modification.

Furthermore, we consider in this paper the case in which a static magnetic field is not present. In the presence of a magnetic field, it is required to modify

the treatment as there appears the Hall field; but the general expression obtained here for the collision drag effect will be valid even in this case.

Here we take an orthodox approach. The interaction Hamiltonian between the conduction electrons and the sound wave is first determined and then its contribution to the elastic constant of the lattice is calculated, the imaginary part of which gives the attenuation of the sound wave.

In § 2, Hamiltonian is discussed in some detail. Then we give the formulae for the elastic constant and the attenuation constant in § 3. § 4 is devoted to the calculation in the second order approximation, which is permissible when the mean free path of conduction electrons is long enough. In § 5, a more general expression, which is not limited to the second order of the scattering, is given by the method of a canonical transformation. In the last section, § 6, we summarize the results and give some arguments, comparing our results with semiclassical theory.

§ 2. Hamiltonian

For simplicity, we consider a monovalent metal with n atoms per unit volume. In the presence of the longitudinal sound wave, the displacement of j-th ion from its equilibrium position $\boldsymbol{R}_j$ is written as

$$\boldsymbol{\xi}_j(t)=[Q(t)\exp(i\boldsymbol{q}\cdot\boldsymbol{R}_j)+Q^*(t)\exp(-i\boldsymbol{q}\cdot\boldsymbol{R}_j)]\boldsymbol{e}_q, \tag{2·1}$$

where $\boldsymbol{q}$ is the wave vector of the sound wave, $\boldsymbol{e}_q$ its polarization vector and Q and Q^* are its amplitude.

Then the Hamiltonian of the total system, consisting of the sound wave and the conduction electrons, is given by

$$H_e+H_Q+H_{eQ}, \tag{2·2}$$

where H_e is the Hamiltonian of the conduction electrons, H_Q is that of the sound wave and H_{eQ} is the interaction Hamiltonian between them.

H_e consists of three parts; i.e.

$$H_e=H_0+H_I+H_c. \tag{2·3}$$

Here H_0 is the periodic part of the one electron Hamiltonian and, in the second quantization representation, is given by

$$H_0=\sum_{\kappa\sigma}\epsilon_\kappa a^*_{\kappa\sigma}a_{\kappa\sigma}, \tag{2·4}$$

where $a^*_{\kappa\sigma}$ and $a_{\kappa\sigma}$ are, respectively, the creation and annihilation operators of electrons with the wave vector $\boldsymbol{\kappa}$ and spin σ. If we take the free electron model,

$$\epsilon_\kappa=\frac{\hbar^2\kappa^2}{2m}, \tag{2·5}$$

where m is the effective mass of the conduction electrons.

H_I is the interaction of electrons with the fixed scatterers. For the case of impurity scattering, provided for simplicity that impurities are all of the same kind, it is given by

$$H_I = \sum_{\boldsymbol{k}} \sum_{\mu} U(\boldsymbol{k}) \exp(i\boldsymbol{k}\cdot\boldsymbol{R}_\mu)\rho_{\boldsymbol{k}}, \tag{2·6a}$$

where

$$\rho_{\boldsymbol{k}} = \sum_{i} \exp(-i\boldsymbol{k}\cdot\boldsymbol{r}_i) = \sum_{\kappa\sigma} a^*_{\kappa-\boldsymbol{k}\sigma}\, a_{\kappa\sigma}, \tag{2·7}$$

which is the $\boldsymbol{k}$ component of the electron density. Here $\boldsymbol{r}_i$ is the position of the i-th electron, $\boldsymbol{R}_\mu$ is the position of the μ-th impurity, and $U(\boldsymbol{k})$ is the matrix element of the impurity potential $U(\boldsymbol{r})$ between the electronic states with the wave vector $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}+\boldsymbol{k}$ and is assumed to be independent of $\boldsymbol{\kappa}$.

When the cause of the scattering is longitudinal thermal phonons, the displacement of the j-th ion by phonons from its equilibrium position $\boldsymbol{R}_j$ is taken to be

$$\boldsymbol{\eta}_j = \sum_{\boldsymbol{k}} [q_{\boldsymbol{k}} \exp(i\boldsymbol{k}\cdot\boldsymbol{R}_j) + q_{\boldsymbol{k}}^* \exp(-i\boldsymbol{k}\cdot\boldsymbol{R}_j)]\boldsymbol{e}_{\boldsymbol{k}}, \tag{2·8}$$

and H_I is given by

$$H_I = n \sum_{\boldsymbol{k}} ikV(\boldsymbol{k})\rho_{\boldsymbol{k}}(q_{-\boldsymbol{k}} + q_{\boldsymbol{k}}^*), \tag{2·6b}$$

where $q_{\boldsymbol{k}}$ and $q_{\boldsymbol{k}}^*$ are the normal coordinates of the longitudinal phonon with the wave vector $\boldsymbol{k}$ and polarization vector $\boldsymbol{e}_{\boldsymbol{k}}$. $V(\boldsymbol{k})$ is the matrix element of the positive ion potential $V(\boldsymbol{r})$, and if positive ions are replaced by a uniform positive background, it becomes

$$V(\boldsymbol{k}) = -\frac{4\pi e^2}{k^2}. \tag{2·9}$$

The last term H_e of (2·3) is the Coulomb interaction between conduction electrons, which is written as

$$H_e = \frac{1}{2} \sum_{\boldsymbol{k}} M(\boldsymbol{k})\rho_{\boldsymbol{k}}\rho_{-\boldsymbol{k}}. \tag{2·10}$$

When the free electron model is taken, the coefficients $M(\boldsymbol{k})$'s become

$$M(\boldsymbol{k}) = \frac{4\pi e^2}{k^2}. \tag{2·11}$$

The interaction H_{eQ} between the conduction electrons and the sound wave consists of two terms $H_{eQ}^{(0)}$ and $H_{eQ}^{(1)}$; i.e.

$$H_{eQ} = H_{eQ}^{(0)} + H_{eQ}^{(1)}, \tag{2·12}$$

where $H_{eQ}^{(0)}$ is the ordinary interaction and $H_{eQ}^{(1)}$ is that which arises from the motion of scatterers by the wave. As long as the amplitude of the wave is

smaller than the wavelength, $H_{eQ}^{(0)}$ takes the same form as (2·6b) ; i.e.

$$H_{eQ}^{(0)}=-inqV(\boldsymbol{q})[\rho_{-q}Q(t)-\rho_q Q^*(t)]. \tag{2·13}$$

$H_{eQ}^{(I)}$ may be called the collision drag interaction, and for the case of impurity scattering, it becomes

$$\begin{aligned} H_{eQ}^{(I)}&=\sum_i\sum_\mu\{U(\boldsymbol{r}_i-\boldsymbol{R}_\mu-\boldsymbol{\xi}_\mu)-U(\boldsymbol{r}_i-\boldsymbol{R}_\mu)\} \\ &=\sum_{\boldsymbol{k}}\sum_\mu U(\boldsymbol{k})\exp(i\boldsymbol{k}\cdot\boldsymbol{R}_\mu)[\exp(i\boldsymbol{k}\cdot\boldsymbol{\xi}_\mu)-1]\rho_{\boldsymbol{k}}, \end{aligned} \tag{2·14a}$$

where $\boldsymbol{r}_i$ is the position of the i-th electron and $\boldsymbol{\xi}_\mu$ is the displacement of the μ-th impurity, which is given by (2·1) if we replace $\boldsymbol{R}_j$ by $\boldsymbol{R}_\mu$. In order to obtain the electron-sound wave interaction in the form linear to Q and Q^*, we must expand the exponential function in the square brackets. Such an expansion may be permissible only when $k\bar{\xi}\ll 1$, $\bar{\xi}$ being the amplitude of the wave. The maximum value of $|\boldsymbol{k}|$ which contributes to (2·14a) is of the order a^{-1}, where a is the force range of the impurity. Therefore it is required for the linearization of (2·14a) that the condition

$$a\gg\bar{\xi} \tag{2·15a}$$

is satisfied. If Q is one of the thermal phonon modes, it may be satisfied, but, when it is the macroscopic sound wave, $\bar{\xi}$ may be a thousand times of the interatomic distance and (2·15a) can hardly be realized. But for a while we confine ourselves to the case $a\gg\bar{\xi}$. It will be shown later that the result obtained in such a way is valid beyond the condition. Then (2·14a) can be linearized as

$$H_{eQ}^{(I)}=-i\sum_{\boldsymbol{k}}\sum_\mu k_x U(\boldsymbol{k})\{\exp[-i(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{R}_\mu]\rho_{-\boldsymbol{k}}Q(t)-\text{c.c.}\}, \tag{2·16a}$$

where the direction of $\boldsymbol{e}_q$ has been taken as x-axis. For the case of phonon scattering, $H_{eQ}^{(I)}$ is given by

$$\begin{aligned} H_{eQ}^{(I)}&=\sum_i\sum_j\{[V(\boldsymbol{r}_i-\boldsymbol{R}_j-\boldsymbol{\xi}_j-\boldsymbol{\eta}_j)-V(\boldsymbol{r}_i-\boldsymbol{R}_j-\boldsymbol{\xi}_j)] \\ &\qquad -[V(\boldsymbol{r}_i-\boldsymbol{R}_j-\boldsymbol{\eta}_j)-V(\boldsymbol{r}_i-\boldsymbol{R}_j)]\} \\ &=i\sum_{\boldsymbol{k}}\sum_{\boldsymbol{f}}\sum_j k(\boldsymbol{k}\cdot\boldsymbol{e})\,V(\boldsymbol{k})\exp[i(\boldsymbol{k}-\boldsymbol{f})\cdot\boldsymbol{R}_j]\rho_{\boldsymbol{k}}(q_{-f}+q_f^*)\,[\exp(i\boldsymbol{k}\cdot\boldsymbol{\xi}_j)-1]. \end{aligned} \tag{2·14b}$$

In order to obtain the interaction linear in Q and Q^*, there occurs the same problem as in the case of impurity scattering. Here the condition for expanding (2·14b) with $\boldsymbol{\xi}_j$ and linearizing the interaction is given by

$$\frac{\hbar v_s}{k_BT}\gg\bar{\xi}, \tag{2·15b}*$$

* The maximum frequency of excited phonons is of the order $(k_BT/\hbar)$ and so the maximum value of the wave number is of the order $(k_BT/\hbar v_s)$. Hence the condition $k\bar{\xi}\ll 1$ becomes (2·15b).

where v_s is the sound velocity. Under this condition, (2·14b) is reduced to

$$H_{eQ}^{(I)}=-n\sum_{k}kk_x V(\boldsymbol{k})\,\rho_k\{(\boldsymbol{k}\cdot\boldsymbol{e}_{k+q})\,(q_{-k-q}+q_{k+q}^{*})Q(t) + (\boldsymbol{k}\cdot\boldsymbol{e}_{k-q})\,(q_{-k+q}+q_{k-q}^{*})Q^{*}(t)\}. \tag{2·16b}$$

§ 3. Elastic constant and attenuation constant

To calculate the attenuation constant α, we consider the motion of the amplitude Q of the sound wave, which obeys the equation of motion

$$\ddot{Q}+\Omega^2 Q=-\frac{1}{d}\left\langle\frac{\partial H_{eQ}}{\partial Q^{*}}\right\rangle_I, \tag{3·1}$$

where d is the density of the crystal, Ω is the unperturbed frequency of the sound wave determined by the Hamiltonian H_Q, the triangle bracket means the average over the distribution of the conduction electrons in the presence of the sound wave, and the subscript I means the average over the distribution of impurities (or thermal phonons).

We write the right-hand side of (3·1) as

$$\left\langle\frac{\partial H_{eQ}}{\partial Q^{*}}\right\rangle_I=q^2 AQ, \tag{3·2}$$

where A is the contribution of the conduction electrons to the elastic stiffness constant, the imaginary part of which causes damping of the sound wave. Then the attenuation constant is given by

$$\alpha=-\frac{q}{dv_s^{2}}\,\mathrm{Im}\,A. \tag{3·3}$$

Thus our problem is reduced to the calculation of the imaginary part of the elastic constant A.

For the case of impurity scattering, we get

$$\frac{\partial H_{eQ}}{\partial Q^{*}}=inqV(\boldsymbol{q})\rho_q+i\sum_{k}\sum_{\mu}k_x U(\boldsymbol{k})\exp[i(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{R}_\mu]\rho_k. \tag{3·4}$$

Therefore A is given as follows:

$$AQ=\frac{1}{q^2}\left\{inqV(\boldsymbol{q})\langle\rho_q\rangle_I+i\sum_{k}k_x U(\boldsymbol{k})\left(\sum_{\mu}\exp[i(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{R}_\mu]\langle\rho_k\rangle\right)_I\right\}. \tag{3·5}$$

From the above expression we can see that it is first required to calculate $\langle\rho_q\rangle$ and $\langle\rho_k\rangle$ $(\boldsymbol{k}\neq\boldsymbol{q})$ in the linear approximation to Q.

The calculations for two cases, impurity scattering and phonon scattering, can be carried out almost in a parallel way, but, when thermal phonons are concerned, equations are more complicated because of the inelasticity of the scattering. So we treat hereafter mainly the case of impurity scattering.

§ 4. Second order approximation—long wavelength limit

We now proceed to the calculation of the contribution of the conduction electrons to the elastic constant according to the general theory of linear response given by Kubo.[10] We denote the density matrix of electrons in the presense of the sound wave by $\sigma'(t)$ and expand it as

$$\sigma'(t)=\sigma+\Delta\sigma(t), \tag{4·1}$$

where σ specifies the equilibrium state in the absence of the wave. $\Delta\sigma(t)$ is calculated in the ordinary way.

In the calculation, one of the most difficult problems is how to deal with the Coulomb interaction H_c. In this case, there can occur two kinds of effects from the Coulomb interaction: The first is the effect of the electron-electron *collision* to the attenuation and the second is the *screening* of various interactions. The first, however, is expected to be small as the mean free path of electrons determined by the electron-electron collision is in most cases much longer than that determined by the electron-impurity interaction.[11] Hereafter, we neglect the first effect.

For the purpose of taking into account the screening effect only, it is sufficient to take the *Hartree approximation* for the Coulomb term. Exchange effect will not be important at least qualitatively. Then

$$\begin{aligned} H_{c\ Hartree}&=\sum_{k} M(\boldsymbol{k})\,(\mathrm{Tr}\,\sigma'(t)\,\rho_k)\,\rho_{-k} \\ &=\sum_{k} M(\boldsymbol{k})\,(\mathrm{Tr}\,\sigma\rho_k)\,\rho_{-k}+\sum_{k} M(\boldsymbol{k})\,(\mathrm{Tr}\,\Delta\sigma(t)\,\rho_k)\,\rho_{-k}. \end{aligned}$$

From the first term of the second expression arises the screening of the interaction between electrons and the fixed scatterers; so, instead of omitting this term, we replace the potential $U(\boldsymbol{k})$ in the Hamiltonian H_I by the screened one $\bar{U}(\boldsymbol{k})$.

In this approximation, the Hamiltonian of the conduction electrons perturbed by the sound wave is given by

$$H+H_{eQ}(t)+H_{eQ}^{c}(t), \tag{4·2}$$

with

$$\begin{aligned} &H=H_0+\bar{H}_I, \\ &H_{eQ}^{c}(t)=\sum_{k} M(\boldsymbol{k})\langle\rho_k\rangle(t)\,\rho_{-k},\quad \langle\rho_k\rangle(t)=\mathrm{Tr}\,\Delta\sigma(t)\,\rho_k. \end{aligned} \tag{4·3}$$

Here $H_{eQ}^{c}(t)$ also gives effectively the interaction between electrons and the sound wave. $\langle\rho_k\rangle$'s should be determined in a self-consistent way.

In the linear approximation, the density matrix $\Delta\sigma(t)$ obeys the equation

$$i\hbar\frac{d\Delta\sigma(t)}{dt}=[H,\,\Delta\sigma(t)]+[H_{eQ}(t)+H_{eQ}^{c}(t),\,\sigma], \tag{4·4}$$

which is solved formally in the ordinary way. Using the solution, $\langle\rho_k\rangle$ is given by

$$\langle\rho_k\rangle=\mathrm{Tr}\,\Delta\sigma(t)\,\rho_k=[-inqV(\boldsymbol{q})\,\langle\rho_k|\rho_{-q}\rangle-i\sum_{k'}\sum_{\mu}k_x'\,U(\boldsymbol{k}')\,\langle\rho_k|\rho_{-k'}\rangle$$
$$\times\exp(-i(\boldsymbol{k}'-\boldsymbol{q})\cdot\boldsymbol{R}_\mu)]Q+\sum_{k'}M(\boldsymbol{k}')\,\langle\rho_k|\rho_{-k'}\rangle\langle\rho_{k'}\rangle, \tag{4·5}$$

with

$$\langle A|B\rangle=\lim_{\varepsilon\to+0}\frac{1}{i\hbar}\int_0^\infty dt\,e^{i\omega t-\varepsilon t}\mathrm{Tr}\,\sigma[A(t),B],$$
$$A(t)=e^{iHt/\hbar}Ae^{iHt/\hbar}. \tag{4·6}$$

Here the frequency ω is not equal to Ω in (3·1), but is renormalized including the real part of the right-hand side of (3·1). In the following calculation, the abbreviation

$$\langle\boldsymbol{k}|\boldsymbol{k}'\rangle\equiv\langle\rho_k|\rho_{-k'}\rangle \tag{4·7}$$

will be used.

In order to determine $\langle\rho_k\rangle$'s in a self-consistent way, we must solve these simultaneous equations of infinite dimension, which is impossible in general. However, in the second order approximation of the electron-impurity interaction, the solution can be found in the following way.

From (3·5), it can be seen that for the purpose of obtaining A to the second order of $U(\boldsymbol{k})$'s we have only to calculate $\langle\rho_q\rangle$ to the second order and $\langle\rho_k\rangle$'s for $\boldsymbol{k}\neq\boldsymbol{q}$ to the first, for the second term of (3·6a) already has the factor $U(\boldsymbol{k})$.

For $\boldsymbol{k}=\boldsymbol{q}$, (4·5) becomes

$$\langle\rho_q\rangle=\{-iq[nV(\boldsymbol{q})+n_iU(\boldsymbol{q})]\langle\boldsymbol{q}|\boldsymbol{q}\rangle-i\sum_{k\neq q,\mu}k_xU(\boldsymbol{k})\,\langle\boldsymbol{q}\,\boldsymbol{k}\rangle\exp[-i(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{R}_\mu]\}Q$$
$$+M(\boldsymbol{q})\,\langle\boldsymbol{q}|\boldsymbol{q}\rangle\langle\rho_q\rangle+\textstyle\sum_{k\neq q}M(\boldsymbol{k})\,\langle\boldsymbol{q}|\boldsymbol{k}\rangle\langle\rho_k\rangle,$$

where n_i is the impurity concentration. In the first term $n_iU(\boldsymbol{q})$ is always much smaller than $nV(\boldsymbol{q})$ and it is only to modify the magnitude of the direct electron-sound wave interaction constant a little, so we ignore it hereafter. It should be noticed here that, if we expand $\langle\boldsymbol{k}|\boldsymbol{k}'\rangle$ with H_I perturbationally, the zeroth order term vanishes unless $\boldsymbol{k}=\boldsymbol{k}'$, as is shown in the Appendix. From this fact, the above expression is reduced, in the second order approximation, to

$$\langle\rho_q\rangle^{(2)}=\{-inqV(\boldsymbol{q})\,\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(2)}-i\sum_{k\neq q,\mu}k_xU(\boldsymbol{k})\,\langle\boldsymbol{q}|\boldsymbol{k}\rangle^{(1)}\exp[-i(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{R}_\mu]\}Q$$
$$+M(\boldsymbol{q})\,\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(2)}\langle\rho_q\rangle^{(2)}+\sum_{k\neq q}M(\boldsymbol{k})\,\langle\boldsymbol{q}|\boldsymbol{k}\rangle^{(1)}\langle\rho_k\rangle^{(1)}, \tag{4·8}$$

where the suffixes (1), (2), ··· mean respectively that the suffixed quantities are exact up to the first order, up to the second order, and so on.

For $\boldsymbol{k}\neq\boldsymbol{q}$, (4·5) becomes to the first order,

$$\langle\rho_k\rangle^{(1)}=\{-inqV(\boldsymbol{q})\langle\boldsymbol{k}|\boldsymbol{q}\rangle^{(1)}-i\sum_{\mu}k_xU(\boldsymbol{k})\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}\exp[-i(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{R}_\mu]\}Q$$
$$+M(\boldsymbol{q})\langle\boldsymbol{k}|\boldsymbol{q}\rangle^{(1)}\langle\rho_q\rangle^{(0)}+M(\boldsymbol{k})\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}\langle\rho_k\rangle^{(1)}. \tag{4·9}$$

We can obtain $\langle\rho_q\rangle^{(0)}$ immediately from (4·8), taking the lowest order, as

$$\langle\rho_q\rangle^{(0)}=\frac{-inqV(\boldsymbol{q})\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(0)}}{1-M(\boldsymbol{q})\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(0)}}. \tag{4·10}$$

From (4·8), (4·9) and (4·10), $\langle\rho_q\rangle^{(2)}$ and $\langle\rho_k\rangle^{(1)}$ $(\boldsymbol{k}\neq\boldsymbol{q})$ are given by

$$\langle\rho_q\rangle^{(2)}=\Big\{-\frac{inqV(\boldsymbol{q})\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(2)}}{1-M(\boldsymbol{q})\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(2)}}-\sum_{\boldsymbol{k}\neq\boldsymbol{q}}\Big[\frac{inqV(\boldsymbol{q})M(\boldsymbol{k})\langle\boldsymbol{q}|\boldsymbol{k}\rangle^{(1)}\langle\boldsymbol{k}|\boldsymbol{q}\rangle^{(1)}}{(1-M(\boldsymbol{q})\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(0)})^2(1-M(\boldsymbol{k})\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)})}$$
$$-\sum_{\mu}\frac{ik_xU(\boldsymbol{k})\langle\boldsymbol{q}|\boldsymbol{k}\rangle^{(1)}\exp[-i(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{R}_\mu]}{(1-M(\boldsymbol{q})\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(0)})(1-M(\boldsymbol{k})\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)})}\Big]\Big\}Q, \tag{4·11}$$

$$\langle\rho_k\rangle^{(1)}=\Big\{-\frac{inqV(\boldsymbol{q})\langle\boldsymbol{k}|\boldsymbol{q}\rangle^{(1)}}{(1-M(\boldsymbol{q})\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(0)})(1-M(\boldsymbol{k})\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)})}$$
$$-\sum_{\mu}\frac{ik_xU(\boldsymbol{k})\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}\exp[-i(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{R}_\mu]}{1-M(\boldsymbol{k})\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}}\Big\}Q. \tag{4·12}$$

Finally, inserting (4·11) and (4·12) in (3·6) and taking the average over the random distribution of impurities, we obtain A in the second order approximation of the electron-impurity interaction. The result is

$$A^{(2)}=\frac{1}{q^2}\left\{\frac{[nqV(\boldsymbol{q})]^2\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(2)}}{1-M(\boldsymbol{q})\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(2)}}\right\}_I+\frac{1}{q^2}\sum_{\boldsymbol{k}\neq\boldsymbol{q}}\Big[\frac{[nqV(\boldsymbol{q})]^2M(\boldsymbol{k})\{\langle\boldsymbol{q}|\boldsymbol{k}\rangle^{(1)}\langle\boldsymbol{k}|\boldsymbol{q}\rangle^{(1)}\}_I}{(1-M(\boldsymbol{q})\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(0)})^2(1-M(\boldsymbol{k})\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)})}$$
$$+\frac{nqV(\boldsymbol{q})k_xU(\boldsymbol{k})\{\sum_\mu(\langle\boldsymbol{q}|\boldsymbol{k}\rangle^{(1)}\exp[i(\boldsymbol{q}-\boldsymbol{k})\cdot\boldsymbol{R}_\mu]+\langle\boldsymbol{k}|\boldsymbol{q}\rangle^{(1)}\exp[i(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{R}_\mu])\}_I}{(1-M(\boldsymbol{q})\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(0)})(1-M(\boldsymbol{k})\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)})}$$
$$+n_i\frac{[k_xU(\boldsymbol{k})]^2\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}}{1-M(\boldsymbol{k})\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}}\Big], \tag{4·13}$$

where the terms in the square brackets have arisen from the motion of impurities, that is, the collision drag and its reaction effect.

Then let us examine the physical meaning of each term in the expression (4·13) and estimate its magnitude. The detailed calculations will be given in the Appendix.

First we consider the zeroth order term of A, which is given by

$$A^{(0)}=\frac{1}{q^2}\frac{[nqV(\boldsymbol{q})]^2\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(0)}}{1-M(\boldsymbol{q})\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(0)}}. \tag{4·14}$$

If we introduce the screened interaction potential $\bar{V}(\boldsymbol{q})$ difined by

$$\bar{V}(\boldsymbol{q})=\frac{V(\boldsymbol{q})}{|1-M(\boldsymbol{q})\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(0)}|}, \tag{4·15}$$

the imaginary part of (4·14) can be written as

$$\mathrm{Im}\,A^{(0)}=\frac{d}{q^2}\,[nq\bar{V}(\boldsymbol{q})]^2\,\mathrm{Im}\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(0)}. \tag{4·16}$$

As q is much smaller than the Fermi wave number k_F, $\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(0)}$ is given by the use of (A·4), (A·5) and the relation $\omega=qv_s$ (v_s : sound velocity) as

$$\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(0)}=-\frac{3n}{2\epsilon_F}\left(1+i\frac{\pi}{2}\frac{v_s}{v_F}\right). \tag{4·17}$$

Hence, using the relation (2·13), we get

$$|1-M(\boldsymbol{q})\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(0)}|\cong 1+3\omega_p^2/q^2v_F^2,\quad \text{where } \omega_p^2=4\pi ne^2/m.$$

We can see from this that the screened potential (4·15) is the same as is obtained in the Thomas-Fermi approximation.[12]

Finally, we obtain $\mathrm{Im}\,A^{(0)}$ and the attenuation constant $\alpha^{(0)}$ at the limit $l\to\infty$ (l : the mean free path of the conduction electrons) as

$$\mathrm{Im}\,A^{(0)}=-\frac{\pi}{3}nmv_Fv_s, \tag{4·18}$$

$$\alpha^{(0)}=\frac{\pi nmv_F\omega}{3dv_s^2}. \tag{4·19}$$

This is equal to the result obtained by Pippard,[3] and is also the same as given in the ordinary transport theory.

Next we remark that in the third term of (4·13) the screened potential of the moving impurities is not

$$\bar{U}(\boldsymbol{k})\equiv\frac{U(\boldsymbol{k})}{|1-M(\boldsymbol{k})\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}|} \tag{4·20}$$

but

$$\tilde{U}(\boldsymbol{k})=\frac{U(\boldsymbol{k})}{1-M(\boldsymbol{k})\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}}. \tag{4·21}$$

As will be shown in the Appendix,

$$\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}=\lim_{\varepsilon\to 0}2\sum_{\kappa}\frac{f(\epsilon_\kappa)-f(\epsilon_{\kappa+k})}{\epsilon_\kappa-\epsilon_{\kappa+k}+\hbar\omega+i\varepsilon}. \tag{4·22}$$

When $\omega=0$, the difference between $\bar{U}(\boldsymbol{k})$ and $\tilde{U}(\boldsymbol{k})$ vanishes and they are equal to the adiabatic screening obtained by Takimoto.[13] When $\omega\neq 0$, however, $\tilde{U}(\boldsymbol{k})$ has the imaginary part ; in other words, there occurs retardation of the motion of electron clouds which screen impurities.

(4·22) becomes for $k\ll k_F$

$$\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}=-\frac{3n}{2\epsilon_F}\left(1+i\frac{\pi}{2}\frac{\omega}{kv_F}\right), \tag{4·23}$$

and this expression shows that the imaginary part is small unless $\omega\gtrsim kv_F$, or

impurities move rapidly compared with the time duration for electrons to pass through the force range of an impurity. In the present case, we can see that $\omega/kv_F=(q/k)\cdot(v_s/v_F)$ is much smaller than unity and such a retardation effect is small and can be ignored.

Now we consider the physical meaning of each term in the square brackets of (4·13). The first term

$$A_1=\frac{1}{q^2}\sum_{k\neq q}[nq\tilde{V}(\boldsymbol{q})]^2\frac{M(\boldsymbol{k})\{\langle\boldsymbol{q}|\boldsymbol{k}\rangle^{(1)}\langle\boldsymbol{k}|\boldsymbol{q}\rangle^{(1)}\}_I}{1-M(\boldsymbol{k})\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}} \tag{4·24}$$

comes from the last term of (4·8) which determines $\langle\rho_q\rangle$, when we insert in $\langle\rho_k\rangle$ the first term of (4·9). $\langle\rho_k\rangle(\boldsymbol{k}\neq\boldsymbol{q})$ is the $\boldsymbol{k}$ component of the density of electron clouds which screen impurities and the first term of (4·9) means that it is affected by the positive charge density wave produced by the sound wave. Therefore A_1 may be regarded as the effect of deformation of the screening electron clouds. There does not appear such an effect in the ordinary treatment, where impurities have a *given* screened potential, or, in other words, electron clouds screen impurities *rigidly*.

The physical meaning of the second term

$$A_2=\frac{1}{q^2}\sum_{k\neq q}nq\tilde{V}(\boldsymbol{q})\,k_x\tilde{U}(\boldsymbol{k})\{\sum_{\mu}\langle\boldsymbol{q}|\boldsymbol{k}\rangle^{(1)}\exp[i(\boldsymbol{q}-\boldsymbol{k})\cdot\boldsymbol{R}_\mu]\}_I \tag{4·25}$$

can be understood easily. This gives the effect of the motion of impurities to the electron density $\langle\rho_q\rangle$; i.e. the collision drag effect.

The next term

$$A_2'=\frac{1}{q^2}\sum_{k\neq q}nq\tilde{V}(\boldsymbol{q})\,k_x\tilde{U}(\boldsymbol{k})\{\sum_{\mu}\langle\boldsymbol{k}|\boldsymbol{q}\rangle^{(1)}\exp[i(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{R}_\mu]\}_I \tag{4·26}$$

has the same form as A_2, but its meaning is a little different; i.e. it is a part of reaction of the collision drag effect.

The last term in the brackets

$$A_3=\frac{n_i}{q^2}\sum_{k\neq q}\frac{[k_xU(\boldsymbol{k})]^2\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}}{1-M(\boldsymbol{k})\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}} \tag{4·27}$$

is also a part of the reaction. The imaginary part of A_3 becomes

$$\mathrm{Im}\,A_3=\frac{n_i}{q^2}\sum_{k\neq q}[k_x\bar{U}(\boldsymbol{k})]^2\,\mathrm{Im}\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}. \tag{4·28}$$

This term may well be regarded as damping of the motion of impurities which run through the viscous fluid of the conduction electrons, where the interaction potential between electrons and impurities is the screened one (4·20).

The calculation of these terms is carried out in the Appendix and the results are as follows:

$$\mathrm{Im}\,A_1=\mathrm{Im}\,A^{(0)}\cdot\mathrm{O}\left(\frac{v_s}{v_F}\frac{1}{k_Fl}\right), \tag{4·29a}$$

$$\mathrm{Im}\,A_2=\mathrm{Im}\,A_2'=-\mathrm{Im}\,A^{(0)}\cdot\mathrm{O}\left(\frac{1}{k_Fl}\right), \tag{4·29b}$$

$$\mathrm{Im}\,A_3=\mathrm{Im}\,A^{(0)}\cdot\mathrm{O}\left(\frac{1}{ql}\right). \tag{4·29c}$$

It is seen from (4·29a) that the deformation of the screening has only a negligible contribution to the attenuation.

Pippard took into account only A_2 in his calculation as the effect of the motion of impurities, and the existence of the terms A_2' and A_3, i.e. the reaction of collision drag was pointed out explicitly by Cohen et al.[5] In their semiclassical treatment, however, these reaction effects are numerically negligible except for very high frequencies, though the collision drag effect has a contribution of the order $1/ql$ compared with the zeroth order term, when $ql\gg1$.

It should be noted here that there is a difference between our result and the semiclassical one: that is, in our case, the most important term among the effects of the motion of impurities is the reaction effect A_3. The difference arises because in the semiclassical treatment the interaction between moving impurities and electrons is taken into account as a fictitious field, which is already averaged out over the distribution of impurities and proportional to $1/\tau$, while in our treatment the interaction is considered as a random force the average of which is to be taken at the end of calculations.

If we neglected the effects of deformation and retardation of the screening electron clouds around impurities, the expression (4·13) can be simplified. It becomes

$$\begin{aligned}\mathrm{Im}\,A^{(2)}=&\frac{1}{q^2}\,\mathrm{Im}\left\{\frac{[nq\bar{V}(\boldsymbol{q})]^2\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(2)}}{1-\bar{M}(\boldsymbol{q})\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(2)}}\right\}_I\\&+\frac{1}{q^2}\sum_{k\neq q}\Big[nq\bar{V}(\boldsymbol{q})\,k_x\bar{\bar{U}}(\boldsymbol{k})\,\mathrm{Im}\{\sum_{\mu}(\langle\boldsymbol{q}|\boldsymbol{k}\rangle^{(1)}\exp[i(\boldsymbol{q}-\boldsymbol{k})\cdot\boldsymbol{R}_\mu]\\&+\langle\boldsymbol{k}|\boldsymbol{q}\rangle^{(1)}\exp[i(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{R}_\mu])\}_I+n_i[k_x\bar{\bar{U}}(\boldsymbol{k})]^2\,\mathrm{Im}\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}\Big],\end{aligned} \tag{4·30}$$

using the screened potential $\bar{V}(\boldsymbol{q})$ and $\bar{\bar{U}}(\boldsymbol{k})$.

§ 5. Method of canonical transformation

Our treatment in the preceding section has some drawback:

(1) In order to linearize the Hamiltonian $H_Q^{(1)}$, we had to assume the condition (2·16), which can never be realized.

(2) As there appears coupling between various components $\langle\rho_k\rangle$'s of the electron density through the Hartree field, the calculation is limited to the second order of the scattering.

At the same time, we have shown that some of the effects obtained from the Hamiltonian (4·2) can be neglected in our problem. Therefore we start

in this section from a simplified Hamiltonian for which such effects are eliminated at the outset, and try to construct a new formalism of the problem by a canonical transformation, which does not require the condition (2·16), and is not limited to the second order.

In the treatment of the preceding section, the screening of the interaction between electrons and moving impurities arises from the Hartree term

$$\sum_{k\neq q} M(\boldsymbol{k})\langle\rho_k\rangle\rho_{-k}$$

in the Hamiltonian (4·3). As we determined there $\langle\rho_k\rangle$'s for $\boldsymbol{k}\neq\boldsymbol{q}$ as well as $\langle\rho_q\rangle$ in a self-consistent way, there occurred retardation and deformation of the screening. As has been shown, however, such effects are small and can be ignored, and so we have only to calculate the screened potential $\bar{U}(\boldsymbol{k})$ in the absence of the sound wave and replace $U(\boldsymbol{k})$ in $H_{eQ}^{(I)}$ by $\bar{U}(\boldsymbol{k})$, instead of adding the term $\sum_{k\neq q} M(\boldsymbol{k})\langle\rho_k\rangle\rho_{-k}$ to the Hamiltonian. Then the Hamiltonian we treat in this section is, in place of (4·2),

$$H_0+\bar{H}_I+H_{eQ}^{(0)}+\bar{H}_{eQ}^{(I)}+M(\boldsymbol{q})\langle\rho_q\rangle\rho_{-q}, \tag{5·1}$$

where $\langle\rho_q\rangle$ must be determined in a self-consistent way.

In (5·1), the electron-sound wave interaction $H_{eQ}^{(I)}$ can not be written in the linear form to Q unless the condition (2·15) is satisfied. To avoid this difficulty, we look for such a canonical transformation $\exp(iS)$ as eliminates $H_{eQ}^{(I)}$: i.e.

$$\exp(-iS)\,(\bar{H}_I+\bar{H}_{eQ}^{(I)})\,\exp(iS)=\bar{H}_I. \tag{5·2}$$

We can see that, if the coordinates of electrons are transformed to a new coordinate system oscillating together with the sound wave, $H_{eQ}^{(I)}$ is eliminated. The explicit form of this transformation can be easily found to be

$$S(t)=-\frac{m}{\hbar}[v_{-q}Q(t)+v_qQ^*(t)], \tag{5·3}$$

where

$$v_q=\frac{\hbar}{m}\sum_{\kappa\sigma}\left(\kappa_x-\frac{1}{2}q\right)a^*_{\kappa-q\sigma}\,a_{\kappa\sigma} \tag{5·4}$$

is the $\boldsymbol{q}$ component of the electron velocity operator in the direction of $\boldsymbol{e}_q$. This transformation is the same as was used by Blount[7] and Tsuneto,[8] but we shall discuss it in some detail in our formalism.

Let us show that (5·3) satisfies (5·2). In the case of impurity scattering, expanding (2·14a) we obtain

$$\bar{H}_I+\bar{H}_{eQ}^{(I)}=\sum_{\mu}\sum_{k}\sum_{n=0}^{\infty}\bar{U}(\boldsymbol{k})\frac{1}{n!}[i(k_x-nq)Q]^n\rho_{k-nq}\exp(-i\boldsymbol{k}\cdot\boldsymbol{R}_\mu).$$

We have also

$$\exp(iS)\,\bar{H}_I\exp(-iS)=\bar{H}_I-i[\bar{H}_I,\,S]+\frac{(-i)^2}{2!}[[\bar{H}_I,\,S],\,S]+\cdots$$

$$=\sum_{\mu}\sum_{k}\sum_{n=0}^{\infty}\bar{U}(\boldsymbol{k})\,\frac{1}{n!}\,[ik_xQ]^n\rho_{k-nq}\exp(-i\boldsymbol{k}\cdot\boldsymbol{R}_\mu),$$

using the relation

$$[\rho_k,\,v_{-q}]=(\hbar k_x/m)\,\rho_{k-q}.$$

Comparing the above, it is seen that terms for $k\gg q$ in the summation $\sum_k$ are nearly equal in two expressions. The most effective terms in the summation are those with k of the order a^{-1}. Therefore if the force range of the impurity is much smaller than the wavelength of the sound wave, or

$$qa\ll 1, \tag{5·5a}$$

the two expressions are nearly equal and we can say that the transformation (5·3) eliminates $H_{eQ}^{(I)}$.

Similarly, in the case of phonon scattering, the condition for such a transformation to be available is given by

$$q(\hbar v_s/k_BT)\ll 1. \tag{5·5b}$$

The discrepancy, which appears when (5·5) is not satisfied, comes from that $H_{eQ}^{(I)}$ in (2·14a) is written by the rigid ion model, while the transformation (5·3) eliminates that given by the deformable ion model. The difference between these two models disappears when (5·5) is satisfied.[14]

Now let us examine how terms other than $\bar{H}_{eQ}^{(I)}+\bar{H}_I$ are transformed. Among them H_0 becomes by the transformation

$$\exp(-iS)\,H_0\exp(iS)=H_0+i[H_0,\,S]+\frac{i^2}{2!}[[H_0,\,S],\,S]+\cdots$$

$$=H_0-i\frac{\hbar^2q}{m}\sum_{\kappa\sigma}\left\{\left(\kappa_x+\frac{1}{2}q\right)^2a^*_{\kappa+q\sigma}\,a_{\kappa\sigma}Q-\text{c.c.}\right\}$$

$$-i\frac{\hbar^2q^2}{m}\sum_{\kappa\sigma}\left\{\left(\kappa_x+\frac{1}{2}\,q\right)\left(\kappa_x+\frac{3}{2}q\right)a^*_{\kappa+2q\sigma}\,a_{\kappa\sigma}Q^2+\cdots\right\}+\cdots.$$

From this expression we see that the higher order terms proportional to Q^n $(n=2,\,3\cdots)$ can be ignored if the amplitude ξ of the sound wave is much smaller than the wavelength or

$$q\xi\ll 1. \tag{5·6}$$

Similarly it is shown that the higher order terms of Q arising from the transformation of $H_{eQ}^{(0)}$ and $M(\boldsymbol{q})\,\langle\rho_q\rangle_l\rho_{-q}$ can be neglected under the same condition.

Therefore the term which arises by the transformation and should be taken into account is the following one: i.e.

$$H_{eQ}^{(1)S}=i[H_0, S]+\hbar\frac{\partial S}{\partial t}=m\{[i\omega v_{-q}-\alpha_{-q}^{(0)}]Q(t)+[i\omega v_q-\alpha_q^{(0)}]Q^*(t)\}, \quad (5\cdot7)$$

where we have introduced an operator $\alpha_{\pm q}^{(0)}$ defined by

$$\alpha_{\pm q}^{(0)}=\frac{1}{i\hbar}[v_{\pm q}, H_0]=\mp iq\left(\frac{\hbar}{m}\right)^2\sum_{\kappa\sigma}\left(\kappa_x-\frac{1}{2}q\right)^2 a_{\kappa\mp q\sigma}^* a_{\kappa\sigma}. \quad (5\cdot8)$$

After all the Hamiltonian in the moving coordinate system is determined as follows:

$$H_0+\bar{H}_I+H_{eQ}^{(0)}+H_{eQ}^{(1)S}+M(\boldsymbol{q})\langle\rho_q\rangle\rho_{-q}. \quad (5\cdot9)$$

Let the transformed density matrix be

$$\sigma_S'(t)=\exp(-iS)\,\sigma'(t)\,\exp(iS). \quad (5\cdot10)$$

Then the motion of $\sigma_S'(t)$ is determined by the Hamiltonian (5·9) and if we expand $\sigma_S'(t)$ as

$$\sigma_S'(t)=\sigma+\Delta\sigma_S(t),$$

$\Delta\sigma_S(t)$ is obtained in the same way as in § 4. As is shown easily, the average $\langle X\rangle\equiv\mathrm{Tr}\,\Delta\sigma X$ is written in the moving coordinate system as

$$\langle X\rangle=\langle X^S\rangle_S+\mathrm{Tr}\,\sigma(X^S-X), \quad (5\cdot11)$$

with

$$X^S=\exp(-iS)\,X\exp(iS), \quad \langle Y\rangle_S=\mathrm{Tr}\,\Delta\sigma_S Y.$$

Now we proceed to the calculation of $\langle\partial H_{eQ}/\partial Q^*\rangle$ to the first order of Q using the expression (5·11). In the calculation of

$$\left\langle\frac{\partial H_{eQ}^{(0)}}{\partial Q^*}\right\rangle=inqV(\boldsymbol{q})\langle\rho_q\rangle, \quad (5\cdot12)$$

putting $X=\rho_q$ in (5·10), we have only to take ${\rho_q}^S\cong\rho_q$ in the first term and ${\rho_q}^S-\rho_q\cong i[\rho_q, S]$ in the second term. Higher terms are smaller under the conditions (5·5) and (5·6). Therefore

$$\langle\rho_q\rangle=\{-inqV(\boldsymbol{q})\langle\!\langle\rho_q|\rho_{-q}\rangle\!\rangle+m\langle\!\langle\rho_q|i\omega v_{-q}-\alpha_{-q}^{(0)}\rangle\!\rangle-inq\}Q+M(\boldsymbol{q})\langle\!\langle\rho_q|\rho_{-q}\rangle\!\rangle\langle\rho_q\rangle. \quad (5\cdot13)$$

By partial integration we can prove the relation

$$\langle\!\langle\rho_q|i\omega v_{-q}\rangle\!\rangle-iq\frac{n}{m}=\langle\!\langle\rho_q|\alpha_{-q}\rangle\!\rangle,$$

where the acceleration operator α_{-q} had been introduced by the definition

$$\alpha_{-q}=\dot{v}_{-q}=\frac{1}{i\hbar}[v_{-q}, H_0+\bar{H}_I]. \quad (5\cdot14)$$

Using the above relation, (5·13) can be rewritten as

$$\langle \rho_q \rangle = \{-inqV(\boldsymbol{q})\langle \rho_q | \rho_{-q} \rangle + \langle \rho_q | m\alpha_{-q}^{(I)} \rangle\} Q + M(\boldsymbol{q})\langle \rho_q | \rho_{-q} \rangle \langle \rho_q \rangle, \qquad (5\cdot15)$$

with

$$\alpha_{-q}^{(I)} = \alpha_{-q} - \alpha_{-q}^{(0)} = \frac{1}{i\hbar}[v_{-q}, \bar{H}_I]. \qquad (5\cdot16)$$

It is of some interest to note that, if we average (5·15) over the impurity distribution, we get the transport equation of the electronic current. Then the correspondence between (5·15) and the transport equation is, except for a constant factor, as follows:

$$-inqV(\boldsymbol{q})Q + M(\boldsymbol{q})\langle \rho \rangle \longrightarrow \textit{electric field},$$
$$\langle \rho_q | \rho_{-q} \rangle \longrightarrow \textit{conductivity},^{10)}$$
$$\langle \rho_q | m\alpha_{-q}^{(I)} \rangle Q \longrightarrow \textit{collision drag effect}.$$

To obtain $\langle \partial H_{eQ}^{(I)} / \partial Q^* \rangle$, we differentiate both sides of (5·2) with Q^*. Then

$$\exp(-iS)\frac{\partial \bar{H}_{eQ}^{(I)}}{\partial Q^*}\exp(iS) - i\left[\frac{\partial S}{\partial Q^*}, \bar{H}_I\right] = 0.$$

Hence

$$\exp(-iS)\frac{\partial \bar{H}_{eQ}^{(I)}}{\partial Q^*}\exp(iS) = \frac{m}{i\hbar}[v_q, \bar{H}_I] = m\alpha_q^{(I)}. \qquad (5\cdot17)$$

From (5·17) and (5·11), we get

$$\left\langle \frac{\partial \bar{H}_{eQ}^{(I)}}{\partial Q^*} \right\rangle = \langle m\alpha_q^{(I)} \rangle_S + \mathrm{Tr}\,\sigma\{m\alpha_q^{(I)} - \exp(iS)\, m\alpha_q^{(I)} \exp(-iS)\}$$
$$= \{-inqV(\boldsymbol{q})\langle m\alpha_q^{(I)} | \rho_{-q} \rangle + \langle m\alpha_q^{(I)} | m\alpha_{-q}^{(I)} \rangle\} Q + M(\boldsymbol{q})\langle m\alpha_q^{(I)} | \rho_{-q} \rangle \langle \rho_q \rangle. \qquad (5\cdot18)$$

Here we have derived the second expression using the relation

$$\langle \alpha_q^{(I)} | i\omega v_{-q} \rangle = -\frac{1}{i\hbar}\,\mathrm{Tr}\,\sigma[\alpha_q^{(I)}, v_{-q}] + \langle \alpha_q^{(I)} | \dot{v}_{-q} \rangle,$$

which is proved by partial integration, and further omitting the higher terms which arise from the second term of the first expression of (5·18), for they vanish when the average is taken over the distribution of scatterers.*

* Expanding the second term of the first expression of (5·18) with Q, we obtain

$$\sum_\mu \sum_k k_x^2 \bar{U}(\boldsymbol{k}) \exp(i\boldsymbol{k}\cdot\boldsymbol{R}_\mu) Q \left\{\mathrm{Tr}(\sigma\rho_k) + \frac{i}{2!}\,\mathrm{Tr}(\sigma\rho_{k-q})\, k_x Q + \cdots\right\},$$

where $\mathrm{Tr}\,\sigma\rho_k$ has the form, in the free electron approximation, as

$$\mathrm{Tr}\,\sigma\rho_k = \sum_\mu D_k \exp(-i\boldsymbol{k}\cdot\boldsymbol{R}_\mu), \quad D_k : \text{real}.$$

Therefore

$$\{\sum_\mu \mathrm{Tr}(\sigma\rho_k) \exp(i\boldsymbol{k}\cdot\boldsymbol{R}_\mu)\}_I = n_i D_k, \ \{\sum_\mu \mathrm{Tr}(\sigma\rho_{k-q}) \exp(i\boldsymbol{k}\cdot\boldsymbol{R}_\mu)\}_I = 0.$$

Solving $\langle\rho_q\rangle$ from (5·15), and inserting it in (5·12) and (5·18), we finally obtain

$$A=\frac{1}{q^2}\left\{\frac{[nqV(\boldsymbol{q})]^2\langle\rho_q|\rho_{-q}\rangle}{1-M(\boldsymbol{q})\langle\rho_q|\rho_{-q}\rangle}+\frac{inqV(\boldsymbol{q})[\langle\rho_q|m\alpha_{-q}^{(I)}\rangle-\langle m\alpha_q^{(I)}|\rho_{-q}\rangle]}{1-M(\boldsymbol{q})\langle\rho_q|\rho_{-q}\rangle}\right.$$
$$\left.+\left[\langle m\alpha_q^{(I)}|m\alpha_{-q}^{(I)}\rangle+\frac{M(\boldsymbol{q})\langle m\alpha_q^{(I)}|\rho_{-q}\rangle\langle\rho_q|m\alpha_{-q}^{(I)}\rangle}{1-M(\boldsymbol{q})\langle\rho_q|\rho_{-q}\rangle}\right]\right\}_I. \qquad (5\cdot19)$$

This expression corresponds to (4·30) in the preceding section, but is not restricted to the second order approximation. Moreover the condition to be required for the validity of the linear approximation is not (2·15) but (5·5) and (5·6).

It is of some interest to see how (5·19) is reduced in the second order approximation. If we rewrite (5·16) explicitly for the case of impurity scattering, we obtain

$$m\alpha_{\pm q}^{(I)}=\pm\sum_{\mu}\sum_{k} i(k_x-q)\,\bar{U}(\boldsymbol{k}-\boldsymbol{q})\exp[\pm i(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{R}_\mu]\rho_{\pm k}, \qquad (5\cdot20)$$

which is nearly equal to the coefficient of Q and Q^* in (2·16a) if the condition (5·5a) is satisfied. Using this expression it is easy to show that (5·19) is reduced to the (4·30) under the condition (5·5a). Though in § 4 calculations were carried out under the condition (2·16), it has been shown that the result is valid under the conditions (5·5) and (5·6), which are expected to be always satisfied.

§ 6. Summary and discussions

We have obtained the expression of the electronic contribution to the elastic constant as (5·19). It is not based on any *a priori* assumption on the collision term and is valid whether the mean free path of electrons may be longer than the wavelength of the sound wave or not. In this expression, effects of the moving scatterers are written by use of the operator $m\alpha_q^{(I)}=(1/ih)[mv_q, H_I]$, which is the force exerted by impurities to electrons. From the general theory of irreversible processes,[10] it is a natural result that the collision drag effect is given by the correlation of this force.

When the electron-impurity interaction is so weak that $ql\gg1$, the second order approximation is permitted. Then (5·19) is reduced to (4·30), which was calculated by the use of the linearized interaction Hamiltonian $H_{eQ}^{(0)}+H_{eQ}^{(I)}$.

Here we must refer to two points about the interaction $H_{eQ}^{(I)}$. The first is the problem of the screening. If the motion of the impurities is rapid, screening electrons can not follow the motion adiabatically, and there occur effects of the nonadiabatic screening. Furthermore, if there exist other forces exerted to electrons, for example, an external electric field, screening electrons are also affected by them. That is, the effective interaction potential between electrons

and moving impurities depend on these circumstances. In the present case, we have seen that these effects are numerically negligible. However, there may be cases such that these effects should be taken into account carefully.

The second problem is the linearization of this interaction. In order to obtain $H_Q^{(I)}$ in the form linear to the amplitude of the sound wave, it is required to assume the unrealistic condition (2·16), as was discussed in § 2. From the theoretical point of view, it is of some interest to see whether it is permitted to calculate the response under the assumption of an infinitesimal small disturbance in the theory of linear response, when the phenomenon is linear in macroscopic scale but not so in microscopic. In this case, our treatment in § 5 has shown that the results obtained under the condition (2·15) is valid beyond the condition.

The discrepancy between our result (4·30) and the semiclassical one given by Pippard and others was pointed out in § 4. Here we discuss this point in more detail.

Now let us calculate the transition probability for a phonon (of the ultrasonic wave) to be absorbed or emitted by the interaction Hamiltonian $H_Q^{(0)}+H_Q^{(I)}$. Then we must calculate the square of the matrix element, which is carried out as follows:

$$|(f|H_{eQ}^{(0)}+H_{eQ}^{(I)}|i)|^2=|(f|H_{eQ}^{(0)}|i)|^2+(f|H_{eQ}^{(0)}|i)(i|H_{eQ}^{(I)}|f)$$
$$+(f|H_{eQ}^{(I)}|i)(i|H_{eQ}^{(0)}|f)+|(f|H_{eQ}^{(I)}|i)|^2.$$

We can see that each term of this expression corresponds to that of (4·30) one to one. That is, the result (4·30) is essentially the same as is obtained in such an ordinary treatment. In the semiclassical treatment, if we expand the result with $(ql)^{-1}$, we can see that contributions from the terms corresponding to the last two terms in the above expression cancel each other in the order $(ql)^{-1}$. As a result, it is permissible to ignore the reaction effects of the collision drag. On the other hand, such a cancellation can not generally be expected in the above expression. In fact our calculation showed that only the last term contributes to the order $(ql)^{-1}$.

Though our treatment is quantum mechanical and that of Pippard semiclassical, we must remark that the discrepancy does not come from the quantum mechanical effect. As was mentioned in § 4, it comes from how the average was taken over the random distribution of impurities.

The problem which has been left to be discussed in future is whether our expression (5·20) agrees with that of Pippard in the case $ql\ll 1$. Tsuneto[8] discussed this problem, but he took the average over the impurity distribution on the stage of Eq. (5·15), which is equivalent to the assumption taken by Pippard. The more careful discussions are required.

Finally we will give some arguments on the magneto-acoustic effect. When a static magnetic field is present, we should modify the treatment taking into

account the Hall field. There is, however, no need to alter the general expression for the collision drag term, if only (2·4) is replaced by the Hamiltonian in a magentic field. Now we consider the magnetic field dependence of the last term in (4·30). This term is the sum of various contributions and each contribution has the form

$$\mathrm{Im}\frac{\langle \boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}}{1-M(\boldsymbol{k})\langle \boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}},$$

which is just the same as the attenuation of the sound wave with the wave vector $\boldsymbol{k}$ except for a constant factor. Therefore, even if each contribution shows any oscillatory behaviour, it may be smeared out by the summation over various $\boldsymbol{k}$'s. From this argument, it is reasonable to expect that the last term of (4·30) depends on the magnetic field monotonically, and that it does not contribute to the oscillatory behaviour of the magnetic field dependence of the attenuation constant though the magnitude of its contribution cannot be ignored. To obtain definite conclusion, however, further investigations are required.

The author wishes to express his sincere thanks to Professor R. Kubo, members of his laboratory and Dr. T. Tsuneto for valuable discussions.

Appendix

In this Appendix, we shall give the perturbational expansion of $\langle\rho_k|\rho_{-k'}\rangle$ to the first order of the electron-impurity interaction in the free electron approximation and estimate the magnitudes of A_1, A_2, A_2' and A_3 in § 4.

The zeroth order term of $\langle \boldsymbol{k}|\boldsymbol{k}'\rangle$ is given by

$$\langle \boldsymbol{k}|\boldsymbol{k}'\rangle^{(0)}=\lim_{\varepsilon\to+0}\frac{1}{i\hbar}\int_0^\infty dt\, e^{i\omega t-\varepsilon t}\,\mathrm{Tr}\,\sigma_0[\rho_k{}^0(t),\,\rho_{-k'}]$$

$$=\lim_{\varepsilon\to+0}\frac{1}{i\hbar}\int_0^\infty dt\, e^{i\omega t-\varepsilon t}2\sum_{\boldsymbol{\kappa}\boldsymbol{\kappa}'}f(\epsilon_\kappa)\{(\boldsymbol{\kappa}|e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}|\boldsymbol{\kappa}')\,(\boldsymbol{\kappa}'|e^{i\boldsymbol{k}'\cdot\boldsymbol{r}}|\boldsymbol{\kappa})\,\exp[i(\epsilon_\kappa-\epsilon_{\kappa'})\,t/\hbar]$$

$$-(\boldsymbol{\kappa}|e^{i\boldsymbol{k}'\cdot\boldsymbol{r}}|\boldsymbol{\kappa}')\,(\boldsymbol{\kappa}'|e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}|\boldsymbol{\kappa})\,\exp[i(\epsilon_{\kappa'}-\epsilon_\kappa)t/\hbar]\},\qquad(\mathrm{A}\cdot1)$$

where $f(\epsilon)$ is the Fermi distribution function and $(\boldsymbol{\kappa}|A|\boldsymbol{\kappa}')$ is the matrix element of A between the electronic states with the wave vectors $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}'$.

As can be seen easily, the expression (A·1) vanishes unless $\boldsymbol{k}=\boldsymbol{k}'$, and when $\boldsymbol{k}=\boldsymbol{k}'$, it becomes

$$\langle \boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}=2\lim_{\varepsilon\to+0}\sum_{\boldsymbol{\kappa}}\frac{f(\epsilon_\kappa)-f(\epsilon_{\kappa+k})}{\epsilon_\kappa-\epsilon_{\kappa+k}+\hbar\omega+i\varepsilon}.\qquad(\mathrm{A}\cdot2)$$

When $\omega\ll kv_F$, the real part is obtained approximately as

$$\mathrm{Re}\langle \boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}=2\sum_{\boldsymbol{\kappa}}\frac{f(\epsilon_\kappa)-f(\epsilon_{\kappa+k})}{\epsilon_\kappa-\epsilon_{\kappa+k}}.\qquad(\mathrm{A}\cdot3)$$

Especially when $k \ll k_F$, it becomes

$$\mathrm{Re}\langle \boldsymbol{k}|\boldsymbol{k}\rangle^{(0)} = \frac{3n}{2\epsilon_F}. \tag{A·4}$$

The imaginary part of (A·2) is given by

$$\mathrm{Im}\langle \boldsymbol{k}|\boldsymbol{k}\rangle^{(0)} = -2\pi\omega \sum_{\kappa} \left(\frac{\partial f}{\partial \epsilon_\kappa}\right) \delta\left[\omega - \frac{1}{\hbar}(\epsilon_{\kappa+k} - \epsilon_\kappa)\right]$$

$$= \begin{cases} -\dfrac{\pi}{2}\dfrac{3n}{2\epsilon_F}\dfrac{\omega}{kv_F}, & \text{for} \quad k < 2k_F, \\ 0, & \text{for} \quad k > 2k_F. \end{cases} \tag{A·5}$$

Proceeding to the calculation of the first order, we obtain

$$\langle \boldsymbol{k}|\boldsymbol{q}\rangle^{(1)} = \langle \boldsymbol{q}|\boldsymbol{q}\rangle^{(0)} \delta_{kq} + \sum_{\mu} \bar{U}(\boldsymbol{q}-\boldsymbol{k})\, F(\boldsymbol{k}, \boldsymbol{q}) \exp[i(\boldsymbol{q}-\boldsymbol{k})\cdot \boldsymbol{R}_\mu], \tag{A·6}$$

where $F(\boldsymbol{k}, \boldsymbol{q})$ is given in the free electron approximation by

$$F(\boldsymbol{k}, \boldsymbol{q}) = \lim_{\varepsilon \to +0} 2 \sum_{\kappa} \left\{ \frac{1}{\epsilon_{\kappa+k} - \epsilon_{\kappa+q}} \left[\frac{f(\epsilon_\kappa) - f(\epsilon_{\kappa+k})}{\epsilon_\kappa - \epsilon_{\kappa+k} + \hbar\omega + i\varepsilon} - \frac{f(\epsilon_\kappa) - f(\epsilon_{\kappa+q})}{\epsilon_\kappa - \epsilon_{\kappa+q} + \hbar\omega + i\varepsilon} \right] \right.$$

$$\left. + \frac{1}{\epsilon_{\kappa-k} - \epsilon_{\kappa-q}} \left[\frac{f(\epsilon_{\kappa-k}) - f(\epsilon_\kappa)}{\epsilon_{\kappa-k} - \epsilon_\kappa + \hbar\omega + i\varepsilon} - \frac{f(\epsilon_{\kappa-q}) - f(\epsilon_\kappa)}{\epsilon_{\kappa-q} - \epsilon_\kappa + \hbar\omega + i\varepsilon} \right] \right\}. \tag{A·7}$$

The calculation of this expression can be carried out in an elementary way, though it is rather tedious. The result is as follows.

When $k < 2k_F$ and $q < 2k_F$, the imaginary part is given by

$$\mathrm{Im}\, F(\boldsymbol{k}, \boldsymbol{q}) = -\frac{\pi\omega}{\hbar}\frac{3n}{2\epsilon_F}\frac{1}{qkv_F^2} \left\{ \frac{S_{\theta_1\theta_1'}(\cos\theta)}{\sqrt{[\cos\theta - \cos(\theta_1+\theta_1')][\cos\theta - \cos(\theta_1-\theta_1')]}} \right.$$

$$\left. - \frac{S_{\theta_2\theta_2'}(\cos\theta)}{\sqrt{[\cos\theta - \cos(\theta_2+\theta_2')][\cos\theta - \cos(\theta_2-\theta_2')]}} \right\}, \tag{A·8}$$

with

$$\theta = \widehat{\boldsymbol{k}\boldsymbol{q}},$$

$$\cos\theta_1 = \frac{\omega}{kv_F} - \frac{k}{2k_F}, \quad \cos\theta_2 = \frac{\omega}{kv_F} + \frac{k}{2k_F},$$

$$\cos\theta_1' = \frac{v_s}{v_F} - \frac{q}{2k_F}, \quad \cos\theta_2' = \frac{v_s}{v_F} + \frac{q}{2k_F}.$$

Here the factor $S_{\theta\theta'}(\cos\theta)$ takes the value $+1$ or 0, and is shown in Fig. 1. In the later calculation, it is required to obtain the average of (A·8) over the direction of $\boldsymbol{k}$. For $q \ll k \ll k_F$, the averages become

$$\{\mathrm{Im}F(\boldsymbol{k}, \boldsymbol{q})\}_{\mathrm{AV}} \simeq \frac{\pi\omega}{\hbar}\frac{3n}{2\epsilon_F}\frac{1}{kqv_F^2}\frac{q}{k_F}, \tag{A·9a}$$

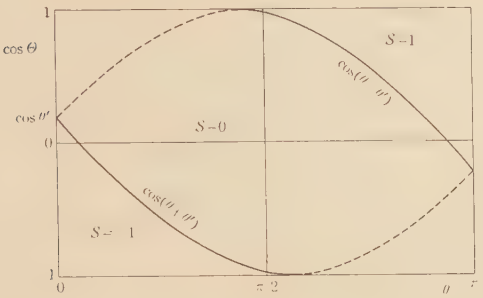

Fig. 1.

$$\{\mathrm{Im}F(\boldsymbol{k},\boldsymbol{q})\cos\theta\}_{\mathrm{AV}}\cong\frac{\pi\omega}{\hbar}\frac{3n}{2\epsilon_F}\frac{1}{kqv_F^2}\frac{2k_Fq}{k^2}. \tag{A·9b}$$

The calculation of the real part cannot be carried out analytically in a general case, so we do it for the case $k\ll k_F$. Then we obtain

$$\mathrm{Re}F(\boldsymbol{k},\boldsymbol{q})\cong-\frac{\omega}{\hbar}\frac{3n}{2\epsilon_F}\frac{2}{kqv_F^2}. \tag{A·10}$$

Now we estimate the magnitudes of the imaginary part of A_1, A_2 and A_3 in § 4, using the above results. By the use of $F(\boldsymbol{k},\boldsymbol{q})$ defined above, they are rewritten as

$$\mathrm{Im}A_1\cong-\frac{1}{q^2}\sum_{\boldsymbol{k}\mp\boldsymbol{q}}[n\boldsymbol{q}\bar{V}(\boldsymbol{q})]^2n_i[\bar{U}(\boldsymbol{k}-\boldsymbol{q})]^2\frac{\mathrm{Im}F(\boldsymbol{q},\boldsymbol{k})\,F(\boldsymbol{k},\boldsymbol{q})}{\mathrm{Re}\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(0)}}, \tag{A·11a}$$

$$\mathrm{Im}A_2\cong\frac{1}{q^2}\sum_{\boldsymbol{k}\mp\boldsymbol{q}}[n\boldsymbol{q}\bar{V}(\boldsymbol{q})]n_ik\bar{U}(\boldsymbol{k})\,\bar{U}(\boldsymbol{q}-\boldsymbol{k})\cos\theta\,\mathrm{Im}F(\boldsymbol{q},\boldsymbol{k}), \tag{A·11b}$$

$$\mathrm{Im}A_3\cong\frac{1}{q^2}\sum_{\boldsymbol{k}}n_i[k\bar{U}(\boldsymbol{k})]^2\cos^2\theta\,\mathrm{Im}\langle\boldsymbol{k}|\boldsymbol{k}\rangle^{(0)}. \tag{A·11c}$$

To estimate the orders of magnitude, we assume that the force range of the impurity potential is of the order k_F^{-1}, and further put $U(\boldsymbol{k})=U=\mathrm{const}$ for $k<k_F$. Moreover, we use the expressions (A·4), (A·9) and (A·10) for $F(\boldsymbol{q},\boldsymbol{k})$ and $\langle\boldsymbol{q}|\boldsymbol{q}\rangle^{(0)}$, though they are valid only for $k\ll k_F$. Then (A·11) are calculated as follows:

$$\mathrm{Im}A_1=-\left(\frac{n_i}{\hbar}|U|^2\frac{3n}{2\epsilon_F}\right)\cdot\frac{\pi n\hbar v_s^2}{3v_F^2}, \tag{A·12a}$$

$$\mathrm{Im}A_2=\left(\frac{n_i}{\hbar}|U|^2\frac{3n}{2\epsilon_F}\right)\cdot\frac{2\pi n\hbar v_s}{v_F}, \tag{A·12b}$$

$$\mathrm{Im}A_3=-\left(\frac{n_i}{\hbar}|U|^2\frac{3n}{2\epsilon_F}\right)\cdot\frac{4nmv_s}{q}. \tag{A·12c}$$

The relaxation time of conduction electrons is given by

$$\frac{1}{\tau}=\frac{2\pi}{\hbar}n_i\sum_{k}|\overline{U}(\boldsymbol{k})|^2(1-\cos\theta)\,\delta(\epsilon_{\kappa}-\epsilon_{\kappa+k}),\quad\text{where }\theta=\widehat{\boldsymbol{\kappa}\,\boldsymbol{\kappa}}+\boldsymbol{k},\qquad(\text{A}\cdot13)$$

and it is calculated in the same approximation as above as

$$\frac{1}{\tau}\cong\frac{\pi}{\hbar}n_i|U|^2\frac{3n}{2\epsilon_F}.$$

Then, making use of the expressions (4·18) for $\mathrm{Im}\,\mathrm{A}^{(0)}$ and putting $v_F\tau=l$, we obtain the results (4·29).

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Subtraction in the Chew-Mandelstam Equation and Determination of the Low-Energy Scattering Amplitude

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The subtracted Chew-Mandelstam equation is studied in the approximation that the contributions from the unphysical branch cut are neglected. The following result is found: When the cut-off function is introduced into the Chew-Mandelstam equation, the subtracted Chew-Mandelstam equation has a new kind of solutions which are not the solutions of the unsubtracted equation. On the other hand, any new solutions are not obtained in the no cut-off case and the subtracted equation becomes equivalent to the unsubtracted one. It is shown in the cut-off case that the new solution has a very different feature from those of the solutions of the unsubtracted equation when the repulsive interaction is introduced by means of the subtraction and that the new solution has at least one zero on the real axis. The dynamical and kinematical resonances appearing in the solutions are also discussed.

§ 1. Introduction

During the last few years; the analytic properties of the scattering amplitude, as a function of energy or momentum transfer, have been deeply studied[1] from the viewpoint of the dispersion theory. The argument is based on the axioms; that is, the Lorentz invariance, the microcausality, the asymptotic condition, and the mass-spectral condition. Using the results of these studies, Mandelstam[2] conjectured the double dispersion relation for the scattering amplitude and gave the program of determining the scattering amplitudes by combining the unitarity relation with this dispersion relation. This program was carried out on the pion-pion scattering problem by Chew and Mandelstam[3] and they obtained the coupled non-linear integral equations from which one could determine the partial wave amplitudes except for some kind of ambiguities. While the Hamiltonian formalism necessitates the complete knowledge of the underlying dynamics from the beginning, the dynamics of the system is not completely given in the Chew-Mandelstam (C–M) theory, because we use only certain fundamental axioms which the amplitude should satisfy. The above mentioned ambiguities correspond to this fact. For the study in the elementary particle physics, this feature of the C–M theory may, however, be particularly appropriate and it is very important to see what kinds of ambiguities arise in the C–M theory.

There are two kinds of these ambiguities: The C–M equation has a similar structure to that of the Chew-Low equation. The latter equation has not a unique solution.[4] This ambiguity is related to the free fields for the unstable particles.[5] Therefore, we have a similar situation for the C–M equation. The other kind of the ambiguity is related to the subtractions for the dispersion relation. One subtraction introduces an arbitrary constant into the C–M equation and consequently we have a number of C–M equations which have different forms from each other, according to the number of the subtractions performed.

In the well-known applications of the dispersion relations as the semi-phenomenological analysis of the experimental data, the number of the subtractions is determined in consideration of the high-energy behaviour of the empirical total cross section. However, we impose usually the low-energy approximation on the C–M equation. In this case, the high-energy behaviour of the solution of the approximate C–M equation has no direct connection with that followed from the empirical knowledge. Furthermore, the above mentioned method gives us only the knowledge about the minimum subtractions, and there is no reason for prohibiting further subtractions. We therefore have not any method in deciding the number of the subtractions *apriori*, when we stand on the viewpoint of the C–M theory. In order to get a definite conclusion for this situation, it may be very important to study how the subtraction changes the low-energy behaviour of the solutions. This problem is the main object of the present paper.

We shall find in § 2 the following result: If the cut-off function is introduced into the C–M equation, the subtracted equation of course has the solutions of new kind which are different from any solutions of the unsubtracted equation. However, these new solutions cannot be obtained in the no cut-off case, and both equations, subtracted and unsubtracted ones, become equivalent to each other. We shall further discuss in § 2 and § 3 some properties of these new solutions for the cut-off theory.

§ 2. The solutions of the subtracted Chew-Mandelstam equation

For brevity, we shall consider a special scattering system in which all four particles in the initial and final states are spinless neutral and have the same mass μ and for which the scattering amplitude does not show the complicated structure such as the anomalous threshold. With the aid of Mandelstam's conjecture, we get the dispersion relation for the s-wave scattering amplitude $F(\nu)$:[3]

$$F(\nu)=\frac{a_0}{\nu+\alpha}-\frac{1}{\pi}\int_0^{\infty}d\nu'\,\frac{\mathrm{Im}\,F(\nu')}{\nu-\nu'}-\frac{1}{\pi}\int_{-\infty}^{-\beta}d\nu'\,\frac{\mathrm{Im}\,F(\nu')}{\nu-\nu'}\,, \qquad (2\cdot1)^*$$

* The amplitude $F(\nu)$ is expressed in terms of the s-wave phase shift as $F(\nu)=-\sqrt{\frac{\nu+\mu^2}{\nu}}\times e^{i\delta}\sin\delta$.

where ν is the square of the three-momentum in the center of mass system and we have assumed the existence of a pole term.

The C–M equation for s-wave amplitude is obtained from this dispersion relation in the following way. The second term of the right-hand side of (2·1) can be modified by using the unitarity condition

$$\mathrm{Im}\, F(\nu) = -\sqrt{\frac{\nu}{\nu+\mu^2}}\,|F(\nu)|^2 \cdot R(\nu), \tag{2·2}$$

where $R(\nu)$ is the ratio of the total to the elastic s-wave cross section, i.e.

$$R(\nu) = \sigma^s_{total}(\nu)/\sigma^s_{elas}(\nu), \tag{2·3}$$

and it is possible for us to put $R(\nu)$ equal to unity in the case of low-energy scattering. The last term of the right-hand side of (2·1) is the contribution from the unphysical region. The knowledge about it is obtained by using the crossing symmetry and the unitarity condition. Through this term, the C–M equation of the s-wave amplitude is coupled with other partial wave amplitudes.

The resulting C–M equations are too complicated to solve analytically. Therefore, we shall neglect the contribution from the unphysical region as a first approximation and get the following uncoupled equation,

$$F(\nu) = \frac{a_0}{\nu+\alpha} + \frac{1}{\pi}\int_0^\infty d\nu' \sqrt{\frac{\nu'}{\nu'+\mu^2}}\,\frac{|F(\nu')|^2}{\nu-\nu'}. \tag{2·4}$$

We can easily have the exact solution for this equation. The disregard of the contribution from the unphysical region causes the strong violation of the crossing symmetry but this fact is not a matter of importance for our purpose.

By changing the variable $\nu+\alpha\to\nu$, Eq. (2·4) reduces to the following form,

$$F(\nu) = \frac{a_0}{\nu} + \frac{1}{\pi}\int_\alpha^\infty d\nu' \cdot K(\nu') f(\nu')\,\frac{|F(\nu')|^2}{\nu-\nu'}, \tag{2·5}$$

where

$$K(\nu) \equiv \sqrt{\frac{(\nu-\alpha)}{(\nu-\alpha)+\mu^2}} \tag{2·6}$$

and we artificially introduced the cut-off function $f(\nu)$, in order to find the important properties of the C–M equation mentioned in the last paragraph of § 1. The subtracted form of (2·5) becomes

$$F(\nu) = a_0\left(\frac{1}{\nu}-\frac{1}{\nu_0}\right) + a_1 + \frac{(\nu_0-\nu)}{\pi}\int_\alpha^\infty d\nu' K(\nu') f(\nu')\,\frac{|F(\nu')|^2}{(\nu-\nu')(\nu_0-\nu')}, \tag{2·7}$$

where we assume $\nu_0 < \alpha$, and the new constant $a_1 \equiv F(\nu_0)$ becomes real.

Our first task is to find the general solution of (2·7), and we shall employ a similar method to that used in the case of Chew-Low equation.[4] Let us consider

the functions $F(z)$ and $X(z)$ of complex variable z defined by

$$F(z)=a_0\left(\frac{1}{z}-\frac{1}{\nu_0}\right)+a_1+\frac{(\nu_0-z)}{\pi}\int_\alpha^\infty d\nu' K(\nu') f(\nu') \frac{|F(\nu')|^2}{(z-\nu')(\nu_0-\nu')}, \quad (2\cdot8)$$

and

$$X(z)\equiv\frac{1}{F(z)}. \quad (2\cdot9)$$

The physical amplitude $F(\nu)$ is the boundary value of this analytic function $F(z)$ when z approaches to the real axis from the upper half plane. For the time being, we shall restrict ourselves to the case $a_0>0$. The function $X(z)$ is the generalized R-function in this case and from the Herglotz theorem,[6] we get the following general form of $X(z)$, [see Appendix 1]:

$$X(z)=\frac{z}{a_0}-\frac{z^2}{\pi}\int_\alpha^\infty d\nu' K(\nu') f(\nu') \frac{1}{\nu'^2(z-\nu')}+\sum_m \frac{\alpha_m z^2}{E_m^2(E_m-z)}, \quad (2\cdot10)$$

where α_m and E_m are real constants which satisfy the conditions*

$$\alpha_m\geq 0, \quad (2\cdot11)$$

$$\frac{1}{a_0}-\frac{1}{\pi}\int_\alpha^\infty d\nu' K(\nu') f(\nu') \frac{1}{\nu'^2}-\sum_m \frac{\alpha_m}{E_m^2}\geq 0. \quad (2\cdot12)$$

The subtraction in (2·7) imposes the restriction

$$\frac{1}{a_1}=X(\nu_0)\equiv\frac{\nu_0}{a_0}-\frac{\nu_0^2}{\pi}\int_\alpha^\infty d\nu' \frac{K(\nu') f(\nu')}{\nu'^2(\nu_0-\nu')}+\sum_m \frac{\alpha_m \nu_0^2}{E_m^2(E_m-\nu_0)}. \quad (2\cdot13)$$

In order to have the general solution of (2·7), it is necessary to discuss the asymptotic behaviour of the function $F(\nu)$ or $X(\nu)$ for large ν. Let us treat this problem separatly for the following two cases, i.e. the cut-off theory and the no cut-off theory.

A. The cut-off theory

We assume that the cut-off function $f(\nu)$ has a simple form:**

$$\begin{cases} f(\nu)=0 & \text{for } \nu>\nu_{max}, \\ f(\nu)=1 & \text{for } \nu<\nu_{max}, \end{cases}$$

* If $1/a_0-1/\pi\int_\alpha^\infty d\nu' K(\nu') f(\nu')/\nu'^2<0$, the solution of (2·7) never exists. This fact corresponds to the appearance of the ghost state. It is easily seen that we have no ghost pole for the wide range of the coupling constant a_0 even in the case of no cut-off, i.e. $f(\nu)=1$.

** We can see that the result in this subsection A is held for the case of the smooth cut-off, as the second term of the right-hand side of (2·17) still converges in the limit $\nu\to\infty$.

and it is obtained that

$$\int_{\alpha}^{\infty} d\nu' K(\nu') f(\nu') \frac{|F(\nu')|^2}{(\nu-\nu')(\nu_0-\nu')} = \frac{A}{\nu} + \frac{B}{\nu^2} + 0\left(\frac{1}{\nu^3}\right), \qquad (2\cdot14)$$

by virtue of the existence of the cut-off factor. By inserting this result into (2·7), the right-hand side becomes

$$\left(-\frac{a_0}{\nu_0} + a_1 - \frac{A}{\pi}\right) + \left(a_0 + \frac{\nu_0 A}{\pi} - \frac{B}{\pi}\right)\frac{1}{\nu} + 0\left(\frac{1}{\nu^2}\right).$$

If the first contribution vanishes,

$$\text{(case (1))} \qquad F(\nu) \sim \frac{1}{\nu} \times \text{const., as } \nu \to \infty, \qquad (2\cdot15)$$

otherwise we have

$$\text{(case (2))} \qquad F(\nu) \sim \text{const., as } \nu \to \infty. \qquad (2\cdot16)$$

These are the conditions which the general solution of (2·7) must satisfy. Expand the right-hand side of (2·10) in powers of $(1/z)$, i.e.

$$X(z) = \left[\frac{1}{a_0} - \frac{1}{\pi}\int_{\alpha}^{\infty} d\nu' \frac{K(\nu') f(\nu')}{\nu'^2} - \sum_m \frac{\alpha_m}{E_m^2}\right] z$$

$$+ \left[-\frac{1}{\pi}\int_{\alpha}^{\infty} d\nu' \frac{K(\nu') f(\nu')}{\nu'} - \sum_m \frac{\alpha_m}{E_m}\right] + 0\left(\frac{1}{z}\right), \qquad (2\cdot17)$$

and we immediately see that the asymptotic conditions (2·15) and (2·16) are automatically satisfied by the function $X(z)$ for the cut-off theory. Thus we can get the general solution of (2·7). If the coefficient of z in the first term of the right-hand side of (2·17) does not vanish, the corresponding $F(\nu)$ has the asymptotic behaviour of (2·15) [case (1)]. If it does vanish, that is,

$$\frac{1}{a_0} - \frac{1}{\pi}\int_{\alpha}^{\infty} d\nu' \frac{K(\nu') f(\nu')}{\nu'^2} - \sum_m \frac{\alpha_m}{E_m^2} = 0, \qquad (2\cdot18)$$

the asymptotic behaviour of $F(\nu)$ behaves as the case (2) :

$$\lim_{\nu\to\infty} F(\nu) = \left[-\frac{1}{\pi}\int_{\alpha}^{\infty} d\nu' \frac{K(\nu') f(\nu')}{\nu'} - \sum_m \frac{\alpha_m}{E_m}\right]^{-1} \equiv C. \qquad (2\cdot19)$$

We shall hereafter call the solution of the former type the *first kind* solution and that of the latter type the *second kind* one.

Consider the unsubtracted equation (2·5). By using the same method as in the case of the subtracted equation, it is seen that the solution of the unsub-

tracted equation also has the general form (2·10). However, the asymptotic behaviour of the general solution of (2·5) for the cut-off theory is only that of the case (1), because of no subtraction. From this fact, we get an important result that *the first kind solutions of the subtracted equation (2·7) coincide with the solutions of the unsubtracted one (2·5), and the second kind ones are new solutions appearing in virtue of the subtraction.* It may be considered that these new solutions give the amplitudes for the scattering due to the different interactions from that considered in the beginning. This means that, in the cut-off theory, it is possible to introduce the new interaction into the system by means of the subtraction.

In order to study the dynamics concerning the second kind solutions, it should be noted that the second kind solutions of (2·7),

$$F(\nu)=a_0\left(\frac{1}{\nu}-\frac{1}{\nu_0}\right)+a_1+\frac{(\nu_0-\nu)}{\pi}\int_\alpha^\infty d\nu' K(\nu')f(\nu')\frac{|F(\nu')|^2}{(\nu-\nu')(\nu_0-\nu')}, \quad (2\cdot7)$$

satisfy another unsubtracted equation

$$F(\nu)=a_0\frac{1}{\nu}+C+\frac{1}{\pi}\int_\alpha^\infty d\nu' K(\nu')f(\nu')\frac{|F(\nu')|^2}{\nu-\nu'}, \quad (2\cdot20)$$

where C is given by (2·19). This fact can be verified by solving (2·20), but we only note here that the subtracted form of (2·20) also coincides with (2·7) and that (2·20) requires the asymptotic behaviour of (2·16) [case (2)]. Eq. (2·20) differs from (2·5) in the existence of the constant term C, which represents the contribution from the new interaction introduced by means of the subtraction.

Before treating the no cut-off case, let us discuss some important property of the simplest second kind solution which is very interesting from the viewpoint of practical use. In order to satisfy the condition (2·18), we cannot take all of a_m's to be zero, except for the special case as $a_0=(1/\pi)\int_\alpha^\infty d\nu' K(\nu')f(\nu')/\nu'^2$, which we leave out of consideration here. The simplest solution of the second kind has one zero at $\nu=E$, i.e.

$$F(\nu)=\left[\frac{\nu}{a_0}-\frac{\nu^2}{\pi}\int_\alpha^\infty d\nu' K(\nu')f(\nu')\frac{1}{\nu'^2(\nu-\nu')}+\frac{\alpha\nu^2}{E(E-\nu)}\right]^{-1}, \quad (2\cdot21)$$

and is uniquely determined, since the parameters α and E are now fixed by the conditions (2·13) and (2·18):

$$\alpha=\bar{\alpha}\equiv\left[\frac{1}{a_0}-\frac{1}{\pi}\int_\alpha^\infty d\nu' K(\nu')f(\nu')\frac{1}{\nu'^2}\right]\bar{E}^2 \quad (2\cdot22)$$

and

$$E=\bar{E}=\frac{1}{a_0}-\frac{1}{\pi}\int_\alpha^\infty d\nu' K(\nu') f(\nu')\frac{1}{\nu'^2} - \frac{1}{a_0\nu_0}+\frac{1}{a_1\nu_0^2}+\frac{1}{\pi}\int_\alpha^\infty d\nu' K(\nu') f(\nu')\frac{1}{\nu'^2(\nu_0-\nu')} \qquad +\nu_0. \quad (2\cdot23)$$

The first kind solution which has one zero is expressed also by the same expression as (2·21) but the parameters α and E have the different values from $\bar{\alpha}$ and $\bar{E}$, respectively. These parameters α and E of the first kind solution are not uniquely determined in contrast to the case of the simplest second kind solution, because we have not such a condition for α/E^2 as (2·18). We, however, see that the first kind solution exists only for the case [the proof will be given in Appendix 2.]

$$\alpha/E\geq 0. \qquad (2\cdot24)$$

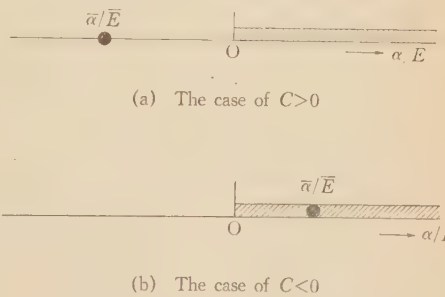
On the other hand, for the simplest second kind solution, the following result is obtained [see Appendix 2]:

$$\bar{\alpha}/\bar{E}<0 \quad \text{for} \quad C>0$$

and

$$\bar{\alpha}/\bar{E}>0 \quad \text{for} \quad C<0.$$

Thus the simplest second kind solution still exists when the $\bar{\alpha}$ and $\bar{E}$ have the negative ratio, i.e. $\bar{\alpha}/\bar{E}<0$. The situation is illustrated in Fig. 1 for the two typical cases of the simplest second kind solution. The shaded region represents the domain in which the first kind solutions (2·21) exist. Fig. 1a corresponds to the case $C>0$ and Fig. 1b to $C<0$. It is seen that the simplest second kind solution for $C>0$ is isolated from the first kind ones, and therefore, this second kind solution may have a very different feature from those of the first kind ones. In § 3, we shall find that the positive C corresponds to the repulsive interaction and the negative C to the attractive one. From these facts, it may be expected that the subtraction essentially changes the dynamical property of the original system when the subtraction introduces the repulsive interaction ($C>0$). Contrary to this case, when $C<0$, the first kind solution continuously tends to the simplest second kind one in the limit of $\alpha\to\bar{\alpha}$ and $E\to\bar{E}$. Therefore, in this case, the significance of the subtraction may considerably decrease.



(a) The case of $C>0$

(b) The case of $C<0$

Fig. 1. The domain of the solution. The shaded region represents the domain in which the first kind solutions (2·21) exist. The black point shows $\bar{\alpha}/\bar{E}$ for the simplest second kind solution.

When we take into account the contribution from the unphysical region, both functions $F(z)$ and $zF(z)$ are not the generalized R-functions. Thus we cannot use the same method as has been employed. However, the result obtained

in the last paragraph reflects the following general situation: The non-uniqueness of the parameters α and E for the first kind solution corresponds to the Castillejo-Dalitz-Dyson ambiguity[4] in the Chew-Low static theory as was mentioned in the Introduction. So the first kind solution of (2·21) contains the contribution from the interaction which is related to the unstable particle.[5] Our condition (2·24) just reflects this last fact. Hence, this condition (2·24) has a wider range of validity, because the above situation is sufficiently general, and the result obtained in the last paragraph is presumably valid for the case of the exact C–M equation.

B. *The no cut-off theory*

In the case of $f(\nu)\equiv 1$, the C–M equation becomes

$$F(\nu)=a_0\left(\frac{1}{\nu}-\frac{1}{\nu_0}\right)+a_1+\frac{(\nu_0-\nu)}{\pi}\int_\alpha^\infty d\nu' K(\nu')\frac{|F(\nu')|^2}{(\nu-\nu')(\nu_0-\nu')}. \qquad (2\cdot 7)'$$

The asymptotic behaviour of the integral

$$\int_\alpha^\infty d\nu' K(\nu')\frac{|F(\nu')|^2}{(\nu-\nu')(\nu_0-\nu')},$$

for large ν depends on the unknown high-energy behaviour of the solution $F(\nu)$ itself. We shall, therefore, discuss the possible high-energy behaviour expected from the expression (2·10). This expression is written as

$$X(\nu)=\left[\frac{1}{a_0}-\frac{1}{\pi}\int_\alpha^\infty d\nu' K(\nu')\frac{1}{\nu'^2}-\sum_m\frac{\alpha_m}{E_m^2}\right]\nu$$

$$+\left[\frac{1}{\pi}\int_\alpha^\infty d\nu' K(\nu')\frac{\nu}{\nu'(\nu'-\nu)}+\sum_m\frac{\alpha_m\nu}{E_m(E_m-\nu)}\right]. \qquad (2\cdot 25)$$

Contrary to the cut-off case, the second term in the right-hand side diverges logarithmically as $\nu\to\infty$. Therefore, we can expect the following two cases as possible high-energy behaviours of $F(\nu)$:

$$\text{(case (1))} \qquad F(\nu)\sim\frac{1}{\nu}\times\text{const., as } \nu\to\infty \qquad (2\cdot 26)$$

and

$$\text{(case (2))} \qquad F(\nu)\sim\frac{1}{\log\nu}\times\text{const., as } \nu\to\infty. \qquad (2\cdot 27)$$

The latter case occurs when the condition (2·18) is satisfied. From these conditions, it is seen that every solution tends to zero as $\nu\to\infty$, in contrast to the cut-off theory. The solution of which high-energy behaviour is constant is never obtained, because the high-energy behaviour of the integral term in the C–M equation becomes independent of the number of the subtractions. This special

feature leads us to the peculiar result which will be shown below.

Let us study the consistency of the conditions (2·26) and (2·27) with the original equation (2·7). Since the problem is trivial as to the imaginary part of $F(\nu)$, we shall consider only the real part, and get the following results for the dispersion integral term of (2·7) [the derivation will be given in Appendix 3]:

$$\frac{(\nu_0-\nu)}{\pi}\,\mathcal{P}\int_\alpha^\infty d\nu'\,\frac{K(\nu')\,|F(\nu')|^2}{(\nu-\nu')\,(\nu_0-\nu')}$$

$$=-\frac{1}{\pi}\int_\alpha^\infty d\nu'\frac{K(\nu')\,|F(\nu')|^2}{\nu_0-\nu'}+\frac{1}{\pi}\,\mathcal{P}\int_\alpha^\infty d\nu'\frac{K(\nu')\,|F(\nu')|^2}{\nu-\nu'},$$

$$\sim\begin{cases}M+0(1/\nu)+0(\log\nu/\nu^2)+\cdots, & \text{for the case (1)}, \qquad (2\cdot28)\\ M+0(1/\log\nu)+0(1/(\log\nu)^3)+\cdots, & \text{for the case (2)}, \qquad (2\cdot29)\end{cases}$$

where M is the constant given by

$$M=-\frac{1}{\pi}\int_\alpha^\infty d\nu'\,\frac{K(\nu')\,|F(\nu')|^2}{\nu_0-\nu'}\,.$$

Here we have used the fact that the unsubtracted form of the dispersion integral [see (2·5)] converges by virtue of the above-mentioned special high-energy behaviour. Inserting (2·28) or (2·29) into (2·7)′, we can see that the right-hand side of (2·7) has the behaviour (2·26) or (2·27), respectively, if and only if $(a_0/\nu_0)-a_1=M$. Therefore, the subtracted equation is reduced to the unsubtracted one and the no cut-off theory does not produce any new solution, even if the subtraction is performed.

Finally, let us give the remarks for the case $a_0<0$. In this case, $F(z)$ is not the generalized R-function. However, the general form (2·10) can be obtained by using the property that $zF(z)$ is still the generalized R-function in each case of the equations (2·5) and (2·7) for $C<0$. The restriction (2·11) now vanishes but the argument in this section 2 is also valid.

§ 3. The physical meaning of the second kind solution

In order to clarify the dynamics implied in the second kind solution of the cut-off theory, we shall discuss the modified Lee model whose Hamiltonian is defined by

$$H=H_0+H_I^{(1)}+H_I^{(2)}, \qquad (3\cdot1)$$

$$H_0=\sum_{\boldsymbol{p}} m\psi_V^{+}(\boldsymbol{p})\psi_V(\boldsymbol{p})+\sum_{\boldsymbol{p}} m\psi_N^{+}(\boldsymbol{p})\,\psi_N(\boldsymbol{p})+\sum_k \omega_k a_k^{+}a_k, \qquad (3\cdot2)$$

$$H_I^{(1)}=\frac{g_0}{\Omega^{1/2}}\cdot\sum_{\substack{k\\(\boldsymbol{p}'+\boldsymbol{k}=\boldsymbol{p})}}\frac{f(k)}{\sqrt{2\omega_k}}[\psi_V^{+}(\boldsymbol{p})\,\psi_N(\boldsymbol{p}')\,a_k+\text{c.c.}]-\sum_{\boldsymbol{p}}\delta m_V\psi_V^{+}(\boldsymbol{p})\,\psi_V(\boldsymbol{p}) \qquad (3\cdot3)$$

and

$$H_I^{(2)}=\frac{f}{\Omega}\cdot\sum_{\substack{k,k'\\(p'+k=p'+k')}}\frac{f(k)\,f(k')}{\sqrt{4\omega_k\,\omega_{k'}}}[\psi_N^+(\boldsymbol{p}')\,\psi_N(\boldsymbol{p})\,a_k^+\,a_k], \tag{3·4}$$

where $\omega_k=\sqrt{k^2+\mu^2}$, μ is the mass of the θ particle and we assume that the physical V particle and N particle have the same mass m. The interaction Hamiltonian $H_I^{(2)}$ is added to the Lee model.

The N-θ scattering amplitude $F(\omega_k)$ is defined by

$$F(\omega_k)=\langle\psi_k^{(-)}|V_q|N\rangle\,\Omega\cdot\left[\frac{f(k)\,f(q)}{\sqrt{4\omega_k\,\omega_q}}\right]^{-1}, \tag{3·5}$$

where $|N\rangle$ and $|\psi_k^{(-)}\rangle$ are the N-particle state and the incoming N-θ scattering state, respectively. V_q is the operator;

$$V_q\equiv\frac{g_0}{\Omega^{1/2}}\frac{f(q)}{\sqrt{2\omega_q}}\sum_{\boldsymbol{p}}\psi_V^+(\boldsymbol{p})\,\psi_N(\boldsymbol{p}-\boldsymbol{q})+\frac{f}{\Omega}\sum_{\substack{k\\(p'+k=p+q)}}\frac{f(q)\,f(k)}{\sqrt{4\omega_q\,\omega_k}}\cdot\psi_N^+(\boldsymbol{p}')\,\psi_N(\boldsymbol{p})\,a_k^+. \tag{3·6}$$

By using the Chew-Low method,[7] the following dispersion relation [the Chew-Low equation] is obtained:

$$F(\omega_k)=\bar{a}_0\frac{1}{\omega_k}+f+\frac{1}{\pi}\int_\mu^\infty d\omega_{m'}\,\bar{K}(\omega_{m'})\,f^2(\omega_{m'})\frac{|F(\omega_{m'})|^2}{\omega_k-\omega_{m'}+i\epsilon}. \tag{3·7}$$

Here we assume that the physical V-particle state and the N-θ scattering states $\psi_k^{(\pm)}$'s form a complete set in the sector considered. [As will be shown afterwards, this assumption is true only for certain restricted values of f. Here we consider only such case.] In (3·7), the coupling constant g is the renormalized one, and

$$L(E)=\frac{1}{\Omega}\sum_k\frac{f^2(k)}{2\omega_k}\cdot\frac{1}{E-m-\omega_k}, \tag{3·8}$$

$$\bar{a}_0=g^2\left(\frac{1}{1-fL(m)}\right)^2 \tag{3·9}$$

and

$$\bar{K}(\omega_k)=\frac{1}{4\pi}[\omega_k{}^2-\mu^2]^{1/2}. \tag{3·10}$$

Eq. (3·7) has a similar structure to that of Eq. (2·20). The asymptotic behaviour of $\bar{K}(\omega)$ as $\omega\to\infty$ is different from that of $K(\nu)$ as $\nu\to\infty$, [see Eq. (2·6)]. However, this difference is not of importance, because the choice of the cut-off function absorbs this difference. Therefore, the solutions of this equation (3·7) may have the similar feature to that of the second kind solutions in

§ 2.* Comparing these two equations, (2·20) and (3·7), we find that the constant C in (2·20) corresponds to the coupling constant f in the interaction Hamiltonian $H_I^{(2)}$.

Next, let us calculate the $F(\omega_k)$ by the Schrödinger method. The result is

$$F(\omega_k)=\frac{g_0^2}{\dfrac{1}{\dfrac{f}{g_0^2}+\dfrac{1}{\omega_k-\delta m_V}}-g_0^2L(\omega_k+m+i\epsilon)}, \tag{3·11}$$

where

$$\delta m_V=g_0^2\left(\frac{L(m)}{1-fL(m)}\right). \tag{3·12}$$

The right-hand side of (3·11) has a pole corresponding to the physical V-particle state at $\omega=0$. Furthermore, in the case of $f<0$, the amplitude (3·11) shows the resonance behaviour or has a singularity corresponding to the appearance of new bound state. The energy of the new pole or the resonances are given as roots of the equation,

$$\frac{1}{(f/g_0^2)+(1/E-\delta m_V)}-g_0^2L^p(E+m)=0,$$

where the $L^p(E+m)$ is the quantity which corresponds to the Cauchy principal value of the integral (3·8). This equation can be rewritten in the more convenient form as

$$[1-fL^p(E+m)]=\frac{(g_0^2/f)}{E+(g_0^2/f)[1-fL(m)]^{-1}}. \tag{3·13}$$

The behaviour of the right-hand side and that of the left-hand side as the function of E are shown in Fig. 2 by the dotted and solid lines, respectively. The intersections of these two lines show the positions of poles and the resonance energy. Here we are interested only in the low-energy resonance, because the

* The variable ν in (2·20) can be written as

$$\nu=\frac{1}{2}\mu(\omega-\mu)+\alpha,$$

where ω is the energy of the incident particle in the *laboratory* system. Eq. (2·20) becomes

$$F(\omega)=\frac{2a_0}{\mu(\omega-\mu)+2\alpha}+C+\frac{1}{\pi}\int_\mu^\infty d\omega' K(\omega')f(\omega')\frac{|F(\omega')|^2}{\omega-\omega'+i\epsilon},$$

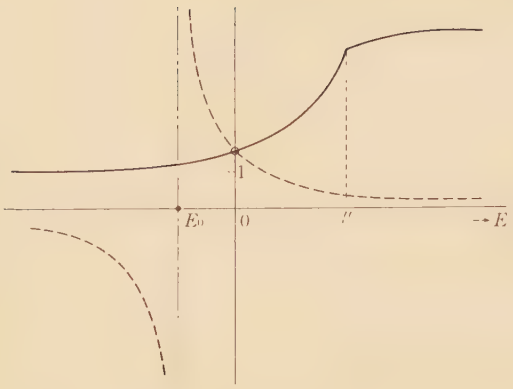
where

$$K(\omega)=\sqrt{\frac{\omega-\mu}{\omega+\mu}}.$$

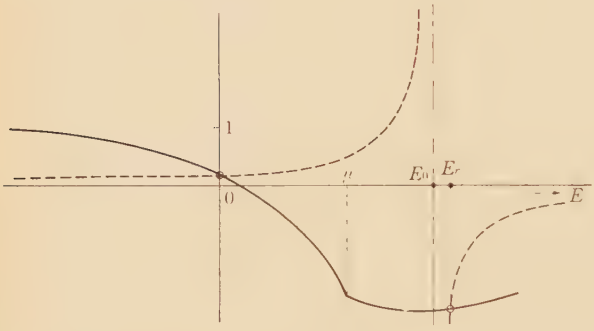
If $\omega\approx\mu$, we get

$$K(\omega)\cong\frac{1}{2\mu}\sqrt{(\omega^2-\mu^2)}\equiv\frac{2\pi}{\mu}\overline{K}(\omega).$$

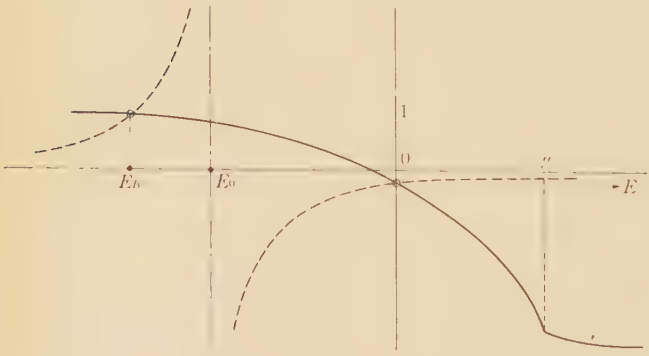
Thus the correspondence of (2·20) to (3·7) is clear.



(a) The case of $f>0$



(b) The case $f<0$ and $[1-f\cdot L(m)]>0$



(c) The case $f<0$ and $[1-f\cdot L(m)]<0$

Fig. 2. The behaviours of both sides of the equation (3·13). The solid lines represent the behaviour of the left-hand side and the dotted lines that of the right-hand side. E_r is the resonance energy, E_B is the energy of the additional bound state and E_0 shows the position of zero of the scattering amplitude.

high-energy behaviour of the left-hand side of (3·13) largely depends on the special choice of the form factor $f(\omega)$. The other special feature of our model is the fact that the scattering amplitude always has one zero. The position is given by the root of the equation,

$$(f_r'/f_0^2)+[E_0-\delta m_r]^{-1}=0,$$

or equivalently the root of the equation

$$E_0+(f_0^2/f)\times[1-fL(m)]^{-1}=0.$$

This energy E_0 just corresponds to the pole in the left-hand side of (3·13). [See Fig. 2.]

Now we shall summarize the low-energy behaviour of the scattering amplitude (3·11):

(a) The case $f>0$: The interaction $H_I^{(2)}$ is repulsive. Accordingly, there appears no new bound state and no resonance, but the amplitude has a zero for the negative ω.

(b) The case that $f<0$ and $[1-fL(m)]>0$: The amplitude shows the resonance behaviour in the low-energy region and otherwise, the new bound state appears in the region $0<\omega<\mu$.

(c) The case that $f<0$ and $[1-fL(m)]<0$: There

appears always the new bound state for the negative ω.

As already mentioned, the constant C in (2·20) corresponds to our coupling constant f. Then the positive value of C means the repulsive interaction and we have, in this case, the scattering amplitude which shows the very strange behaviour, namely it has no resonance but has one zero.

§ 4. Discussion

As was shown in § 2, the C–M equation changes its charactor by introducing the cut-off factor. Therefore, it is very important to investigate whether the introduction of the cut-off factor has a physical background. One possibility may be that the cut-off factor is introduced as that reflecting the effects of the inelastic processes which were neglected because of the assumption $R(\nu)=1$. However, we have not yet gotten any definite conclusion for this possibility. If we consider $R(\nu)$ as a given function, $R(\nu)$ does not play a role of the cut-off function, because $R(\nu)\geq 1$ [see (2·3)].

Any way, the cut-off procedure makes it possible for us to introduce the new dynamics into the theory by means of the subtraction. When the attractive interaction is introduced, we may have the new solution (the simplest second kind solution) showing the resonance behaviour in the low-energy region. We can also get for the unsubtracted equation the solution which has the same feature as that of the above-mentioned new solution, because we can get the first kind solutions tending continuously to the second kind one in this case. However, the resonance of the first kind solution corresponds to the so-called "*kinematical*"[3] resonance which is seen in the Dyson model[5] and for which the energy and the width of the resonance are the free parameters. In the resonance of the simplest second kind solution, however, these two quantities are determined by fixing only one parameter a_1 in Eq. (2·7). Thus this kind of resonance is the "*dynamical*". It may be very difficult to distinguish to which types the empirical resonance belongs. If the width of the empirical resonance is considerably different from that given by the second kind solution for which the energy of the resonance is taken to be the empirical value, this resonance may correspond to a *kinematical* one. However, when this theoretical width is consistent with the empirical value, we can interpret this resonance from the both points of view, as mentioned in the conclusion of § 2.

When the subtraction introduces the repulsive interaction, the second kind solution has the feature which is quite different from that of the first kind ones.

Acknowledgments

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Appendix 1

The derivation of (2·10)

From the definitions (2·8) and (2·9), the following properties of the function $X(z)$ are obtained:

(a) $X(z)$ is the meromorphic function of z with a branch cut along the real axis from α to $+\infty$, and it has a zero at origin.

(b) From (2·8), we get the relation

$$\mathrm{Im}\,F(z)=\left[-\frac{a_0}{|z|}-\frac{1}{\pi}\int_{\alpha}^{\infty}d\nu' K(\nu')f(\nu')\frac{|F(\nu')|^2}{|z-\nu'|^2}\right]\mathrm{Im}\,z,$$

and we see that, in the case of $a_0>0$, $X(z)$ is the generalized R-function, i.e. $\mathrm{Im}\,X(z)>0$ for $\mathrm{Im}\,z>0$.

(c) For the boundary value ν, we have $\mathrm{Im}\,X(\nu)=K(\nu)f(\nu)$.

(d) $X(\nu)/\nu|_{\nu=0}=1/a_0$.

(e) $X(\nu_0)=1/a_1$.

According to the Herglotz theorem,[6] we get the following general form for the function $X(z)$ which satisfies the conditions (a), (b) for $a_0>0$ and (c):

$$X(z)=A+\frac{B}{z}+Dz+\frac{1}{\pi}\int_{\alpha}^{\infty}d\nu'\left(\frac{z}{\nu'}\cdot\frac{K(\nu')f(\nu')}{(\nu'-z)}\right)+\sum_m\frac{\alpha_m z}{E_m(E_m-z)},\quad(\mathrm{A}\cdot1\cdot1)$$

where A, B, D, α_m and E_m are all real constants and satisfy the conditions

$$B\leq0,\ D\geq0 \text{ and } \alpha_m\geq0.$$

The constants A, B and D are determined by the condition (d) as follows:

$$A=0,\quad B=0$$

and

$$D=\frac{1}{a_0}-\frac{1}{\pi}\int_{\alpha}^{\infty}\frac{K(\nu')f(\nu')}{\nu'^2}d\nu'-\sum_m\frac{\alpha_m}{E_m^2}\geq0.\qquad(2\cdot12)$$

Inserting these values into (A·1·1), we obtain the general form (2·10). The restriction (2·13) is derived from the condition (e).

Appendix 2

The derivation of (2·24)

Let us consider the function $zF(z)$. Here we shall denote the first kind solution by $F^{(1)}(z)$ and the second kind one by $F^{(2)}(z)$. From (2·5), it is obtained that

$$\mathrm{Im}[zF^{(1)}(z)]=\left\{-\frac{1}{\pi}\int_{\alpha}^{\infty}d\nu' K(\nu')f(\nu')\frac{\nu'|F^{(1)}(\nu')|^2}{|z-\nu'|^2}\right\}\mathrm{Im}\,z.$$

The quantity in the curly brackets has definite sign, and this means that the function $zF^{(1)}(z)$ is also the generalized R-function whose property requires the important condition

$$\alpha_m/E_m \geq 0. \qquad \text{Q. E. D.} \tag{2·24}$$

On the other hand, for the second kind solution, we have the following relation from (2·20) :

$$\mathrm{Im}[zF^{(2)}(z)] = \left\{C - \frac{1}{\pi}\int_{\alpha}^{\infty} d\nu' K(\nu') f(\nu') \frac{\nu' |F^{(2)}(\nu')|^2}{|z-\nu'|^2}\right\} \mathrm{Im}\, z.$$

Therefore, $zF^{(2)}(z)$ is not the generalized R-function for $C>0$. Consequently, it is obtained for the simplest second kind solution that $\bar{\alpha}/\bar{E}<0$ for $C>0$ and $\bar{\alpha}/\bar{E}>0$ for $C<0$.

Appendix 3

The derivation of (2·28) and (2·29)

We shall consider the asymptotic behaviour of the integral

$$H(\nu) \equiv \mathcal{P}\int_{\alpha}^{\infty} d\nu' K(\nu') \frac{|F(\nu')|^2}{\nu-\nu'}$$

for large ν, when the behaviour (2·26) or (2·27) of $F(\nu)$ is assumed. Separate the integral region into two parts by γ which is sufficiently large but is smaller than ν, and we get

$$H(\nu) \sim \int_{\alpha}^{\gamma} d\nu' K(\nu') \frac{|F(\nu')|^2}{\nu-\nu'} + \mathcal{P}\int_{\gamma}^{\infty} d\nu' \frac{|F(\nu')|^2}{\nu-\nu'},$$

where we use the fact that $K(\nu)\sim 1$ as $\nu\to\infty$. The first term in the right-hand side damps as fast as $1/\nu$ for $\nu\to\infty$. For the second term, the following results are obtained :

(1) For the case $F(\nu)\sim(1/\nu)$,

$$\text{the second term} = \mathcal{P}\int_{\gamma}^{\infty} d\nu' \frac{1}{(\nu-\nu')\nu'^2} \sim \text{const.}\times\frac{1}{\nu} + 0(\log\nu/\nu^2).$$

(2) For the case $F(\nu)\sim(1/\log\nu)$,

$$\text{the second term} = \lim_{\epsilon\to 0}\left[\int_{\nu+\epsilon}^{\infty} d\nu' \frac{1}{(\nu-\nu')(\log\nu')^2} + \int_{\gamma}^{\nu-\epsilon} d\nu' \frac{1}{(\nu-\nu')(\log\nu')^2}\right]$$

$$= \lim_{\epsilon\to 0}\left[-\int_{\nu+\epsilon}^{\infty} d\nu' \frac{1}{\nu'}\left[1+\left(\frac{\nu}{\nu'}\right)+\left(\frac{\nu}{\nu'}\right)^2+\cdots\right]\frac{1}{(\log\nu')^2} + \right.$$

$$+\int_{\gamma}^{\nu-\epsilon} d\nu' \frac{1}{\nu}\left[1+\left(\frac{\nu'}{\nu}\right)+\left(\frac{\nu'}{\nu}\right)^2+\cdots\right]\frac{1}{(\log\nu')^2}\Bigg].$$

By using the formula

$$\int \frac{d\nu}{\nu^n(\log\nu)^2} = -\frac{1}{n-1}\left(\frac{1}{\nu^{n-1}(\log\nu)^2}\right)+\frac{2}{(n-1)^2}\left(\frac{1}{\nu^{n-1}(\log\nu)^3}\right)$$
$$-\frac{3}{(n-1)^3}\left(\frac{1}{\nu^{n-1}(\log\nu)^4}\right)+\cdots$$

for $n\neq 1$, the above expression becomes

$$=\lim_{\epsilon\to 0}\Bigg[\Bigg\{\left|\frac{1}{\log\nu'}\right|_{\nu+\epsilon}^{\infty}+\nu\left|\frac{1}{\nu'(\log\nu')^2}-\frac{2}{\nu'(\log\nu')^3}+\cdots\right|_{\nu+\epsilon}^{\infty}$$
$$+\nu^2\left|\frac{1}{2}\cdot\frac{1}{\nu'^2(\log\nu')^2}-\frac{2}{4}\cdot\frac{1}{\nu'^2(\log\nu')^3}+\cdots\right|_{\nu+\epsilon}^{\infty}+\cdots\Bigg\}$$
$$+\Bigg\{\frac{1}{\nu}\left|\frac{\nu'}{(\log\nu')^2}+2\frac{\nu'}{(\log\nu')^3}+\cdots\right|_{\gamma}^{\nu-\epsilon}$$
$$+\frac{1}{\nu^2}\left|\frac{1}{2}\cdot\frac{\nu'^2}{(\log\nu')^2}+\frac{2}{4}\cdot\frac{\nu'^2}{(\log\nu')^3}+\cdots\right|_{\gamma}^{\nu-\epsilon}+\cdots\Bigg\}\Bigg]$$
$$=\left[-\frac{1}{\log\nu}+\frac{4}{(\log\nu)^3}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)+\cdots\right]+0\left(\frac{1}{\nu}\right)$$

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Dispersion Relations in Nucleon-Nucleon Scattering and Resonances in Many-Pion System

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The two-pion contribution to the absorptive part of the forward nucleon-nucleon scattering amplitude in the unphysical region is calculated numerically in Cini-Fubini approximation. The magnitude of the contribution to the absorptive part from multi-pion (three and more-pion) intermediate states is evaluated from the relations used as sum rules and it is found that a resonance appears either in $T=0$, $J=1$, odd parity state (neutral vector boson) or in $T=0$, $J=0$, odd parity state (neutral pseudoscalar boson) or in both states of the multi-pion system. If the resonance is caused by a neutral vector boson with 3 pion-mass, the magnitude of the rationalized vector coupling constant is $f_V^2/4\pi=1.2$.

§ 1. Introduction

Yukawa theory of pion-nucleon interaction has been successfully applied to many phenomena. For example, the existence of one-pion exchange process between two nucleons has been verified by looking for the pole of the S-matrix element[1] and by investigating the outer part of the nuclear potential.[2] π-N scattering and the iso-vector part of the electromagnetic structure of the nucleon may be explained if we assume a π-π resonance in the $T=1$, $J=1$ state.[3),4)] Using this knowledge of π-N scattering, a nuclear potential due to two-pion exchange process has been calculated and this potential is qualitatively in agreement with the intermediate part of the (semi-)phenomenological potential.[5)] The above processes are those in which only π-N vertex and an annihilation of a nucleon pair into two-pions and π-π scattering take part.

There are still many unsolved problems——for example, the iso-scalar part of the electromagnetic structure of the nucleon and the multi-pion exchange process in nucleon-nucleon scattering which is responsible for the inner part of the nuclear potential (hard core and L-S force of short range, and so on).

The big size of the iso-scalar part of the mean square radius of the charge distribution of the nucleon and the small size of the iso-scalar part of the magnetic moment of the nucleon suggest the existence of a resonance in the $T=0$, $J=1$ state of three-pion system which may be approximated by a vector boson strongly interacting with a nucleon through vector coupling. We cannot

yet find a resonance solving three-body problem, though there are many theories predicting the existence of a neutral vector boson (or resonance) and a neutral pseudoscalar boson (or resonance). If there exist such resonances, they have effect on nucleon-nucleon scattering.[6]

Whether there is a big contribution from multi-pion exchange processes to nucleon-nucleon scattering or not is important if we want to solve nucleon-nucleon scattering dispersion-theoretically. Recently, several authors[7] have proposed to deal with the nucleon-nucleon scattering on the Mandelstam's two-dimensional representation, deriving dispersion relations (integral equations) for the partial wave amplitudes. The kernel of these integral equations is determined from the absorptive part of the amplitude in the unphysical region. However, as we stated above, the information that is available about the absorptive part in the unphysical region is limited only to low-mass intermediate states contributions. As to three- and more-pion contributions, it is impossible to calculate. Some authors considered that they were suppressed by subtractions and that their effects came in only through low-energy parameters such as scattering lengths.

The aim of this work is to evaluate the contribution to the absorptive part of the nucleon-nucleon scattering amplitude in the unphysical region from multi-pion intermediate states and to see if the multi-pion contribution to the forward nucleon-nucleon scattering amplitude and the iso-scalar part of the electromagnetic structure of the nucleon can be explained in terms of the multi-pion resonance consistently.

In § 2 we calculate the contribution to the absorptive part of the forward nucleon-nucleon scattering amplitude in the unphysical region from two-pion intermediate states. For the calculation, a spectral representation of π-N scattering is necessary, and we use one in Cini-Fubini approximation.[8]

In § 3 we apply the relation to unpolarized p-p and n-p scattering, and to spin singlet p-p scattering. We substitute experimental data into the both sides of the relation regarded as a sum rule (the relation with one- and two-pion absorptive part and without three- and more-pion absorptive part in the unphysical region). By comparing the both sides of the relation, we evaluate the magnitude of the contribution to the absorptive part from multi-pion intermediate states. In § 4 we try to answer the question: "In what state of the multi-pion system does a resonance appear?" There we approximate resonances by bosons. The effect of the resonance on the iso-scalar part of the electromagnetic structure of the nucleon is also discussed. In the Appendix we explain the details of the calculation of the contribution to the absorptive part from two-pion intermediate state.

§ 2. The dispersion relation[9]

As is well known, the dispersive part, $D(E)$, and the absorptive part,

$A(E)$, of the S-matrix element of the forward nucleon-nucleon scattering satisfy the dispersion relation,

$$D(E)=\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{A(E')\,dE'}{E'-E}\,, \tag{2·1}$$

where E stands for the energy of the incident nucleon in the laboratory system. $D(E)$ and $A(E)$ are related to the S-matrix element in the following way,

$$\begin{aligned}\langle p, q|S-1|q, p\rangle &= (2\pi)^4 i\delta^4(0)\left[\frac{m^4}{(2\pi)^{12}p_0^{\,2}q_0^{\,2}}\right]^{1/2}(D(E)+iA(E)), \quad E\geq m\\ &= (2\pi)^4 i\delta^4(0)\left[\frac{m^4}{(2\pi)^{12}p_0^{\,2}q_0^{\,2}}\right]^{1/2}(D(E)-iA(E)), \quad E<m\end{aligned} \tag{2·2}$$

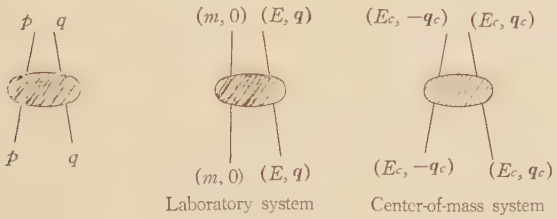

Fig. 1.

and in the physical region related to the total cross sections of nucleon-nucleon scattering and nucleon-antinucleon scattering in the following way,

$$A(E)=\frac{q_c E_c}{m^2}\sigma_{NN}(E_c), \qquad E\geq m$$

and (2·3)

$$A(E)=-\frac{q_c E_c}{m^2}\sigma_{N\bar{N}}(E_c), \quad E\leq -m$$

where E_c and q_c are energy and momentum of the incident nucleon in the center of mass system.* $D(E)$ is known experimentally if the differential cross section for forward scattering is known for spin singlet and triplet states, respectively,

$$D_i(E)^2+A_i(E)^2=\frac{16\pi^2 E_c^{\,2}}{m^4}\frac{d\sigma_i(E_c, 0^\circ)}{d\Omega}. \tag{2·4}**$$

As experiments seem to indicate $\lim_{E\to\infty}\sigma(E)=\text{const}$, two subtractions may be sufficient, and we use the following dispersion relation,

* $2E_c^{\,2}=m(m+E)$, $E_c^{\,2}=m^2+q_c^{\,2}$.

** The index i stands for singlet or triplet.

$$D(E)=D(m)+(E-m)\left.\frac{dD}{dE}\right|_{E=m}+(E-m)^2\lim_{\substack{E_1\to m\\ E_2\to m}}\frac{P}{\pi}\int_m^\infty\frac{A(E')\,dE'}{(E'-E_1)(E'-E_2)(E'-E)}$$

$$+\frac{(E-m)^2}{\pi}\int_{-m}^{m}\frac{A(E')\,dE'}{(E'-m)^2(E'-E)}+\frac{(E-m)^2}{\pi}\int_{-\infty}^{-m}\frac{A(E')\,dE'}{(E'-m)^2(E'-E)}\,. \tag{2·5}$$

As experiments seem also to indicate

$$\sigma_{NN}(E)=\sigma_{N\bar{N}}(E),\quad \text{for } E\gtrsim 30\ \text{Bev},$$

we can use the following dispersion relation,

$$D(E)=D(m)+(E-m)\lim_{E_1\to m}\frac{P}{\pi}\int_m^\infty\frac{A(E')\,dE'}{(E'-E_1)(E'-E)}$$

$$+\frac{(E-m)}{\pi}\int_{-m}^{m}\frac{A(E')\,dE'}{(E'-m)(E'-E)}+\frac{(E-m)}{\pi}\int_{-\infty}^{-m}\frac{A(E')\,dE'}{(E'-m)(E'-E)}\,. \tag{2·6}$$

All terms except for the fourth term in the right-hand side of Eq. (2·5) and the 3rd term in Eq. (2·6) are expressed by observable quantities through Eqs. (2·3) and (2·4). The fourth (third) term, however, is an integral over the unphysical region, and there $A(E)$ is not directly related to observable quantities. In this unphysical region, $A(E)$ can be expressed formally as

$$A(E)=\pi(2\pi)^6\left(\frac{p_0}{m}\right)\overline{u_\alpha}(q)\sum_n\langle p|\overline{f_\beta}|n\rangle\langle n|f_\alpha|p\rangle\delta^4(p_n-p+q)\,u_\beta(q), \tag{2·7}$$

where intermediate states are states with nucleon number zero, i.e. one-pion, two-pion and more-pion states -the n-pion state has a spectrum running from $m-n^2/2m$ to $-\infty$. In this section we calculate the contributions from one-pion and two-pion intermediate states. At present, we do not know how to calculate the contributions from multi-pion intermediate states. Though the importance of multi-pion contributions is reduced by subtraction(s), they do not seem to be negligible. We estimate their magnitude in the next section, making use of the dispersion relations.

One-pion contribution to $A(E)$, $A_{1\pi}(E)$, is exactly given by[10]

$$A_{1\pi}(E)=-\frac{g^2}{4\pi}\frac{\pi^2}{2m^3}\left(\frac{3}{2}-\frac{1}{2}\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2\right)\left(\frac{1}{2}-\frac{1}{2}\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2+\boldsymbol{\sigma}^1\cdot\boldsymbol{e}\,\boldsymbol{\sigma}^2\cdot\boldsymbol{e}\right)$$

$$\times\delta\left(E-m+\frac{1}{2m}\right). \tag{2·8}*$$

* g is the pseudoscalar coupling constant.

The two-pion contribution to $A(E)$, $A_{2\pi}(E)$, can be calculated using the spectral representation for π-N scattering. In this work we use Cini-Fubini approximation,[4)8)] i.e. we include π-N pole, S and P wave π-N scattering and S and P wave π-π scattering——we supply the contribution form π-π scattering to the π-N dispersion relation by Chew, Goldberger, Low and Nambu.[11)]* We neglect Coulomb interaction and assume charge independence. Through the investigations by many workers the presence of a resonance in P wave π-π scattering has been found favourable[3)4)] and we assume the resonance and we use the results of the phenomenological analysis by Bowcock et al.[4)]** As to S wave π-π scattering, we have not so detailed knowledge as we have on P wave. S wave π-π scattering may, however, be much smaller than P wave because resonances do not appear in S wave in chain approximation.[12)] Therefore, we neglect S wave π-π scattering. We assume for π-N scattering that the (3, 3) resonance gives the only important contribution to the integral and we neglect small phase shifts in π-N scattering. Thus, we leave only π-N pole, (3, 3) resonance and P wave π-π resonance. We use zero-width approximation to (3, 3) resonance.

With these assumptions we can calculate $A_{2\pi}$, and the results are tabulated in Table I (with π-π resonance) and in Table II (without π-π resonance).*** The result in Table I is qualitatively in agreement with the previous result[10)] based on the Chew-Low equation. The derivation of these results is given in the Appendix. The notation is

$$A_{2\pi}=A_1^{(2\pi)}+\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2 A_2^{(2\pi)}+\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2 A_3^{(2\pi)}+\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2 A_4^{(2\pi)}+\boldsymbol{\sigma}^1\cdot\boldsymbol{e}\boldsymbol{\sigma}^2\cdot\boldsymbol{e}A_5^{(2\pi)}$$
$$+\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2\boldsymbol{\sigma}^1\cdot\boldsymbol{e}\boldsymbol{\sigma}^2\cdot\boldsymbol{e}A_6^{(2\pi)}. \qquad (2\cdot9)$$

Table I. $A_{2\pi}(E)$

Δ*	$A_{2\pi}^{(1)}$	$A_{2\pi}^{(2)}$	$A_{2\pi}^{(3)}$	$A_{2\pi}^{(4)}$	$A_{2\pi}^{(5)}$	$A_{2\pi}^{(6)}$
2.00	0.00	0.00	0.00	0.00	0.00	0.00
2.05	−0.07	0.66	0.03	0.60	0.10	−0.06
2.2	−0.41	1.08	−0.04	0.81	0.36	−0.27
2.6	−1.08	1.98	−0.06	1.28	1.02	−0.70
3.0	−1.87	2.84	−0.04	1.79	1.83	−1.08
4.0	−8.5	6.5	−1.3	3.6	7.2	−3.0
5.0	2.9	4.2	3.1	3.5	0.2	−0.6

* $E=m-\dfrac{\Delta^2}{2m}$

* Legendre expansion does not converge for $t=-(\text{momentum transfer})^2\lesssim-26$ and we put $A_{2\pi}=0$ for $t<-26$.

** See the Appendix.

*** We use the units $\hbar=c=m_\pi=1$.

Table II. $A_{2\pi}(E)$ (without π-π scattering)

A	$A_{2\pi}^{(1)}$	$A_{2\pi}^{(2)}$	$A_{2\pi}^{(3)}$	$A_{2\pi}^{(4)}$	$A_{2\pi}^{(5)}$	$A_{2\pi}^{(6)}$
2.00	0.00	0.00	0.00	0.00	0.00	0.00
2.05	−0.03	0.65	0.06	0.59	0.09	−0.06
2.2	−0.30	1.04	0.02	0.79	0.31	−0.25
2.6	−0.64	1.83	0.12	1.22	0.76	−0.61
3.0	−0.84	2.50	0.33	1.67	1.17	−0.86
4.0	−1.1	4.1	1.1	2.8	2.2	−1.3
5.0	−1.3	5.6	2.0	3.9	3.3	−1.7

The evaluation of errors is given in the Appendix.

§ 3. Application to nucleon-nucleon scattering

In this section we substitute experimental data in both sides of the dispersion relation with the absorptive part calculated in the previous section but without that from multi-pion intermediate states. By comparing the magnitudes of the both sides of the relation as a sum rule, we can estimate the magnitude of the contribution from multi-pion intermediate states to the absorptive part in the unphysical region.

For forward scattering amplitudes the dispersion relation holds for singlet scattering and triplet scattering, respectively.[9] In this paper we discuss singlet p-p scattering and unpolarized p-p and n-p scattering. $D(E)$ and $A(E)$ are

$$\begin{aligned} D(E) &= \frac{8\pi E_c}{m^2 q_c}\left(\sin{}^1\delta_S^0 \cos{}^1\delta_S^0 + 5\sin{}^1\delta_D^2 \cos{}^1\delta_D^2 + \cdots\right), \\ A(E) &= \frac{8\pi E_c}{m^2 q_c}\left(\sin^2{}^1\delta_S^0 + 5\sin^2{}^1\delta_D^2 + \cdots\right), \end{aligned} \tag{3·1}$$

for singlet p-p scattering and

$$\begin{aligned} D(E) = \frac{2\pi E_c}{m^2 q_c}&\left(\sin{}^1\delta_S^0 \cos{}^1\delta_S^0 + \sin{}^3\delta_P^0 \cos{}^3\delta_P^0 + 3\sin{}^3\delta_P^1 \cos{}^3\delta_P^1\right. \\ &\left. + 5\sin{}^3\delta_P^2 \cos{}^3\delta_P^2 + 5\sin{}^1\delta_D^2 \cos{}^1\delta_D^2 + \cdots\right), \\ A(E) = \frac{2\pi E_c}{m^2 q_c}&\left(\sin^2{}^1\delta_S^0 + \sin^2{}^3\delta_P^0 + 3\sin^2{}^3\delta_P^1 + 5\sin^2{}^3\delta_P^2 + 5\sin^2{}^1\delta_D^2 + \cdots\right) \end{aligned} \tag{3·2}$$

for unpolarized p-p scattering and

$$D(E) = \frac{\pi E_c}{m^2 q_c}\left(\sin{}^1\delta_S^0 \cos{}^1\delta_S^0 + 3\sin{}^1\delta_P^1 \cos{}^1\delta_P^1 + 5\sin{}^1\delta_D^2 \cos{}^1\delta_D^2 + \cdots\right)$$

$$+\frac{\pi E_c}{m^2 q_c}(3\sin^3\delta_S^1\cos^3\delta_S^1+\sin^3\delta_P^0\cos^3\delta_P^0+3\sin^3\delta_P^1\cos^3\delta_P^1$$
$$+5\sin^3\delta_P^2\cos^3\delta_P^2+\cdots), \tag{3·3}$$
$$A(E)=\frac{\pi E_c}{m^2 q_c}(\sin^2{}^1\delta_S^0+3\sin^2{}^1\delta_P^1+5\sin^2{}^1\delta_D^2+\cdots)$$
$$+\frac{\pi E_c}{m^2 q_c}(3\sin^2{}^3\delta_S^1+\sin^2{}^3\delta_P^0+3\sin^2{}^3\delta_P^1+5\sin^2{}^3\delta_P^2+\cdots)$$

for unpolarized n-p scattering.

As we neglect Coulomb force and assume π-N interaction is charge independent, nucleon-nucleon scattering is also charge independent. Hence, only the precise experimental data of n-p scattering is sufficient for our purpose. n-p scattering experiment, however, is not so precise as p-p scattering experiment except at thermal energy. There are no reliable phase shift analysis of n-p scattering. On the other hand, there are unique sets of phase shifts at several energies for p-p scattering.[13] Two protons, however, interact through Coulomb force besides through nuclear force,* and at present it is impossible to get rid of the effect of the Coulomb force from phase shifts of p-p scattering completely. In this paper we use Stapp's nuclear bar phase shifts[14] as true nuclear phase shifts, and we use the scattering lengths, the effective ranges and the shape parameter obtained from low-energy n-p scattering.

For p-p scattering between 10 Mev and 310 Mev we have precise knowledge on $D(E)$ and $A(E)$ from phase shift analysis.[13),15] Of course, $D(E)$ and $A(E)$ depend on the choice of the sets of phase shifts, but the dependence is rather small. This may be because they are closely related to observable quantities. Above 310 Mev we use for σ_{tot} the data collected by Hess[16] and the data obtained since then.[17] At low energies we consider σ_{tot} approaches σ_{eff}, the cross section in the effective range approximation. The latest data on scattering lengths, effective ranges, and shape parameters of n-p scattering is as follows:**[18]

$$\begin{aligned}
&a_s=(-23.74\pm0.09)\times10^{-13}\,\text{cm}\\
&r_{0s}=(2.670\pm0.023)\times10^{-13}\,\text{cm}\\
&P_s=0.017\pm0.028\\
&a_t=(5.39\pm0.03)\times10^{-13}\,\text{cm}\\
&r_{0t}=(1.704\pm0.028)\times10^{-13}\,\text{cm}\\
&P_t=0\ \text{(by assumption)}.
\end{aligned} \tag{3·4}$$

* The effect of the electromagnetic interaction on n-p scattering may be negligible.

** $q_c\cot\delta=-\frac{1}{a}+\frac{1}{2}r_0 q_c^2-Pr_0^3 q_c^4$.

We use the data reported in references 17) and 19) for total cross section of nucleon-antinucleon scattering.

At first we apply the relation (2·6) to unpolarized n-p scattering. For the sake of convenience, we rewrite the relation:

$$D(E)=D(m)+\frac{(E-m)}{\pi}\lim_{E_1\to m}P\int_m^\infty\frac{A_{eff}(E')\,dE'}{(E'-E_1)\,(E'-E)}$$
$$+\frac{12\pi\gamma^3(m-E)}{m^2(1-\gamma r_{0t})\,(m-2B+B^2/2m-E)\,(2B-B^2/2m)}$$
$$+\frac{E-m}{\pi}\int_m^\infty\frac{A(E')-A_{eff}(E')}{(E'-m)\,(E'-E)}dE'+\frac{E-m}{\pi}\int_{-\infty}^{-m}\frac{A(E')\,dE'}{(E'-m)\,(E'-E)}$$
$$+\frac{E-m}{\pi}\int_{-m}^{m}\frac{A(E')\,dE'}{(E'-m)\,(E'-E)} \qquad (3\cdot5)$$

where A_{eff} is the absorptive part in the effective range approximation, the second line in (3·5) is the contribution from deuteron intermediate state, and γ is related to the binding energy of the deuteron B through $B=\gamma^2/m$. As there is no reliable result of phase shift analysis of n-p scattering, we use $D(E)$ calculated from phases which are obtained from (semi-)phenomenological potential method (by Gammel, Christian and Thaler[20] and by Hamada[21]). As scattering lengths and effective ranges, we use

$$a_s=-16.78 \quad a_t=3.809$$
$$r_{0s}=1.887 \quad r_{0t}=1.204$$

which are the same as (3·4), and we neglect P_s and P_t as the dependence of the result on P is negligible.* For high energy scattering, we assume

$$\sigma_{NN}(E)=\sigma_{N\bar{N}}(E)=40\text{ mb}, \quad E>20\text{ Bev}. \qquad (3\cdot6)^{**}$$

For the absorptive part in the last integral in (3·5), we use

$$A(E)=A_{1\pi}(E)+A_{2\pi}(E), \quad m>E\gtrsim m-26/2m$$
$$A(E)=0, \quad m-26/2m\gtrsim E>-m. \qquad (3\cdot7)$$

Among the neglected intermediate states, low mass intermediate states (three-pion, etc.) may contribute to $D(E)$, but the contribution from massive intermediate states may be small as

$$\lim_{E\to-m-0}A(E)\approx-2.$$

* See the application to unpolarized p-p scattering.

** There is no significant reason for the value 20 Bev. The dependence of the result on this energy is small as the integral in the 3rd line in (3·5) is logarithmic divergent.

The result is tabulated in Table III and shown in Fig. 2.

Table III. Unpolarized *n-p* scattering

GCT and H stand for "calculated using phase shifts from potentials by Gammel, Christian and Thaler[20] and by Hamada[21] ", respectively. (We use phase shifts from semi-phenomenological potential as there is no reliable phase shift analysis.)

E_{lab} (Mev)	left side of (3·5)		1st and 2nd line in (3·5)	3rd line	4th line		Sum of right side
	GCT	H			1π	2π	
10	−0.19	−0.16	0.32	0.03	−0.49	−0.27	−0.41
20	0.43	0.49	1.34	0.06	−0.66	−0.49	0.25
30	0.68	0.64	1.87	0.08	−0.75	−0.69	0.51
40	0.73	0.70	2.20	0.11	−0.80	−0.87	0.64
60	0.78	0.74	2.62	0.15	−0.86	−1.16	0.75
90	0.74	0.69	2.98	0.21	−0.90	−1.48	0.81
120	0.69	0.64	3.20	0.25	−0.92	−1.75	0.78
160	0.62	0.54	3.39	0.28	−0.95	−2.07	0.65

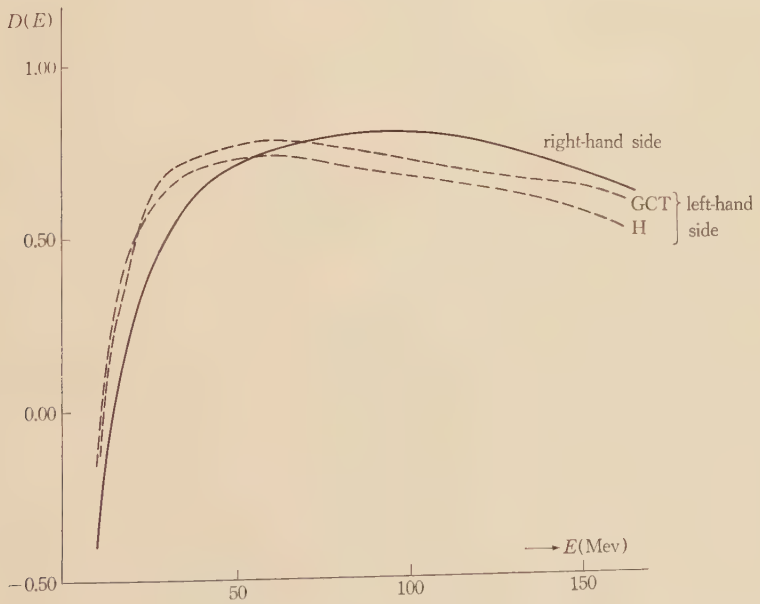

Fig. 2. Unpolarized n-p scattering

Solid line is theoretical curve without 3 or more pion contributions (right-hand side of (3·5)).
Broken lines are experimental curves (left-hand side of (3·5)).

We can conclude that the contribution from three- and more-pion intermediate states to the unpolarized *n-p* scattering is small.

Next, we apply the relation (2·6) to unpolarized *p-p* scattering.

$$D(E)=D(m)+\frac{E-m}{\pi}\lim_{E_1\to m}P\int_m^\infty\frac{A_{eff}(E')}{(E'-E_1)(E'-E)}dE'$$
$$+\frac{E-m}{\pi}\int_m^\infty\frac{A(E')-A_{eff}(E')}{(E'-m)(E'-E)}dE'+\frac{E-m}{\pi}\int_{-\infty}^{-m}\frac{A(E')dE'}{(E'-m)(E'-E)}$$
$$+\frac{E-m}{\pi}\int_{-m}^{m}\frac{A(E')dE'}{(E'-m)(E'-E)}. \qquad (3\cdot8)$$

The assumptions (3·6) and (3·7) are also assumed here. We use the scattering length, the effective range and the shape parameter obtained from *n-p* scattering assuming charge independence,

$$a=-16.78,\quad r=1.887,\quad P=0.017. \qquad (3\cdot9)$$

Table IV. Unpolarized *p-p* scattering

D_{exp} is experimental value (left side of Eq. (3·8)).
"Assumed 3π resonance" stands for the contribution from one vector boson exchange process (mass=3 pion mass with rationalized vector coupling constant $f_V^2/4\pi=1.2$).

E_{lab} (Mev)	D_{exp}			1st line in (3·8)	2nd line	3rd line		Sum	Assumed 3π resonance	Sum
	exp.	GT	H			1π	2π			
9.69	0.92	0.90	1.03	1.06	0.04	−0.24	0.04	0.90	−0.08	0.82
14.16	0.81	0.82	0.93	1.05	0.05	−0.29	0.05	0.86	−0.11	0.75
19.8	0.72	0.76	0.87	1.05	0.05	−0.33	0.06	0.83	−0.14	0.69
25.63	0.77	0.74	0.83	1.06	0.06	−0.36	0.08	0.84	−0.18	0.66
39.4	0.81	0.72	0.77	1.11	0.09	−0.40	0.10	0.90	−0.25	0.65
68.3	0.7	0.70	0.74	1.19	0.12	−0.44	0.15	1.02	−0.35	0.67
95	0.7	0.69	0.73	1.25	0.14	−0.46	0.19	1.12	−0.42	0.70
145	0.7	0.67	0.70	1.31	0.17	−0.47	0.20	1.21	−0.51	0.70

The results are tabulated in Table IV and shown in Fig. 3. In order to see the effect of the change in the magnitude of parameters, we consider the following three cases,

(a) $a=-12.4$ $r=2.02$ $P=0.016$

(b) $a=-11.0$ $r=1.87$ $P=0.00$

(c) $a=-11.0$ $r=1.87$ $P=0.017$.

The results are tabulated in Table V. The dependence of the results on a, r and P is small. We find from these results that multi-pion intermediate states give rather big contribution to unpolarized *p-p* scattering.

As there is no experimental data of high energy polarized nucleon-nucleon scattering and polarized nucleon-antinucleon scattering, we cannot apply the

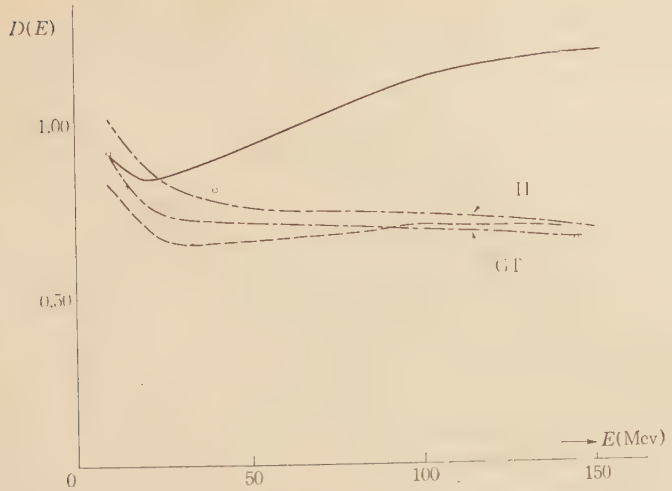

Fig. 3. Unpolarized p-p scattering.

——— theoretical curve without 3- or more-pion contributions (right-hand side of (3·8)),

— — — right-hand side of (3·8) with 3π resonance (mass=3 pion mass and $f_V^2/4\pi=1.2$),

—·—·— experimental curve on potential method, circles are experimental points.

Table V. Theoretical values assuming various sets of low energy parameters (3·9), (*a*), (*b*), (*c*).

E_{lab} (Mev)	Right side of (3·8) without multi-pion contribution			
	(3·9)	(*a*)	(*b*)	(*c*)
9.69	0.90	0.97	0.91	0.92
14.16	0.86	0.91	0.84	0.84
19.8	0.83	0.88	0.79	0.79
25.63	0.84	0.89	0.81	0.80
39.4	0.90	0.94	0.85	0.84
68.3	1.02	1.04	0.95	0.95
95	1.12	1.12	1.03	1.03
145	1.21	1.24	1.10	1.10

relation (2·6) to polarized scattering. In the twice subtracted relation (2·5), the contribution from high energy scattering and nucleon-antinucleon scattering is suppressed, but the contribution from low-energy scattering is enhanced. As we have little knowledge on low-energy P wave scattering, we can apply the twice subtracted relation (2·5) only to singlet p-p scattering. With the same assumptions as were made in the last application of (2·6), we obtain the result tabulated in Table VI and shown in Fig. 4. The result is very dependent on

the choice of the low-energy parameters a, r and P and we find that the experimental error in P_s, ± 0.028, prevents us from deducing a conclusion from polarized scattering (see Fig 4).

Table VI. Spin singlet p-p scattering

D_{exp} is experimental value (left side of Eq. (2·5)). D_{theor} is the right side of Eq. (2·5) (without 3- or more-pion contribution).
"Assumed 3π resonance" stands for the contribution from one vector boson exchange process (mass=3 pion mass with rationalized vector coupling constant $f_V^2/4\pi=1.2$).

E_{lab} (Mev)	D_{exp}	D_{theor} without 3 or more pion contribution				1π	2π	Assumed 3π resonance
		(3·9)	(a)	(b)	(c)			
70	2.1	2.8	1.9	4.6	3.8	5.7	0.9	−0.6
100	1.9	2.9	1.9	5.5	4.4	8.7	1.6	−1.0
200	1.6	3.4	0.7	9.0	6.5	18.3	4.3	−2.4

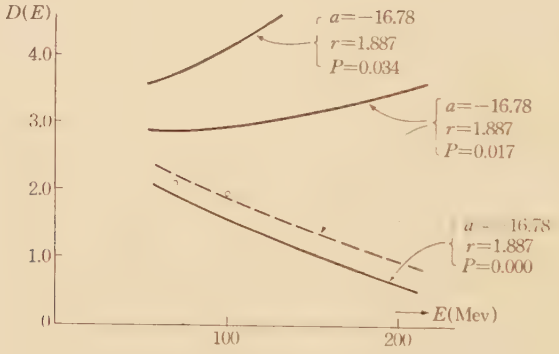

Fig. 4. Singlet p-p scattering.

Circles are experimental points. Solid lines are theoretical curves without 3- or more-pion contribution (right-hand side of Eq. (2·5)). Broken line is the theoretical curve with multi-pion resonance (mass=3 pion mass and $f_V^2/4\pi=1.2$). The experimental error in P prevents us from evaluating the magnitude of the multi-pion contribution to singlet p-p scattering.

§ 4. Discussions and conclusion

Let us assume there is a boson not yet discovered or there is a resonance which can be approximated by a boson. The interaction Lagrangean density is

$$L=f_s\bar{\psi}\psi\phi \quad \text{or} \quad ig_s\bar{\psi}\gamma_\mu\psi\frac{\partial}{\partial x_\mu}\phi, \quad \text{for scalar boson,}$$

$$L=if_p\bar{\psi}\gamma_5\psi\phi+ig_p\bar{\psi}\gamma_5\gamma_\mu\psi\frac{\partial}{\partial x_\mu}\phi, \quad \text{for pseudoscalar boson,}$$

$$L=if_V\bar{\psi}\gamma_\mu\psi\phi_\mu+\frac{1}{2}g_V\bar{\psi}\sigma_{\mu\nu}\psi\phi_{\mu\nu}, \quad \text{for vector boson,} \tag{4·1}*$$

or

$$L=if_A\bar{\psi}\gamma_5\gamma_\mu\psi\phi_\mu \quad \text{or} \quad \frac{i}{2}g_A\bar{\psi}\gamma_5\sigma_{\mu\nu}\psi\phi_{\mu\nu}, \quad \text{for axial vector boson,}$$

where

$$\sigma_{\mu\nu}=\frac{1}{2i}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu),$$

and

$$\phi_{\mu\nu}=\partial\phi_\nu/\partial x_\mu-\partial\phi_\mu/\partial x_\nu.$$

As to the isospin state of the boson, $I=0$ and 1 are possible. The Hamiltonian (4·1) is for $T=0$. For $T=1$, ϕ in (4·1) should be replaced by $\tau_\alpha\phi_\alpha$.

The one boson exchange contribution to the absorptive part of p-p scattering is

$$A=\frac{\pi}{2m}f_s^2\frac{1+\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2}{2}\delta(E-E_r), \quad \text{(for scalar boson),}$$

$$A=-\frac{\pi}{2m}\left(\frac{\mu}{2m}f_p+g_p\right)^2\left(\frac{1}{2}+\boldsymbol{\sigma}^1\cdot\boldsymbol{e}\boldsymbol{\sigma}^2\cdot\boldsymbol{e}-\frac{1}{2}\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2\right)\delta(E-E_r),$$

(for pseudoscalar boson)

$$A=-\frac{\pi}{2m}f_V^2\frac{1+\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2}{2}\delta(E-E_r)-\frac{\pi g_V^2\mu^2}{2m}(1-\boldsymbol{\sigma}^1\cdot\boldsymbol{e}\boldsymbol{\sigma}^2\cdot\boldsymbol{e})\,\delta(E-E_r),$$

(for vector boson), (4·2)

and

$$A=\frac{\pi f_A^2}{2m}\left(\frac{3}{2}-\frac{\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2}{2}\right)\delta(E-E_r) \quad \text{or} \quad \frac{\pi g_A^2\mu^2}{2m}\left(\frac{1}{2}+\boldsymbol{\sigma}^1\cdot\boldsymbol{e}\boldsymbol{\sigma}^2\cdot\boldsymbol{e}\right.$$

$$\left.-\frac{1}{2}\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2\right)\delta(E-E_r), \quad \text{(for axial vector boson),}$$

for both $T=0$ and $T=1$ bosons, where we have neglected the terms of the order of magnitude $O(\mu/m)$ and where μ is the mass of the boson and

$$E_r=m-\mu^2/2m.$$

For unpolarized n-p scattering, (4·2) should be multiplied by a factor 2 in the case of iso-vector boson and should be multiplied by zero in the case of iso-scalar (neutral) boson.

---

* We neglect the possibility that the boson is with higher angular momentum as it seems less probable.

The boson (or resonance) appears in other phenomena. If the G-parity of the boson is odd, the iso-scalar vector boson with vector coupling contributes to the iso-scalar part of the charge distribution of the nucleon, and the iso-scalar vector boson with tensor coupling contributes to the iso-scalar part of the charge distribution of the nucleon.

In the following we assume that the multi-pion contribution can be approximated by the contribution from multi-pion resonances which are approximated by bosons, and we assume that the number of the resonances is either one or two. Then, it is deduced from unpolarized n-p scattering that the boson(s) must be neutral as the contribution from multi-pion resonance is small and as neutral bosons do not contribute to (charge exchange) n-p scattering. It is deduced from unpolarized p-p scattering that either a (neutral) vector boson or a (neutral) pseudoscalar boson or both bosons are necessary as the multi-pion intermediate states have to give negative contribution to unpolarized p-p forward

Table VII. The sign of the contribution to $D(E)$ from bosons

Spin parity (coupling type)	unpolarized p-p	singlet p-p
S	+	+
PS	−	+
V(V)	−	−
V(T)	−	+
A	+	−

scattering (Table VII). In the case of pseudoscalar boson we can not distinguish, in our approximation, pseudoscalar coupling and pseudovector coupling due to the equivalence theorem. In the case of vector boson, vector coupling and tensor coupling gives different contribution to polarized scattering. However, experimental errors in low-energy parameters (especially that of P) prevent us from distinguishing between pseudoscalar boson, vector boson with vector coupling and vector boson with tensor coupling. If we neglect experimental errors, both relations for unpolarized p-p scattering and for singlet p-p scattering are satisfied by neutral vector boson with rationalized vector coupling constant,

$$\frac{f_V^2}{4\pi}=1.2$$

and with 3 pion mass.*

* Unpolarized p-p scattering alone gives

$$\frac{1}{4\pi}\left(\frac{\mu}{2m}f_p+g_p\right)^2+\frac{f_V^2}{4\pi}+\frac{2}{4\pi}\,g_V^2\mu^2=1.2\pm0.2,$$

where we have neglected the possible resonances which can be approximated by a scalar boson and by an axial vector boson. If they exist, the magnitude, 1.2, increases.

Iso-scalar vector boson with vector coupling contributes to the iso-scalar part of the charge distribution of the nucleon. The mean square radius is given in the lowest order by[22]

$$\langle r_1^2\rangle=\frac{f_V^2}{4\pi}D,$$

where

$$D=\frac{2}{\pi\mu^2}\left(\frac{5}{6}+\frac{1}{i\pi^2}\int\frac{d^4q}{(q^2-m^2)^2}\right). \tag{4·3}$$

As the integral in (4·3) is divergent, we restrict the domain of integration to values of q for which $q^2\leq M^2$. With $f_V^2/4\pi=1.2$ and $\mu=3m_\pi$

$$\begin{aligned}\sqrt{\langle r_1^2\rangle}&=0.41\times(10^{-13}\text{ cm}) \quad \text{for } M^2=m^2\\ &=0.53 \qquad\qquad\qquad\quad =4m^2\\ &=0.60 \qquad\qquad\qquad\quad =9m^2\\ &=0.92 \qquad\qquad\qquad\quad =16m^2.\end{aligned} \tag{4·4}$$

The experimental value of $\sqrt{\langle r_1^2\rangle}$ is 0.8×10^{-13} cm.

The author would like to express his gratitude to Prof. H. Miyazawa and Dr. W. Watari for discussions and hearty encouragements. He is also grateful to Dr. T. Hamada and Dr. R.J.N. Phillips for informing him of the most recent situation regarding the shape parameter.

Appendix

The calculation of $A_{2\pi}(E)$

The two-pion contribution to $A(E)$, $A_{2\pi}(E)$, is calculated by using the spectral representation for π-N scattering in Cini-Fubini approximation.[4),8)] We assume that for π-N scattering the (3, 3) resonance gives the only important contribution to the integral, and suppose that π-π scattering is dominated by a resonance in the $T=1$, $J=1$ state. In this approximation the T-matrix for π-N scattering may be represented as follows,*

$$\begin{aligned}&((2\pi)^9 4k_0l_0p_0/m)^{1/2}\bar{u}_\alpha(q)\langle \boldsymbol{ki}, \boldsymbol{lj}|f_\alpha|p\rangle\\ &=\bar{u}(q)\left\{\left[A^{(+)}(s,\bar{s},t)+\frac{1}{2}i\gamma\cdot(l-k)\,B^{(+)}(s,\bar{s},t)\right]\delta_{ij}\right.\\ &\left.+\left[A^{(-)}(s,\bar{s},t)+\frac{1}{2}i\gamma\cdot(l-k)\,B^{(-)}(s,\bar{s},t)\right]\frac{1}{2}[\tau_j,\tau_i]\right\}u(p),\end{aligned} \tag{A·1}$$

and

* $x^2=x_0^2-\boldsymbol{x}^2$

$$A^{(+)}(s,\bar{s},t)=a^{(+)}(t)+\frac{\alpha^{+}(t)}{s_r-s}+\frac{\alpha^{+}(t)}{s_r-\bar{s}},$$

$$A^{(-)}(s,\bar{s},t)=(s-\bar{s})\,a^{(-)}(t)+\frac{\alpha^{-}(t)}{s_r-s}-\frac{\alpha^{-}(t)}{s_r-\bar{s}}, \qquad (A\cdot 2)$$

$$B^{(+)}(s,\bar{s},t)=\frac{g_r^2}{m^2-s}-\frac{g_r^2}{m^2-\bar{s}}+\frac{\beta^{+}}{s_r^2-s}-\frac{\beta^{+}}{s_r-\bar{s}},$$

$$B^{(-)}(s,\bar{s},t)=b^{(-)}(t)+\frac{g_r^2}{m^2-s}+\frac{g_r^2}{m^2-\bar{s}}+\frac{\beta^{-}}{s_r-s}+\frac{\beta^{-}}{s_r-\bar{s}},$$

where

$$a^{(+)}(t)=0,$$

$$a^{(-)}(t)=-\frac{6\pi\times 0.272}{22.4-t-0.376iq^3},$$

$$b^{(-)}(t)=\frac{12\pi(1+0.544m)}{22.4-t-0.376iq^3},$$

$$\alpha^{+}(t)=\frac{4g_r^2q_r^2}{9m}\left[3\,\frac{W_r+m}{E_r+m}\left(1+\frac{t}{2q_r^2}\right)+\frac{W_r-m}{E_r-m}\right], \qquad (A\cdot 3)$$

$$\beta^{+}=-\frac{8}{9}g_r^2,$$

$$\alpha^{-}(t)=-\frac{1}{2}\alpha^{+}(t),\quad \beta^{-}=-\frac{1}{2}\beta^{+},$$

where $s=(p-k)^2$, $\bar{s}=(p-l)^2$, $t=(k+l)^2$, W, E and q are total energy, total nucleon energy and magnitude of meson momentum in the center-of-mass system, and index r stands for "resonance."

With the above spectral representation, we calculate the two-pion contribution to the absorptive part, using the technique given in the previous paper.[10]

$$A_{2\pi}=\pi(2\pi)^{-3}\frac{1}{4k_0l_0}$$

$$\times\sum_{k,l,ij}\bar{u}(q)\left\{\left[A^{(+)}(s_1,\bar{s}_1,t_1)+\frac{i}{2}\gamma\cdot(l-k)\,B^{(+)}(s_1,\bar{s}_1,t_1)\right]\delta_{ij}\right.$$

$$\left.+\left[A^{(-)}(s_1,\bar{s}_1,t_1)+\frac{i}{2}\gamma\cdot(l-k)B^{(-)}(s_1,\bar{s}_1,t_1)\right]\frac{1}{2}[\tau_i,\tau_j]\right\}u(p)$$

$$\times\bar{u}(p)\left\{\left[A^{(+)}(s_2,\bar{s}_2,t_2)+\frac{i}{2}\gamma\cdot(l-k)\,B^{(+)}(s_2,\bar{s}_2,t_2)\right]\delta_{ij}\right.$$

$$\left.+\left[A^{(-)}(s_2,\bar{s}_2,t_2)+\frac{i}{2}\gamma\cdot(l-k)\,B^{(-)}(s_2,\bar{s}_2,t_2)\right]\frac{1}{2}[\tau_j,\tau_i]\right\}u(q)$$

$$=\pi(2\pi)^{-3}\frac{1}{4k_0 l_0}\sum_{k,l}\left\{3\bar{u}(q)\left[A_1^{(+)}+\frac{i}{2}\gamma\cdot(l-k)\,B_1^{(+)}\right]u(p)\right.$$

$$\times\bar{u}(p)\left[A_2^{(+)}+\frac{i}{2}\gamma\cdot(l-k)\,B_2^{(+)}\right]u(q)$$

$$+2\bar{u}(q)\left[A_1^{(-)}+\frac{i}{2}\gamma\cdot(l-k)\,B_1^{(-)}\right]\tau u(p)$$

$$\left.\times\bar{u}(p)\left[A_2^{(-)}+\frac{i}{2}\gamma\cdot(l-k)\,B_2^{(-)}\right]\tau u(q)\right\}$$

$$\equiv-\frac{3}{2}\,(I^{+}+2I^{-})+\frac{1}{2}\,(3I^{+}-2I^{-})\,\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)} \qquad (A\cdot4)$$

where

$$s_1=(p+l)^2,\quad \bar{s}_1=(p+k)^2,\quad t_1=(k+l)^2,$$
$$s_2=(q-k)^2,\quad \bar{s}_2=(q-l)^2,\quad t_2=(k+l)^2,$$

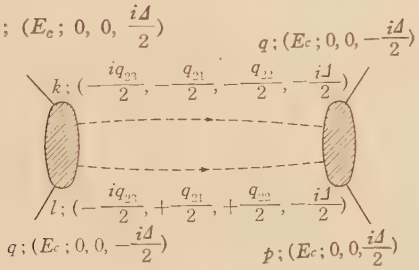

Fig. 5.

and

$$I^{\pm}=\frac{\pi}{2}\,(2\pi)^{-3}\,\frac{1}{4\Delta}\int dq_{21}\,dq_{22}\,dq_{23}\,\delta(\Delta^2-\boldsymbol{q}_2^2)$$

$$\times\bar{u}(q)\left[A^{(\pm)}-\frac{i}{2}\,(\gamma_1 q_{21}+\gamma_2 q_{22}+\gamma_4 q_{23})\,B^{(\pm)}\right]u(p)$$

$$\times\bar{u}(p)\left[A^{(\pm)}-\frac{i}{2}\,(\gamma_1 q_{21}+\gamma_2 q_{22}+\gamma_4 q_{23})\,B^{(\pm)}\right]u(q). \qquad (A\cdot5)$$

Substituting (A2) in (A5),

$$32\pi I^{\pm}=\bar{u}(p)\,u(q)\,\bar{u}(q)\,u(p)\,\frac{|\boldsymbol{q}_2|}{\Delta}$$

$$\times\left\{(a^{\pm})^2\left\{\frac{2}{x^2+y^2}\pm\frac{2}{xy}\tan^{-1}\frac{y}{x}\right\}+\left(\begin{array}{l}\dfrac{4\alpha^{+}a^{(+)}}{y}\tan^{-1}\dfrac{y}{x}+(a^{(+)})^2,\\ \text{or}\\ -\dfrac{4}{3}y^2(a^{(-)})^2-8a^{(-)}\alpha^{-}\left(1-\dfrac{x}{y}\tan^{-1}\dfrac{y}{x}\right)\end{array}\right)\right\}$$

$$+(\bar{u}(p)\gamma_4 u(q)\,\bar{u}(q)\,u(p)+(\bar{u}p)\,u(q)\,u(q)\,\gamma_4 u(p))\times\frac{|\boldsymbol{q}_2|^2}{J}$$

$$\times\left\{\begin{array}{l}\dfrac{\alpha^{\pm}g_r^2}{y^2}\left\{\dfrac{x}{x'-x}\tan^{-1}\dfrac{y}{x}-\dfrac{x'}{x'-x}\tan\dfrac{y}{x'}\mp\dfrac{x}{x'+x}\tan^{-1}\dfrac{y}{x}\pm\dfrac{x'}{x'+x}\tan^{-1}\dfrac{y}{x'}\right\}\\ +\dfrac{\alpha^{\pm}\beta^{\pm}}{y}\left\{\dfrac{x}{x^2+y^2}-\dfrac{1}{y}\tan^{-1}\dfrac{y}{x}\right\}\end{array}\right.$$

$$-\left\{\begin{array}{l}\dfrac{a^{(+)}}{y}\left(g_r^2+\beta^{+}-\dfrac{x'g_r^2}{y}\tan^{-1}\dfrac{y}{x'}-\dfrac{\beta^{-}x}{y}\tan^{-1}\dfrac{y}{x}\right),\\ \text{or}\\ \dfrac{y}{3}\,a^{(-)}b^{(-)}+\dfrac{2}{y}a^{(-)}\beta^{-}x\left(1-\dfrac{x}{y}\tan^{-1}\dfrac{y}{x}\right)+\dfrac{b^{(-)}}{y}\alpha^{-}\left(1-\dfrac{x}{y}\tan^{-1}\dfrac{y}{x}\right)\\ \qquad+\dfrac{2}{y}a^{(-)}g_r^2x'\left(1-\dfrac{x'}{y}\tan^{-1}\dfrac{y}{x'}\right)\end{array}\right.$$

$$+\bar{u}(p)\gamma_4 u(q)\,\bar{u}(q)\gamma_4 u(p)\times\frac{|\boldsymbol{q}_2|^3}{2J}$$

$$\times\left[\frac{g_r^4}{y^2}\left\{1-\frac{2x'}{y}\tan^{-1}\frac{y}{x'}+\frac{x'^2}{x'^2+y^2}\pm1\mp\frac{x'}{y}\tan^{-1}\frac{y}{x'}\right\}\right.$$

$$+\frac{(\beta^{\pm})^2}{y^2}\left\{1-\frac{2x}{y}\tan^{-1}\frac{y}{x}+\frac{x^2}{x^2+y^2}\pm1\mp\frac{x}{y}\tan^{-1}\frac{y}{x}\right\}$$

$$+\frac{2g_r^2\beta^{\pm}}{y^2}\left\{1+\frac{x'^2}{(x-x')y}\tan^{-1}\frac{y}{x'}-\frac{x^2}{(x-x')y}\tan^{-1}\frac{y}{x}\right.$$

$$\left.\pm1\mp\frac{x'^2}{(x+x')y}\tan^{-1}\frac{y}{x'}\mp\frac{x^2}{(x+x')y}\tan^{-1}\frac{y}{x}\right\}$$

$$\left.-\frac{2b^{(-)}}{y^2}\left\{g_r^2x'\left(1-\frac{x'}{y}\tan^{-1}\frac{y}{x'}\right)+\beta^{-}x\left(1-\frac{x}{y}\tan^{-1}\frac{y}{x}\right)\right\}-\frac{1}{6}(b^{(-)})^2\right]$$

$$+(\bar{u}(p)\gamma_1 u(q)\,\bar{u}(q)\gamma_1 u(p)+\bar{u}(p)\gamma_2 u(q)\,\bar{u}(q)\gamma_2 u(p))\times\frac{|\boldsymbol{q}_2|^3}{4J}$$

$$\times\left[\frac{g_r^4}{y^2}\left\{-2+\frac{2x'}{y}\tan^{-1}\frac{y}{x'}\mp1\pm\frac{x'^2+y^2}{x'y}\tan^{-1}\frac{y}{x'}\right\}\right.$$

$$+\frac{(\beta^{\pm})^2}{y^2}\left\{-2+\frac{2x}{y}\tan^{-1}\frac{y}{x}\mp1\pm\frac{x^2+y^2}{xy}\tan^{-1}\frac{y}{x}\right\}$$

$$+\frac{2g_r^2\beta^{\pm}}{y^2}\left\{-1-\frac{x'^2+y^2}{(x-x')y}\tan^{-1}\frac{y}{x'}+\frac{x^2+y^2}{(x-x')y}\tan^{-1}\frac{y}{x}\right.$$

$$\mp 1 \pm \frac{x^2+y^2}{(x+x')\,y}\tan^{-1}\frac{y}{x} \pm \frac{x'^2+y^2}{(x+x')\,y}\tan^{-1}\frac{y}{x'}\Big\}$$
$$-\frac{2b^{(-)}}{y^2}\left\{g_r^2 x'\left(-1+\frac{x'^2+y^2}{x'y}\tan^{-1}\frac{y}{x'}\right)+\beta^- x\left(-1+\frac{x^2+y^2}{xy}\tan^{-1}\frac{y}{x}\right)\right\}$$
$$-\frac{1}{3}(b^{(-)})^2\Big], \qquad (A\cdot 6)$$

where

$$x=\frac{\Delta^2}{2}-1+s_r-m^2,$$
$$x'=\frac{\Delta^2}{2}-1,$$
$$y=E_c\,c_2|,$$
$$|q_2|^2=q_{21}^2+q_{22}^2+q_{23}^2=\Delta^2-4,$$

and

$$\bar{u}(p)\,u(q)\,\bar{u}(q)\,u(p)=\frac{1+\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}}{2}\left(\frac{E_c}{m}\right)^2,$$
$$\bar{u}(p)\,\gamma_4 u(q)\,\bar{u}(q)\,u(p)+\bar{u}(p)\,u(q)\,\bar{u}(q)\,\gamma_4 u(p)=(1+\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\frac{E_c}{m},$$
$$\bar{u}(p)\,\gamma_4 u(q)\,\bar{u}(q)\,\gamma_4 u(p)=\frac{1+\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}}{2}, \qquad (A\cdot 7)$$
$$\bar{u}(p)\,\gamma_1 u(q)\,\bar{u}(q)\,\gamma_1 u(p)+\bar{u}(p)\,\gamma_2 u(q)\,u(q)\,\gamma_2 u(p)=(1-\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{e}\,\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{e})\frac{\Delta^2}{4m^2}.$$

We use the following values for nucleon mass, m, meson mass, m_π, resonance energy (kinetic energy of pion in laboratory system), E_r, and rationalized pseudoscalar coupling constant, $g_r^2/4\pi$,

$$m=939\text{ Mev},$$
$$m_\pi=140\text{ Mev},$$
$$E_r=180\text{ Mev},$$
$$g_r^2/4\pi=14.4.$$

The contribution from S-wave π-π scattering, $a^{(+)}(t)$, is evaluated using Table VIII. We can find the contribution to $A_{2\pi}$ by

$$A_{2\pi}^{(t)}=J^{(t)}(a^{(+)})^2+K^{(t)}a^{(+)}.$$

As the magnitude of $a^{(+)}$ is supposed to be

$$|a^{(+)}|\sim 1,$$

the contribution from $a^{(+)}(t)$ may be negligible.

Table VIII

Δ	$J^{(1)}=J^{(2)}=J^{(3)}=J^{(4)}$*	$K^{(1)}=K^{(2)}=K^{(3)}=K^{(4)}$
2.05	0.0016	0.00026
2.2	0.0030	0.00038
2.6	0.0046	0.00060
3.0	0.0053	0.00078
4.0	0.0059	0.0012
5.0	0.0059	0.0015

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* $J^{(5)}=J^{(6)}=K^{(5)}=K^{(6)}=0$

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On $\pi+N\to Y+\pi+K$ Reaction

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Reaction $\pi+N\to Y+\pi+K$ at high energy is studied by taking into account the effects of Y^* and K^*. Since the isotopic spins of Y_1^*, Y_0^* and K^* seem to be equal to 1, 0 and 1/2 respectively, we may expect that, for example, $\pi^-+p\to\Sigma^0+\pi^-+K^+$ and $\pi^-+p\to\Sigma^+ +\pi^-+K^0$ take place mainly through the processes $\pi^-+p\to\Sigma^0+K^*\to\Sigma^0+\pi^-+K^+$ and $\pi^-+p\to Y_0^*+K^0\to\Sigma^+ +\pi^-+K^0$ respectively. The cross sections and angular distributions of $\pi+N\to Y^*+K$ and $\pi+N\to Y+K^*$ processes are examined, and the effects due to Y^* are compared with those due to K^*.

§ 1. Introduction

As is well known the recent measurements on K^--p collision at 1.15 Bev/c have shown the remarkable results for the existence of the Y_1^*, Y_0^* and K^* which are the isobaric states in Λ-π, Σ-π and K-π systems respectively.[1),2),3)] In order to study K meson interactions at high energy we think it worth while to examine how these isobaric states have the large effects on K meson reactions and to compare the effects due to the K^* with those due to the Y^*. In this paper we should like to study $\pi+N\to Y+\pi+K$ reaction.

Let us pay attention to the π^--p collision at high energy. In $\pi^-+p\to\Sigma^+ +\pi^-+K^0$ reaction, it may be impossible to expect the effect of K^* because the isotopic spin of K^* seems to be equal to 1/2. In $\pi^-+p\to\Sigma^0+\pi^-+K^+$ reaction, it may be impossible to expect the effect of Y_0^* because the isotopic spin of Y_0^* is equal to zero. Therefore these reactions take place mainly through the following processes.

$$\pi^-+p\to Y_0^*+K^0\to\Sigma^+ +\pi^-+K^0,$$

$$\pi^-+p\to\Sigma^0+K^{*0}\to\Sigma^0+\pi^-+K^+.$$

On the other hand, in the study for $\pi^-+p\to\Lambda+\pi^-+K^+$ it is necessary to take into account the following two processes:

$$\pi^-+p\to Y_1^{*-}+K^+\to\Lambda+\pi^-+K^+$$

and

$$\pi^-+p\to\Lambda+K^{*0}\to\Lambda+\pi^-+K^+.$$

It is interesting for us to examine which process plays the more important role in $\pi^-+p\to\Lambda+\pi^-+K^+$ reaction.

It seems that the spin of Y_0^* may be equal to $1/2$[3] and that the experimental results are consistent with $J=1/2$ or $J=3/2$ (J indicates the spin of Y_1^*).[4),5)] We now assume for simplicity $J=1/2$. With respect to the spin of K^*, both $S=1$ and $S=0$ cases (S indicates the spin of K^*) are taken into consideration although it has been shown in the previous paper[6] that the assumption of $S=1$ rather than that of $S=0$ is consistent with the experimental results for K^--p collision at 1.15 Bev/c.

First of all $Y_1^*\Lambda\pi$- and $Y_0^*\Sigma\pi$-interactions are investigated along the same line with our study for $K^*\pi K$-interaction,[6] and the values of coupling constants for these interactions are estimated. In § 2 a detailed account of determination of coupling constant for $Y_1^*\Lambda\pi$-interaction is given. By making use of the results for coupling constants, the following processes at $\omega=2$ Bev and 3 Bev are studied, although any reliable experimental data have not yet been reported, where ω means the total energy of incident pion in the laboratory system.

(i) $\pi^-+p\rightarrow Y_0^*+K^0$,

(ii) $\pi^-+p\rightarrow Y_1^{*-}+K^+$,

(iii) $\pi^-+p\rightarrow \Sigma^0+K^{*0}$,

(iv) $\pi^-+p\rightarrow \Lambda+K^{*0}$.

In § 3 it is shown that how the cross sections and angular distributions of processes (i) and (ii) depend on the parity of Y_0^*(or Y_1^*) and on the type of $N\Sigma K$ or $N\Lambda K$ interaction. In § 4 the processes (iii) and (iv) are studied on the assumption of $S=0$ or $S=1$. In § 5 the processes $\pi^++p\rightarrow Y+\pi+K$ together with $\pi^-+p\rightarrow Y+\pi+K$ are discussed. § 6 is devoted to the conclusions and discussions.

§ 2. π-Λ scattering and coupling constant of $Y_1^*\Lambda\pi$-interaction

When the spin of Y_1^* is equal to $1/2$, the Hamiltonian of $Y_1^*\Lambda\pi$-interaction is given as follows:

$$H_{Y_1^*\Lambda\pi}=F_\Lambda[\bar{\Psi}^+O\psi_\Lambda\phi^+ +\bar{\Psi}^0O\psi_\Lambda\phi^0+\bar{\Psi}^-O\psi_\Lambda\phi^-]+\text{h. c.}, \tag{1}$$

$$O=\begin{cases}1,\\ i\gamma_5,\end{cases} \tag{2}$$

where Ψ, ψ_Λ and ϕ are the wave functions of the Y_1^*, Λ and π respectively. The propagation function of Y_1^* is modified along the same line with our study for K^*.[6] Then its expression can be written down in the following form.

$$S_F(k)=\frac{1}{2\pi i}\cdot\frac{(1+\hat{\varepsilon})(i\gamma k)-(M+m_\Lambda O^2\gamma_l)}{(1+\hat{\varepsilon})^2k^2+(M+m_\Lambda O^2\gamma_l)^2}, \tag{3}$$

$$\hat{\varepsilon}=\frac{i}{4}\left(\frac{F_\Lambda^2}{4\pi}\right)\frac{A}{\sqrt{-k^2}}\left[1-\frac{B}{\sqrt{-k^2}}\right], \tag{4}$$

$$\eta=-\frac{i}{4}\left(\frac{F_A^2}{4\pi}\right)\frac{A}{\sqrt{-k^2}}, \tag{5}$$

$$A=\sqrt{\frac{[k^2+(m_A+\mu)^2][k^2+(m_A-\mu)^2]}{-4k^2}}, \tag{6}$$

$$B=\sqrt{\frac{(k^2+m_A^2-\mu^2)^2}{-4k^2}}=\sqrt{A^2+\mu^2}, \tag{7}$$

where $m_A(m_\Sigma)$, M, μ, m and m_K are the masses of $\Lambda(\Sigma)$, Y^*, π, nucleon and K meson respectively. For simplicity we adopt the following approximation. i) The ξ and η in the numerator of $\bar{S}_F(k)$ are neglected. ii) In the expression for the denominator of $\bar{S}_F(k)$, the damping effect is taken into account up to the first order of ξ and η. Then

$$\bar{S}_F(k)=\frac{1}{2\pi i}\cdot\frac{i\gamma k-M}{k^2+M^2+2\xi k^2+2Mm_AO^2\eta}. \tag{3}'$$

As was pointed out in the previous paper,[6] the imaginary part of denominator of $2\pi i\bar{S}_F(k)$ should be multiplied by the factor 2 in order to obtain the correct exprcssion for the modified propagation function. Thus we get the following $\bar{\bar{S}}_F(k)$ as the expression of modified propagation function of Y_1^*.

$$\bar{\bar{S}}_F(k)=\frac{1}{2\pi i}\cdot\frac{i\gamma k-M}{k^2+M^2+4\xi k^2+4Mm_AO^2\eta}. \tag{3}''$$

It can easily be seen from the following consideration that this $\bar{\bar{S}}_F(k)$ gives the correct value of cross section for π-Λ scattering at the just resonance energy. Since the spin of Y_1^* is equal to 1/2, the angular distribution for π-Λ scattering ought to be nearly isotropic. We now examine the differential cross section at 0°. At the just resonance energy, $k^2+M^2=0$ and the quantity A turns out to be equal to the magnitude of momentum of pion p^* in the center of mass system. Employing the $\bar{\bar{S}}_F(k)$ in Eq. (3)'', we obtain the following expressions for the differential cross sections at 0°.

i) In the case of $O=1$ (see Eqs. (1) and (2)),

$$(d\sigma_{resonance}(\pi\text{-}\Lambda)/d\Omega)_{at\ 0^\circ}=\frac{[(M+m_A)^2-\mu^2]^2}{4M^2p^{*2}(\sqrt{p^{*2}+m_A^2}+m_A)^2}. \tag{8}$$

ii) In the case of $O=i\gamma_5$,

$$(d\sigma_{resonance}(\pi\text{-}\Lambda)/d\Omega)_{at\ 0^\circ}=\frac{[(M-m_A)^2-\mu^2]^2}{4M^2p^{*2}(\sqrt{p^{*2}+m_A^2}-m_A)^2}. \tag{9}$$

First let us examine the expression of (8). At the resonance energy $\sqrt{p^{*2}+m_A^2}=1.01645\,m_A$. If we set approximately $\sqrt{p^{*2}+m_A^2}\simeq m_A$, $M\simeq m_A$ and $\mu^2/m_A^2\simeq 0$, Eq. (8) turns out to be

$$(d\sigma_{resonance}(\pi\text{-}\Lambda)/d\Omega)_{at\ 0^\circ}=\frac{1}{p^{*2}}. \tag{10}$$

Next let us examine the expression of (9). Since $p^{*2}/m_\Lambda{}^2$ is a small quantity,

$$\sqrt{p^{*2}+m_\Lambda{}^2}-m_\Lambda\cong 1/2(p^{*2}/m_\Lambda)\cong 1/2(p^{*2}/M).$$

By means of the relation

$$p^*=\sqrt{[M^2-(m_\Lambda+\mu)^2][M^2-(m_\Lambda-\mu)^2]}/4M^2,$$

$$(M-m_\Lambda)^2-\mu^2=4M^2p^{*2}/[(M+m_\Lambda)^2-\mu^2]\cong 4M^2p^{*2}/4M^2=p^{*2}.$$

Thus Eq. (9) can be expressed approximately as follows,

$$(d\sigma_{resonance}(\pi\text{-}\Lambda)/d\Omega)_{at\ 0^\circ}=\frac{1}{p^{*2}}. \tag{11}$$

It is needless to say that the values mentioned in Eqs. (10) and (11) agree with those obtained by kinematical study.

We try to estimate the value of coupling constant. The denominator of $\bar{\bar{S}}_F(k)$ can be written as

$$k^2+M^2+4\xi k^2+4Mm_\Lambda O^2\eta=(E_r+E^*)\left[(E_r-E^*)+\frac{-4\xi E^{*2}+4Mm_\Lambda O^2\eta}{E_r+E^*}\right]. \tag{12}$$

From this it follows that

$$\frac{\Gamma}{2}\cong\frac{(F_\Lambda^2/4\pi)A}{2E_r}\left[E_r-B+\frac{Mm_\Lambda O^2}{E_r}\right], \tag{13}$$

where E_r and E^* stand for the resonance energy and the total energy of the colliding system respectively, and $\Gamma/2$ indicates the half width of Y_1^*. Putting the experimental value $\Gamma/2=15$ Mev, we obtain the following results.

i) In the case of $O=1$,

$$F_\Lambda^2/4\pi=0.0907. \tag{14}$$

ii) In the case of $O=i\gamma_5$,

$$F_\Lambda^2/4\pi=11.3. \tag{15}$$

The value of coupling constant for the following $Y_0^*\Sigma\pi$-interaction can be estimated by the same way as in the case of $Y_1^*\Lambda\pi$-interaction.

$$H_{Y_0^*\Sigma\pi}=F_\Sigma[\bar{\Psi}O\psi_{\Sigma^-}\phi^+ +\bar{\Psi}O\psi_{\Sigma^0}\phi^0+\bar{\Psi}O\psi_{\Sigma^+}\phi^-]+\text{h. c.} \tag{16}*$$

Taking into account the factor which is derived from the consideration about

* It must be noted that any assumption with respect to the relative $\Sigma\Lambda$ parity (or the relative Y_1^* Y_0^* parity) is not introduced because the O' in Eq. (19) (or the O in Eq. (1)) can be chosen independently of the O' in Eq. (20) (or the O in Eq. (16)).

isotopic spin and assuming that the half width of Y_0^* is equal to 15 Mev, we obtain the following results.

i) In the case of $O=1$,

$$F_\Sigma^2/4\pi=0.039. \tag{17}$$

ii) In the case of $O=i\gamma_5$,

$$F_\Sigma^2/4\pi=9.83. \tag{18}$$

§ 3. $\pi^-+p\to Y_0^*+K^0$ and $\pi^-+p\to Y_1^{*-}+K^+$ reactions

As the K meson-hyperon-nucleon interactions, the usual interaction Hamiltonians are used, that is,

$$H_{N\Sigma K}=g_\Sigma(\sqrt{2}\,\bar{\psi}_{\Sigma^+}O'\psi_p\varphi_{K^0}^*+\sqrt{2}\,\bar{\psi}_{\Sigma^-}O'\psi_n\varphi_{K^+}^*+\bar{\psi}_{\Sigma^0}O'\psi_p\varphi_{K^+}^*-\bar{\psi}_{\Sigma^0}O'\psi_n\varphi_{K^0}^*)$$
$$+\text{h. c.}, \tag{19}*$$

$$H_{N\Lambda K}=g_\Lambda(\bar{\psi}_\Lambda O'\psi_n\varphi_{K^0}^*+\bar{\psi}_\Lambda O'\psi_p\varphi_{K^+}^*)+\text{h. c.}, \tag{20}*$$

$$O'=\begin{cases}1,\\ i\gamma_5,\end{cases} \tag{21}$$

where $\varphi_{K^+}^*$ corresponds to emission of K^+ meson or absorption of K^- meson, and similarly for $\varphi_{K^0}^*$. In the study for K meson reactions at high energy, the recoil of nucleons may not be neglected and it may be necessary to adopt the relativistic method in our calculation. In this paper the cross sections for these reactions are evaluated by means of perturbation theory. Fig. 1 shows the diagram corresponding to the lowest order perturbation.**

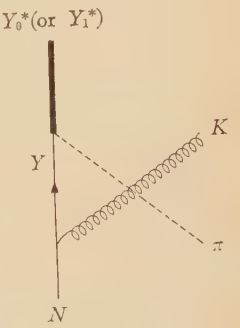

Fig. 1.

When we denote the four momenta of proton, pion, Y_0^* and K meson as $\hat{I}$, $\hat{p}$, $\hat{F}$ and $\hat{q}$, respectively, the differential cross sections for $\pi^-+p\to Y_0^*+K^0$ can be expressed as follows.

$$\frac{d\sigma}{d\Omega}=\frac{4\pi^2q}{p(\sqrt{p^2+m^2}+\sqrt{p^2+\mu^2})^2}|\langle f|I|i\rangle|^2, \tag{22}$$

$$|\langle f|I|i\rangle|^2=-\frac{4(F_\Sigma^2/4\pi)(g_\Sigma^2/4\pi)}{(4\pi)^2[-m^2-m_K^2-2(\hat{I}\hat{q})+m_\Sigma^2]^2}\Big[2(\hat{I}\hat{q})(\hat{F}\hat{q})+(m_K^2-C^2m^2)(\hat{I}\hat{F})$$
$$+2\epsilon_1C\{m^2(\hat{F}\hat{q})+\epsilon_2mM(\hat{I}\hat{q})\}+\epsilon_3mM(m_K^2+C^2m^2)\Big]. \tag{23}$$

* See the footnote on the preceding page.

** If some Y_0^*NK (or Y_1^*NK) interaction is introduced, another process must be taken into account.

Table I.

Coefficient \ Interaction type	$O'=1$		$O'=i\gamma_5$	
	$O=1$	$O=i\gamma_5$	$O=1$	$O=i\gamma_5$
C	$-(1+m_\Sigma/m)$	$-(1+m_\Sigma/m)$	$1-m_\Sigma/m$	$1-m_\Sigma/m$
ϵ_1	-1	-1	1	1
ϵ_2	1	-1	-1	1
ϵ_3	1	-1	-1	1

C, ϵ_1, ϵ_2 and ϵ_3 are given in Table I. Hereafter the coupling constant in the case of $O=i\gamma_5$(or $O'=i\gamma_5$) is discriminated from that in the case of $O=1$(or $O'=1$) by the notation of prime ($'$).

The differential cross section of $\pi^-+p\to Y_1^{*-}+K^+$ can similarly be expressed. In this case it is necessary to take into account the factor which is derived from the consideration about isotopic spin. The total cross sections for $\pi^-+p\to Y_0^*+K^0$ and $\pi^-+p\to Y_1^{*-}+K^+$ at $\omega=2$ Bev and 3 Bev are shown in Table II and Table III, where the values of coupling constants $F_\Sigma^2/4\pi=0.039$, $F_\Sigma'^2/4\pi=9.83$, $F_A^2/4\pi=0.0907$ and $F_A'^2/4\pi=11.3$ obtained in § 2 are adopted.

Table II. The total cross sections for $\pi^-+p\to Y_0^*+K^0$

Interaction type	Cross section	$\sigma(\pi^-+p\to Y_0^*+K^0)$	
		$\omega=2$ Bev	$\omega=3$ Bev
$O'=1$	$O=1$	0.0463 $(g_\Sigma^2/4\pi)$ mb	0.0271 $(g_\Sigma^2/4\pi)$ mb
	$O=i\gamma_5$	6.02 $(g_\Sigma^2/4\pi)$ mb	6.13 $(g_\Sigma^2/4\pi)$ mb
$O'=i\gamma_5$	$O=1$	0.0070 $(g_\Sigma'^2/4\pi)$ mb	0.0131 $(g_\Sigma'^2/4\pi)$ mb
	$O=i\gamma_5$	4.77 $(g_\Sigma'^2/4\pi)$ mb	5.27 $(g_\Sigma'^2/4\pi)$ mb

Table III. The total cross sections for $\pi^-+p\to Y_1^{*-}+K^+$

Interaction type	Cross section	$\sigma(\pi^-+p\to Y_1^{*-}+K^+)$	
		$\omega=2$ Bev	$\omega=3$ Bev
$O'=1$	$O=1$	0.0604 $(g_A^2/4\pi)$ mb	0.0343 $(g_A^2/4\pi)$ mb
	$O=i\gamma_5$	4.08 $(g_A^2/4\pi)$ mb	3.94 $(g_A^2/4\pi)$ mb
$O'=i\gamma_5$	$O=1$	0.0107 $(g_A'^2/4\pi)$ mb	0.0177 $(g_A'^2/4\pi)$ mb
	$O=i\gamma_5$	3.02 $(g_A'^2/4\pi)$ mb	3.24 $(g_A'^2/4\pi)$ mb

From these results we can see that the values of cross sections in the case of $O=i\gamma_5$ are about 100 times as large as those in the case of $O=1$ independently of the type of $N\Sigma K$ (or NIK)-interaction. This is mainly due to the difference of magnitude of coupling constant. Since the angular distributions

of K^+ for $\pi^-+p\to Y_1^{*-}+K^+$ have similar forms to those of K^0 for $\pi^-+p\to Y_0^*+K^0$, we show in Fig. 2 and Fig. 3 only the angular distributions for the latter process.

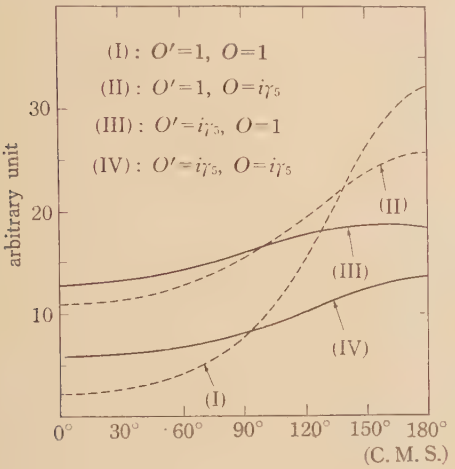

Fig. 2. The angular distributions of K^0 for $\pi^-+p\to Y_0^*+K^0$ at 2 Bev.

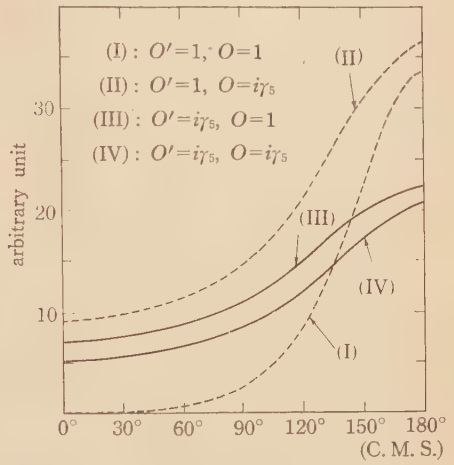

Fig. 3. The angular distributions of K^0 for $\pi^-+p\to Y_0^*+K^0$ at 3 Bev.

§ 4. $\pi^-+p\to\Sigma^0+K^{*0}$ and $\pi^-+p\to\Lambda+K^{*0}$ processes

We have previously studied K meson-pion interaction[6] taking into account the effects of K^*. And the values of coupling constants for $K^*\pi K$-interactions have been estimated as follows. When the interaction Hamiltonian in the case of $S=0$ is given by the form

$$H_1=m_K G_s \Phi(x)\,\tau_\alpha\varphi(x)\,\phi_\alpha(x), \tag{24}$$

$$G_s^2/4\pi=(0.15\sim0.20). \tag{25}$$

Φ, φ and ϕ_α are the wave functions of K^*, K meson and pion respectively. When the interaction Hamiltonian in the case of $S=1$ is given by the form

$$H_2=G_v\Phi_\mu\tau_\alpha\left(\varphi\frac{\partial\phi_\alpha}{\partial x_\mu}-\frac{\partial\varphi}{\partial x_\mu}\phi_\alpha\right), \tag{26}$$

$$G_v^2/4\pi=(0.35\sim0.45). \tag{27}$$

Adopting these interaction Hamiltonians we try to evaluate the cross sections for $\pi^-+p\to\Sigma^0+K^{*0}$ and $\pi^-+p\to\Lambda+K^{*0}$ reactions at $\omega=2$ Bev and 3 Bev. The differential cross sections for the former process are expressed as follows.

i) In the case of $S=0$,

$$\frac{d\sigma}{d\Omega}=\frac{k}{p(\sqrt{p^2+m^2}+\sqrt{p^2+\mu^2})^2}\left(\frac{m_K}{m}\right)^2\left(\frac{G_s^2}{4\pi}\right)\left(\frac{g_\Sigma^2}{4\pi}\right)$$

$$\times \frac{\left[-\frac{(\hat{I}\hat{F})}{m^2}+O'^2\frac{m_\Sigma}{m}\right]}{\left[-1-\left(\frac{m_\Sigma}{m}\right)^2+\left(\frac{m_K}{m}\right)^2-2\frac{(\hat{I}\hat{F})}{m^2}\right]^2}. \qquad (28)^*$$

ii) In the case of $S=1$,

$$\frac{d\sigma}{d\Omega}=\frac{k}{p(\sqrt{p^2+m^2}+\sqrt{p^2+\mu^2})^2}\left(\frac{G_v^2}{4\pi}\right)\left(\frac{g_\Sigma^2}{4\pi}\right)$$
$$\times \frac{\left[-\frac{(\hat{I}\hat{F})}{m^2}+O'^2\frac{m_\Sigma}{m}\right]}{\left[-1-\left(\frac{m_\Sigma}{m}\right)^2+\left(\frac{m_K}{m}\right)^2-2\frac{(\hat{I}\hat{F})}{m^2}\right]^2}\cdot\frac{(4p^2+k^2+4pk\cos\theta)}{m^2}. \qquad (29)^*$$

$\hat{I}$ and $\hat{F}$ are the four momenta of proton and Σ^0 respectively, and p and k are the momenta of pion and K^* respectively. θ is the angle between the direction of incident pion and that of Σ^0 in the final state.

Similarly we obtain the expressions for differential cross sections of $\pi^-+p\to\Lambda+K^{*0}$. The total cross sections for $\pi^-+p\to\Sigma^0+K^{*0}$ and $\pi^-+p\to\Lambda+K^{*0}$ reactions are shown in Table IV and Table V, where the values of coupling constants $G_s^2/4\pi=0.15$ and $G_v^2/4\pi=0.35$ are adopted.

Table IV. The total cross sections for $\pi^-+p\to\Sigma^0+K^{*0}$

Interaction type	Cross section	$\sigma(\pi^-+p\to\Sigma^0+K^{*0})$	
		$\omega=2$ Bev	$\omega=3$ Bev
$S=0$	$O'=1$	0.0371 $(g_\Sigma^2/4\pi)$ mb	0.0357 $(g_\Sigma^2/4\pi)$ mb
	$O'=i\gamma_5$	0.0052 $(g_\Sigma'^2/4\pi)$ mb	0.0052 $(g_\Sigma'^2/4\pi)$ mb
$S=1$	$O'=1$	0.990 $(g_\Sigma^2/4\pi)$ mb	1.17 $(g_\Sigma^2/4\pi)$ mb
	$O'=i\gamma_5$	0.148 $(g_\Sigma'^2/4\pi)$ mb	0.223 $(g_\Sigma'^2/4\pi)$ mb

Table V. The total cross sections for $\pi^-+p\to\Lambda+K^{*0}$

Interaction type	Cross section	$\sigma(\pi^-+p\to\Lambda+K^{*0})$	
		$\omega=2$ Bev	$\omega=3$ Bev
$S=0$	$O'=1$	0.0513 $(g_\Lambda^2/4\pi)$ mb	0.0377 $(g_\Lambda^2/4\pi)$ mb
	$O'=i\gamma_5$	0.0067 $(g_\Lambda'^2/4\pi)$ mb	0.0053 $(g_\Lambda'^2/4\pi)$ mb
$S=1$	$O'=1$	1.27 $(g_\Lambda^2/4\pi)$ mb	1.18 $(g_\Lambda^2/4\pi)$ mb
	$O'=i\gamma_5$	0.189 $(g_\Lambda'^2/4\pi)$ mb	0.226 $(g_\Lambda'^2/4\pi)$ mb

* With respect to O', see Eq. (21).

From these results we can derive the following conclusion. The values of cross sections in the case of $S=1$ are $20\sim40$ times as large as those in the case of $S=0$ independently of $N\Sigma K$ (or $N\Lambda K$)-interaction.

Since the angular distributions of Λ for $\pi^-+p\to\Lambda+K^{*0}$ have similar forms to those of Σ^0 for $\pi^-+p\to\Sigma^0+K^{*0}$, we show in Figs. 4 and 5 only the angular distributions for the latter process.

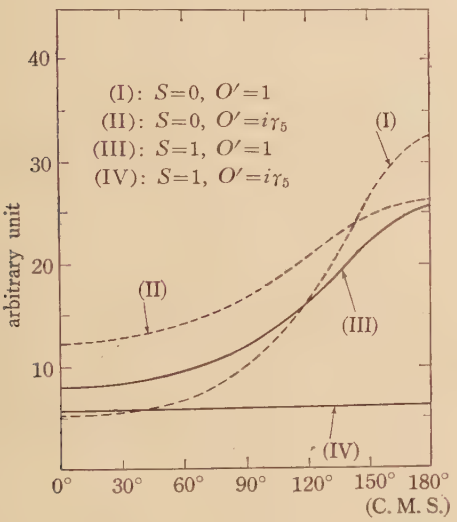

Fig. 4. The angular distributions of Σ^0 for $\pi^-+p\to\Sigma^0+K^{*0}$ at $\omega=2$ Bev.

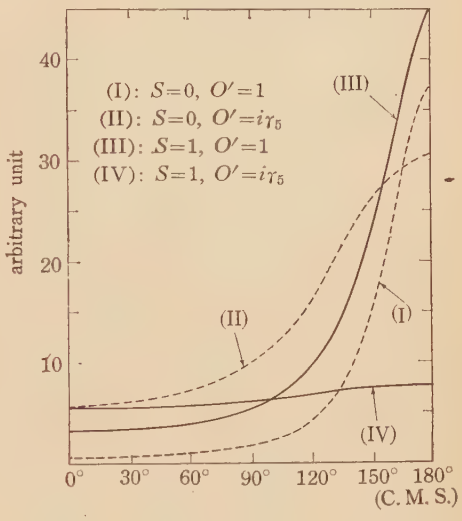

Fig. 5. The angular distributions of Σ^0 for $\pi^-+p\to\Sigma^0+K^{*0}$ at $\omega=3$ Bev.

§ 5. Some remarks on $\pi+N\to Y+\pi+K$ reactions

In this section not only $\pi^-+p\to Y+\pi+K$ but also $\pi^++p\to Y+\pi+K$ reactions are studied on the basis of Y^* or K^* model. Since the isotopic spins of Y_1^*, Y_0^* and K^* are equal to 1, 0 and 1/2 respectively, it may be said that $\pi+p\to Y+\pi+K$ reactions take place mainly through the following processes.

$$\pi^++p\to Y_1^{*+}+K^+\to\Lambda+\pi^++K^+, \tag{A}$$

$$\pi^++p\to\Sigma^++K^{*+}\to\Sigma^++\pi^0+K^+ \tag{B}$$

$$\searrow\Sigma^++\pi^++K^0, \tag{C}$$

$$\pi^-+p\to\Sigma^0+K^{*0}\to\Sigma^0+\pi^-+K^+, \tag{D}$$

$$\pi^-+p\to\Sigma^-+K^{*+}\to\Sigma^-+\pi^0+K^+. \tag{E}$$

From this model it follows that

$$\sigma(\pi^++p\to\Sigma^++\pi^++K^0)=2\sigma(\pi^++p\to\Sigma^++\pi^0+K^+). \tag{30}*$$

* Moreover, perturbation calculation gives the following relation:
$\sigma(\pi^++p\to\Sigma^++\pi^0+K^+)=\sigma(\pi^-+p\to\Sigma^0+\pi^-+K^+)$.

Now let us make some discussion about the process $\pi^-+p\to\Sigma^-+K^{*+}$ in reaction (E). When we denote the R-matrices for the states of isotopic spin $I=3/2$ and $I=1/2$ as $R_{3/2}$ and $R_{1/2}$ respectively, the lowest order perturbation calculation gives the following result because the matrix element of the lowest order ($g_\Sigma G_s$ or $g_\Sigma G_v$) for this $\pi^-+p\to\Sigma^-+K^{*+}$ process turns out to be zero.*

$$R_{3/2}=-2R_{1/2}. \tag{31}$$

From (31) it follows that

$$\sigma(\pi^-+p\to\Sigma^0+K^{*0})/\sigma(\pi^++p\to\Sigma^++K^{*+})=1/2. \tag{32}$$

In the process $\pi+N\to Y^*+K\to Y+\pi+K$ (for example, $\pi^-+p\to Y_0^*+K^0\to\Sigma^++\pi^-+K^0$) or $\pi+N\to Y+K^*\to Y+\pi+K$ (for example, $\pi^-+p\to\Sigma^0+K^{*0}\to\Sigma^0+\pi^-+K^+$), it is conventional to discuss Y^* or K^* data in terms of a simplified model in which they are produced and decay "isolated".** Then the main features of angular distributions of K^0 and Σ^0 for $\pi^-+p\to\Sigma^++\pi^-+K^0$ and $\pi^-+p\to\Sigma^0+\pi^-+K^+$ may respectively be represented by the forms of those of K^0 and Σ^0 for the processes $\pi^-+p\to Y_0^*+K^0$ and $\pi^-+p\to\Sigma^0+K^{*0}$. Thus we may say as follows. The angular distributions and energy spectra of Σ^+ in reaction (B) or (C) have similar forms to those of Σ^0 in reaction (D). If the mass difference between Y_1^* and Y_0^* is neglected, we may expect the similar situation between reaction (A) and $\pi^-+p\to Y_0^*+K^0\to\Sigma^++\pi^-+K^0$.

§ 6. Conclusions and discussions

It has been shown in the previous paper[6] that $S=1$ rather than $S=0$ is consistent with the experimental results for K^--p collision at 1.15 Bev/c. On the basis of this result our discussion is developed under the assumption of $S=1$. Comparing the results mentioned in Table III with those in Table V, we may say as follows.

i) If the $Y_1^*\Lambda\pi$-interaction is of scalar type (that is, $O=1$), the effects of K^* on $\pi^-+p\to\Lambda+\pi^-+K^+$ reaction overwhelm those of Y_1^*.

ii) When the $Y_1^*\Lambda\pi$-interaction is of pseudoscalar type (that is, $O=i\gamma_5$), the results depend on the $N\Lambda K$-interaction type. If $O'=1$, the effects of Y_1^* are comparable to (or slightly larger than) those of K^*. If $O'=i\gamma_5$, the effects of Y_1^* overwhelm those of K^*.

Next let us discuss the angular distributions for $\pi+N\to Y^*+K$ (for example, $\pi^-+p\to Y_0^*+K^0$) and $\pi+N\to Y+K^*$ (for example, $\pi^-+p\to\Sigma^0+K^{*0}$) pro-

* If $\sigma(\pi^-+p\to\Sigma^0+\pi^-+K^+)\cong\sigma(\pi^-+p\to\Sigma^-+\pi^0+K^+)$ is confirmed by experiment, the result may mean that the perturbation theory cannot be applied to such problems.

** The "isolated" model in $K^-+p\to\Lambda+\pi^++\pi^-$ reaction seems to be far from realistic. But this may be due to the interference effects arising from the requirement of Bose statistics for the final pions.[5] In the $\pi+p\to Y+\pi+K$ reactions, there is no need to take into account such interference effects.

cesses. Each process is of course one of the processes in which nucleon core is changed into hyperon core. But there is a remarkable difference between them. The K^* corresponds to a resonant state for pion cloud or K meson cloud of baryon, while the Y^* corresponds to a resonant state for the core in the center part. Therefore it can be expected in π-p collision at high energy that the angular distribution of hyperon in the latter process rather than that of Y^* in the former process has a peak in the backward direction. Such a property as this may be seen through Fig. 2$\sim$Fig. 5. Since Fig. 2 and Fig. 3 show the angular distributions of K^0 for $\pi^-+p\to Y_0^*+K^0$ process, the angular distributions of Y_0^* are larger in the forward than in the backward direction. Although the angular distributions of Σ^0 are remarkably larger in the backward than in the forward direction, it must be noted that they are nearly isotropic in the case of $O'=i\gamma_5$. Thus we may say as follows. If the experimental data show that the angular distribution of Σ^0(or Σ^+) for $\pi^-+p\to\Sigma^0+\pi^-+K^+$ (or $\pi^++p\to\Sigma^++\pi^++K^0$) reaction at 2 Bev$\sim$3 Bev is isotropic, then the assumption of $O'=i\gamma_5$ is consistent with the experimental results.

Let us try to make some discussion about the case of $O'=i\gamma_5$. In this case, the mechanism of $\pi^-+p\to\Sigma^0+K^{*0}$ process is similar to that of $K^-+N\to N+\bar{K}^*$ so far as the type of interaction is concerned. As was shown in the previous paper,[6] the angular distributions of protons in $K^-+p\to p+K^{*-}$ reaction at 1.15 Bev/c are isotropic and those at 2 Bev/c have a peak in the backward direction. Since $K^-+p\to p+K^{*-}$ at 1.15 Bev/c corresponds to the reaction in the neighborhood of threshold energy, it may easily be expected that K^{*-} is produced predominantly in s-state. The angular distributions of protons at 2 Bev/c may be interpreted as follows. i) In high energy region, a number of partial waves contribute to this reaction. ii) K^* will be emitted by the collision in periphery of the nucleon. On the other hand, not only the angular distributions of Σ^0 for $\pi^-+p\to\Sigma^0+K^{*0}$ at $\omega=2$ Bev but also those at 3 Bev are nearly isotropic when $O'=i\gamma_5$-type interaction is adopted. This is mainly due to the following situation. The $\pi^-+p\to\Sigma^0+K^{*0}$ reaction takes place through the interaction caused by a K meson exchange instead of a pion exchange. As $m_K\cong 3\mu$, it may be said that the K^{*0} in $\pi^-+p\to\Sigma^0+K^{*0}$ will be emitted by the collision in the inner region of nucleon compared with K^{*-} production in $K^-+p\to p+K^{*-}$ reaction. If the mass of a virtual K meson were equal to that of pion, the angular distributions of Σ^0 for $\pi^-+p\to\Sigma^0+K^{*0}$ at $\omega=3$ Bev would have a peak in the backward direction. As the pion energy is increased further, it may probably be expected that the Σ^0 turns out to be emitted predominantly in the backward direction.

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Dynamical Theory of Ultrasonic Attenuation. II
——*Note on the Weinreich Relation*——

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The Weinreich relation for longitudinal sound waves is derived from a general standpoint and it is pointed out that in weak collision cases an anisotropic structure of the Fermi surface reveals itself in the anisotropy of the acousto-electric effect. It is then shown that the same results follow for transverse cases when an allowance is made for the fact that the local Hall fields generated by a sound wave can yield a net electric current in the propagation direction of the wave.

§ 1. Introduction

The ultrasonic attenuation in a conduction electron system is attended with a secondary effect. Since a sound wave loses a part of its energy, the electron system necessarily gains the energy. At first sight it seems plausible to think in a simple manner that the energy is transferred directly to the electron system in a form of the kinetic energy in the propagation direction of the sound wave, as a result of which there comes out a net current of the electrons in that direction, which is the secondary effect in question. Starting from a phenomenological energy balance equation based on this standpoint, many authors[1] have calculated the steady field, the acousto-electro motive force, keeping this current, and some of them have derived a fundamental relation, the Weinreich relation, between the e.m.f. and the attenuation.*

However, the situation is not so simple, when viewed from a microscopic standpoint. The electric field under consideration is the steady part of the space-time average of the local field in the medium which the individual electron feels. The local electric field generated by a longitudinal sound wave (in an isotropic case) certainly points in the propagation direction and it is natural

* Further processes are of course possible in principle. Namely, the current will generate other sound waves in the medium by an interaction with the thermal lattice vibrations. The repetition of such processes will result in infinite (eventually non-convergent) series for the e. m. f. as well as for the attenuation constant. However, we shall not be concerned in this article with this rather academic problem, and shall confine ourselves to the first process, the attenuation of the original sound waves, and to the second process, the e. m. f. directly associated with the first one.

that a net steady field arises in that direction. However, the local electric field which is induced by a transverse sound wave and *which is directly connected with the attenuation of the wave* points, on the other hand, in a direction perpendicular to the propagation direction and it is unnatural from a symmetry reasoning that a net steady field arises in the propagation direction, or else in a particular perpendicular direction, if we disregard the effect of the magnetic fields also generated by the sound wave.

It is the purpose of the present article to clear up these points with the use of the general method developed in the author's previous article,[2] hereafter referred to as I, in which particular attention was paid on the screening of the electromagnetic interactions between the sound waves and the conduction electrons. In § 2 it is shown that the Weinreich relation holds in longitudinal cases between the attenuation and the steady electric field under mild restrictions. In § 3 it is shown that in transverse cases the steady electric field vanishes in isotropic cases, but the Weinreich relation holds between the attenuation and the steady Lorentz force.*

§ 2. Acousto-electric effect in longitudinal cases

The same notation and terminology as were used in I will be used also throughout this article. To be calculated is the steady electric field exerted on the conduction electrons in the presence of propagating longitudinal sound waves. The electric field $-e\boldsymbol{E}(\boldsymbol{x}, t)$ is given by

$$-e\boldsymbol{E}(\boldsymbol{x}, t)=e\left[\operatorname{grad}\phi(\boldsymbol{x}, t)+\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{x}, t)\right],$$

or, in Fourier components,

$$-e\boldsymbol{E}(\boldsymbol{\kappa}, t)=\sum_{\omega}\boldsymbol{F}(\boldsymbol{\kappa}, \omega)\, e^{-i\omega t} \tag{2·1}$$

with

$$\boldsymbol{F}(\boldsymbol{\kappa}, \omega)=i\left[\boldsymbol{\kappa}\phi(\boldsymbol{\kappa}, \omega)-\frac{\omega}{c}\boldsymbol{A}(\boldsymbol{\kappa}, \omega)\right]\delta_0(\boldsymbol{\kappa}), \tag{2·2}$$

where $\delta_0(\boldsymbol{\kappa})$ is the charge density operator, and $\phi(\boldsymbol{\kappa}, \omega)$ and $\boldsymbol{A}(\boldsymbol{\kappa}, \omega)$ are respectively the effective scalar and vector potentials in the deformable medium, related to an incident sound wave through Maxwell's equation. The sum in Eq. (2·1) is extended over $(\boldsymbol{\kappa}, \omega)$ of the propagating sound waves, that is, over (κ, ω) and $(-\kappa, -\omega)$ in our case.

Now, the average of the field has to be taken over the distribution function

$$f=f_0+f_1+\cdots$$

* We owe to Prof. S. Nakajima for calling our attention to it.

where f_0 is the equilibrium distribution in the absence of the sound waves and f_1 is the first order change in their presence whose expression in a matrix form is represented by Eq. (3·2) of I. Since we are dealing only with the steady part of the average, the average over f_0 does not concern us, and the lowest order terms are

$$\begin{aligned} \boldsymbol{F}^0 &= \langle -e\boldsymbol{E}(\boldsymbol{x}, t)\rangle_{steady} \\ &= \sum_{\kappa,\omega}\sum_{p,q}\{f_0(E_p)-f_0(E_q)\}\langle q|H'(\boldsymbol{\kappa}, \omega)|p\rangle D_{pq}(\omega)\, F(-\boldsymbol{\kappa}, -\omega), \end{aligned} \tag{2·3}$$

where

$$H'(\boldsymbol{\kappa}, \omega) = -\delta_\nu(\boldsymbol{\kappa})\, A_\nu(\boldsymbol{\kappa}, \omega) + \delta_0(\kappa)\,\phi(\boldsymbol{\kappa}, \omega) \tag{2·4}$$

with $\delta_\nu(\boldsymbol{\kappa})$ denoting the current density operator is the perturbation due to the sound waves.

If we suppose that sound velocities suffer negligible anisotropic influences from the conduction electrons,* the propagation direction of the sound waves in the medium is parallel to the incident direction provided that there occurs no anisotropy in them from other origins.

In the longitudinal case the field parallel to the propagation direction (taken to be z-direction throughout) is

$$\begin{aligned} F_z^0 = \sum_{\kappa,\omega}\sum_{p,q}\{f_0(E_p)-f_0(E_q)\}\, i\kappa|\langle p|\,\delta_0(\kappa)\,|q\rangle|^2 \\ \times D_{pq}(\omega)\left(1-\frac{\omega^2}{\kappa^2 c^2}\right)^2|\phi(\kappa, \omega)|^2, \end{aligned}$$

where p and q specify the eigenstates of the total Hamiltonian of the system, and the Lorentz condition together with the same sum-rule type identity as was used in deriving Eq. (3·10) of I is again used. It will be shown later that the above expression describes the dominant effect. In terms of the longitudinal electric susceptibility $\Gamma(\kappa, \omega)$ defined by Eq. (8·2a) of I, which is a sufficiently general expression for our purpose provided that a diffraction of the incident sound waves does not occur, we have

$$\begin{aligned} F_z^0 &= -\sum_{\kappa,\omega} i\kappa\Gamma(\kappa, \omega)\left(1-\frac{\omega^2}{\kappa^2 c^2}\right)^2|\phi(\kappa, \omega)|^2 \\ &= \sum_{\kappa,\omega}\kappa\,\mathrm{Im}\,\Gamma(\kappa, \omega)\left(1-\frac{\omega^2}{\kappa^2 c^2}\right)^2|\phi(\kappa, \omega)|^2, \end{aligned} \tag{2·5}$$

the last equation being due to the identity $\Gamma(-\kappa, -\omega)=\Gamma^*(\kappa, \omega)$. Substituting the solution of Eq. (3·14a) of I for $\phi(\kappa, \omega)$, we obtain

$$F_z^0 = 2\kappa\,\mathrm{Im}\left(\frac{-1}{\kappa^2+4\pi\Gamma(\kappa, \omega)}\right)4\pi|\rho^0(\kappa, \omega)|^2. \tag{2·6}$$

* That this is the case is shown in § 7 of I for weak collision cases ($\kappa l \gg 1$), but strong collision cases ($\kappa l \ll 1$) have not been examined yet.

The attenuation constant, on the other hand, can be derived from the dispersion relation ((5·3a) of I):

$$\omega^2=\left[\omega_p{}^2\frac{m}{M}\frac{1}{\kappa^2+4\pi\Gamma(\kappa,\omega)}+(u_l{}^0)^2\right]\kappa^2. \tag{2·7}$$

Owing to the actually valid inequalities:*

$$\mathrm{Re}\,\Gamma(\kappa,\omega)\simeq\mathrm{Re}\,\Gamma(0,0)\gg(\kappa^2,\ \mathrm{Im}\,\Gamma(\kappa,\omega))$$

and

$$\omega\gg u_l\alpha_l, \tag{2·8}$$

where u_l is the longitudinal sound velocity given by

$$u_l{}^2=(u_l{}^0)^2+\omega_p{}^2\frac{m}{M}\frac{1}{4\pi\,\mathrm{Re}\,\Gamma(0,0)}, \tag{2·9}$$

we have

$$\alpha_l=\mathrm{Im}\,\kappa=-\frac{m\omega\omega_p{}^2}{2Mu_l{}^3}\,\mathrm{Im}\left(\frac{1}{\kappa^2+4\pi\Gamma(\kappa,\omega)}\right). \tag{2·10}$$

Comparison of Eq. (2·10) with Eq. (2·6) gives

$$\frac{\alpha_l}{F_z{}^0}=\frac{m\omega_p{}^2}{4Mu_l{}^2 4\pi|\rho^0(\kappa,\omega)|^2}$$

$$=\frac{u_l}{2Q},$$

where Q is the energy flow density of the sound wave

$$Q=2nMu_l\left|\frac{u_l\rho^0(\kappa,\omega)}{-ne}\right|^2.$$

If we define an equivalent electric field $E_z{}^0$ by

$$F_z{}^0=-neE_z{}^0,$$

where n is the electron number density, then we obtain the Weinreich relation

$$\frac{\alpha_l}{|E_z{}^0|}=\frac{neu_l}{2Q}. \tag{2·11}$$

The factor 2 arises from the difference in the definition of the attenuation from the conventional one (attenuation of the amplitude instead of that of the energy). This relation is quite general, being valid independently of the collision mechanisms as well as of the degree of the degeneracy of the conduction electrons.

From now on, a scattering-free degenerate electron case will be discussed somewhat in detail. For a case of degenerate Bloch electrons of a single band

* Otherwise, sound waves cannot exist in the medium.

with mean free path longer than the acoustic wavelength, α_l is expressed in terms of an integral of certain geometrical quantities on the Fermi surface. The Weinreich relation implies that the same is the case also for the accusto-electric effect. In other words, the anisotropic structure of the Fermi surface reveals itself also in the anisotropy of the acousto-electro motive force. An analysis of the Fermi surface with this effect seems in principle more powerful than the analysis with the attenuation, for the present effect will be to a considerable extent free from the excess anisotropies mentioned in § 7 of I.*

Now, we shall examine whether the anisotropic structure of the Fermi surface can yield a remarkable steady e.m.f. in a direction perpendicular to the propagation one. To see this we have to evaluate the steady electric field in, say, x-direction (one of the principal axes), which is from Eqs. (2·2) and (2·3)

$$F_x^0=-i\sum_{\kappa,\omega}\frac{\omega}{c}\Gamma(\kappa,\omega)\left(1-\frac{\omega^2}{\kappa^2c^2}\right)\phi(-\kappa-\omega)\,A_x(\kappa,\omega).$$

Thus the ratio E_x^0 to E_z^0 is equal to that of $(\omega/\kappa c)A_x(\kappa,\omega)$ to $\phi(\kappa,\omega)$. From (AII·3) of I this ratio is given approximately by

$$-4\pi\mathrm{K}_{x0}(\kappa,\omega)\Big/\left(\kappa^2-\frac{\omega^2}{c^2}-4\pi\mathrm{K}_{xx}(\kappa,\omega)\right), \tag{2·12}$$

where $\mathrm{K}_{ij}(\kappa,\omega)$ is defined by (AII·2) of I. $\mathrm{K}_{x0}(\kappa,\omega)$ can be reduced as follows.

$$\begin{aligned}\mathrm{K}_{x0}(\kappa,\omega)&=i\sum_{\boldsymbol{k}}\{f_0(E_{\boldsymbol{k}})-f_0(E_{\boldsymbol{k}+\boldsymbol{\kappa}})\}\langle\boldsymbol{k}|\delta_0(-\boldsymbol{\kappa})|\boldsymbol{k}+\boldsymbol{\kappa}\rangle\langle\boldsymbol{k}+\boldsymbol{\kappa}|\delta_x(\boldsymbol{\kappa})|\boldsymbol{k}\rangle\\&\quad\times\left\{\frac{1}{E_{\boldsymbol{k}}-E_{\boldsymbol{k}+\boldsymbol{\kappa}}+\hbar\omega}-i\pi\delta(E_{\boldsymbol{k}}-E_{\boldsymbol{k}+\boldsymbol{\kappa}}+\hbar\omega)\right\}\\&=\frac{ie^2}{c}\sum_{\boldsymbol{k}}\frac{\partial f_0}{\partial E_{\boldsymbol{k}}}v_x(\boldsymbol{k})\{1+i\pi\hbar\omega\delta(E_{\boldsymbol{k}}-E_{\boldsymbol{k}+\boldsymbol{\kappa}}+\hbar\omega)\}+O\left(\frac{u_l}{\langle v_0\rangle}\right),\end{aligned}$$

where $v_x(\boldsymbol{k})=(i/\hbar)(\partial E_{\boldsymbol{k}}/\partial k_x)$, $\langle v_0\rangle$ is the average Fermi velocity and interband effects are dropped out. The symmetry relation $v_x(-\boldsymbol{k})=-v_x(\boldsymbol{k})$ simplifies the above expression to

$$\mathrm{K}_{x0}(\boldsymbol{\kappa},\omega)\simeq-\frac{\pi e^2}{c}\hbar\omega\sum_{\boldsymbol{k}}\frac{\partial f_0}{\partial E_{\boldsymbol{k}}}v_x(\boldsymbol{k})\,\delta(E_{\boldsymbol{k}}-E_{\boldsymbol{k}+\boldsymbol{\kappa}}+\hbar\omega).$$

Using the same notation as in § 7 of I, we have

$$\mathrm{K}_{x0}(\boldsymbol{\kappa},\omega)\propto\int_\gamma\frac{v_x(\boldsymbol{k})}{|\mathrm{grad}\,E_{\boldsymbol{k}}|^2}\rho\sin\theta\,ds, \tag{2·13}$$

where γ is the path on the Fermi surface having the property that

* A numerical estimation for a typical metal shows that this effect is strong enough for being detected (see also Mikoshiba's paper of ref. 1)).

$$\cos\beta=\frac{\omega}{|\kappa|v_0} \tag{2·14}$$

holds everywhere on the path. If β, an angle between a velocity on γ and z-direction, were supposed to be $\pi/2$, in which case we denote the path by γ', then the integration vanishes on account of the relation $v_x(-\boldsymbol{k})=-v_x(\boldsymbol{k})$. Hence the magnitude of the integral depends, besides on the anisotropic structure, also on the deviation of γ from γ'. However, this deviation is extremely small in practice so that if the Fermi surface is moderately smooth, the quantity under consideration may be quite small. The restriction on the smoothness is from Eqs. (2·12) and (2·13)

$$\left|\int_\gamma \frac{cv_x(\boldsymbol{k})}{|\mathrm{grad}\,E_k|^2}\rho\sin\theta\,ds\right| \ll \left|\int_\gamma \frac{v_x^2(\boldsymbol{k})}{|\mathrm{grad}\,E_k|^2}\rho\sin\theta\,ds\right|, \tag{2·15}$$

except for extremely high frequencies. Only under this restriction is the procedure in deriving Eq. (2·6) justified and the Weinreich relation valid.

§ 3. Transverse cases and discussions

The aim of this section is first to show by the same procedure as before that the steady electric fields in the presence of a transverse sound wave arise only from an anisotropic structure of the Fermi surface, and then to show, after calculating the steady part of the Lorentz force, that the Weinreich relation holds between the attenuation and the Lorentz force.

The steady electric field in the direction of one of the principal axes is from Eqs. (2·2) and (2·3)

$$\begin{aligned} F_x^0 &= -\sum_{\kappa,\omega}\sum_{k}\frac{i\omega}{c}\{f_0(E_k)-f_0(E_{k+\kappa})\}\langle\boldsymbol{k}|\delta_0(-\boldsymbol{\kappa})|\boldsymbol{k}+\boldsymbol{\kappa}\rangle\langle|\boldsymbol{k}+\boldsymbol{\kappa}|\delta_x(\boldsymbol{\kappa})|\boldsymbol{k}\rangle \\ &\quad\times\left\{\frac{P}{E_k-E_{k+\kappa}+\hbar\omega}-i\pi\delta(E_k-E_{k+\kappa}+\hbar\omega)\right\}|A_x(\boldsymbol{\kappa},\,\omega)|^2 \\ &\simeq\sum_{\kappa,\omega}\frac{\hbar\omega^2e^2}{4\pi^2c^2\kappa}\int_\gamma\frac{v_x(\boldsymbol{k})}{|\mathrm{grad}\,E_k|^2}\rho\sin\theta\,ds|A_x(\boldsymbol{\kappa},\,\omega)|^2. \end{aligned} \tag{3·1}$$

It is evident that the above integral vanishes in isotropic cases, and, barring exceptional complexities, will be very small in anisotropic cases. Meanwhile, the steady electric field in z-direction has the same structure as E_x^0 in the previous section, and hence the same conclusion follows. Thus, so far as we confine ourselves only in the electric fields, no appreciable steady field is present in any direction, especially none in isotropic cases.

Now, we turn our attention to the Lorentz force, which is

$$F_z^L=-\frac{e}{mc}\left[\boldsymbol{p}+\frac{e}{c}\boldsymbol{A},\ \mathrm{rot}\,\boldsymbol{A}\right]_z$$

$$=\sum_{\kappa,\omega}\left\{i\kappa\delta_x(\kappa)\,A_x(\kappa,\omega)\,e^{-i\omega t}-\frac{i\kappa e^2}{mc^2}|A_x(\kappa,\omega)|^2\right\},\qquad(3\cdot2)$$

where we have retained only the steady part of the second order terms. Taking the average of the above expression, and retaining only the steady parts, we have

$$F_z^L=\sum_{\kappa,\omega}(i\kappa)\left\{\frac{ne^2}{mc^2}+\sum_{p,q}(f_0(E_p)-f_0(E_q))\,\langle p|\delta_x(-\kappa)|q\rangle|^2 D_{pq}(\omega)\right\}|A_x(\kappa,\omega)|^2$$

$$=\sum_{\kappa,\omega}(i\kappa)\,\Delta(\kappa,\omega)|A_x(\kappa,\omega)|^2=-\sum_{\kappa,\omega}\kappa\,\mathrm{Im}\,\Delta(\kappa,\omega)|A_x(\kappa,\omega)|^2,$$

where $\Delta(\kappa,\omega)$ is the transverse susceptibility defined by Eq. (8·2b) of I. Substituting the solution of Eq. (3·14b) of I for $A_x(\kappa,\omega)$, we have

$$F_z^L=2\kappa\,\mathrm{Im}\left(\frac{1}{\kappa^2-\omega^2/c^2+4\pi\Delta(\kappa,\omega)}\right)\frac{4\pi}{c^2}|j_x(\kappa,\omega)|^2.\qquad(3\cdot3)$$

The attenuation constant for the transverse case, on the other hand, can be derived from the dispersion relation ((5·3b)) of I)

$$\omega^2\left[1+\frac{m}{M}\,\frac{\omega_p^2/c^2}{\kappa^2-\omega^2/c^2+4\pi\Delta(\kappa,\omega)}\right]=(u_t^0)^2\kappa^2.$$

Assuming that the sound velocity suffers no appreciable influence from the conduction electrons, we obtain

$$\alpha_t=\frac{m\omega\omega_p^2}{2Mu_tc^2}\,\mathrm{Im}\left(\frac{1}{\kappa^2-\omega^2/c^2+4\pi\Delta(\kappa,\omega)}\right).\qquad(3\cdot4)$$

Comparison of Eq. (3·3) with Eq. (3·4) gives

$$\frac{\alpha_t}{F_z^L}=\frac{u_t}{2Q},\qquad(3\cdot5)$$

where Q is the energy flow density of the sound wave

$$Q=2nMu_t\left|\frac{j_t^0(\kappa,\omega)}{ne}\right|^2.$$

If we define an equivalent electric field E_z^0 by

$$F_z^L=-neE_z^0,$$

then the Weinreich relation follows

$$\frac{\alpha_t}{|E_z^0|}=\frac{neu_t}{2Q}.\qquad(3\cdot6)$$

Eq. (3·6) is indirectly proved in n-type Germanium; namely it is used in a successful order estimation of the inter-valley scattering rates in that substance

(see the paper of G. Weinreich et al. of ref. 1)).

In what follows our results will be discussed in weak collision cases. Let us consider a free electron case for simplicity. As has been referred to in I, the attenuations in weak collision cases are determined by dynamical properties of those electrons whose velocities are such that the electric field they feel is steady as a result of the Doppler effect. Such electrons make a closed circle on the Fermi surface. Owing to the sinusoidal spatial distribution of the electric field with a definite wavelength, there occurs a steady deceleration as well as acceleration of the electrons. However, on account of the slight inclination of their velocity vectors with xy-plane, there remains the second order effect in the electron system. That is to say, in the longitudinal case the electrons gain the energy from the sound waves directly in a form of the kinetic energy in z-direction, and there follows accordingly a net current of them in $+z$-direction. The driving force of this current is nothing but E_z^0 obtained in § 2. The situation differs a little in the transverse case though the results are the same. As a direct consequence of the interaction the energy of the sound wave is partly transferred to the kinetic energy of the electron system in the direction of the induction force with no net current nevertheless followed because of the symmetry, but a net current arises in the propagation direction as a result of a superposition of the local Hall currents.

Finally, it should be noted that the steady field E_z^0 in the longitudinal case is just the same as the frictional force that gives rise to a resistance to an electronic motion interacting with the thermal lattice vibrations. The results obtained hitherto show that the transverse sound waves should have effects of the same order on the electric resistance as the longitudinal ones, at least in a frequency range in which the attenuations for both waves are the same in order. According to the results of I, this is the case with ω's less than 10^8 or so, so that under ordinary conditions no appreciable effects are expected for natural thermal lattice vibrations, but it is not so with artificial vibrations. The same discussion is given independently by Stoltz.[3]

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Alpha-Particle Reduced Widths in Heavy Nuclei

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The α-particle reduced widths γ_α^2 for the ground state in Po^{210} and Po^{212} are calculated on the basis of the nuclear shell model. The calculations are made taking the boundary condition in an approximate way into consideration. The effects of the configuration mixing of the parent and daughter nucleus wave functions on γ_α^2 are examined and it is found that they give quite large contributions to γ_α^2. Some features of the distortion of the two nucleon wave functions arising from the configuration mixing are discussed graphically. It is shown that the wave function of relative motion with the mixing of the level which is lowered from the upper band by spin-orbit force would correspond to the bound electron-pair in the superconducting metals. Although there are some unavoidable uncertainties in the course of the calculations, it is concluded that the wave functions derived by conventional shell model calculations can explain the major part of the experimental values of γ_α^2.

§ 1. Introduction

It is well known that the independent particle model is remarkably successful in explaining the various problems of the nucleus, including the nuclear reactions. The justifications of the independent particle model on the basis of the strong nuclear forces between nucleons have been made by Brueckner et al.[1] However, their theories treat only the case of the infinite nuclear matter and do not describe the behavior of nucleons in the low density region of nuclear surface. Although there are some studies for a dilute Fermi gas,[2] it is not clear whether the conclusions from the studies of the endless dilute gas are applicable to the structure of nuclear surface without any modification. Perhaps the true theory of nuclear surface will have the finite size effect as a matter of first consideration.[3] Unfortunately, there are scarcely any clear-cut experimental facts concerning the characteristic nucleon-correlation at the nuclear surface.[4]

As is well known, the physical quantity which gives a simple measure of nucleon-correlation at the nuclear surface is the reduced widths for the composite particle channels, such as for deuteron or α-particle. In this paper we shall examine to what extent the experimental values of γ_α^2 are explained by the independent particle model within a finite well, such as harmonic oscillator shell model. In the case of light nuclei, we can expect large γ_α^2 also from the independent particle model,[5] since both protons and neutrons are in the same

orbit and the size of the α-particle and light nuclei are not so different. Therefore, we shall examine the cases of heavy α-decaying nuclei in which one might be able to expect large difference between γ_α^2(I.P.M.) and γ_α^2(exp.) if the nucleus has nucleon-correlation at the nuclear surface.

Recently, Mang[6] has calculated the γ_α^2's for nuclei around Pb^{208} on the basis of the nuclear shell model, and obtained good agreement with the experimental data. However, he discussed only the relative magnitude of γ_α^2 and not the absolute values, because he neglected the boundary condition at the nuclear surface. Since for the evaluation of the absolute magnitude of γ_α^2 the boundary condition at the nuclear surface is very important, we have tried to calculate γ_α^2 taking into account the boundary condition as far as possible.

Moreover, we have examined the effect of configuration mixing on the value of γ_α^2 and found that it gives quite large contributions to the γ_α^2. For the case of Po^{210} decay, it is concluded that the nuclear shell model with configuration mixing can explain the experimental value of γ_α^2 satisfactorily, and we need not assume any other mechanism of "clustering" of particles at the nuclear surface.

§ 2. Method of calculation

The most general theory of α-decay is developed by Thomas[7] from the time-independent R-matrix theory of nuclear reactions. He derived the relation between the decay constant λ and the reduced widths γ_α^2 in the following form:

$$\lambda = \frac{2\gamma_{\alpha L}^2}{\hbar} P_L \tag{1}$$

where P_L is the familiar barrier penetrability. The reduced widths amplitude $\gamma_{\alpha L}$ can be expressed as

$$\gamma_{\alpha L} = \sqrt{\frac{\hbar^2 R_0}{2M}} \int \varphi_{JM}^* \chi_\alpha(x_\alpha) \sum_m (Ljm M-m\, JM)\, Y_L^m\left(\begin{matrix} \boldsymbol{R} \\ R \end{matrix}\right) \Psi_{jM-m}(x_\kappa)\, dx_\kappa\, dx_\alpha\, d\Omega_R \tag{2}$$

where φ_{JM} and ψ_{jM-m} are the wave functions of the parent and the daughter nuclei. χ_α and $Y_L^m(\boldsymbol{R}, R)$ are the internal wave function and the angular part of the centre-of-mass motion of α-particle, respectively.

As is well known, the barrier penetrability P_L is very sensitive to R_0, beyond which α-particle is not influenced by the nuclear force. However, the recent optical model analyses of α-nucleus scatterings have determined the shape of the nuclear potential at the nuclear surface fairly definitely. In Fig. 1 we reproduce the nuclear optical potential of Pb^{208} for 40 Mev α-particle determined by Igo.[8] Due to the strong absorbent of nucleus for α-particle, the inner parts of the potential are not determined, but all potential-sets which have the same form at the nuclear surface can give good agreement with the experimental data. Assuming that the shape of the potential-tail will not be sensitive to the energy, although the main body of the optical potential is, we have determined

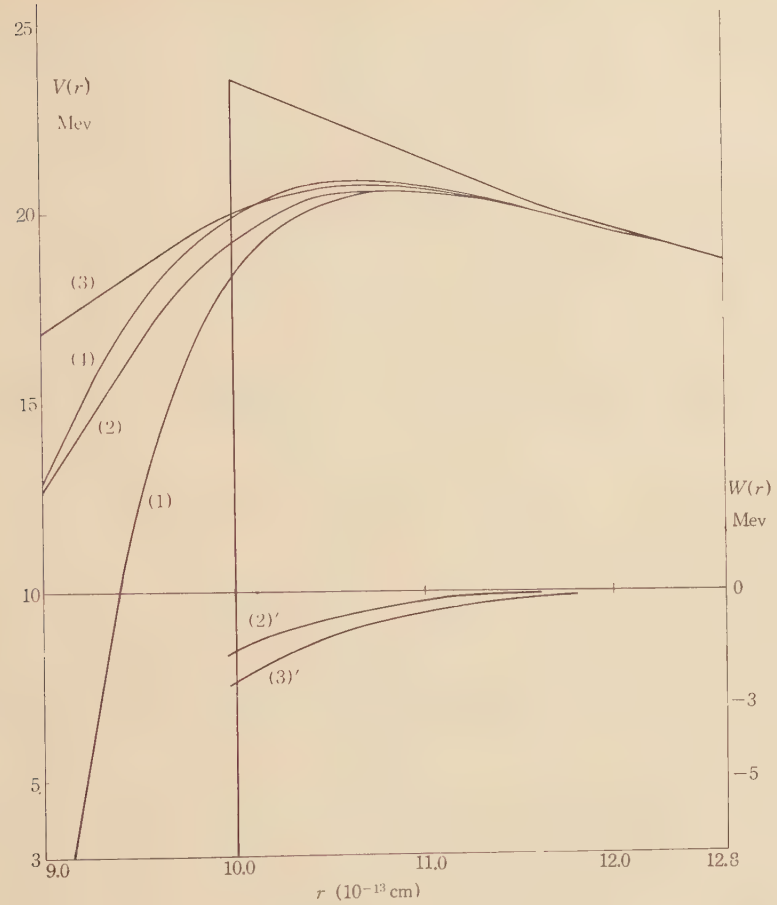

Fig. 1. Various optical potentials of Pb^{208} for 40 Mev α-particle which can give equally good agreement with the experimental data. For real and imaginary parts the left and right scales should be taken, respectively. These have been determined by G. Igo.[8]

the interaction radius R_0 as 10 fermi ($=1.4A^{1/3}+1.6$ fermi) for the cases of α-decay of Po^{210} and Po^{212} in which the energies of α-particle are several Mev. Idealizing the nuclear potential as the schematic one drawn in Fig. 1, we calculate the barrier penetrability P_L, and then from the experimental values of the decay constant, we derive the α-particle reduced widths for ground state of heavy nuclei which are to be compared with the theoretical values.*

* The reduced widths δ^2 given by Rasmussen[9] are derived under a different definition from ours. His definition and its relation to ours are

$$\lambda=(\delta^2/h)e^{-2G},$$
$$\delta^2=4\pi kR_0\gamma_\alpha{}^2\simeq 120\,\gamma_\alpha{}^2$$

where
$$k=\frac{\sqrt{2M_\alpha E_\alpha}}{\hbar}.$$

As mentioned in the Introduction, our aim is to calculate the $\gamma_{\alpha L}$ by using Eq. (2) on the basis of the nuclear shell model. For this purpose, the following assumptions are made for the wave functions.*

(1) The α-particle wave function is of Gaussian type.

$$\chi_\alpha(1234)=\left(\frac{2\beta^{3/2}}{(1/2)!}\right)^{3/2}\exp\left[-\frac{\beta}{2}(\xi_1^2+\xi_2^2+\xi_3^2)\right](4\pi)^{-3/2}\chi_0^0(12)\chi_0^0(34), \tag{3}$$

$$\xi_1=\frac{1}{\sqrt{2}}(r_1-r_2),\quad \xi_2=\frac{1}{\sqrt{2}}(r_3-r_4),\quad \xi_3=\frac{1}{2}(r_1+r_2-r_3-r_4)$$

χ_0^0: spin singlet function.

The constant β is chosen so that the *rms* radius of the charge density is equal to the measured value (1.6 fermi).[10] From the equation

$$R_\alpha^2=\frac{1}{4}\sum_i\langle\chi_\alpha(r_i-R)^2\chi_\alpha\rangle, \tag{4}$$

we have

$$\beta=0.44\ \text{fermi}^{-2}.$$

(2) The wave functions of the parent and daughter nucleus are shell model wave functions with seniority zero (in this paper, we treat only even-even nuclei: Po^{210}, Po^{212}). For the single particle wave functions, harmonic oscillator functions are used:

$$\varphi_{\nu lj}^m=\sqrt{\frac{2\nu!\,b^{3/2}}{(\nu+l+1/2)!}}L_\nu^{l+1/2}(br^2)(\sqrt{b}\,r)^l e^{-(1/2)br^2}\sum_\mu\left(l\ \frac{1}{2}\ \mu m-\mu,jm\right)Y_l^\mu\left(\frac{r}{r}\right)\chi_{1/2}^{m-\mu}, \tag{5}$$

where

$$L_\nu^{l+1/2}(br^2)=\sum_{h=0}^{\nu}\binom{\nu+l+1/2}{\nu-h}\frac{(-)^h}{h!}(br^2)^h.$$

With the above assumptions, we get the reduced widths[11] for the ground state transitions of even-even nuclei, in which cases all angular momenta of the parent, daughter nuclei and of the outgoing α-particle are equal to zero. It is

$$\gamma_\alpha\Big/\sqrt{\frac{\hbar^2R_0}{2M}}=\left(\frac{A_1!}{2!\,(A_1-2)!}\,\frac{A_3!}{2!\,(A_3-2)!}\right)^{1/2}$$
$$\times\sqrt{\frac{2j_1+3-A_1}{(A_1-1)(2j_1+1)}}\sqrt{\frac{2j_3+3-A_3}{(A_3-1)(2j_3+1)}}\times\mathcal{O}(R_0). \tag{6}$$

$A_1(A_3)$ is the number of protons (neutrons) in the unfilled subshell with quantum number $\nu_1l_1j_1(\nu_3l_3j_3)$ in the parent nucleus. The second factor is the double

* We adopt the same notation as Mang's.

parentage coefficient.[12] The last factor is the overlap integral between four nucleons in the j_1 and j_3 orbits and the α-particle leaving the daughter nucleus with the orbital angular momentum zero:

$$\mathcal{O}(R_0)=\int \psi_p^*(j_1^2,\ J_p=0)\psi_n^*(j_3^2,\ J_n=0)\chi_\alpha(1234)\,Y_0^0\left(\frac{\boldsymbol{R}}{R}\right)dx_\alpha d\Omega_R. \qquad (7)$$

Integration implies also the summation over spin variables. Since the $\chi_\alpha(1234)$ includes the spin-singlet function, we must transform the j–j coupling representation of each two-nucleon wave function to the L–S coupling representation. Then

$$\mathcal{O}(R_0)=T(00\,;\,j_1j_1)\,T(00\,;\,j_3j_3)\int \psi'_p{}^*(L=0,\ S=0)\,\psi'_n{}^*(L=0,\ S=0)\chi_\alpha\,Y_0^0\,dx_\alpha d\Omega_R, \qquad (8)$$

where

$$T(L=0,\ S=0\,;\,j_1j_1)=\frac{(-)^{(j_1+1/2-j_1)}}{2}(2j_1+1)\,W\left(j_1l_1j_1l_1\,;\,\frac{1}{2}0\right). \qquad (9)$$

By using Talmi transformation coefficients $\langle NL,\ nl\,;\,\bar{L}|N_1l_1,\ N_2l_2\,;\,\bar{L}\rangle$,[13] the following expression is derived,

$$\begin{aligned}
\mathcal{O}(R_0)&=T(00\,;\,j_1j_1)\,T(00\,;\,j_3j_3)\int \sum_{Nn_1n_2}\langle NSn_3s\,;\,0|N_1SN_2S\,;\,0\rangle\\
&\times\langle N_1Sn_1s\,;\,0|\nu_1l_1\nu_1l_1\,;\,0\rangle\langle N_2Sn_2s\,;\,0|\nu_3l_3\nu_3l_3\,;\,0\rangle\\
&\times\psi'_{n_1s}(\xi_1)\,\psi'_{n_2s}(\xi_2)\,\psi'_{n_3s}(\xi_3)\,\psi'_{NS}(R_0)\\
&\times\left(\frac{2\beta^{3/2}}{(1/2)!}\right)^{3/2}\exp\left[-\frac{\beta}{2}(\xi_1^2+\xi_2^2+\xi_3^2)\right](4\pi)^{-3/2}\\
&\times\xi_1^2\xi_2^2\xi_3^2\,d\xi_1\,d\xi_2\,d\xi_3(4\pi)^3\,d\Omega_R\\
&=T(00\,;\,j_1j_1)T(00\,;\,j_3j_3)\sum_{Nn_1n_2}\langle NSn_3s\,;\,0|N_1SN_2S\,;\,0\rangle\\
&\times\langle N_1Sn_1s\,;\,0|\nu_1l_1\nu_1l_1\,;\,0\rangle\langle N_2Sn_2s\,;\,0|\nu_3l_3\nu_3l_3\,;\,0\rangle\\
&\times\left(\frac{1}{2}!\right)^{-3/2}\left[\frac{(n_1+1/2)!\,(n_2+1/2)!(n_3+1/2)!}{n_1!\,n_2!\,n_3!}\right]^{1/2}\\
&\times\left(\frac{2\sqrt{\beta b}}{\beta+b}\right)^{9/2}\left(\frac{\beta-b}{\beta+b}\right)^{n_1+n_2+n_3}\sqrt{\frac{2\cdot N!\,(4b)^{3/2}}{(N+1/2)!}}\,L_N^{1/2}(4bR_0^2)\,e^{-2bR_0^2}\\
&\equiv\sum_N C_N\left(\frac{\beta-b}{\beta+b}\right)^{N_m-N}\psi_{NS}(R_0),
\end{aligned} \qquad (10)$$

where N_m is the maximum of N and is given by

$$N_m=2\nu_1+l_1+2\nu_3+l_3.$$

Table I.

	$x=\frac{1}{2}R_0^3\psi_{NS}^2(R_0)$	
NL	$Po^{210}(S=-18.5)$*	$Po^{210}(S=-16.7)$*
11 S	0.785	0.855
10 S	0.756	0.803
9 S	0.719	0.757

* S is the well-known shift factor and is expressed as

$$S=-R_0\frac{\sqrt{2M_\alpha(V-E_\alpha)}}{\hbar}$$

in the W. K. B. approximation.

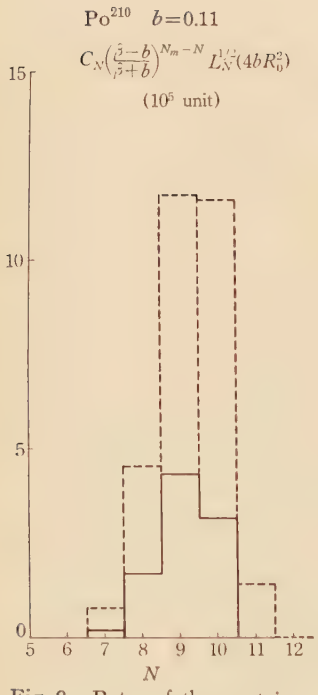

Fig. 2. Rates of the contribution to γ_α from each $\psi_{NS}(R_0)$ in Po^{210}. The full and dotted lines are the ones for pure configuration and mixed configuration, respectively.

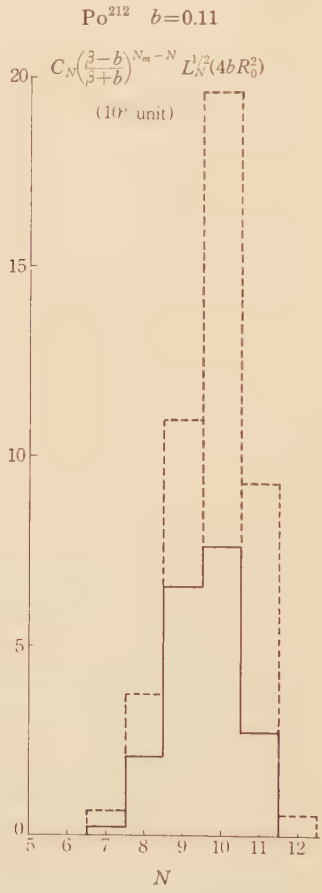

Fig. 3. Rates of the contribution to γ_α from each $\psi_{NS}(R_0)$ in Po^{212}. The full and dotted lines are the ones for pure configuration and mixed configuration, respectively.

The size parameter b of the harmonic oscillator is determined as follows. The eigenstate in the infinite harmonic oscillator well which is regular at infinity cannot have an open channel. Therefore it is obvious that the shell model wave function does not satisfy the boundary condition which we normally impose upon a compound state in resonance reaction theory. Then, instead of fitting the logarithmic derivative of the internal wave function to the external one, we determine the size parameter b so that the value of the harmonic oscillator wave function at R_0 is the same as that of the square well wave function satisfying the boundary condition at R_0 exactly. At this stage we regard the size parameter b as a free parameter which adjusts only the magnitude of wave function at R_0, and forget its relation to the eigenenergy $\hbar\omega$.

The values of the square well wave func-

tions at R_0, which satisfy the boundary conditions, are easily calculated[14] and are given in Table I. On the other hand, we see in Eq. (10) that γ_α is written as a sum over $\psi_{NS}(R_0)$'s with numerical coefficients dependent N. The relative amount of the contribution to γ_α from each term is mainly determined by transformation coefficients, but it also depends on the size parameter b through the factor $\left(\frac{\beta-b}{\beta+b}\right)^{N_m-N}\psi_{NS}(R_0)$. However, it is found that the latter dependence on b is rather weak within the reasonable range of b, and thus could be neglected. In Figs. 2 and 3 we show the magnitude of each term of γ_α as a function of N when $b=0.11$, as an example. It is thus seen that the maximum contributing wave function to γ_α may be taken as $\psi_{10S}(R_0)$ in both cases of Po^{210} and Po^{212}, and b is determined as 0.106, so that the above boundary condition is satisfied for this partial wave, which gives 0.8 for $x=1/2\cdot R_0^3\psi_{10S}^2(R_0)$.

§ 3. Numerical calculation for the cases of Po^{210} and Po^{212}

To evaluate $\mathcal{O}(R_0)$ in Eq. (10) we have first to calculate a number of Talmi transformation coefficients which appear in the coefficients C_N. We carry out these tedious calculations using the formula derived by Arima and Terasawa.[13] The Talmi transformation coefficient which appears as the first factor in the summand of Eq. (10) may be written in a rather compact form:

$$\langle NS, ns; 0|N_1S, N_2S; 0\rangle = \frac{1}{2^{N+n}}\sqrt{\frac{N!\,(2N+1)!!}{N_1!\,N_2!\,(2N_1+1)!!\,(2N_2+1)!!}}\sqrt{\frac{n!}{(2n+1)!!}}\,I_n,$$

$$I_n=2^n\left\{\sum_{i=0}^{n}\frac{1}{i!\,(n-i)!}\,\frac{(2N_1+1)!!}{\{2(N_1-i)+1\}!!}\,\frac{N_1!}{(N_1-i)!}\times\frac{(2N_2+1)!!}{\{2(N_2-n+i)+1\}!!}\,\frac{N_2!}{(N_2-n+i)!}\right\}-\sum_{i=0}^{n-1}\frac{1}{(n-i)!}\,\frac{\{2(N-i)+1\}!!}{\{2(N-n)+1\}!!}\,\frac{(N-i)!}{(N-n)!}\,I_i,$$

$$I_0=1.$$

The numbers of the sets (N, n_1, n_2, n_3) that satisfy the condition of the energy conservation are 216 and 273 for the cases of Po^{210} and Po^{212}, even when pure configurations are assumed. The experimental and calculated values of $\gamma_\alpha{}^2$ are given in Table II.

It is worth while to point out that in the case of pure configuration the calculated C_N's change their signs alternately with N. Consequently, all the summands in Eq. (10) have a same sign and thus are added without any can-

Table II.

	Po^{210}		Po^{212}	
E_α	5.33 Mev		8.81 Mev	
P_0	1.22×10^{-24}		5.31×10^{-12}	
S	−18.51		−16.66	
$\gamma^2_{exp.}$	15.7 ev		143 ev	
$\gamma^2_{exp.}\Big/\frac{\hbar^2}{MR_0^2}$	0.88×10^{-4}		0.80×10^{-3}	
$\gamma^2_{theor.}$	0.31 ev	(3.3 ev)	1.3 ev	(7.1 ev)
$\gamma^2_{theor}/\gamma^2_{exp}$	$\frac{1}{50}$	$\left(\frac{1}{4.7}\right)$	$\frac{1}{110}$	$\left(\frac{1}{20}\right)$

cellation, because $\zeta'_{NS}(R)$ also changes its sign successively with N at the surface.* Nevertheless, it is seen from the last row of Table II that the γ^2_{theor} is rather too small compared with γ^2_{exp}.

In order to see whether the configuration mixing will increase the γ^2_{theor}, we take the case of $Po^{210}\to Pb^{206}$ decay, in which case the elaborate shell model calculations have been done by several authors.[15] ** Employing the wave functions obtained by them:

$$|Po^{210}) = 0.943|(h_{9/2})^2 J=0) + 0.101|(f_{7/2})^2 J=0) - 0.317|(i_{13/2})^2 J=0),$$

$$|Pb^{206}) = 0.865|(p_{1/2})^2 J=0) + 0.308|(f_{5/2})^2 J=0) + 0.377|(p_{3/2})^2 J=0) - 0.122|(i_{13/2})^2 J=0),$$

lengthy calculations are made with the result shown in parentheses in Table II. It is seen that the γ^2_{theor} is increased by almost one order of magnitude compared with the case without configuration mixing.

In the course of the numerical calculations, we found that all the products of C_N's and the mixing coefficients have the same sign as C_N's for the pure configuration, thus the summand in Eq. (10) are again added without any cancellation. To see more clearly, this strikingly large contributions caused by the configuration mixing, it will be interesting to compare the relative wave functions of two protons and two neutrons for the cases with and without the configuration mixing. We do this for the wave functions in which the centre-of-mass motion lies in $5S$ and $4S$ state, since in these cases overlapping with

* On the contrary, in the inner region the summation consists of terms partially cancelling each other.

** Strictly speaking, the nuclear force (Serber, Gauss) and the size parameter used in the calculations are slightly different in each case. Following the procedure of Horie and Sasaki,[16] we recalculate the Po^{210} case taking the same parameter as in Pb^{206}, and the result is

$$|Po^{210}) = 0.946|(h_{9/2})^2 J=0) + 0.143|(f_{7/2})^2 J=0) - 0.290|(i_{13/2})^2 J=0).$$

The minor changes of mixing coefficients which resulted from this recalculation are not important, since there are inherent uncertainties in the experimental reduced widths.

the α-particle wave function becomes maximum. They are written as:

proton $(5S)$

$$\begin{cases} \text{no mixing}: \psi(r) = \underset{0.0630}{\langle 5S,\ 0s|0h,\ 0h\rangle}\psi_{0s}(r) \\ \text{with mixing}: \psi(r) = 0.943\underset{0.0630}{\langle 5S,\ 0s|0h,\ 0h\rangle}\psi_{0s}(r) + 0.101\underset{0.2764}{\langle 5S,\ 0s|1f,\ 1f\rangle}\psi_{0s}(r) \\ \qquad\qquad - 0.317\underset{-0.1677}{\langle 5S,\ 1s|0i,\ 0i\rangle}\psi_{1s}(r), \end{cases}$$

proton $(4S)$

$$\begin{cases} \text{no mixing}: \psi(r) = \underset{-0.2697}{\langle 4S,\ 1s|0h,\ 0h\rangle}\psi_{1s}(r) \\ \text{with mixing}: \psi(r) = 0.943\underset{-0.2697}{\langle 4S,\ 1s|0h,\ 0h\rangle}\psi_{1s}(r) + 0.101\underset{-0.4088}{\langle 4S,\ 1s|1f,\ 1f\rangle}\psi_{1s}(r) \\ \qquad\qquad - 0.317\underset{0.3934}{\langle 4S,\ 2s|0i,\ 0i\rangle}\psi_{2s}(r), \end{cases}$$

neutron $(5S)$

$$\begin{cases} \text{no mixing}: \psi(r) = \underset{0.4071}{\langle 5S,\ 0s|2p,\ 2p\rangle}\psi_{0s}(r) \\ \text{with mixing}: \psi(r) = 1.242\underset{0.4071}{\langle 5S,\ 0s|2p,\ 2p\rangle}\psi_{0s}(r) + 0.308\underset{0.2764}{\langle 5S,\ 0s|1f,\ 1f\rangle}\psi_{0s}(r) \\ \qquad\qquad - 0.122\underset{-0.1677}{\langle 5S,\ 1s|0i,\ 0i\rangle}\psi_{1s}(r), \end{cases}$$

neutron $(4S)$

$$\begin{cases} \text{no mixing}: \psi(r) = \underset{0.03169}{\langle 4S,\ 1s|2p,\ 2p\rangle}\psi_{1s}(r) \\ \text{with mixing}: \psi(r) = 1.242\underset{0.03169}{\langle 4S,\ 1s|2p,\ 2p\rangle}\psi_{1s}(r) + 0.308\underset{-0.4088}{\langle 4S,\ 1s|1f,\ 1f\rangle}\psi_{1s}(r) \\ \qquad\qquad - 0.122\underset{0.3934}{\langle 4S,\ 2s|0i,\ 0i\rangle}\psi_{2s}(r). \end{cases}$$

The numbers written below the transformation coefficients are their numerical values. These relative wave functions are shown in Fig. 4 and Fig. 5, together with the wave function of the α-particle for the purpose of comparison.

From Fig. 4, we see that the configuration mixing makes the overlap approximately two times as large as that of the case of no mixing for $5S$ centre-of-mass motion, and that the overlap is rather insensitive to the adopted size parameter, which is seen by looking at the α-particle wave functions given for $\beta = 2b$ and $4b$ in Fig. 4. The relative wave function for $4S$ centre-of-mass motion undergoes less distortion due to the configuration mixing than that for $5S$ one, since it has already a large $1s$ component, i.e. $(4S,\ 1s|0h,\ 0h)$ is large, even in the case of no mixing.

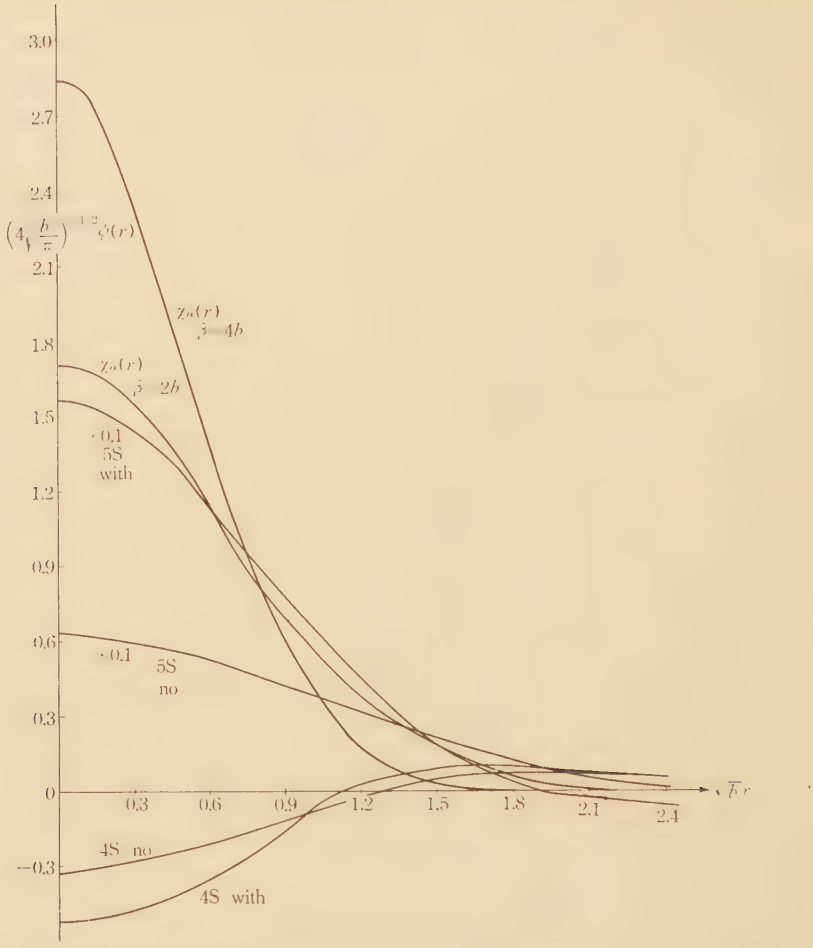

Fig. 4. The relative wave function of two protons.

Similar situations are seen in the case of two-neutrons. In this case, however, the relative wave function for $4S$ centre-of-mass motion changes its sign from plus to minus due to the configuration mixing. Because of the two facts (i) that the centre-of-mass wave function $4S$ and $5S$ have opposite signs at the nuclear surface R_0, and (ii) that the relative wave function corresponding to the $5S$ centre-of-mass has the plus sign, this change of sign is favorable for the construction of di-neutron.

It is noteworthy, in both $4S$ and $5S$ cases, the distortion of the relative wave function, by which the latter resembles more to the α-particle wave function than no mixing case, is mainly caused by the mixing of the $(i_{13/2})^2$ configuration which inherently belongs to the upper band, $6\hbar\omega$. Owing to this, mixed relative wave function contains high frequency wave. This situation corresponds to the bound electron-pairs[17] in the superconducting metals.

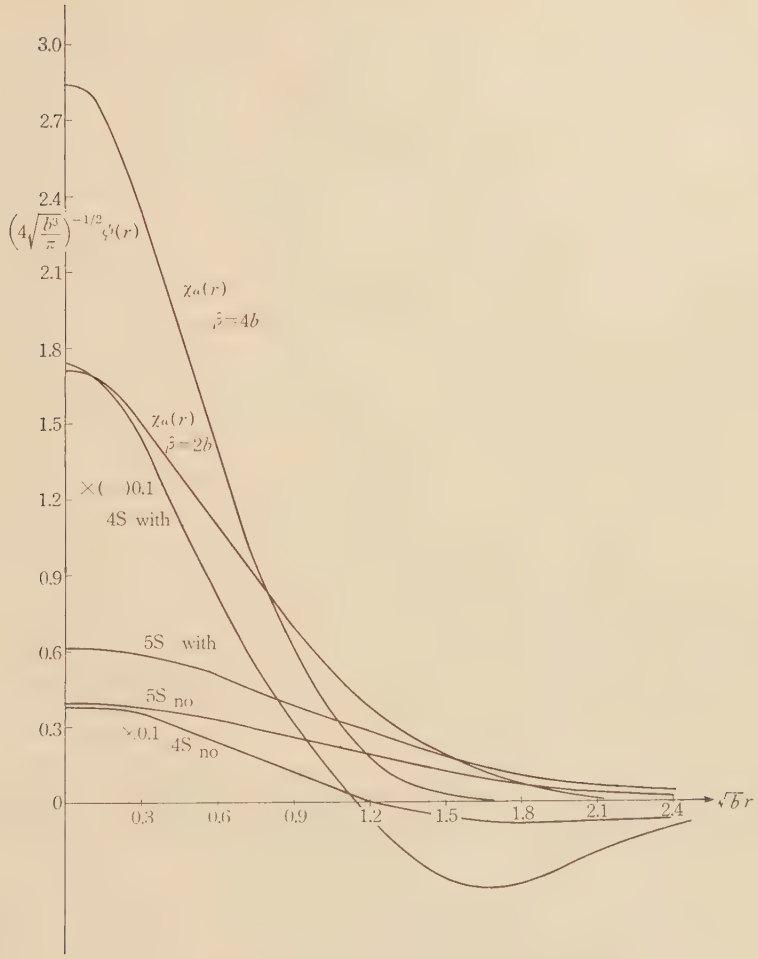

Fig. 5. The relative wave function of two neutrons.

For the case of $Po^{210}\rightarrow Pb^{208}$ decay, a reliable wave function of Po^{212} is not available because of the lack of definite knowledge about the position of the single particle neutron levels which belong to the $6\hbar\omega$ band. For example, the $j_{15/2}$ level predicted by Nilsson[18] has not yet been observed experimentally.[19] Banerjee and Zeh calculated the Po^{210} and Pb^{210} wave function assuming the Serber type δ-force for p–p and n–n force by neglecting the mixture of $(j_{15/2})^2$. Their results are:

$$|Po^{210}) = 0.914(h_{9/2})^2 + 0.256(f_{7/2})^2 - 0.317(i_{13/2})^2,$$

$$|Pb^{210}) = 0.935(g_{9/2})^2 + 0.325(i_{11/2})^2 + 0.141(d_{5/2})^2.$$

Although the wave function $|Pb^{210})$ is less certain in view of the above-mentioned circumstance, in order to obtain a rough estimation for the effect of mixing

for Po^{212} we take $|Po^{210}) \times |Pb^{210})$, each factor being defined above, for $|Po^{212})$ as the first approximation and calculate γ_α^2. This amounts to neglecting the n–p force. The result is also shown in parentheses in Table II.

In this case, the agreement with the experimental value is not so good compared to the Po^{210} case. However, there is a plausible reason for it, that is, the neglect of the mixing of the $j_{15/2}$ level and the n–p force. As is seen in the Po^{210} case, the $j_{15/2}$ level is expected to play an important role for the construction of the neutron-pair. Presumably, the n–p force will act partly for the formation of p–p and n–n pairs, and partly against it. The latter effect will contribute to the formation of n–p pairs. The questions relevant to the n–p force will be studied in a later paper.

§ 4. Discussion and conclusion

It is found from the calculation of the α-particle reduced widths that the wave function derived by the usual shell model calculation with a limited amount of configuration mixing can account for the major part of the experimental values γ_α^2, if we further take into consideration the boundary condition at the nuclear surface appropriately. In other words it is found that the configuration mixing strongly reinforces the formation of α-particle in nucleus. The reason why the configuration mixing has a large effect on the γ_α^2 is that only the s-wave part of the relative motion contributes to γ_α^2, and the amplitude of this s-wave part is much increased under the influence of the short-range nuclear force which caused the configuration mixing. In particular, it is stressed that the mixing of the level which is lowered from the upper band by the spin-orbit force plays an important role for nucleon-clustering. It may thus be concluded that, although some unavoidable uncertainties enter into the course of the calculations, the independent particle model with mixed configurations can explain the experimental values of γ_α^2.

Finally we should discuss R_0 dependence of the experimental and theoretical γ_α^2. For reference, we give the numerical values of penetrability P_0 and shift factor S at the $R_0 = 10$ fermi and 10.5 fermi in the following Table.

Table III.

	R_0	Po^{210}	Po^{212}
P_0	10.0	1.2×10^{-24}	0.5×10^{-11}
	10.5	1.8×10^{-23}	2.7×10^{-11}
S	10.0	−18.5	−16.6
	10.5	−18.8	−16.8

From the above Table it can be seen that γ_α^2(exp.) is very sensitive to R_0, but γ_α^2(theor.) is not so sensitive. Therefore our quantitative conclusion does depend

rather critically on the assumption of $R_0 = 10$ fermi. However, since one cannot possibly take a smaller value than 10 fermi for R_0 (see Fig. 1), which means that our $\gamma^2_{theor}/\gamma^2_{exp}$ given in Table II would be the lower limit, the conclusion in this paper need not be changed.

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Nonstatic One-Boson-Exchange Potentials*

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A list of the nonstatic one-boson-exchange potentials between two nucleons is given. The potentials are obtained exactly in the momentum space. In the coordinate space they are given up to the order M^{-2} in the M^{-1} expansion. (M is the nucleon mass.)

§ 1. Introduction

The existence of resonance or bound states with isospin and angular momentum zero or one has recently been anticipated from both experiments[1] and theoretical speculations.[2] It also results from the Sakata model,[3] especially from the hypothesis of the full symmetry between p, n and Λ.[4,5] If such a resonance is really present, it will more or less affect the two-nucleon force. Actually several authors[6-9] have discussed on this problem. At present, of cource, no definite conclusion has been drawn because of our poor knowledge about the nature of such states. It may still be hoped, however, that the two-nucleon phenomena could give useful information about the above-mentioned states.

The aim of this note is to calculate extensively the nonstatic two-nucleon potentials due to the exchange of a boson, assuming that the resonance or bound state with definite spin and parity can be approximated by an actual particle with the same spin and parity. These potentials are expected to serve as a basis or guide in the more systematic analysis in the future investigations.

In this note we would like to consider spin zero boson and spin one boson. The interaction Lagrangian density is

$$L' = \begin{cases} g_S \bar{\psi}\psi\phi, & (S) \\ \dfrac{f_S}{\mu}\bar{\psi} i\gamma_\mu \psi \partial_\mu \phi, & (V) \end{cases}$$

for scalar boson,

$$L' = g_P \psi i\gamma_5 \psi\phi + \frac{f_P}{\mu}\psi i\gamma_5\gamma_\mu \psi \partial_\mu \phi,$$

for pseudoscalar boson,

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$$L'=g_V\bar{\psi}i\gamma_\mu\psi\phi_\mu+\frac{f_V}{2\mu}\bar{\psi}\sigma_{\mu\nu}\psi\phi_{\mu\nu},$$

for vector boson,

and

$$L'=\begin{cases} g_A\bar{\psi}i\gamma_5\gamma_\mu\psi\phi_\mu, & (PV) \\ \dfrac{f_A}{2\mu}\bar{\psi}i\gamma_5\sigma_{\mu\nu}\psi\phi_{\mu\nu}, & (PT) \end{cases}$$

for pseudovector boson,

where

$$\sigma_{\mu\nu}=(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)/2i$$

and

$$\phi_{\mu\nu}=\partial_\mu\phi_\nu-\partial_\nu\phi_\mu.$$

For iso-triplet boson, ϕ or ϕ_μ is replaced by $\tau_i\phi_i$ or $\tau_i\phi_{\mu i}$. Mixing of (S) and (V) for scalar field and mixing of (PV) and (PT) for pseudovector field are not allowed, assuming that the theory is invariant under charge conjugation or time reflection.[10]

In the next section we give a list of the nonstatic one-boson-exchange potentials (OBEP) calculated from the above Lagrangian. These potentials are derived following the method adopted in reference 11). The two-boson-exchange potentials are left untouched throughout this paper, because they are expected to have little effects on the outside of the force range in which we are mainly interested.

§ 3 is devoted to the following discussion. First, some features of the present potentials are mentioned. Secondly, remarks about the attempts to explain the origin of $\boldsymbol{L}\cdot\boldsymbol{S}$ force by ρ meson are briefly given. Finally, we refer to a possible method of investigating the bosons which have been predicted by the full symmetry theory of the Sakata model.

§ 2. One-boson-exchange potentials

i) The potential is defined in the momentum space as the interaction kernel in the integral equation for the two-nucleon system. Then we can decompose the potential, where two incoming nucleons with momenta $\boldsymbol{p}$, $-\boldsymbol{p}$ are scattered into a state with $\boldsymbol{p}'$, $-\boldsymbol{p}'$, as

$$V=V_0+\frac{(i\boldsymbol{S}\cdot\boldsymbol{k}\times\boldsymbol{q})}{\mu^2}V_1+\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})}{\mu^2}V_2+\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})}{\mu^2}V_3$$
$$+\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k}\times\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k}\times\boldsymbol{q})}{\mu^4}V_4+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})V_5, \qquad (1)$$

where

$$V_i=V_i(\boldsymbol{k}^2,\ \boldsymbol{q}^2,\ (\boldsymbol{k}\times\boldsymbol{q})^2),$$

and

$$\boldsymbol{k}=\boldsymbol{p}'-\boldsymbol{p},\quad \boldsymbol{q}=(\boldsymbol{p}'+\boldsymbol{p})/2,\quad \boldsymbol{S}=(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})/2.$$

μ is the mass of a boson under consideration. The whole expression of V_t corresponding to the one-boson-exchange process is given in the Appendix.

One merit of working out the calculation in the momemtum space[12] may be to make the M^{-1} expansion of the potential unnecessary, M being the mass of a nucleon. This point would become more important when we consider heavy bosons ($\mu \gtrsim M$). In spite of this advantage, however, we are still going to expand the potential with respect to $\boldsymbol{q}/M$ and $\boldsymbol{k}/M$, in order to transform it into coordinate space. (μ/M-expansion need not be made here.) The validity of this M^{-1} expansion has been discussed in reference 11). It has been concluded that the $\boldsymbol{q}/M$-expansion is valid for $q \lesssim 4\mu_\pi c$, μ_π being the mass of a pion.

This criterion may also remain valid for other boson cases. Now we tabulate OBEP up to the order of M^{-2}:

α) scalar boson

$$V_0=-\frac{g_S^2}{\omega^2}\left(1-\frac{\boldsymbol{q}^2}{M^2}+\frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2}{M^2\omega^2}\right)+O\left(\frac{1}{M^4}\right), \tag{2a}$$

$$V_1=g_S^2\frac{\mu^2}{2M^2}\frac{1}{\omega^2}+O\left(\frac{1}{M^4}\right), \tag{2b}$$

$$V_4=O\left(\frac{1}{M^4}\right). \tag{2c}$$

As is well known the vector interaction of the scalar boson has no effects on the nuclear force.

β) pseudoscalar boson

$$V_2=-\left(f_P+\frac{\mu}{2M}g_P\right)^2\frac{1}{\omega^2}\left\{1-\frac{\boldsymbol{k}^2}{4M^2}-\frac{\boldsymbol{q}^2}{M^2}+\frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2}{M^2\omega^2}+O\left(\frac{1}{M^4}\right)\right\}+\frac{\boldsymbol{q}^2}{\mu^2}V_{kq}, \tag{3a}$$

$$V_3=\left(f_P+\frac{\mu}{2M}g_P\right)^2 O\left(\frac{1}{M^4}\right)+\frac{\boldsymbol{k}^2}{\mu^2}V_{kq}, \tag{3b}$$

$$V_4=V_{kq}, \tag{3c}$$

$$V_5=-\frac{(\boldsymbol{k}\times\boldsymbol{q})^2}{\mu^4}V_{kq}, \tag{3d}$$

with

$$V_{kq}=\left(\left(\frac{\mu}{2M}g_P\right)^2-f_P^2\right)\frac{\mu^2}{2M^2}\frac{1}{\omega^2}+\frac{\mu}{2M}g_Pf_PO\left(\frac{1}{M^4}\right). \tag{3e}$$

γ) vector boson

$$V_0=\frac{g_V^2}{\omega^2}\left\{1-\frac{\boldsymbol{k}^2}{4M^2}+\frac{\boldsymbol{q}^2}{M^2}+\frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2}{M^2\omega^2}\right\}-f_V g_V\frac{\boldsymbol{k}^2}{\mu M}\frac{1}{\omega^2}+f_V^2\frac{\boldsymbol{k}^4}{4\mu^2M^2}\frac{1}{\omega^2}$$
$$+g_V^2O\left(\frac{1}{M^4}\right)+f_V g_V O\left(\frac{1}{M^3}\right)+f_V^2O\left(\frac{1}{M^4}\right), \tag{4a}$$

$$V_1=\frac{3}{2}g_V^2\frac{\mu^2}{M^2}\frac{1}{\omega^2}+4f_V g_V\frac{\mu}{M}\frac{1}{\omega^2}-\frac{3}{2}f_V^2\frac{\boldsymbol{k}^2}{M^2}\frac{1}{\omega^2}$$
$$+g_V^2O\left(\frac{1}{M^4}\right)+f_V g_V O\left(\frac{1}{M^3}\right)+f_V^2O\left(\frac{1}{M^4}\right), \tag{4b}$$

$$V_2=\frac{1}{4}g_V^2\frac{\mu^2}{M^2}\frac{1}{\omega^2}+f_V g_V\frac{\mu}{M}\frac{1}{\omega^2}+f_V^2\frac{1}{\omega^2}\left\{1-\frac{\boldsymbol{k}^2}{4M^2}+\frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2}{M^2\omega^2}\right\}$$
$$+\frac{\boldsymbol{q}^2}{\mu^2}V_{kq}+g_V^2O\left(\frac{1}{M^4}\right)+f_V g_V O\left(\frac{1}{M^3}\right)+f_V^2O\left(\frac{1}{M^4}\right), \tag{4c}$$

$$V_3=f_V^2\frac{\boldsymbol{k}^2}{M^2}\frac{1}{\omega^2}+\frac{\boldsymbol{k}^2}{\mu^2}V_{kq}+g_V^2O\left(\frac{1}{M^6}\right)+f_V g_V O\left(\frac{1}{M^5}\right)+f_V^2O\left(\frac{1}{M^4}\right), \tag{4d}$$

$$V_4=-f_V^2\frac{\mu^2}{M^2}\frac{1}{\omega^2}+V_{kq}+g_V^2O\left(\frac{1}{M^4}\right)+f_V g_V O\left(\frac{1}{M^3}\right)+f_V^2O\left(\frac{1}{M^4}\right), \tag{4e}$$

$$V_5=-\frac{1}{4}g_V^2\frac{\boldsymbol{k}^2}{M^2}\frac{1}{\omega^2}-f_V g_V\frac{\boldsymbol{k}^2}{\mu M}\frac{1}{\omega^2}-f_V^2\frac{\boldsymbol{k}^2}{\mu^2\omega^2}\left\{1-\frac{\boldsymbol{k}^2}{4M^2}+\frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2}{M^2\omega^2}\right\}$$
$$-\frac{(\boldsymbol{k}\times\boldsymbol{q})^2}{\mu^4}V_{kq}+g_V^2O\left(\frac{1}{M^4}\right)+f_V g_V O\left(\frac{1}{M^3}\right)+f_V^2O\left(\frac{1}{M^4}\right), \tag{4f}$$

with

$$V_{kq}=-f_V^2\frac{\mu^2}{2M^2}\frac{1}{\omega^2}+g_V^2O\left(\frac{1}{M^4}\right)+f_V g_V O\left(\frac{1}{M^3}\right)+f_V^2O\left(\frac{1}{M^4}\right). \tag{4g}$$

δa) pseudovector boson (PV-coupling)

$$V_0=O\left(\frac{1}{M^4}\right), \tag{5a}$$

$$V_1=\frac{1}{2}g_A^2\frac{\mu^2}{M^2}\frac{1}{\omega^2}+O\left(\frac{1}{M^4}\right), \tag{5b}$$

$$V_2=-g_A^2\frac{1}{\omega^2}\left\{1-\frac{\boldsymbol{k}^2}{4M^2}-\frac{\boldsymbol{q}^2}{M^2}-\frac{\mu^2}{4M^2}+\frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2}{M^2\omega^2}\right\}+\frac{\boldsymbol{q}^2}{\mu^2}V_{kq}+O\left(\frac{1}{M^4}\right), \tag{5c}$$

$$V_3=-2g_A^2\frac{\mu^2}{M^2}\frac{1}{\omega^2}+\frac{\boldsymbol{k}^2}{\mu^2}V_{kq}+O\left(\frac{1}{M^4}\right), \tag{5d}$$

$$V_4=V_{kq}+O\left(\frac{1}{M^4}\right), \tag{5e}$$

$$V_5=-\frac{g_A^2}{\omega^2}\left\{1-\frac{\boldsymbol{q}^2}{M^2}+\frac{(\boldsymbol{k}\cdot\boldsymbol{q})}{M^2\omega^2}\right\}-\frac{(\boldsymbol{k}\times\boldsymbol{q})^2}{\mu^4}V_{kq}+O\left(\frac{1}{M^4}\right), \tag{5f}$$

with

$$V_{kq}=-g_A^2\frac{\mu^2}{2M^2}\frac{1}{\omega^2}+O\left(\frac{1}{M^4}\right). \tag{5g}$$

∂b) *pseudovector boson (PT-coupling)*

$$V_2=\frac{f_A^2}{\omega^2}\left\{1+\frac{\boldsymbol{q}^2}{M^2}+\frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2}{M^2\omega^2}\right\}+\frac{\boldsymbol{q}^2}{\mu^2}V_{kq}+O\left(\frac{1}{M^4}\right), \tag{6a}$$

$$V_3=\frac{\boldsymbol{k}^2}{\mu^2}V_{kq}, \tag{6b}$$

$$V_4=V_{kq}, \tag{6c}$$

$$V_5=-\frac{(\boldsymbol{k}\times\boldsymbol{q})^2}{\mu^4}V_{kq}, \tag{6d}$$

with

$$V_{kq}=-\frac{3}{2}f_A^2\frac{\mu^2}{M^2}\frac{1}{\omega^2}+O\left(\frac{1}{M^4}\right). \tag{6e}$$

If we consider iso-triplet bosons, all V_i's given above should be replaced by $(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})V_i$. In the case that the ρ meson interacts with nucleon after decaying into pions, g and f in the above equations will be replaced by $gF_1(k^2)$ and $fF_2(k^2)$, $F_\alpha(k^2)$ being the nucleon form factors.

It would be in order here to remark the following. As is known, the form of the nonstatic part of V_2, V_3, V_4 and V_5 is not unique. Because of the identity

$$(\boldsymbol{k}\cdot\boldsymbol{q})\{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})\}=\boldsymbol{q}^2(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})$$
$$+\boldsymbol{k}^2(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k}\times\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k}\times\boldsymbol{q})-(\boldsymbol{k}\times\boldsymbol{q})^2(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}), \tag{7}$$

the V_{kq}-terms on the right-hand side of the expressions for V_2, V_3, V_4 and V_5 can be summarized as

$$V_{kq}(\boldsymbol{k}\cdot\boldsymbol{q})\{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})\}/\mu^4. \tag{8}$$

This term together with the retardation term, $g^2(\boldsymbol{k}\cdot\boldsymbol{q})^2/M^2\omega^4$, can be transformed into the fourth order potential by a suitable unitary transformation. The static part and $\boldsymbol{L}\cdot\boldsymbol{S}$ potential, on the other hand, are free from this ambiguity up to the second order approximation.

In this note we use, for the sake of simplicity, the representation in which the term (8) together with the retardation term, $g^2(\boldsymbol{k}\cdot\boldsymbol{q})^2/M^2\omega^4$, are transformed into the fourth order. We therefore omit these terms from Eqs. (2)$\sim$(6) in what follows.

ii) *List of the potentials in the coordinate representation*

The potentials in the coordinate space are given by the Fourier transforms of $V(\boldsymbol{k}, \boldsymbol{q})$, Eq. (1) :

$$V(r, \boldsymbol{q}) = \int \frac{d\boldsymbol{k}}{(2\pi)^3} V(\boldsymbol{k}, \boldsymbol{q})\, e^{i\boldsymbol{k}\boldsymbol{r}}. \tag{9}$$

$\boldsymbol{q}$ in $V(r, \boldsymbol{q})$ means a differential operator on the two-nucleon wave fuction, $-i(\vec{\nabla}+\overleftarrow{\nabla})/2$. Then $V(r, \boldsymbol{q})$ can be expressed as a sum of the static part, $\boldsymbol{L}\cdot\boldsymbol{S}$ force and other nonstatic forces, each of which is given below.

α) The static limit $W(r)$ of $V(r, \boldsymbol{q})$ is well known. Here we reproduce it for the sake of completeness :

$$W = -\mu\frac{g_S^2}{4\pi}\phi(x), \tag{10a}$$

for scalar boson,

$$W = \mu\frac{1}{4\pi}\left(f_P + \frac{\mu}{2M}g_P\right)^2\left\{\frac{1}{3}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\,\phi(x) + S_{12}\chi(x)\right\}, \tag{10b}$$

for pseudoscalar boson,

$$W = \mu\frac{g_V^2}{4\pi}\phi(x) + \mu\frac{f_V^2}{4\pi}\left\{\frac{2}{3}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\,\phi(x) - S_{12}\chi(x)\right\}, \tag{10c}$$

for vector boson,

$$W = \mu\frac{g_A^2}{4\pi}\left\{-\frac{2}{3}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\,\phi(x) + S_{12}\chi(x)\right\}, \tag{10d}$$

for pseudovector boson (PV-coupling),

and

$$W = -\mu\frac{f_A^2}{4\pi}\left\{\frac{1}{3}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\,\phi(x) + S_{12}\chi(x)\right\}, \tag{10e}$$

for pseudovector boson (PT-coupling),

where

$$\phi(x) = \frac{e^{-x}}{x}, \quad \chi(x) = \frac{1}{3}\left(1+\frac{3}{x}+\frac{3}{x^2}\right)\frac{e^{-x}}{x}, \tag{11}$$

with $x=\mu r$, μ being the mass of a boson under consideration.

β) The $\boldsymbol{L}\cdot\boldsymbol{S}$ force, $V_{LS}(r)\,(\boldsymbol{L}\cdot\boldsymbol{S})$, is given by

$$V_{LS} = -\mu\frac{g_S^2}{4\pi}\frac{\mu^2}{2M^2}\left(1+\frac{1}{x}\right)\frac{e^{-x}}{x^2}, \tag{12a}$$

for scalar boson,

$$V_{LS}=-\mu\left(\frac{g_V^2}{4\pi}\frac{3}{2}\frac{\mu^2}{M^2}+\frac{f_V g_V}{4\pi}4\frac{\mu}{M}+\frac{f_V^2}{4\pi}\frac{3}{2}\frac{\mu^2}{M^2}\right)\left(1+\frac{1}{x}\right)\frac{e^{-x}}{x^2}, \tag{12b}$$

for vector boson,

and

$$V_{LS}=-\mu\frac{g_A^2}{4\pi}\frac{\mu^2}{2M^2}\left(1+\frac{1}{x}\right)\frac{e^{-x}}{x^2}, \tag{12c}$$

for pseudovector boson.

γ) Other types of the nonstatic potentials.

γ-1) The quadratic spin-orbit force, $V_{LL}(r)\,(\boldsymbol{\sigma}^{(1)}\widetilde{\boldsymbol{L}})\,(\boldsymbol{\sigma}^{(2)}\widetilde{\boldsymbol{L}})$* is expressed as

$$V_{LL}=\mu\frac{f_V^2}{4\pi}\frac{\mu^2}{M^2}\left(1+\frac{3}{x}+\frac{3}{x^2}\right)\frac{e^{-x}}{x^3}, \tag{13}$$

for vector boson.

γ-2) $V_q(r)\,(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})\,(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})/\mu^2$ is given by

$$V_q=\mu\frac{f_V^2}{4\pi}\frac{\mu^2}{M^2}\left(-1+\frac{1}{x}+\frac{1}{x^2}\right)\frac{e^{-x}}{x}, \tag{14a}$$

for vector boson,

$$V_q=-\mu\frac{g_A^2}{4\pi}\frac{2\mu^2}{M^2}\phi(x), \tag{14b}$$

for pseudovector boson.

δ) Corrections to the static central and tensor forces:

$$W'=\mu\frac{g_S^2}{4\pi}\frac{\boldsymbol{q}^2}{M^2}\phi(x), \tag{15a}$$

for scalar boson,

$$W'=\mu\frac{1}{4\pi}\left(f_P+\frac{\mu}{2M}g_P\right)^2\left(\frac{\mu^2}{4M^2}-\frac{\boldsymbol{q}^2}{M^2}\right)\left\{\frac{1}{3}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\,\phi(x)+S_{12}\chi(x)\right\}, \tag{15b}$$

for pseudoscalar boson,

$$W'=\mu\left\{\frac{g_V^2}{4\pi}\left(\frac{\mu^2}{4M^2}+\frac{\boldsymbol{q}^2}{M^2}\right)+\frac{f_V g_V}{4\pi}\frac{\mu}{M}+\frac{f_V^2}{4\pi}\frac{\mu^2}{4M^2}\right\}\phi(x)$$
$$+\mu\left\{\frac{g_V^2}{4\pi}\frac{\mu^2}{4M^2}+\frac{f_V g_V}{4\pi}\frac{\mu}{M}+\frac{f_V^2}{4\pi}\frac{\mu^2}{4M^2}\right\}\left\{\frac{2}{3}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\,\phi(x)-S_{12}\chi(x)\right\}$$

* $(\boldsymbol{\sigma}^{(1)}\cdot\widetilde{\boldsymbol{L}})\,(\boldsymbol{\sigma}^{(2)}\cdot\widetilde{\boldsymbol{L}})\equiv(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{L})\,(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{L})-1/4(\boldsymbol{x}\cdot\overrightarrow{\boldsymbol{\nabla}}-\overleftarrow{\boldsymbol{\nabla}}\cdot\boldsymbol{x})\,(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})+1/4\{(\boldsymbol{\sigma}^{(1)}\cdot\overrightarrow{\boldsymbol{\nabla}})\,(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{x})-(\boldsymbol{\sigma}^{(1)}\cdot\overleftarrow{\boldsymbol{\nabla}})\times(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{x})\}$.[11]

$$-\mu\frac{f_V^2}{4\pi}\frac{\boldsymbol{q}^2}{M^2}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\left(1+\frac{1}{x}\right)\frac{e^{-x}}{x^2}, \tag{15c}$$

for vector boson,

$$W'=\mu\frac{g_A^2}{4\pi}\frac{\boldsymbol{q}^2}{M^2}\left\{\frac{2}{3}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\,\phi(x)-S_{12}\chi(x)\right\}, \tag{15d}$$

for pseudovector boson (PV-coupling),

and

$$W'=-\mu\frac{f_A^2}{4\pi}\frac{\boldsymbol{q}^2}{M^2}\left\{\frac{1}{3}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\,\phi(x)+S_{12}\chi(x)\right\}, \tag{15e}$$

for pseudovector boson (PT-coupling).

The δ-function terms are neglected in the above expressions.

These formulas will be valid only in the outside of the core region. Near the core region the M^{-1}-expansion cannot be allowed, because in this region the contribution from high momentum part, $k\gtrsim Mc$, is not small. Moreover, in this region, the higher order effects should be taken into account.

§ 3. Discussions

i) *Characteristics of the nonstatic OBEP*

As shown in Eqs. (12a) ~ (12c) the $\boldsymbol{L}\cdot\boldsymbol{S}$ potential is obtained as an attractive force (multiplied by $(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})$ in the case of iso-triplet bosons) with the magnitude of order $(\mu/M)^2$ in the case of scalar, vector and pseudovector bosons. The $f_V g_V$-term in the case of vector boson is only exceptional and has the magnitude of order μ/M. No $\boldsymbol{L}\cdot\boldsymbol{S}$ force is obtained from pseudoscalar boson or pseudovector boson with pseudotensor-coupling in the f^2-approximation.

As to the other nonstatic forces we only mention that their magnitude is, in general, of the same order as that of the $\boldsymbol{L}\cdot\boldsymbol{S}$ force.

ii) *Some remarks about the attempts to explain the origin of* $\boldsymbol{L}\cdot\boldsymbol{S}$ *force by* ρ *meson*

High-energy p-p scattering seems to indicate that the large $\boldsymbol{L}\cdot\boldsymbol{S}$ type potentials are probably present. Breit[6a] proposes to regard this $\boldsymbol{L}\cdot\boldsymbol{S}$ force and the repulsive core as originating in the interaction through vector meson with mass$\approx 9\mu_\pi\sim 12\mu_\pi$. The coupling constant, $g_V^2/4\pi$, is estimated to be 300~1700, which seems to be unreasonably large. Furthermore, Sakurai[7] claims that this presence of a large $\boldsymbol{L}\cdot\boldsymbol{S}$ force can be completely accounted for if there exists a strongly interacting neutral vector meson (or a sharp resonance in the $I=0$, $J=1$ state of the three-pion system) of mass $\approx 3\mu_\pi\sim 4\mu_\pi$, with the vector coupling to the nucleon of the order $g_V^2/4\pi\approx 4\sim 7$. However, his $\boldsymbol{L}\cdot\boldsymbol{S}$ force, if it be present, necessarily accompanies a very strong repulsive central force (about $(M/\mu)^2$-times

larger than V_{LS}) as is easily seen from Eqs. (10 c) and (12 b). The tail of this repulsive force extends into one-pion-exchange region and appears therefore to be improbably large as it has been pointed out by Breit.[8] Furthermore, Breit[8] and Ohnuma[8] criticize Sakurai's conjecture from other point of view.

Gupta[6] considers the $\boldsymbol{L}\cdot\boldsymbol{S}$ force arising from the neutral scalar meson with $\mu\approx 2\mu_\pi$ which is coupled strongly to the nucleon, $g_S^2/4\pi\approx 14$. In this case, too, the strong (attractive) central force necessarily appears in the one-pion-exchange region, and it may be inconsistent with the available low energy data.

Fujii[9] calculates the potential arising from the two-pion P-wave resonance ($I=1$, $J=1$) at an energy of about $4.3\,\mu_\pi$. This resonance corresponds to a vector meson with $I=1$ (ρ-meson). The coupling constants, g_V and f_V in Eqs. (4a) $\sim$ (4g), can be replaced by $hF_1(k^2)$ and $hF_2(k^2)$, where $F_\alpha(k^2)$ are the nucleon electromagnetic form factors and h is the ρ meson-pion coupling constant, $h^2/4\pi\sim 5$. Then it is estimated at $k^2=0$ that $g_V^2/4\pi\sim 0.4$, $f_V^2/4\pi\sim 5.5$. In his potential, the $f_V g_V$-term in Eq. (12b) gives a main contribution. The nature of the spacial extension of his potential is influenced by $F_\alpha(k^2)$.

iii) *A possible method of investigating the bosons predicted by the full symmetry theory of the Sakata model*

In the Sakata model, p, n and Λ are assumed to be fundamental particles. Furthermore, the full symmetry between these particles has been assumed by Ogawa.[4] Then iso-singlet π_0'-(and π_0''-) meson* is predicted to exist. This boson would contribute to the nuclear force in the outside of the core region. Several resonance states** with $I=0$, 1, which are obtained in the system of two baryons and two antibaryons, would also contribute to the nuclear force. In order to investigate whether these bosons really exist or not, the following approach*** may be possible:

α) The one- and two-pion-exchange potentials with the contribution from pion-pion force neglected, (OPEP+TPEP), are subtracted from the phenomenological potential, V_{phen}, which can reproduce all the available experimental data. We then investigate whether or not the remaining part may be identified with the potentials due to the exchange of bosons mentioned above,

$$V_{phen}-(\mathrm{OPEP}+\mathrm{TPEP})=\Sigma'(\mathrm{OBEP}), \tag{16}$$

or more simply

β) $V_{phen}-(\mathrm{OPEP})=\Sigma'(\mathrm{OBEP})$,

according to whether $\Sigma'(\mathrm{OBEP}) \lesssim \mathrm{TPEP}$ or $\Sigma'(\mathrm{OBEP}) \gtrsim \mathrm{TPEP}$ in the region under consideration. Finally, we mention that this approach should be applied to the phenomena on which the potential in the core region has little effect.

* The mass formula of elementary particles by Matumoto[5] and by Sawada and Yonezawa[5] gives the mass of π_0' (π_0'') to be 615 Mev (377 Mev).

** The mass formula[5] predicts the energy of these levels to be 753 Mev and 849 Mev for $I=1$ state, 1135 Mev for $I=0$ state.

*** The detailed analysis is made in reference 13).

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Appendix

Exact expression for the nonstatic one-boson-exchange potentials in the momentum representation

In this Appendix we give an exact expression of V_i's in Eq. (1).

(α) *scalar boson*

$$V_0=-g_S^2\left(1-\frac{\boldsymbol{p}'\cdot\boldsymbol{p}}{(E_{p'}+M)(E_p+M)}\right)^2\frac{(E_{p'}+M)(E_p+M)}{4E_{p'}E_p}\bar{J}, \tag{A·1}$$

$$V_1=g_S^2\left(1-\frac{\boldsymbol{p}'\boldsymbol{p}}{(E_{p'}+M)(E_p+M)}\right)\frac{\mu^2}{2E_{p'}E_p}\bar{J}, \tag{A·2}$$

$$V_4=g_S^2\frac{\mu^4}{4E_{p'}E_p(E_{p'}+M)(E_p+M)}\bar{J}, \tag{A·3}$$

where $E_p=(\boldsymbol{p}^2+M^2)^{1/2}$ and $\bar{J}=P\dfrac{1}{\boldsymbol{k}^2+\mu^2-(E_{p'}-E_p)^2}$. P stands for principal value.

(β) *pseudoscalar boson*

$$V_2=-\left(f_P+\frac{\mu}{2M}g_P\right)^2\left(\frac{M}{E_{p'}+M}+\frac{M}{E_p+M}\right)^2\frac{(E_{p'}+M)(E_p+M)}{4E_{p'}E_p}\bar{J}$$
$$+\frac{\boldsymbol{q}^2}{\mu^2}V_{kq}+(E_{p'}-E_p)^2V_p, \tag{A·4}$$

$$V_3=-\left(f_P+\frac{\mu}{2M}g_P\right)^2\left(\frac{M}{E_{p'}+M}-\frac{M}{E_p+M}\right)^2\frac{(E_{p'}+M)(E_p+M)}{E_{p'}E_p}\bar{J}$$
$$+\frac{\boldsymbol{k}^2}{\mu^2}V_{kq}+(E_{p'}-E_p)^2 4V_p, \tag{A·5}$$

$$V_4=V_{kq}, \tag{A·6}$$

$$V_5=-\frac{(\boldsymbol{k}\times\boldsymbol{q})^2}{\mu^4}V_{kq}, \tag{A·7}$$

with

$$V_{kq}=\left\{\left(\frac{\mu}{2M}g_P\right)^2M(E_{p'}+E_p+2M)-\frac{\mu}{2M}g_Pf_P(\boldsymbol{p}'^2+\boldsymbol{p}^2)\right.$$
$$\left.-f_P^2(E_{p'}^2+E_p^2+M(E_{p'}+E_p))\right\}\frac{\mu^2M}{E_{p'}E_p(E_{p'}+M)(E_p+M)(E_{p'}+E_p)}\bar{J}, \tag{A·8}$$

$$V_p=f_P\left(f_P+\frac{\mu}{2M}g_P\right)\frac{M(E_{p'}+E_p+2M)}{4E_{p'}E_p(E_{p'}+M)(E_p+M)}\bar{J}. \tag{A·9}$$

(γ) *vector boson*

$$V_0=\Big[g_V^2\Big\{\Big(1+\frac{\boldsymbol{p}'\cdot\boldsymbol{p}}{(E_{p'}+M)(E_p+M)}\Big)^2+\Big(\frac{\boldsymbol{p}'}{E_{p'}+M}+\frac{\boldsymbol{p}}{E_p+M}\Big)^2\Big\}$$
$$+4\frac{f_V g_V}{\mu}\Big\{-(E_{p'}+E_p-2M)+M\Big(\frac{\boldsymbol{p}'}{E_{p'}+M}+\frac{\boldsymbol{p}}{E_p+M}\Big)^2$$
$$+\frac{(E_{p'}+E_p+2M)(\boldsymbol{p}'\cdot\boldsymbol{p})^2}{(E_{p'}+M)^2(E_p+M)^2}+\frac{(E_p-E_{p'})^2M}{(E_{p'}+M)(E_p+M)}\Big\}$$
$$+\frac{f_V^2}{\mu^2}\Big\{\Big(\frac{(\boldsymbol{p}'\cdot\boldsymbol{p}'-\boldsymbol{p})}{E_{p'}+M}-\frac{(\boldsymbol{p}\cdot\boldsymbol{p}'-\boldsymbol{p})}{E_p+M}\Big)^2+\frac{(\boldsymbol{p}'-\boldsymbol{p})^2(\boldsymbol{p}'\times\boldsymbol{p})^2}{(E_{p'}+M)^2(E_p+M)^2}$$
$$-(E_{p'}-E_p)^2\Big(\frac{\boldsymbol{p}'}{E_{p'}+M}-\frac{\boldsymbol{p}}{E_p+M}\Big)^2\Big\}\Big]\times\frac{(E_{p'}+M)(E_p+M)}{4E_{p'}E_p}\,\bar{\Delta}, \tag{A·10}$$

$$V_1=\Big[g_V^2\Big(3+\frac{\boldsymbol{p}'\cdot\boldsymbol{p}}{(E_{p'}+M)(E_p+M)}\Big)+4\frac{f_V g_V}{\mu}\Big(2M+\frac{(E_{p'}+E_p+2M)(\boldsymbol{p}'\cdot\boldsymbol{p})}{(E_{p'}+M)(E_p+M)}\Big)$$
$$-\frac{f_V^2}{\mu^2}\Big\{(\boldsymbol{p}'-\boldsymbol{p})^2\Big(1+\frac{(E_{p'}+E_p+2M)^2+2\boldsymbol{p}'\cdot\boldsymbol{p}}{2(E_{p'}+M)(E_p+M)}\Big)$$
$$-(E_{p'}-E_p)^2\Big(2+\frac{(E_{p'}+E_p+2M)(E_{p'}+E_p)}{2(E_{p'}+M)(E_p+M)}\Big)\Big\}\Big]\frac{\mu^2}{2E_{p'}E_p}\,\bar{\Delta}, \tag{A·11}$$

$$V_2=\Big[g_V^2\frac{(E_{p'}+E_p+2M)^2}{4(E_{p'}+M)^2(E_p+M)^2}+\frac{f_V g_V}{\mu}$$
$$\times\frac{M(E_{p'}+E_p+2M)^2-(E_{p'}-E_p)^2(E_{p'}+E_p+2M)/2}{(E_{p'}+M)^2(E_p+M)^2}$$
$$+\frac{f_V^2}{\mu^2}\Big\{\Big(1+\frac{\boldsymbol{p}'\cdot\boldsymbol{p}}{(E_{p'}+M)(E_p+M)}\Big)^2+\frac{(E_{p'}-E_p)^4}{4(E_{p'}+M)^2(E_p+M)^2}\Big\}\Big]$$
$$\times\frac{(E_{p'}+M)(E_p+M)\mu^2}{4E_{p'}E_p}\,\bar{\Delta}+\frac{\boldsymbol{q}^2}{\mu^2}V_{kq}, \tag{A·12}$$

$$V_3=\Big[g_V^2\frac{(E_{p'}-E_p)^2}{4(E_{p'}+M)(E_p+M)}-\frac{f_V g_V}{\mu}\frac{(E_{p'}-E_p)^2(E_{p'}+E_p)}{2(E_{p'}+M)(E_p+M)}$$
$$+\frac{f_V^2}{\mu^2}\Big\{(\boldsymbol{p}'-\boldsymbol{p})^2\Big(1-\frac{(\boldsymbol{p}'-\boldsymbol{p})^2}{4(E_{p'}+M)(E_p+M)}\Big)$$
$$-(E_{p'}-E_p)^2\frac{M(E_{p'}+E_p+M)}{4(E_{p'}+M)(E_p+M)}\Big\}\Big]\frac{\mu^2}{E_{p'}E_p}\,\bar{\Delta}+\frac{\boldsymbol{k}^2}{\mu^2}V_{kq}, \tag{A·13}$$

$$V_4=-\Big[g_V^2+4\frac{f_V g_V}{\mu}(E_{p'}+E_p+2M)+\frac{f_V^2}{\mu^2}(E_{p'}+E_p+2M)^2\Big]$$
$$\times\frac{\mu^4\bar{\Delta}}{4E_{p'}E_p(E_{p'}+M)(E_p+M)}+V_{kq}, \tag{A·14}$$

$$V_5=-\Big[g_V^2\Big(\frac{\boldsymbol{p}'}{E_{p'}+M}-\frac{\boldsymbol{p}}{E_p+M}\Big)^2+f_Vg_V\frac{4M}{\mu}\Big\{\Big(\frac{\boldsymbol{p}'}{E_{p'}+M}-\frac{\boldsymbol{p}}{E_p+M}\Big)^2$$
$$+\frac{(E_{p'}-E_p)^2}{(E_{p'}+M)(E_p+M)}\Big\}+\frac{f_V^2}{\mu^2}\Big\{(\boldsymbol{p}'-\boldsymbol{p})^2\Big(1+\frac{\boldsymbol{p}'\cdot\boldsymbol{p}}{(E_{p'}+M)(E_p+M)}\Big)^2$$
$$-(E_{p'}-E_p)^2\Big(\frac{\boldsymbol{p}'}{E_{p'}+M}+\frac{\boldsymbol{p}}{E_p+M}\Big)^2\Big\}\Big]\frac{(E_{p'}+M)(E_p+M)}{4E_{p'}E_p}\,\overline{\Delta}$$
$$-\frac{(\boldsymbol{k}\times\boldsymbol{q})^2}{\mu^4}V_{kq}, \tag{A·15}$$

with

$$V_{kq}=\Big[-g_V^2(E_{p'}+E_p+2M)+2\frac{f_Vg_V}{\mu}(\boldsymbol{p}'^2+\boldsymbol{p}^2)-\frac{f_V^2}{\mu^2}$$
$$\times\Big\{2(E_{p'}+M)(E_p+M)(E_{p'}+E_p)\Big(1+\frac{\boldsymbol{p}'\cdot\boldsymbol{p}}{(E_{p'}+M)(E_p+M)}\Big)$$
$$-(E_{p'}-E_p)^2(E_{p'}+E_p+2M)\Big\}\Big]\frac{\mu^4\overline{\Delta}}{4E_{p'}E_p(E_{p'}+M)(E_p+M)(E_{p'}+E_p)}. \tag{A·16}$$

(δa) *pseudovector boson (PV-coupling)*

$$V_0=g_A^2\frac{(\boldsymbol{p}'\times\boldsymbol{p})^2}{4E_{p'}E_p(E_{p'}+M)(E_p+M)}\overline{\Delta}, \tag{A·17}$$

$$V_1=g_A^2\Big\{1-\frac{\boldsymbol{p}'\cdot\boldsymbol{p}}{(E_{p'}+M)(E_p+M)}\Big\}\frac{\mu^2}{2E_{p'}E_p}\overline{\Delta}, \tag{A·18}$$

$$V_2=-g_A^2\Big\{\frac{M^2(E_{p'}+E_p+2M)^2}{\mu^2(E_{p'}+M)(E_p+M)}-1$$
$$+\frac{(\boldsymbol{p}'+\boldsymbol{p})^2+(E_{p'}-E_p)^2(1-4M(E_{p'}+E_p+2M)/\mu^2)}{4(E_{p'}+M)(E_p+M)}\Big\}\frac{\mu^2}{4E_{p'}E_p}\overline{\Delta}+\frac{\boldsymbol{q}^2}{\mu^2}V_{kq}, \tag{A·19}$$

$$V_3=-g_A^2\Big\{1+\frac{(E_{p'}+E_p+2M)^2}{4(E_{p'}+M)(E_p+M)}+\frac{(\boldsymbol{p}'-\boldsymbol{p})^2}{(E_{p'}+M)(E_p+M)}$$
$$-\frac{M(E_{p'}+E_p+M)(E_{p'}-E_p)^2}{\mu^2(E_{p'}+M)(E_p+M)}\Big\}\frac{\mu^2}{E_{p'}E_p}\overline{\Delta}+\frac{\boldsymbol{k}^2}{\mu^2}V_{kq}, \tag{A·20}$$

$$V_4=V_{kq}, \tag{A·21}$$

$$V_5=-g_A^2\Big\{1-\frac{\boldsymbol{p}'\cdot\boldsymbol{p}}{(E_{p'}+M)(E_p+M)}\Big\}^2\frac{(E_{p'}+M)(E_p+M)}{4E_{p'}E_p}\overline{\Delta}-\frac{(\boldsymbol{k}\times\boldsymbol{q})^2}{\mu^4}V_{kq}, \tag{A·22}$$

with

$$V_{kq}=-g_A^2\frac{\mu^2[2\{E_{p'}^2+E_p^2+M(E_{p'}+E_p)\}M-(E_{p'}+E_p+M)\mu^2]}{2E_{p'}E_p(E_{p'}+M)(E_p+M)(E_{p'}+E_p)}\overline{\Delta}. \tag{A·23}$$

(∂b) *pseudovector boson (PT-coupling)*

$$V_2=f_A^2\left\{\frac{1}{2}(E_{p'}+E_p)(E_{p'}+E_p+2M)-(E_{p'}-E_p)^2\right\}$$
$$\times\frac{(E_{p'}+E_p)(E_{p'}+E_p+2M)}{8E_{p'}E_p(E_{p'}+M)(E_p+M)}\bar{J}+\frac{q^2}{\mu^2}V_{kq}, \tag{A·24}$$

$$V_3=-f_A^2\left[\frac{(\boldsymbol{p}'-\boldsymbol{p})^2(E_{p'}+E_p+2M)^2}{4(E_{p'}+M)(E_p+M)}-(E_{p'}-E_p)^2\right.$$
$$\left.\times\left\{1-\frac{(\boldsymbol{p}'-\boldsymbol{p})^2-(E_{p'}+E_p)^2/4}{4(E_{p'}+M)(E_p+M)}\right\}\right]\frac{1}{E_{p'}E_p}\bar{J}+\frac{k^2}{\mu^2}V_{kq}, \tag{A·25}$$

$$V_4=-f_A^2\frac{\mu^2(E_{p'}+E_p+2M)^2}{4E_{p'}E_p(E_{p'}+M)(E_p+M)}\bar{J}+V_{kq}, \tag{A·26}$$

$$V_5=\frac{f_A^2}{\mu^2}\left\{(\boldsymbol{p}'-\boldsymbol{p})^2\left(\frac{\boldsymbol{p}'}{E_{p'}+M}+\frac{\boldsymbol{p}}{E_p+M}\right)^2-(E_{p'}-E_p)^2\right.$$
$$\left.\times\left(1+\frac{\boldsymbol{p}'\cdot\boldsymbol{p}}{(E_{p'}+M)(E_p+M)}\right)^2\right\}\frac{(E_{p'}+M)(E_p+M)}{4E_{p'}E_p}\bar{J}-\frac{(\boldsymbol{k}\times\boldsymbol{q})^2}{\mu^4}V_{kq}, \tag{A·27}$$

with

$$V_{kq}=-f_A^2\left[(E_{p'}+E_p)(E_{p'}+E_p+2M)\left\{1-\frac{(\boldsymbol{p}'-\boldsymbol{p})^2}{(E_{p'}+E_p)^2}\right\}-(E_{p'}-E_p)^2\right]$$
$$\times\frac{\mu^2\bar{J}}{4E_{p'}E_p(E_{p'}+M)(E_p+M)}. \tag{A·28}$$

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Magnetic Ordering in the Rare-Earth Metals with More than Half-Filled 4*f* Shells

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The effect of the anisotropy energy on the screw spin structure is investigated. Various spin structures found in dysprosium, holmium and erbium are shown to be stable if the anisotropy constants of second, fourth and sixth orders have appropriate magnitudes and signs. The anisotropy energies decrease with rising temperature at different rates according to their orders. This temperature variation of the anisotropy energy causes the magnetic transitions between different spin structures in these rare-earth metals. It is shown that the state with the oscillation of the z component of the spin which has been observed in the high temperature phase of erbium is always stable just below the Néel temperature when the second order anisotropy makes the easy axis of the spin parallel to the c axis.

The change of the pitch of the screw with temperature due to the quadrupole interaction is also calculated on the basis of the molecular field approximation and it is shown to be too small compared with the experimental result.

§ 1. Introduction

The neutron diffraction experiments on the single crystals of dysprosium, holmium and erbium done by Koehler, Wilkinson, Wollan and Cable[1)-3)] at Oak Ridge have revealed that the magnetic structures in these rare-earth metals show characteristics of a screw structure. According to their experimental results, dysprosium takes a helical arrangement below 179°K and undergoes a transition to the ferromagnetic state at 87°K. The direction of the spin is in the c plane and the pitch of the screw is temperature-dependent. Holmium shows a similar magnetic behavior in the whole temperature region below 133°K, but a small component of the magnetic moment parallel to the c axis appears in addition to the screw structure in the c plane below 19°K. In erbium, only the z component of the spin oscillates in the temperature region from 80°K to 52°K and, superposed to this oscillation, the oscillation of the perpendicular component is found below 52°K. Below 20°K, the magnetic structure of erbium changes to a conical structure in which the z component aligns parallel to the hexagonal axis but the xy component still takes a screw arrangement. The details of the experimental results obtained by Koehler et al. are summarized in Table I.

In a previous paper[4)] with a similar title to that of the present paper, we have considered the magnetic behavior of dysprosium and holmium on the basis

Table I. Ordered spin structures of Dy, Ho and Er observed by Koehler, Wilkinson, Wollan and Cable.

			T_c				T_N	
Dy	T		87°K				179°K	
	$\mu_{\parallel}$	0			0			0
	$\mu_{\perp}$	ferro			screw			0
	φ		26.5°				43.2°	
Ho	T		19°K		35°K		133°K	
	$\mu_{\parallel}$	ferro		0		0		0
	$\mu_{\perp}$	screw		screw		screw		0
	φ		36°		36°		50°	
Er	T		20°K		52°K		80°K	
	$\mu_{\parallel}$	ferro		oscillate		oscillate		0
	$\mu_{\perp}$	screw		screw		0		0
	φ		43.4°–45°		51°		51°	

$\mu_{\parallel}$ and $\mu_{\perp}$ represent the parallel and perpendicular components of the magnetic moment to the c axis. φ is the interplaner turn angle.

of the spin wave approximation and have discussed a possibility of a magnetic transition from the ferromagnetic state to the screw state and the temperature dependence of the pitch of the screw. There, we have assumed that the Fourier transform of the exchange integral $J(\boldsymbol{q})$ has two maxima at $\boldsymbol{q}=0$ and $\boldsymbol{q}=\boldsymbol{Q}_0$ and put all the types of the anisotropy energy out of our considerations except the axial anisotropy energy of the lowest order which keeps the spins in the c plane. However, the anisotropy energy is important, as stated in the previous paper, and can never be neglected in rare-earth metals. The purpose of this paper is to investigate the effect of the anisotropy energy on a spin system with the exchange interaction which in itself stabilizes a simple screw structure, and to explain magnetic behavior of dysprosium, holmium and erbium.

The anisotropy energy relevant to the hexagonal close packed structure can be decomposed into the following spherical harmonics:

$$H_a=V_2^0+V_4^0+V_6^0+V_6^6, \tag{1·1}$$

$$V_2^0=D_0\{3S_z^2-S(S+1)\}, \tag{1·2}$$

$$V_4^0=E_0\{35S_z^4-30S(S+1)S_z^2+25S_z^2+3S^2(S+1)^2-6S(S+1)\}, \tag{1·3}$$

$$V_6^0=F_0\{231S_z^6-315S(S+1)S_z^4+735S_z^4+105S^2(S+1)^2S_z^2 -525S(S+1)S_z^2+294S_z^2-5S^3(S+1)^3+40S^2(S+1)^2-60S(S+1)\}, \tag{1·4}$$

$$V_6^6=G_0\,\frac{1}{2}\,\{S_+^6+S_-^6\}. \tag{1·5}$$

In rare-earth metals, all these parts of the anisotropy energy are not always negligibly small even for heavier elements. If all of the anisotropy constants of axial parts D_0, E_0 and F_0 are negative, the easy direction of the spin will be along the c axis. However, if signs of these constants are not all negative, the easy direction of the spin is not necessarily along the c axis nor in the c plane but it is possible that the easy direction is between these two directions. In § 4, we shall consider what kinds of spin structure will be stable for the spin system with the anisotropy energy of Eq. (1·1) and the exchange interaction which stabilizes a simple screw structure, and the transition from a conical spin arrangement to a screw state or other spin arrangements found at low temperatures for holmium and erbium will be discussed.

In dysprosium, V_6^6 which represents the anisotropy in the c plane will also play an important part in its magnetic behavior. The effect of this anisotropy energy on the screw spin structure will be considered in § 2. The anisotropy energy in the c plane will stabilize the ferromagnetic state in the first order of the anisotropy constant G_0, but for the screw state this will be averaged out and will affect the screw state only in the second order in G_0. Accordingly, this anisotropy will effectively play a role of the first maximum in $J(\boldsymbol{q})$ curve at $\boldsymbol{q}=0$. Thus, if $J(Q_0)$ is smaller than the sum of $J(0)$ and the energy gain at the absolute zero of temperature by this anisotropy energy, a transition from the ferromagnetic state to the screw state will take place even for the case in which $J(\boldsymbol{q})$ curve has one maximum at $\boldsymbol{q}=\boldsymbol{Q}_0$. The magnetic transition in dysprosium will also be considered on the basis of the idea mentioned here in § 2. In connection with this consideration, the fact that the ferromagnetic state is realized in gadolinium will mean that here $J(\boldsymbol{q})$ curve has its maximum at $\boldsymbol{q}=0$. This will be reasonable because we can see in Table I a tendency that the pitch of the screw increases from erbium to dysprosium.

The change of the pitch of the screw with temperature may arise from several origins. The exchange interaction itself will give rise to a temperture dependence of the pitch of the screw, but as shown in the previous paper the calculated temperature dependence did not seem to agree with the experimental results. The quadrupole interaction will also give rise to a temperature dependence of the pitch. The effect of this interaction on the screw spin arrangement will be calculated in § 3. The obtained result on the change of the pitch with temperature agrees with the experimental results in its temperature dependence but it seems to be too small in its magnitude.

When an easy direction is along the c axis as in the case of erbium, the spin structure will be variant according to the nature of the anisotropy energy. For the one-ion anisotropy energy, the temperature at which the periodic modulation of each of three components of the moment sets in will differ for the parallel component from that for the perpendicular component to the c axis because of the anisotropy energy. This situation will be investigated in § 5 on the basis

of a simplified Weiss approximation in connection with the magnetic orderings which appear in erbium in its high temperature phases. In the temperature region in which only the z component of the moment is periodically modulated, there will appear the components of higher harmonics with the frequencies of odd multiples of the fundamental frequency corresponding to Q_0. These higher harmonics will have a tendency of squaring a simple sinusoidal modulation in accordance with the experimental results.

In the final section, the function of each part of the anisotropy energy will be summarized for each element of heavy rare-earth metals and a brief discussion on the exchange interaction will also be presented.

§ 2. The effect of the anisotropy energy in the c plane

According to the neutron diffraction experiments by Wilkinson, Koehler, Wollan and Cable,[1] dysprosium has a screw spin arrangement in which the screw axis is along the c axis and the direction of the spin moment is in the c plane in the temperature region from 179°K to 87°K and it shows ferromagnetism below 87°K.

We have considered in our previous paper, which will hereafter be referred to as YM,[4] this transition from the ferromagnetic state to the screw state on the basis of the following Hamiltonian:

$$H=-\sum_i\sum_j J(\boldsymbol{R}_j-\boldsymbol{R}_i)\boldsymbol{S}_j\cdot\boldsymbol{S}_i+3D_0\sum_i S_{iz}^2, \qquad (2\cdot1)$$

where $J(\boldsymbol{R}_j-\boldsymbol{R}_i)$ represents the exchange integral between two spins $\boldsymbol{S}_j$ and $\boldsymbol{S}_i$ on the lattice sites $\boldsymbol{R}_j$ and $\boldsymbol{R}_i$, and D_0 represents the anisotropy constant. The sign of D_0 was assumed to be positive in accordance with the experimental fact that the easy direction of the moment is in the c plane for dysprosium. With this simple Hamiltonian it has been shown that the ferro-to-screw transition can take place if the Fourier transform of the exchange integral

$$J(\boldsymbol{q})=\sum_j J(\boldsymbol{R}_j-\boldsymbol{R}_i)\,e^{i\boldsymbol{q}\cdot(\boldsymbol{R}_j-\boldsymbol{R}_i)} \qquad (2\cdot2)$$

has two maxima at $\boldsymbol{q}=0$ and $\boldsymbol{q}=\boldsymbol{Q}_0$ and $J(0)$ is larger than $J(Q_0)$. However, if we take into account the anisotropy in the c plane, the assumption that $J(\boldsymbol{q})$ has a maximum at $\boldsymbol{q}=0$ will not necessarily be needed in order to explain the magnetic transition. Therefore, in the following we shall assume that $J(\boldsymbol{q})$ has one maximum at $\boldsymbol{q}=\boldsymbol{Q}_0$. With this assumption, we can show that the spin system with the Hamiltonian (2·1) takes a screw spin arrangement specified by $\boldsymbol{Q}_0$ in which the screw axis is along the c axis and the direction of the spin moment is in the c plane at the absolute zero of temperature. In this section we shall consider the influence of the anisotropy energy in the c plane on a simple screw spin arrangement.

The anisotropy energy in the c plane is represented, in a hexagonal crystal, by

$$V_6^6=\frac{G_0}{2}\{(S_{ix}+iS_{iy})^6+(S_{ix}-iS_{iy})^6\}, \tag{2·3}$$

where the anisotropy constant G_0 will be negative for dysprosium and positive for holmium. The axial anisotropy energy represented by the second term of Eq. (2·1) keeps the spin moment in the c plane. Therefore, the direction of each spin at the absolute zero of temperature is specified by the angle from the x axis in the c plane as follows:

$$S_{ix}=S\cos\varphi_i, \quad S_{iy}=S\sin\varphi_i, \tag{2·4}$$

where S represents the magnitude of the spin. φ_i is determined by the following equation which is obtained by minimizing the sum of the exchange interaction and the anisotropy energy expressed by Eq. (2·3):

$$-2\sum_j J(\boldsymbol{R}_j-\boldsymbol{R}_i)\sin(\varphi_j-\varphi_i)-6G_0S^4\sin 6\varphi_i=0. \tag{2·5}$$

In the absence of the anisotropy energy in the c plane, φ_i is given by

$$\varphi_i{}^0=\boldsymbol{Q}\cdot\boldsymbol{R}_i+\varphi_0, \tag{2·6}$$

where φ_0 is an arbitrary phase angle. A solution of Eq. (2·5) can be obtained in a power series of G_0 as follows:

$$\varphi_i=\varphi_i{}^0+\delta\sin 6\varphi_i{}^0+\cdots, \tag{2·7}$$

$$\delta=3\widetilde{G}/[J(Q)], \tag{2·8}$$

where $\widetilde{G}$ and $[J(Q)]$ are defined by

$$\widetilde{G}=G_0S^4, \quad [J(Q)]=J(Q)-\frac{1}{2}J(5Q)-\frac{1}{2}J(7Q). \tag{2·9}$$

The energy gain by this change in φ_i becomes

$$-NS^2\delta^2[J(Q)]/2. \tag{2·10}$$

With the use of Eq. (2·7), $\exp(i\varphi_i)$ can be calculated as

$$e^{i\varphi_i}=e^{i\varphi_i{}^0}(1+i\delta\sin 6\varphi_i{}^0+\cdots)=e^{i\varphi_i{}^0}+\frac{\delta}{2}(e^{7i\varphi_i{}^0}-e^{-5i\varphi_i{}^0})+\cdots. \tag{2·11}$$

This result means that the higher order satellites corresponding to the 7th and the 5th harmonics appear in neutron reflections. The ratio of the intensity of these two reflections to the primary satellite is given by $\delta^2/4$. This quantity will be an order of 10^{-3} for dysprosium and will be larger than this by a factor of about 4 for holmium. Therefore, it is not unlikely that the higher order satellite corresponding to the 5th harmonic, which has been observed in holmium by Koehler et al.,[2] originates from the anisotropy energy in the c plane. However, this anisotropy energy does not give rise to the second harmonic which has also been observed in holmium.

A spin wave calculation for a screw spin arrangement has been done in

YM. According to this paper, the Hamiltonian (2·1) is expressed in this spin wave approximation by

$$H=-NS(S+1)J(Q)+\sum_k\{A_k(a_k^*a_k+a_ka_k^*)-B_k(a_ka_{-k}+a_k^*a_{-k}^*)\}, \tag{2·12}$$

$$A_k=\frac{S}{2}\left[2J(\boldsymbol{Q})-\frac{1}{2}\{J(\boldsymbol{Q}+\boldsymbol{k})+J(\boldsymbol{Q}-\boldsymbol{k})\}-J(\boldsymbol{k})+3D_0\right], \tag{2·13}$$

$$B_k=\frac{S}{2}\left[\frac{1}{2}\{J(\boldsymbol{Q}+\boldsymbol{k})+J(\boldsymbol{Q}-\boldsymbol{k})\}-J(\boldsymbol{k})+3D_0\right], \tag{2·14}$$

where $\boldsymbol{k}$ is the the wave vector of the spin wave. This Hamiltonian can easily be diagonalized and the spin wave frequency is obtained by

$$\hbar\omega_k=2(A_k^2-B_k^2)^{1/2}. \tag{2·15}$$

The change of the spin wave frequency due to the anisotropy energy in the c plane represented by Eq. (2·3) can be calculated in an expansion with respect to the anisotropy constant G_0. The obtained result is as follows:

$$(\hbar\omega_k)^2=4(A_k^2-B_k^2)+2(41A_k+31B_k-72A_0)S\delta^2[J(Q)]+\cdots \quad (\text{for } \boldsymbol{k}\to 0). \tag{2·16}$$

As seen from this expression, the spin wave frequency is affected by this anisotropy energy only in its second order and this second order term in δ or G_0 also vanishes in the limit of the long wavelength. Therefore, there is no energy gap for spin wave excitation. This fact will generally hold since the anisotropy energy can never fix the arbitrary phase angle φ_0.

The spin wave frequency in the ferromagnetic state can also be calculated for the Hamiltonian consisting of Eq. (2·1) and Eq. (2·3). The result is as follows:

$$\hbar\omega_k=2S[\{18|\widetilde{G}|+J(0)-J(\boldsymbol{k})\}\{3D_0+3|\widetilde{G}|+J(0)-J(\boldsymbol{k})\}]^{1/2}. \tag{2·17}$$

This is a special case of the result obtained by Niira.[5] In the ferromagnetic case, the excitation energy for the spin wave will have an energy gap

$$\Delta=2S[54|\widetilde{G}|(D_0+|\widetilde{G}|)]^{1/2}. \tag{2·18}$$

The energy of the ferromagnetic ground state is given by

$$E_f=-NS(S+1)J(0)-N|\widetilde{G}|S^2+\frac{1}{2}\sum_k\hbar\omega_k. \tag{2·19}$$

Thus, the energy lowering is proportional to the anisotropy constant G_0 in this case, while it is proportional to the square of G_0 in the screw state. Therefore, if $J(Q)-J(0)$ which is assumed to be positive is smaller than $|\widetilde{G}|$, the ferromagnetic state will be more stable than the screw state at low tem-

peratures. However, since the excitation energy for the spin wave is larger in the ferromagnetic state because of the energy gap, the entropy of this state is smaller than that of the screw state. Accordingly, a transition of the first kind between these two states will be expected.

In order to calculate the free energies of these two states, we approximate the spin wave spectrum for the screw state by its expansion for a long wavelength,

$$\hbar\omega_s(\boldsymbol{k})=[c_{\parallel}^2 k_z^2+c_{\perp}^2(k_x^2+k_y^2)]^{1/2}, \tag{2·20}$$

where $c_{\parallel}$ and $c_{\perp}$ are calculated with the use of Eq. (2·15) as

$$c_{\parallel}^2=2S^2\{J(Q)-J(0)+3D_0\}\,|J_{zz}(Q)|, \tag{2·21}$$

$$c_{\perp}^2=2S^2\{J(Q)-J(0)+3D_0\}\,|J_{xx}(Q)|. \tag{2·22}$$

For the ferromagnetic state, the frequency spectrum is approximated by

$$\begin{aligned}\hbar\omega_f(\boldsymbol{k})&=\Delta, \qquad \text{for } \hbar\omega_s(\boldsymbol{k})\leq\Delta\\ &=\hbar\omega_s(\boldsymbol{k}), \quad \text{for } \hbar\omega_s(\boldsymbol{k})>\Delta,\end{aligned} \tag{2·23}$$

where Δ is given by Eq. (2·18). Then the difference between free energies of the ferromagnetic state and the screw state can be written as

$$\begin{aligned}F_f-F_s=&NS(S+1)\{J(Q)-J(0)\}-N\left\{|\widetilde{G}|-\frac{9}{2}\widetilde{G}^2/[J(Q)]\right\}S^2\\ &+\frac{1}{2}\sum_{\substack{\boldsymbol{k}\\ \hbar\omega_s(\boldsymbol{k})\leq\Delta}}\{\Delta-\hbar\omega_s(\boldsymbol{k})\}\\ &+kT\sum_{\substack{\boldsymbol{k}\\ \hbar\omega_s(\boldsymbol{k})\leq\Delta}}\{\ln(1-e^{-\Delta/kT})-\ln(1-e^{-\hbar\omega_s(\boldsymbol{k})/kT})\},\end{aligned} \tag{2·24}$$

where the first and the second terms are, respectively, the difference in the classical exchange energy and that in the classical anisotropy energy, the third term the difference in the zero point energy and the fourth term the difference in the free energy arising from the spin wave excitation. Eq. (2·24) can be calculated for $kT>\Delta$ as follows:

$$\begin{aligned}(F_f-F_s)/N=&S(S+1)\{J(Q)-J(0)\}-\left\{|\widetilde{G}|-\frac{9}{2}\widetilde{G}^2/[J(Q)]\right\}S^2\\ &+\frac{1}{8}\gamma+\gamma\left(\frac{kT}{3\Delta}-\frac{1}{8}+\cdots\right),\end{aligned} \tag{2·25}$$

and

$$\gamma=\frac{\Omega\Delta^4}{6\pi^2 c_{\perp}^2 c_{\parallel}}, \tag{2·26}$$

where Ω is the atomic volume. From Eq. (2·25), the transition temperature T_c is given by

$$T_c=3\frac{\mathcal{J}}{k\tilde{\gamma}}\left[\{|\widetilde{G}|-9\widetilde{G}^2/2[J(Q)]\}S^2-S(S+1)\{J(Q)-J(0)\}\right], \quad (2\cdot27)$$

and the critical magnetic field at which the transition from the screw state to the ferromagnetic state takes place is given by

$$gS\mu_B H_c=\frac{1}{3}\frac{\tilde{\gamma}}{\mathcal{J}}k(T-T_c). \quad (2\cdot28)$$

Experimentally, this critical field is expressed in the case of dysprosium by

$$H_c=120(T-T_c) \quad \text{oersteds} \quad (2\cdot29)$$

according to Behrendt, Legvold and Spedding.[6] Comparing Eq. (2·29) with Eq. (2·28) and using $gS\sim9$, we obtain a value of $\tilde{\gamma}/\mathcal{J}$ as follows:

$$\tilde{\gamma}/\mathcal{J}=0.22. \quad (2\cdot30)$$

The values of $J(0)$ and $J(Q)$ in dysprosium can be estimated from the Néel temperature and two susceptibilities parallel and perpendicular to the c axis above the Néel temperature as

$$J(0)=3.6°\text{K} \quad \text{and} \quad J(Q)=3.9°\text{K}. \quad (2\cdot31)$$

If we use these values and the experimental value of the transition temperature 87°K together with (2·30) in Eq. (2·27), we obtain

$$\widetilde{G}=-0.45°\text{K}. \quad (2\cdot32)$$

On the other hand, the anisotropy constant $\widetilde{G}$ can be estimated from the experimental data obtained by Behrendt, Legvold and Spedding[6] on the magnetization curve in the c plane as

$$\widetilde{G}=-0.1°\text{K}. \quad (2\cdot33)$$

Thus, the value of $\widetilde{G}$ derived from the relation (2·27) is too large compared with (2·33). If we take a value of (2·33) as $\widetilde{G}$, the screw state will be stable even at the absolute zero of temperature. This fact implies that the value of $J(Q)-J(0)$ decreases with decreasing temperature. Therefore, in order to take into account this effect, we express $J(\boldsymbol{q})$ for $\boldsymbol{q}$ parallel to the c axis by the following conventional form:

$$J(q)=J_0+J_1\cos qc'-J_2\cos 2qc', \quad (2\cdot34)$$

where J_0, J_1 and J_2 are the three parameters proportional to the exchange integrals and c' is the distance between adjacent layers. Then, $J(Q)-J(0)$ is given by

$$J(Q)-J(0)=J_1(2J_2/J_1+J_1/8J_2-1), \quad (2\cdot35)$$

where

$$\cos Qc'=J_1/4J_2. \quad (2\cdot36)$$

The interplaner turn angle of dysprosium has been measured by Wilkinson, Koehler, Wollan and Cable.[1] According to their results, the turn angle is temperature-dependent and

$$Qc' = 43.2^\circ \quad \text{at the Néel temperature } T_s$$

and

$$Qc' = 26.5^\circ \quad \text{at the transition temperature } T_c.$$

Inserting these values into Eq. (2·35), we obtain

$$J(Q) - J(0) = 0.051\, J_1 \quad \text{at} \quad T_s,$$

and

$$J(Q) - J(0) = 0.006\, J_1 \quad \text{at} \quad T_c.$$

Therefore, if we assume that J_1 is nearly independent of temperature, the value of $J(Q) - J(0)$ at the absolute zero of temperature will be smaller nearly by a factor of ten than that at the Néel temperature. Accordingly, if we assume that $J(Q) - J(0)$ is 0.04°K at the absolute zero of temperature, the anisotropy constant $\widetilde{G}$ will be given by

$$\widetilde{G} = -0.16^\circ\text{K} \tag{2·37}$$

which seems to be rather reasonable compared with the value evaluated from the magnetization curve.

§ 3. The effect of the quadrupole interaction

According to the neutron diffraction experiments, the pitch of the screw varies with temperature. A simple molecular field approach for the spin system with only the exchange interaction leads to the result that the pitch of the screw is independent of temperature. In the previous paper YM, we have considered the temperature dependence of the pitch of the screw on the basis of the spin wave theory. Besides the exchange interaction itself, the anisotropy energy in the c plane will be expected to have an effect of varying the pitch with temperature since this anisotropy energy adds the terms including $J(5Q)$ and $J(7Q)$ to the free energy as shown in the previous section. However, this effect will be small compared with the experimental change of the pitch of the screw with temperature. Another source of changing the pitch with temperature will be the multipole interaction. In this section, we shall consider the effect of the quadrupole interaction on the screw spin structure.

The quadrupole interaction between electrons in two atoms can be expressed as

$$\frac{e^2}{R_{12}^5} \sum_{ij} \Big[\frac{3}{2} (3\zeta_i^2 - \rho_i^2)(3\zeta_j^2 - \rho_j^2) + 12\{\zeta_i\zeta_j - (\boldsymbol{\rho}_i \cdot \boldsymbol{\rho}_j)\}\zeta_i\zeta_j + \frac{3}{4}\{(\rho_i^2 - \zeta_i^2)(\rho_j^2 - \zeta_j^2) - 2(\xi_i\eta_j - \eta_i\xi_j)^2\}\Big], \tag{3·1}$$

where R_{12} represents the distance between the atom 1 and the atom 2. The summations with respect to i and j are, respectively, taken over $4f$ electrons belonging to the atom 1 and the atom 2. ξ_i, η_i and ζ_i are the components of the position vector ρ_i of the i-th electron parallel to the coordinate axes ξ, η and ζ referred to the atom 1 and the ζ axis is taken in the direction of R_{12}. ξ_i, η_i and ζ_i can be expressed in terms of x_i, y_i and z_i which are the components towards the coordinate axes referred to the crystal as follows:

$$\begin{aligned}\xi_i &= l_1 x_i + l_2 y_i + l_3 z_i,\\ \eta_i &= m_1 x_i + m_2 y_i + m_3 z_i,\\ \zeta_i &= n_1 x_i + n_2 y_i + n_3 z_i,\end{aligned} \tag{3·2}$$

where (l_1, l_2, l_3), (m_1, m_2, m_3) and (n_1, n_2, n_3) are, respectively, the direction cosines of the ξ, η and ζ axes with respect to the x, y and z axes.

Since the magnetic moments in each c plane perpendicular to the hexagonal axis are all parallel in the ordered state of the rare-earth metals which we are now considering, we can deal with all the spins on the same c layer as equivalent. Then if we insert the relation (3·2) in Eq. (3·1) and take a summation over the nearest neighbor atoms of the atom 1 on the upper or the lower layer with respect to the layer on which the atom 1 is situated, the quadrupole interaction of the atom 1 with its nearest neighbor atoms on the upper or the lower layer can be expressed as

$$\begin{aligned}\sum_i \sum_j \Big\{ &-\frac{1}{3}(A_0+B_0) r_i^2 r_j^2 + A_0(x_i^2 x_j^2 + y_i^2 y_j^2)\\ &+ (A_0+B_0) z_i^2 z_j^2 + B_0(x_i^2 y_j^2 + y_i^2 x_j^2)\\ &- \frac{4}{3}(A_0+B_0)(y_i z_i y_j z_j + z_i x_i z_j x_j) - \frac{4}{3}(A_0-2B_0) x_i y_i x_j y_j \Big\},\end{aligned} \tag{3·3}$$

and

$$\begin{aligned}A_0 &= \frac{e^2}{R^5}\left(-\frac{105}{2}\sum n_1^2 n_3^2 - \frac{105}{4}\sum n_1^2 n_2^2 + \frac{21}{2}\sum n_1^2 + \frac{21}{4}\sum n_3^2\right)\\ &= \frac{63}{32}\frac{e^2}{R^5}(35 n_3^4 - 30 n_3^2 + 3),\\ B_0 &= \frac{e^2}{R^5}\left\{\frac{105}{4}(\sum n_1^2 n_2^2 - \sum n_1^2 n_3^2) + \frac{15}{4}(\sum n_3^2 - n_1^2 \sum)\right\}\\ &= \frac{45}{32}\frac{e^2}{R^5}(35 n_3^4 - 30 n_3^2 + 3),\end{aligned} \tag{3·4}$$

where the summation in front of the direction cosines is taken over the atoms on the upper or the lower layer and R represents the distance between the nearest neighbors. In deriving Eq. (3·3) we have omitted those terms which are non-

zero in this stage but will vanish after taking an average with respect to the equilibrium spin state.

As Stevens[7] has shown, in the subspace in which the states of a magnetic atom are eigenstates of the total angular momentum which is the vector sum of the orbital angular momentum and the spin angular momentum, the following operator equivalence holds:

$$\sum_i x_i^2=\alpha\langle r^2\rangle S_x^2,\quad \sum_i x_i y_i=\frac{1}{2}\alpha\langle r^2\rangle(S_xS_y+S_yS_x),\cdots, \qquad (3\cdot5)$$

where α is a constant value determined by the magnitudes of the total angular momentum, the orbital angular momentum and the spin angular momentum and its value has been calculated for rare-earth metal ions by Stevens.[7] $\langle r^2\rangle$ means the average of r^2 of an electron over the $4f$ radial wave function. With the use of the relation (3·5) in Eq. (3·3), we obtain the quadrupole interaction as follows:

$$\begin{aligned}H_Q=&\frac{1}{2}\sum_i\sum_j\Big[-\frac{1}{3}(A+B)\,S^2(S+1)^2+A(S_{ix}^2S_{jx}^2+S_{iy}^2S_{jy}^2)\\&+(A+B)\,S_{iz}^2S_{jz}^2+B(S_{ix}^2S_{jy}^2+S_{iy}^2S_{jx}^2)\\&-\frac{1}{3}(A-2B)(S_{ix}S_{iy}+S_{iy}S_{ix})(S_{jx}S_{jy}+S_{jy}S_{jx})\\&-\frac{1}{3}(A+B)\Big\{(S_{iy}S_{iz}+S_{iz}S_{iy})(S_{jy}S_{jz}+S_{jz}S_{jy})\\&+(S_{iz}S_{ix}+S_{ix}S_{iz})(S_{jz}S_{jx}+S_{jx}S_{jz})\Big\}\Big],\end{aligned} \qquad (3\cdot6)$$

$$A=A_0\alpha^2\langle r^2\rangle^2,\quad B=B_0\alpha^2\langle r^2\rangle^2, \qquad (3\cdot7)$$

where the summation with respect to i is taken over all spins, and that over j is taken over the upper and the lower layers.

The contribution of this quadrupole interaction to the free energy is, to its first order, given by its thermal average $\langle H_Q\rangle$. In the screw state of dysprosium and holmium, each spin $\boldsymbol{S}_i$ is expressed as

$$\begin{aligned}&S_{iz}=S_{i\eta},\\&S_{iy}=S_{i\zeta}\cos\varphi_i-S_{i\xi}\sin\varphi_i,\\&S_{ix}=S_{i\zeta}\sin\varphi_i+S_{i\xi}\cos\varphi_i,\\&\varphi_i=\boldsymbol{Q}\cdot\boldsymbol{R}_i+\varphi_0,\end{aligned} \qquad (3\cdot8)$$

where the ζ_i axis is taken in the average direction of the i-th spin. If we insert the relation of (3·8) into Eq. (3·6) and use the Weiss approximation, the thermal average of the quadrupole interaction (3·6) can be calculated as

$$\langle H_Q\rangle=\frac{1}{24}N(A+B)\left(1+\frac{1}{2}\cos 2Qc'\right)\{3\langle S_\zeta^2\rangle-S(S+1)\}^2, \qquad (3\cdot9)$$

where $\langle S_\zeta^2 \rangle$ means the average of the square of the spin component parallel to the molecular field in the presence of the molecular field. Thus, the contribution of the quadrupole interaction to the free energy will increase in its magnitude with decreasing temperature as M^6 at low temperatures and as M^4 at high temperatures near the Néel temperature.

The free energy of the spin system in the absence of the quadrupole interaction can be calculated on the basis of the Weiss approximation as follows:

$$F_0 = NJ(Q)\langle S_\zeta \rangle^2 - NkT \ln \mathrm{Tr}\left[\exp \frac{2J(Q)}{kT}\langle S_\zeta \rangle S_\zeta\right], \qquad (3\cdot10)$$

where $\langle S_\zeta \rangle$ means the average value of the spin component parallel to the molecular field.

Minimizing the sum of Eqs. (3·10) and (3·9) with respect to Q, we obtain, to the first order in $(A+B)$,

$$-\langle S_\zeta \rangle^2 \frac{dJ(Q)}{dQ} - \frac{1}{24}(A+B)\, c' \sin 2Qc' \{3\langle S_\zeta^2 \rangle - S(S+1)\}^2 = 0. \qquad (3\cdot11)$$

Expanding $J(Q)$ in the Taylor series around its maximum value $J(Q_0)$ as

$$J(Q) = J(Q_0) + \frac{1}{2}\left(\frac{d^2 J(Q)}{dQ^2}\right)_{Q=Q_0}(Q-Q_0)^2 + \cdots \qquad (3\cdot12)$$

and inserting this expression into Eq. (3·11), we obtain the following result for the temperature dependence of Q:

$$Q - Q_0 = \frac{(1/24)(A+B)\, c' \sin 2Q_0 c'}{-(d^2 J(Q)/dQ^2)_{Q=Q_0}} \frac{\{3\langle S_\zeta^2 \rangle - S(S+1)\}^2}{\langle S_\zeta \rangle^2}. \qquad (3\cdot13)$$

With the use of Eqs. (3·4) and (3·7), $(A+B)$ becomes

$$A+B = \frac{27}{8}\frac{e^2}{R^5}(35 n_3^4 - 30 n_3^2 + 3)\, \alpha^2 \langle r^2 \rangle^2. \qquad (3\cdot14)$$

Since this value is negative for the crystal structure of dysprosium and holmium, we can see from Eq. (3·13) that the quadrupole interaction has a tendency of decreasing the turn angle of the spin for a smaller turn angle than 90° and increasing the turn angle for a larger angle than 90°. The temperature dependence is described by the last factor of (3·13). This factor increases with decreasing temperature almost linearly. Therefore, Eq. (3·13) leads to the almost linear decrease of the turn angle with lowering temperature. This result seems to be consistent with the experimental results.

Below and near the Néel temperature, $\langle S_\zeta^2 \rangle$ is approximated by

$$\langle S_\zeta^2 \rangle = \frac{1}{3}S(S+1) + \frac{1}{2}\left(\frac{2J(Q)}{kT}\right)^2 \langle S_\zeta \rangle^2 \left\{\frac{4}{45}S^2(S+1)^2 - \frac{1}{15}S(S+1)\right\}, \qquad (3\cdot15)$$

and

$$\langle S_\zeta \rangle^2 = \frac{10}{3}\left(\frac{T}{T_s}\right)^2 \frac{T_s - T}{T_s} \frac{S^2(S+1)^2}{S^2+(S+1)^2}. \tag{3·16}$$

Inserting Eqs. (3·15) and (3·16) into (3·13), we obtain the temperature dependence of the turn angle near the Néel temperature T_s as

$$\frac{(Q-Q_0)c'}{T_s-T} = \frac{1}{5} \frac{\{S(S+1)-3/4\}^2}{S^2+(S+1)^2} \frac{1}{T_s} \frac{A+B}{-(1/c'^2)(d^2J(Q)/dQ^2)_{Q=Q_0}} \sin 2Q_0c'. \tag{3·17}$$

For dysprosium,

$$S=15/2, \quad T_s=179°\mathrm{K} \quad \text{and} \quad \sin 2Q_0c' \cong 1. \tag{3·18}$$

$-(1/c'^2)(d^2J(Q)/dQ^2)_{Q=Q_0}$ can be evaluated by using Eq. (2·34) and the value of (2·31) as

$$-\frac{1}{c'^2}\left(\frac{d^2J(Q)}{dQ^2}\right)_{Q=Q_0} = 3.7°\mathrm{K}. \tag{3·19}$$

The lattice parameters of dysprosium are given by

$$a=3.5903\,\mathrm{A} \quad \text{and} \quad c=5.6457\,\mathrm{A}$$

and

$$\alpha = 2/(9\times 35).$$

With the use of these values and the value of $\langle r^2 \rangle$ calculated by Kasuya and Watabe[8]

$$\langle r^2 \rangle = 0.33\,\mathrm{A}^2, \tag{3·20}$$

$(A+B)$ is evaluated as follows:

$$(A+B) = -8\times 10^{-3}\,°\mathrm{K}. \tag{3·21}$$

Inserting (3·21), (3·19) and (3·18) into Eq. (3·17), we obtain the following value for the temperature dependence of the turn angle in dysprosium:

$$\frac{(Q-Q_0)c'}{T_s-T} = -0.74\times 10^{-4}\,°\mathrm{K}^{-1}. \tag{3·22}$$

The experimental value for this is given by

$$\left[\frac{(Q-Q_0)c'}{T_s-T}\right]_{\mathrm{exp}} \sim -3\times 10^{-3}\,°\mathrm{K}^{-1}. \tag{3·23}$$

Therefore, the theoretical value amounts only to one fortieth of the experimental value. For holmium, the situation is even worse because of a smaller value of α.

Thus, we can conclude that the change of the pitch of the screw in rare-earth metals is not due to the pure quadrupole interaction. In connection with

this conclusion, it is noted that the change of the pitch of the same order as that in rare-earth metals has also been found in $MnAu_2$[9] in which the quadrupole interaction is considered to be small.

However, considering the fact that the temperature dependence and the sign of (3·13) are in agreement with the experimental results, we could not deny that the indirect quadrupole interaction would be a possible source of the change of the pitch with temperature.

We can also calculate the contribution of the quadrupole interaction to the free energy in the high temperature phase of erbium in which only the z component of the moment oscillates because of the axial anisotropy which tends to make the moment point to the c axis. The result is that this contribution does not vanish but gives rise to a similar term to Eq. (3·9). Therefore, the quadrupole interaction is expected to change the period with temperature also for this case. However, according to the experimental results obtained by Cable, Wollan, Koehler and Wilkinson,[3] the period in this temperature region is temperature-independent. Therefore, we can not explain the case of erbium by the pseudo-quadrupole interaction. An indirect interaction of the other type than the quadrupole interaction seems to be an origin of the temperature variation of the pitch.

§ 4. The effect of the higher order anisotropy energy

At low temperatures, a conical spin arrangement has been found in holmium and erbium. This spin structure suggests that the easy axis of the spin moment is in an intermediate direction between the c axis and a direction in the c plane. The orientation of the spin moment in the intermediate direction is considered to result from the competing action of the higher order anisotropy energies against the second order energy. In this section, we shall consider the effect of the axial anisotropy energies of higher order, that is, E_0- and F_0-terms in the Hamiltonian (1·1). Since the total angular momentum for the heavy rare-earth ions is large ($J=6$ at least), we shall, for simplicity, treat spins classically in this and the following sections. This simplification will not cause any essential changes in the results except at very low temperatures.

Now, the axial anisotropy energy for the classical spin can be expressed as

$$V(\theta_i)=DP_2(\cos\theta_i)+EP_4(\cos\theta_i)+FP_6(\cos\theta_i), \qquad (4\cdot1)$$

where θ_i denotes the angle between the direction of the i-th spin and the c axis, and $P_n(\cos\theta)$ is the Legendre polynomial of the n-th order. Coefficients D, E and F stand, respectively, for $2D_0S^2$, $8E_0S^4$ and $16F_0S^6$, where D_0, E_0 and F_0 were defined in Eqs. (1·2)–(1·4). These constants vary from element to element not only in their magnitudes but also in their signs. If we assume that the anisotropy energy for a single ion has its origin in the crystalline field

which is produced by the positive point charges located at the lattice points, we can deduce the signs of these constants.

As to their absolute magnitudes, there are some ambiguities in calculating the expansion coefficients of spherical harmonics for the crystalline field and in converting them into the coefficients in the spin Hamiltonian (1·1). As to their signs, on the other hand, the deduction is less ambiguous. So, we calculate the signs of the crystalline field coefficients taking only nearest neighbor lattice points into account. Multiplying them by the conversion factors α, β and γ for the trivalent ions given by Stevens,[7] we obtain the final result as shown in Table II. In this table, the sign of G is chosen so that the x axis coincides with one of the a axes of the crystal.

Table II. The signs of the anisotropy constants

Ions	Tb^{3+}	Dy^{3+}	Ho^{3+}	Er^{3+}	Tm^{3+}
D	+	+	+	−	−
E	−	+	+	−	−
F	−	+	−	+	−
G	+	−	+	−	+

First, we shall investigate the behavior of the anisotropy energy V as the angle θ varies. The second order anisotropy simply makes the direction of the spin moment either parallel or perpendicular to the c axis according to its sign. However, $V(\theta)$ given by Eq. (4·1) generally has four extrema in the range $0 \leqq \theta \leqq \pi/2$. Therefore, if we take into account the higher order anisotropy energy, the intermediate direction is also expected to become a preferred direction of the spin.

The relation between the behavior of $V(\theta)$ and the ratios of E and F to D is shown in Fig. 1. In this figure, $E/D-F/D$ plane is divided into twelve regions and the angular variation of V/D in each region is shown schematically. The dashed lines are the loci of the points at which the function $V(\theta)$ has one of its extrema for some particular values of θ, i. e. 15°, 30°,···etc.

As seen from Table II, the representative points for dysprosium and thulium lie in the first quadrant, for holmium and erbium in the fourth quadrant and for terbium in the third quadrant of Fig. 1. In fact, this is the main reason why these metals apparently have fairly different magnetic orderings from each other.

Now, we shall consider the effect of the anisotropy of (4·1) on the spin system with the exchange interaction such that a screw arrangement of the spins with the screw axis along the c axis be stable. In this case, we assume that the exchange interaction is much larger than the anisotropy energy, but that the difference in energy between the screw and the ferromagnetic spin

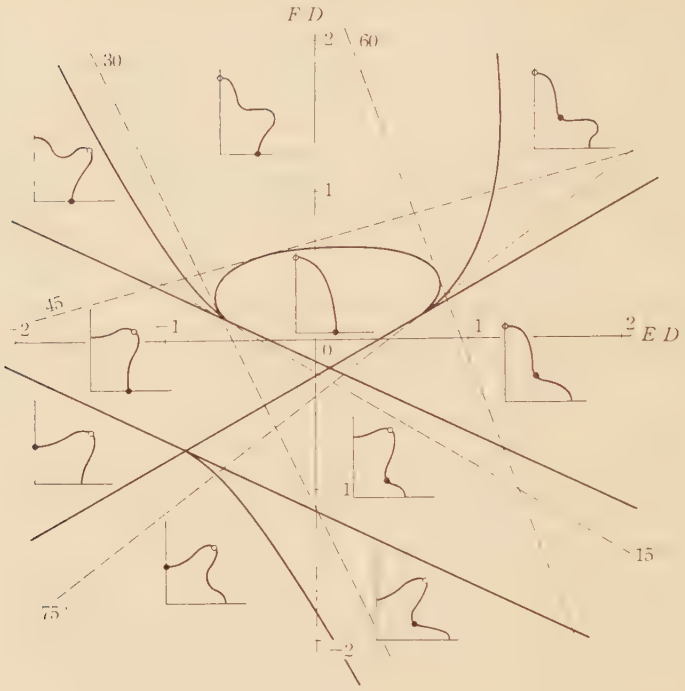

Fig. 1. The relation between the angular variation of the anisotropy energy and the ratios of the anisotropy constants. E/D (abscissa) $-F/D$ (ordinate) plane is divided into twelve regions, where $V(\theta)$ behaves in different ways (two of them are outside of this figure). In each region, a polar diagram for V/D is shown schematically. The vertical and the horizontal axes represent, respectively, the directions parallel and perpendicular to the c axis. A solid circle indicates the preferred direction of the spin moment for the case $D>0$, and an open circle for the case $D<0$. A dashed line is a locus of the points at which the preferred angle takes a particular value indicated there.

arrangements is of the same order of magnitude as the anisotropy energy. These assumptions seem to be valid to a certain extent in rare-earth metals under consideration. For this spin system, the various kinds of spin arrangement may be conceivable. However, we take up the following three typical spin structures and calculate the energy of each structure to the first order of the anisotropy energy as a function of the characteristic angle θ which will be defined below:

i) The ferromagnetic spin arrangement in which three spin components are described as

$$\begin{aligned} S_{ix} &= S\sin\theta, \\ S_{iy} &= 0, \\ S_{iz} &= S\cos\theta. \end{aligned} \tag{4·2}$$

ii) The pure screw spin arrangement in which three spin components are described as

$$\begin{aligned} S_{ix} &= S\cos\theta\cos\boldsymbol{Q}\cdot\boldsymbol{R}_i, \\ S_{iy} &= S\sin\boldsymbol{Q}\cdot\boldsymbol{R}_i, \\ S_{iz} &= S\sin\theta\cos\boldsymbol{Q}\cdot\boldsymbol{R}_i. \end{aligned} \tag{4·3}$$

iii) The conical spin arrangement in which three spin components are described as

$$\begin{aligned} S_{ix} &= S\sin\theta\cos\boldsymbol{Q}\cdot\boldsymbol{R}_i, \\ S_{iy} &= S\sin\theta\sin\boldsymbol{Q}\cdot\boldsymbol{R}_i, \\ S_{iz} &= S\cos\theta. \end{aligned} \tag{4·4}$$

These three spin structures are shown in Fig. 2.

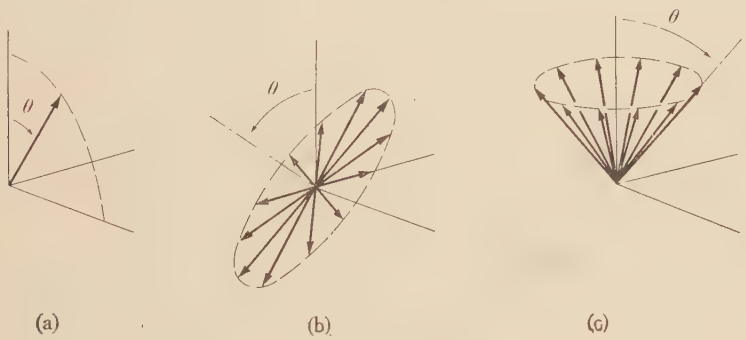

Fig. 2. Spin arrangements in (a) ferromagnetic, (b) screw and (c) conical states.

Arrows which represent the spin moments on the successive layers are drawn from a common center.

The energy of each state is given by

$$E_{\text{ferro}}(\theta)/N = -J(0)\,S^2 + DP_2(\cos\theta) + EP_4(\cos\theta) + FP_6(\cos\theta), \tag{4·5}$$

$$E_{\text{screw}}(\theta)/N = -J(Q)\,S^2 - \frac{1}{2}DP_2(\cos\theta) + \frac{3}{8}EP_4(\cos\theta) - \frac{5}{16}FP_6(\cos\theta), \tag{4·6}$$

$$E_{\text{cone}}(\theta)/N = -J(Q)\,S^2 + \{J(Q) - J(0)\}\,S^2\cos^2\theta + DP_2(\cos\theta) + EP_4(\cos\theta) + FP_6(\cos\theta). \tag{4·7}$$

It should be noted that the energy difference between two of these structures is of the order of the anisotropy energy. The characteristic angle θ for each structure is determined by the minimum condition and the structure with the smallest energy among these three structures will be realized.

Next, we shall consider the change in the spin structure with temperature. In order to do this, we have to compare the free energies of various spin arrangements, the calculation of which will be made later in the molecular field approximation. But, in order to clarify the essential features of the problem, we shall here give a qualitative explanation basing on the energy expressions given by Eqs. (4·5)–(4·7), because the free energies would be expressed in those forms similar to Eqs. (4·5)–(4·7), in which coefficients S^2, D, E and F are regarded as temperature-dependent.

The exchange and the second order terms vary most slowly with temperature, and the higher the order of anisotropy, the more rapid its temperature variation. Therefore, as the temperature is lowered from the paramagnetic region, a screw arrangement will set in at the Néel temperature. Although the axis of the screw arrangement (the direction of translation) is assumed to be fixed to the c axis, the axis of rotation of the spin moments may point to any direction unless the anisotropy is taken into account. Down to a fairly low temperature, the D-term is only appreciable, and the spins rotate in the c plane ($\theta=0$) or in a plane which contains the c axis ($\theta=\pi/2$), according as D is positive or negative. The former was named the "proper screw" structure and the latter the "cycloidal" structure by Yoshimori.[10] These proper screw and cycloidal structures are corresponding to the states in the central region in Fig. 1, because E and F can be neglected compared with D in this temperature range. At the lower temperatures the higher order anisotropy energies grow up, and if they have appropriate magnitudes and signs the anisotropy parameters of the spin system will depart from the central region of Fig. 1, and consequently, the plane in which the direction of the spin lies will begin to tilt from the original position. This transition will usually be of the second kind, that is, θ varies continuously from zero (or $\pi/2$) in proportion to $(T_c-T)^{1/2}$. In this arrangement some spins are along the preferred direction and others are not.

When the anisotropy energies increase further, a transition to the ferromagnetic state might occur if the difference between $J(Q)$ and $J(0)$ is not large. In the ferromagnetic state every spin gains in its anisotropy energy to the detriment of the exchange energy. By comparing Eq. (4·5) with Eq. (4·7), however, the conical arrangement is shown to be more favorable than the ferromagnetic one because of its less disadvantage in the exchange energy. Thus, a transition between the screw arrangement and the conical one takes place as a result of the competition between the exchange energy and the higher order anisotropy energy. This transition is of the first kind except the case in which the screw has its rotation axis in the c direction where the transition is of the second kind. This seems to be the case for holmium. It is also possible that a screw-cone transition occurs before the screw axis inclines. Similarly to the case of dysprosium, the six-fold anisotropy in the c plane or the external magnetic field will stabilize the ferromagnetic state more than the conical state.

Although other spin arrangements than the three which have been considered here should be taken into consideration and, in the actual situation, the anisotropy energies may not necessarily be small corrections to the exchange energy, we can understand, from the explanation given here, how the higher order anisotropy energy affects the screw spin structures.

Because of the largest value of the orbital angular momentum, the higher order anisotropy will be most important for holmium and erbium among the rare-earth metals which have more than half-filled $4f$ shells. Actually, in the lowest temperature phase, the spins of these metals align on a cone, and in holmium the conical spin arrangement can easily be transformed to the ferromagnetic state by external magnetic field. Due to the different decreasing rates among the anisotropy energies of various orders, the vertical angle of the cone increases with temperature as observed by Koehler et al.[2] in holmium from the intensities of the $(10\bar{1}0)$ and $(11\bar{2}0)$ neutron reflections. In the higher temperature range, holmium has a proper screw structure and erbium has a cycloidal structure modified by the second order anisotropy energy. The neutron diffraction data show obviously that the transition from the conical spin arrangement is of the second kind in holmium and of the first kind in erbium. These facts can be understood as a result of the higher order anisotropy energy from the considerations mentioned above. Whether there exists a tilted screw arrangement in a certain temperature range in erbium is not made clear by the neutron diffraction experiment. The oscillation of the z component of the spin moment found in the highest temperature phase of erbium will be considered in the next section.

In addition to the appearance of a conical arrangement, there is another experimental evidence which suggests that the easy direction of the spin moment in holmium and erbium inclines to the c axis. In ordinary experiments of measuring the saturation moment in a polycrystalline sample, a magnetic field of the order of 10^4 oersteds is used. The magnetic field of this order is considered to be strong enough to align all the spins ferromagnetically, but not so strong enough to make this aligned moment free from the easy direction. This easy direction will be in the six equivalent directions on the cone if we take into account the six-fold anisotropy in the c plane. Assuming that the aligned moment in each crystallite points to the easy direction nearest to that of the applied field, the magnetic moment induced along the field can easily be calculated as follows:

$$M/Ng\mu_B S=\frac{6}{\pi}\int_0^{\pi/2} d\theta \int_0^{\pi/6} d\varphi \sin\theta\,(\cos\theta_0\cos\theta+\sin\theta_0\sin\theta\cos\varphi)$$

$$=\frac{1}{2}\cos\theta_0+\frac{3}{4}\sin\theta_0, \qquad (4\cdot8)$$

where θ_0 represents the vertical angle of the cone.

Rhodes, Legvold and Spedding[11] observed the saturation moment of holmium metal which amounts to 91% or, if the g-value determined from paramagnetic susceptibility is used, 89% of the theoretical value for complete saturation. This corresponds to the cone with an angle of 45°–70°, while the neutron data give about 77° for this angle. On the other hand, according to the measurement by Elliott, Legvold and Spedding[12] for erbium, the saturation moment amounts to 60–89% of the theoretical value. This value corresponds to an angle of 10°–50°. The neutron data give 30° and recent data on single crystals reported by Green, Legvold and Spedding[13] imply an angle of about 17°. In view of the facts that the assumption made here is simple and that the vertical angle of the cone should be a little wider than the preferred angle, these data would leave no doubt as to the inclination of the ferromagnetic moment. By the way, thulium shows about 50% saturation[14] that is equivalent to $\theta_0=0$.

Now, we shall calculate the free energies of various screw-type structures on the basis of the molecular field approximation, and provide a sound basis to the arguments made on the temperature effect. Instead of the condition of self-consistency we shall adopt as a guiding principle minimization of the free energy with respect to order parameters. The exchange Hamiltonian can be divided into the molecular field Hamiltonian H_0 and the remainder H'_{ex} as follows:

$$H_{ex}=H_0+H'_{ex}, \tag{4·9}$$

$$H_0=\sum_i\sum_j J(\boldsymbol{R}_j-\boldsymbol{R}_i)\boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_i-2\sum_i\sum_j J(\boldsymbol{R}_j-\boldsymbol{R}_i)\boldsymbol{\sigma}_j\cdot\boldsymbol{S}_i, \tag{4·10}$$

$$H'_{ex}=-\sum_i\sum_j J(\boldsymbol{R}_j-\boldsymbol{R}_i)\,(\boldsymbol{S}_j-\boldsymbol{\sigma}_j)\cdot(\boldsymbol{S}_i-\boldsymbol{\sigma}_i), \tag{4·11}$$

where $\boldsymbol{\sigma}_i$'s were introduced as usual to represent the thermal averages of the spins but here they are rather regarded as order parameters which will be determined later.

Considering H_0 as an unperturbed Hamiltonian we calculate the free energy to the first order of the anisotropy energy as follows:

$$F=-kT\ln Z_0+\sum_i\langle V(\theta_i)\rangle_0+\langle H'_{ex}\rangle_0+\cdots, \tag{4·12}$$

where Z_0 is the partition function for the Hamiltonian H_0 and given by

$$Z_0=(4\pi)^{-N}\int\cdots\int e^{-H_0/kT}\Pi_i\,d\Omega_i, \tag{4·13}$$

and $\langle\cdots\rangle_0$ denotes the canonical average with respect to H_0,

$$\langle A\rangle_0=(4\pi)^{-N}Z_0^{-1}\int\cdots\int A\,e^{-H_0/kT}\Pi_i\,d\Omega_i. \tag{4·14}$$

We introduce here a vector $\boldsymbol{u}_i$ which is related to the molecular field acting on the i-th spin and is defined by

$$\boldsymbol{u}_i = \frac{2S}{kT} \sum_j J(\boldsymbol{R}_j - \boldsymbol{R}_i)\boldsymbol{\sigma}_j . \tag{4·15}$$

We denote the magnitude of $\boldsymbol{u}_i$ by u_i and its direction by (θ_i, φ_i)* referring to the polar coordinate system with the c axis of the crystal as the polar axis. Then, by changing the coordinate system for each spin to a new one which has its polar axis in the direction of $\boldsymbol{u}_i$, and making use of the addition theorem for the Legendre polynomial, the averages in Eq. (4·12) can be calculated explicitly as follows:

$$\begin{aligned} F = {} & \sum_i \sum_j J(\boldsymbol{R}_j - \boldsymbol{R}_i)\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_i - kT \sum_i \ln(\sinh u_i / u_i) \\ & + D \sum_i \frac{1}{2} (3\cos^2\theta_i - 1)\, \Phi_2(u_i) \\ & + E \sum_i \frac{1}{8} (35\cos^4\theta_i - 30\cos^2\theta_i + 3)\, \Phi_4(u_i) \\ & + F \sum_i \frac{1}{16} (231\cos^6\theta_i - 315\cos^4\theta_i + 105\cos^2\theta_i - 5)\, \Phi_6(u_i) \\ & - \sum_i \sum_j J(\boldsymbol{R}_j - \boldsymbol{R}_i)\, (\langle \boldsymbol{S}_j \rangle_0 - \boldsymbol{\sigma}_j) \cdot (\langle \boldsymbol{S}_i \rangle_0 - \boldsymbol{\sigma}_i), \end{aligned} \tag{4·16}$$

where $\Phi_n(u)$ is the following average of the Legendre polynomial:

$$\Phi_n(u) = \int_0^\pi P_n(\cos\theta)\, e^{u\cos\Theta} \sin\theta\, d\theta \Big/ \int_0^\pi e^{u\cos\Theta} \sin\theta\, d\theta . \tag{4·17}$$

Expressions for these functions and their asymptotic forms will be given in the Appendix.

The free energy (4·16) should be minimized with respect to the order parameters $\boldsymbol{\sigma}_i$'s in order to determine the values of these parameters. For practical purposes we have to assume a trial form of $\boldsymbol{\sigma}_i$ which contains only a few parameters. When we choose $\boldsymbol{\sigma}_i = \boldsymbol{\sigma} = \text{const.}$ (for all i), the equation for the condition of self-consistency results from the minimum principle, and consequently, the last term in Eq. (4·16) vanishes. Thus, we are led to the ordinary Weiss theory of ferromagnetism.

In our case, however, where $J(\boldsymbol{q})$ has its maximum at $\boldsymbol{q}$ which is not equal to zero, there are many other ordered states which have less free energy than the ferromagnetic state. In most of these states, it is impossible to satisfy the condition of self-consistency for every spin if our trial form of $\boldsymbol{\sigma}_i$ is simple enough to deal with. However, the order parameters which give the minimum value for Eq. (4·16) will be regarded as the actually established in our approximation.

* It should be noted that θ_i and φ_i do not specify the direction of the i-th spin but that of $\boldsymbol{u}_i$.

If we have sufficient number of exchange and anisotropy constants which could be determined from measurements of the magnetic properties on single crystals, we can follow the magnetic behavior of the spin system by explicit calculations of the free energies under the above-mentioned scheme.

In order to investigate the properties of the spin system under the external magnetic field $\boldsymbol{H}$, we add the Zeeman energy

$$g\mu_B \boldsymbol{H}\cdot\boldsymbol{S} \tag{4·18}$$

to the Hamiltonian. Then, the free energy (4·16) is supplemented by

$$g\mu_B S\boldsymbol{H}\cdot\sum_i \frac{\boldsymbol{u}_i}{u_i}\Phi_1(u_i), \tag{4·19}$$

unless the magnetic field is very strong. In this case σ_i's will have a uniform component, from which we can calculate the susceptibility, the critical field for the screw-ferro transition and other quantities.

§ 5. Magnetic ordering in the intermediate and high temperature phases of erbium

In dysprosium and holmium where the second order anisotropy constant D has a positive sign, the spin arrangement just below the Néel temperature takes a simple screw structure in which the rotation axis coincides with the translation axis, namely the c axis. However, when D has an opposite sign the magnetic ordering becomes complicated. Even if the anisotropy is very small, the cycloidal arrangement will be modified to some extent. In this section we shall treat this case of erbium in the approximation valid near the Néel temperature.

For the sake of simplicity, we shall omit the thorough discussion and consider only a few cases which interest us and which really give the minimum free energy, that is, the metastable state at least.

First, we put

$$\begin{aligned}\sigma_{ix}&=Sm_x\cos\boldsymbol{Q}\cdot\boldsymbol{R}_i,\\ \sigma_{iy}&=Sm_y\sin\boldsymbol{Q}\cdot\boldsymbol{R}_i,\\ \sigma_{iz}&=Sm_z\cos\boldsymbol{Q}\cdot\boldsymbol{R}_i.\end{aligned} \tag{5·1}$$

The hodograph of the spin moment for this structure is an ellipse. Expanding the free energy (4·15) with respect to m's and retaining terms up to the fourth power, we obtain

$$\begin{aligned}f=&\frac{3}{4}\left(\frac{T_s^0}{T}\right)^2\left\{\left(\frac{kT^2}{T_s^0}-kT+\frac{2}{5}D\right)m_z^2+\left(\frac{kT^2}{T_s^0}-kT-\frac{1}{5}D\right)(m_x^2+m_y^2)\right\}\\ &+\frac{27}{160}\left(\frac{T_s^0}{T}\right)^4\left\{\left(kT-\frac{8}{7}D+\frac{4}{21}E\right)m_z^4+\left(kT+\frac{4}{7}D+\frac{1}{14}E\right)(m_x^4+m_y^4)\right.\end{aligned}$$

$$+\left(\frac{2}{3}kT-\frac{4}{21}D-\frac{4}{21}E\right)m_z^2m_y^2+\left(2kT-\frac{4}{7}D-\frac{4}{7}E\right)m_z^2m_x^2$$

$$+\left(\frac{2}{3}kT+\frac{8}{21}D+\frac{1}{21}E\right)m_x^2m_y^2\Big\}. \tag{5·2}$$

In this equation, T_s^0 is defined by

$$T_s^0=2J(Q)\,S^2/3k, \tag{5·3}$$

and it is the temperature at which a screw-magnetism would set in if there were no anisotropy energies. It is noted that the last term in Eq. (4·16) which offers a measure of self-consistency contributes in the sixth order.

Since at high temperatures all coefficients in Eq. (5·2) are positive, the values of m's which minimize the free energy are all zero. As the temperature is lowered the coefficient of m_z^2 first becomes zero at $T=T_z$, which is given by a quadratic equation,

$$\frac{kT_z^2}{T_s^0}-kT_z+\frac{2}{5}D=0, \tag{5·4}$$

and at this temperature the sinusoidal oscillation of the z component of the spin moments begins, while the x and the y components are still left as zero. This corresponds to the transition at 80°K in erbium. As Q enters the expression (5·2) only through T_s^0, we can readily see that the value of Q which makes $J(Q)$ maximum will simultaneously make the free energy minimum and, therefore, the period of the oscillation will not change with temperature in this approximation. Below the transition temperature, we have, in our approximation,

$$m_z^2=-\frac{20}{9}\left(\frac{T}{T_s^0}\right)^2\left(\frac{T^2}{T_s^0}-T+\frac{D}{k}\right)\Big/\left(T-\frac{8}{7}\frac{D}{k}+\frac{4}{21}\frac{E}{k}\right). \tag{5·5}$$

Near T_z, the expression which characterizes the second kind phase transition is obtained as follows:

$$m_z^2\approx\frac{20}{9}\left(\frac{T_z}{T_s^0}\right)^2\frac{T_z-T}{T_z}. \tag{5·6}$$

So long as D is negative, the ordered phase just below the transition is of this type of spin arrangement. This is true even if the second order anisotropy energy is very small. When the anisotropy energy is large, we can safely adopt the Ising model. Then, the molecular field approximation leads to the oscillation of the spin which has inherently the z component only.

The temperature at which the modulation of the x or the y component appears is not simply given by

$$\frac{T_y^2}{T_s^0}-T_y-\frac{1}{5}\frac{D}{k}=0, \tag{5·7}$$

in consequence of the previous establishment of the z component modulation. Instead, T_y for the y component modulation is reduced and determined by the equation

$$\frac{T_y^2}{T_s^0}-T_y-\frac{1}{5}\frac{D}{k}+\frac{9}{40}\left(\frac{T_s^0}{T_y}\right)^2\left(\frac{2}{3}T_y-\frac{4}{21}\frac{D}{k}-\frac{4}{21}\frac{E}{k}\right)(m_z^2)_{T=T_y}=0. \tag{5·8}$$

For the x component it is generally reduced further, as seen from Eq. (5·2). In the temperature range $T<T_y$, the elliptical hodograph has the c axis as its major axis. m_y grows with decreasing temperature as $(T_y-T)^{1/2}$, namely the transition which corresponds to the one at 52°K in erbium is again of the second kind. m_z suffers only a little change from this transition.

As to the x component, the situation is somewhat different. When the coefficient of m_x^2 reaches zero, the ellipse begins to tilt. However, this would be realized only when E and F have appropriate values as was discussed in the previous section. But at such a low temperature as the last transition will be expected, the approximation used here would lose its validity. The usual neutron diffraction experiment can not distinguish the last structure from the second, because six equivalent structures exist simultaneously in an actual crystal because of the six-fold anisotropy neglected here, although a possibility that we can align them in one direction by an external magnetic field would not be denied.

Next, we shall give an explanation for higher harmonics in z-component oscillation. Here we put

$$\begin{aligned}&\sigma_{iz}=S(m_z\cos\boldsymbol{Q}\cdot\boldsymbol{R}_i+m'\cos 3\boldsymbol{Q}\cdot\boldsymbol{R}_i),\\&\sigma_{ix}=\sigma_{iy}=0.\end{aligned} \tag{5·9}$$

The free energy is calculated to the fourth power as

$$\begin{aligned}f=&\frac{3}{4}\left(\frac{T_s^0}{T}\right)^2\left(\frac{kT^2}{T_s^0}-kT+\frac{2}{5}D\right)m_z^2+\frac{27}{160}\left(\frac{T_s^0}{T}\right)^4\left(kT-\frac{8}{7}D+\frac{4}{21}E\right)m_z^4\\&+\frac{3}{4}\left(\frac{T_{3Q}}{T}\right)^2\left(\frac{kT^2}{T_{3Q}}-kT+\frac{2}{5}D\right)m'^2+\frac{9}{40}\left(\frac{T_s^0}{T}\right)^3\left(\frac{T_{3Q}}{T}\right)\left(kT-\frac{8}{7}D+\frac{4}{21}E\right)m_z^3m'.\end{aligned} \tag{5·10}$$

Here, we have put $T_{3Q}=2J(3Q)S^2/3k$ and have assumed that $J(3Q)$ is positive, otherwise the last term in Eq. (4·16) should be treated carefully. In this equation the terms proportional to $m_z^2m'^2$ and m'^4 are neglected. In the high temperature phase of erbium, m_z is given by Eq. (5·6). Once m_z takes a non-zero value, m' becomes also non-zero as seen from Eq. (5·10). The amplitude of the overtone whose frequency is triple of the fundamental one is given by

$$m' = -\frac{3}{20}\left(\frac{T_s^0}{T}\right)^3 m_z^3 \left(1-\frac{8}{7}\frac{D}{kT}+\frac{4}{21}\frac{E}{kT}\right)\Big/\left\{1-\frac{T_{3Q}}{T}\left(1-\frac{2}{5}\frac{D}{kT}\right)\right\}, \tag{5·11}$$

and varies as $(T_z-T)^{3/2}$ near the Néel temperature. It has an opposite sign to m_z. Accordingly, the sinusoidal modulation is gradually squared. Similarly, we also expect other odd-multiple overtones, although they are much smaller.

Since the free energy (5·10) depends on Q through $J(3Q)$ as well as $J(Q)$, the period of the fundamental oscillation may change as m' grows. In an attempt to see the order of magnitude of this temperature variation, we neglect all the anisotropy terms in Eq. (5·10) and obtain for the temperature $T \lesssim T_z$,

$$f \approx -\frac{5}{6}kT\left(1-\frac{T}{T_s}\right)^2 - \frac{5}{27}kT_{3Q}\left(1-\frac{T}{T_s}\right)^3\Big/\left(1-\frac{T_{3Q}}{T}\right). \tag{5·12}$$

Differentiating with respect to Q and using the expansion (3·12), we get

$$Q-Q_0 \approx -\frac{1}{3}\frac{J'(3Q_0)}{J''(Q_0)}\left(\frac{T_z-T}{T_z}\right)^2. \tag{5·13}$$

Assuming the form of Eq. (2·34) for $J(Q)$ and adopting $Q_0 c' = 51°$ from Table I, we have

$$(Q-Q_0)\,c' \approx -0.4\left(\frac{T_z-T}{T_z}\right)^2. \tag{5·14}$$

Though the estimate has an ambiguity in determining the values of $J'(3Q_0)$ and $J''(Q_0)$, it leads to about 5 percent increase of the period from 80°K to 52°K for erbium. However, this change due to higher harmonics is considerably smaller than that which is observed in dysprosium, holmium and in the intermediate phase of erbium. On the other hand, Cable et al.[3] observed no change in the period in this temperature range.

Thus, apart from the change in the period, we can understand the spin structure of erbium and its variation with temperature above 20°K as a natural consequence of the fact that the second order anisotropy constant D is negative.

§ 6. Conclusion

We have investigated the effect of the anisotropy energy on the spin system with the exchange interaction which makes a pure screw structure stable, and have discussed the stability of various spin structures found in dysprosium, holmium and erbium and the magnetic transitions between these structures. In these rare-earth metals, the exchange interaction itself makes a screw structure stable, namely its Fourier transform $J(\boldsymbol{q})$ has a maximum at $\boldsymbol{q}=\boldsymbol{Q}_0$. The anisotropy energy determines the rotation axis of the spin moment and further it sometimes modifies the screw structure itself.

In the case of dysprosium, the axial anisotropy energy including higher order terms tends to keep the spin moment in the c plane. Therefore, at high temperatures, a pure screw structure is realized. However, as the temperature is lowered, the anisotropy energy of higher order grows. The sixth order energy which describes the anisotropy in the c plane stabilizes the ferromagnetic state more than the screw state. Therefore, a transition from the screw state to the ferromagnetic state will occur at a certain temperature if this intraplaner anisotropy energy is large enough compared with $J(Q)-J(0)$. Thus, the magnetic transition in dysprosium can be explained by a large anisotropy energy in the c plane.

In the case of holmium, the anisotropy energy including fourth and sixth order terms will make the spin moment parallel to an intermediate direction between the c axis and the c plane. Therefore, taking into account the exchange interaction, it can be shown that a conical spin structure is stable at low temperatures. However, as the temperature rises, the fourth order energy which makes the spin moment incline at an angle to the c plane in this case becomes smaller compared with the second order anisotropy energy which makes the direction of the spin perpendicular to the c axis. It is, therefore, expected that the conical structure becomes flat with temperature and that at a certain temperature this structure is transformed into a pure screw structure.

In the case of erbium, it is expected that the second order anisotropy energy makes the spin moment parallel to the c axis and in competition with this term, the sixth order axial term, i. e., F-term, tends to make the direction of the spin deviate from the c axis. Therefore, similarly to the case of holmium, the spin structure in the low temperature phase will be a conical structure. As the temperature rises, it is expected that this conical structure changes to a screw structure in which the axis of rotation of the spin moment deviates from the c axis and then it changes to a modified cycloidal structure, or that the conical structure is transformed directly into a cycloidal structure. At present, however, it is a difficult thing to distinguish experimentally between the cycloidal structure and the screw structure with an inclined rotation plane.

In the high temperature phase of erbium, only the z component of the moment has been found to oscillate. In connection with this experimental result, it can be shown on the basis of the molecular field approximation that the temperature at which the oscillation of the z component of the spin moment sets in becomes higher than that for the oscillation of the x or the y component and therefore that the phase in which only the z component oscillates is the most stable near the Néel temperature when the second order anisotropy energy makes the c axis be the easy direction of the spin. The oscillation of the z component of the spin can also be expected on the basis of the Ising model. However, it is to be noted that there appears no perpendicular component at all in the case of the Ising model, whereas in our treatment the perpendicular

components are only averaged out by the thermal motion.

Besides these three metals, the magnetic properties of terbium and thulium can also be understood from the same standpoint as considered in the present paper, though the spin arrangements in their ordered phases have not so far been made clear. Terbium will probably behave in almost the same way as dysprosium. Namely, it is expected that it has a pure screw spin arrangement in the c plane below the Néel temperature, and that at low temperatures it has a ferromagnetic arrangement with the direction of easy magnetization parallel to $[10\bar{1}0]$ in the c plane, because the sign of G for terbium is opposite to that for dysprosium. In thulium, the preferred axis for the spin moment will always be parallel to the c axis. Therefore, we expect that it has a similar spin arrangement to that in erbium and shows similar temperature effects. But, instead of a conical structure, it can become a c-axis ferromagnet at low temperatures or in external magnetic field, if $J(Q)-J(0)$ is sufficiently small.

Concerning the temperature dependence of the pitch of the screw, we have shown that the quadrupole interaction would be its possible source. But, although the quadrupole interaction gives rise to a correct temperature dependence, the change in the pitch obtained from this effect is too small in its magnitude compared with the experimental result. Therefore, the indirect quadrupole interaction might be considered to be a possible source of the temperature dependence of the pitch. The indirect quadrupole interaction between spins will be caused by the Coulomb interaction between $4f$ electrons and conduction electrons and also by the interaction between spins and lattice phonons, but it could not be expected that it would be much stronger than the direct interaction. Further, the quadrupole interaction will also change the period of the oscillation of the z component according to our theory, whereas, experimentally, the period of this oscillation is found to be independent of temperature. Therefore, at present the origin of the change of the pitch with temperature is not clear.

As for the anisotropy energy, the appropriate values have been assumed for each case. The signs of the anisotropy constants assumed in accordance with the experimental results can be confirmed by a simple calculation of the crystalline field on the screened point charge model. However, these calculated magnitudes are sometimes found to be small compared with the values required by the experiments. This suggests that the anisotropy energy will be affected by the interaction between $4f$ electrons and conduction electrons and possibly lattice phonons.

The exchange interaction in rare-earth metals is considered to arise mainly from the intermediary action of the conduction electrons. The Fourier transform of the exchange integral from this origin can be expressed as

$$J(\boldsymbol{q})=(g_J-1)^2N^{-1}\sum_{\boldsymbol{K}_n}|\mathcal{J}(\boldsymbol{q}+\boldsymbol{K}_n)|^2\sum_{k}^{k_F}\frac{1}{E(\boldsymbol{k}+\boldsymbol{q}+\boldsymbol{K}_n)-E(\boldsymbol{k})}, \quad (6\cdot1)$$

where $\mathcal{J}(\boldsymbol{q})$ represents the Fourier transform of the exchange integral between conduction electrons and $4f$ electrons, $E(\boldsymbol{k})$ represents the band energy for the conduction electron with the wave vector $\boldsymbol{k}$, k_F the Fermi maximum of the wave vector and $\boldsymbol{K}_n$ the vector representing the reciprocal lattice point. If we treat the conduction electrons as free, the last sum gives a monotonically decreasing function of $\boldsymbol{q}+\boldsymbol{K}_n$ and so for this case, even if $\mathcal{J}(\boldsymbol{q})$ is also monotonically decreasing with $\boldsymbol{q}$, the sum over $\boldsymbol{K}_n$ might be possible to make $J(\boldsymbol{q})$ maximum at $\boldsymbol{q}\neq 0$. It may also be possible for $\mathcal{J}(\boldsymbol{q})$ itself to have its maximum at a non-zero value of $\boldsymbol{q}$ and further we could not deny that the last summation has a maximum at $\boldsymbol{q}=\boldsymbol{Q}_0\neq 0$ for general band structures.

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Appendix

Explicit functional forms and their asymptotic expressions for $\ln(\sinh u/u)$, Φ_1, Φ_2, Φ_4 and Φ_6, which have been defined in § 4, can be calculated as follows:

$$\ln(\sinh u/u) \sim \frac{u^2}{6}-\frac{u^4}{180}+\frac{u^6}{2835}-\cdots \quad (u\to 0) \tag{A·1}$$

$$\sim u-\ln 2u-\cdots \quad (u\to\infty), \tag{A·2}$$

$$\Phi_1(u)=\coth u-\frac{1}{u}\sim\frac{u}{3}-\frac{u^3}{45}+\cdots \quad (u\to 0) \tag{A·3}$$

$$\sim 1-\frac{1}{u}+2e^{-2u}+\cdots \quad (u\to\infty), \tag{A·4}$$

$$\Phi_2(u)=(3+u^2-3u\coth u)/u^2\sim\frac{u^2}{15}-\frac{2u^4}{315}+\cdots \quad (u\to 0) \tag{A·5}$$

$$\sim 1-\frac{3}{u}+\frac{3}{u^2}-\cdots \quad (u\to\infty), \tag{A·6}$$

$$\Phi_4(u)=\{(105+45u^2+u^4)-(105u+10u^3)\coth u\}/u^4$$

$$\sim\frac{u^4}{945}-\frac{4u^6}{31185}+\cdots \quad (u\to 0) \tag{A·7}$$

$$\sim 1-\frac{10}{u}+\frac{45}{u^2}-\cdots \quad (u\to\infty), \tag{A·8}$$

$$\Phi_6(u)=\{(10395+4725u^2+210u^4+u^6)-(10395u+1260u^3+21u^5)\coth u\}/u^6$$

$$\sim \frac{u^6}{135135} - \cdots \quad (u \to 0) \tag{A·9}$$

$$\sim 1 - \frac{21}{u} + \frac{210}{u^2} - \cdots \quad (u \to \infty). \tag{A·10}$$

It can easily be seen that Φ_n varies as the n-th power of Φ_1 for $u \to 0$ (near the Néel temperature) and as the $n(n+1)/2$-th power of Φ_1 for $u \to \infty$ (at low temperatures).

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Correlation between Measurements in Quantum Theory

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An attempt to extend the postulational basis of quantum theory by introducing correlations between the results of measurements, developed in this article, leads to negative joint probabilities for otherwise meaningful sets of measured values. Since, so far as can be seen, the attempt made is the only one compatible with the theory of random variables, and is incompatible with the structure of Hilbert space, we conclude that correlations are absent. This result, though it is tantamount to a denial of von Neumann's projection postulate and the "reduction of wave packets" on measurement, is nevertheless shown to be entirely satisfactory from the physical point of view.

§ 1. Introduction

It has become customary in textbooks and courses on quantum mechanics to allege that a measurement of an observable whose operator is X, when performed on a physical system in a state ψ', will leave the system in the eigenstate ψ'_{x_i} which corresponds to the measured value x_i. If ψ' is regarded as a vector in Hilbert space, the measurement is represented by an operator M which, when acting on ψ', projects it in the direction of ψ'_{x_i}; the index i is unpredictable on the basis of the usual axioms of the quantum theory, although the change in state can be formalized by assigning to M a definite mathematical operator. This proposition, seemingly first advanced by von Neumann,[1] will be called the *projection postulate.*

The context in which it arose is an interesting one. There are, he suggests, three stages of causal or acausal description. The first renders measurement a totally statistical act, allowing only the assignment of relative frequencies to measured values without implying what value a second measurement, performed immediately after the first, will yield. The second stage permits a variance of measured values only before an initial measurement, which then forces the outcome of later measurements of the same observable in strict deterministic fashion. According to the third or highest version of causality, every measurement is completely determined, as it is in classical physics. The second stage, von Neumann holds, is the one upon which quantum mechanics operates, and to insure it he introduces the projection postulate.

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Now it is a curious fact that in the applications of quantum mechanics to any physical situation this postulate is never needed, even though it is occasionally invoked—uselessly to be sure—in the explanation of cloud chamber tracks.[2] Among its less desirable consequences are the Einstein-Podolski-Rosen paradox[3] and the so-called reduction of the wave packet. The latter phenomenon, which is the contraction of a wave packet spread through space into a delta function when a precise position measurement is made, has given rise to the query whether quantum mechanics conveys an objective description of physical reality or is merely a subjective vehicle for an observer's knowledge. Such grave implications, together with important recent reconsiderations[4] of the measurement problem, have motivated the present analysis which tends to show an inconsistency of a formal sort in the unqualified acceptance of the projection postulate. While our discussion will be limited to elementary, non-relativistic quantum theory, no obstacle is apparent which would preclude its extension to four-dimensional analysis and to field theory.

Aside from the fact that the projection postulate is almost never included among the formal axioms of the theory, its use being limited to collateral expositions of a vague philosophic kind and of no apparent relevance to physical problems, it invites fundamental suspicion for the following reason. The so-called "Copenhagen interpretation" is said to be a radically statistical one, permitting no connection with the prediction-potent, individual events of classical physics. Now in every normal theory of probability or random variables it is understood that a single observation cannot establish a probability distribution, a large sample of observations being required for this end. Yet the customary quantum theory, if it includes the postulate in question, claims to be a radical probability theory wherein "God plays dice" and at the same time pretends to perform the most unorthodox feat of creating, in a single act of measurement, knowledge of an entire probability distribution, thus setting its competence high above what is regarded as normal by statisticians. It is for this reason among others that one of us has in the past directed criticism against the projection postulate (understood in von Neumann's sense),[5] and the strictures were based upon the observation that there are many good measurements, fully accredited by physicists, which empirically violate the postulate.[6] In the present article an attempt is made to incorporate the postulate in the foundations of the quantum theory. The success is only partial, but the effort is interesting inasmuch as it shows that negative probabilities are introduced when the requisite axiomatic extension is made.

A set of basic propositions sufficient for all of quantum mechanical analysis is the following.[7]

1. To every observable there corresponds an operator.
2. The operand ψ of the operator represents a physical state.
3. The numerical values which a measurement upon an observable with oper-

ator P can yield are the eigenvalues of P.

4. When a physical system is in a state ψ, the expectation value of a sequence of measurements on the observable whose operator is P is given by

$$E(P) \equiv \bar{p} = \int \psi^* P\psi \, d\tau \equiv (\psi^*, P\psi).$$

This axiom is written in the coordinate (Schrödinger) representation; its isomorphic forms in other representations are well known.[8]

5. States change in time according to the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi$$

or its equivalents in other representations. Added to this list are symmetry requirements, like the exclusion principle, which are of perhaps a less fundamental or even derivative status. They will not be included here. Perhaps it should be said, however, that a separate premise concerning single-valuedness and normalizability of states is quite unnecessary, since such items follow from 4. The uncertainty principle, of course, is implied by the set when 1 is implemented by a specific assignment of operators which are empirically found to be satisfactory. Finally, 4 can be shown to entail the statement: $|(\psi^*, \psi_{x_i})|^2$ is the probability that the value x_i (whose correlated eigenstate is ψ_{x_i}) will occur in a measurement of X on a system in state ψ.

§ 2. Covariance among measurements

The preceding axioms are in complete conformity with the normal procedures in random-variable analysis.[9] What they say ignores correlations between the results of different measurements. On the other hand, the projection postulate stipulates a very specific correlation between measurements, and it cannot be discussed unless our set is suitably enlarged to provide for correlations. What one would like to know is the probability that a joint measurement of two random variables, X and Y, shall yield values x_i and y_j, provided the system is known to be in some well-defined quantum state ψ. The latter will be taken for simplicity to be a pure case, not a mixture. By joint measurement we mean either two simultaneous measurements of X and Y or, as an interesting subclass of such procedures, the measurement of X at time t_1 and of Y at time t_2. The variables X and Y can of course be identical.

This specification contradicts the claim, often encountered in the literature, that certain variables, like position and momentum, cannot be measured simultaneously. If this were true, such measurements must be ruled out either by the axioms of quantum mechanics or by the empirical contingencies of experimental physics. The latter is certainly not the case, and the former is impossible

because the axioms fail to speak of correlations. The joint probability in question, as defined in the foregoing paragraph, will be designated as

$$P(x_i, y_j ; \psi).$$

Briefly, it is the probability for the occurrence of x_i and y_j on the evidence ψ. It must satisfy the relations

$$\sum_{ij} P(x_i, y_j ; \psi) = 1, \tag{1}$$

$$\sum_i P(x_i, y_j ; \psi) = P(y_j ; \psi), \quad \sum_j P(x_i, y_j ; \psi) = P(x_i ; \psi). \tag{2}$$

The ordinary probabilities, $P(x_i ; \psi)$ and $P(y_j ; \psi)$ are indeed given by our axioms, viz:

$$P(x_i ; \psi) = |(\psi^*, \psi_{x_i})|^2 \tag{3}$$

with a similar relation for $P(y_j ; \psi)$.

The quantity thus introduced must be carefully distinguished from the probability $W(x_i, y_j)$ that, when x_i is known to occur with certainty, y_j will ensue. It, too, is given by the axiom set:

$$W(x_i, y_j) = |(\psi^*_{x_i}, \psi_{y_j})|^2. \tag{4}$$

If the projection postulate were valid, the following relation might be thought to hold:

$$P(x_i, y_j ; \psi) = |(\psi^*, \psi_{x_i})|^2 W(x_i, y_j), \tag{5}$$

provided y_j is observed immediately after x_i. This version, however, cannot be accepted because it violates one of the relations (2).

A possible definition of $P(x_i, y_j ; \psi)$, indeed one which is compatible with and almost suggested by axioms 1–5, is this:

$$P(x_i, y_j ; \psi) = |(\psi^*, \psi_{x_i})(\psi, \psi^*_{y_j})|^2. \tag{6}$$

The closure relation for complete sets of eigenstates may be used to show that this formulation satisfies relations 1, 2 and 3, but it is clearly different from Eq. (5) and it does not allow measurements on Y to depend in any way on the outcome of measurements on X. For this reason, one might look upon Eqs. (5) and (6) as imperfect and search for another possible definition of P, perhaps one closer to Eq. (5), at any rate one which introduces correlations. This is best done via the concept of covariance, as follows:

Let X and Y be random variables. Then

$$\mathrm{Cov}(XY) = \mathrm{E}(XY) - \bar{x}\bar{y}. \tag{7}$$

If X and Y are identical, $\mathrm{Cov}(XY)$ becomes the variance of X, $\mathrm{Var}(X)$. The structure of our axioms, particularly 4, suggests at once that we postulate for a quantum mechanical ensemble of measurements on a state ψ

$$\mathrm{Cov}(XY)=(\psi^*, XY\psi)-(\psi^*, X\psi)(\psi^*, Y\psi). \tag{8'}$$

This interpretation differs from the one which (illegitimately, it seems) takes the operator XY to be associated with an observation of X followed immediately by an observation of Y.

Assuming that X and Y are hermitean, one can at once establish the following reasonable facts.

If ψ is an eigenstate of X

$$\mathrm{Cov}(XY)=0,$$

for in that instance

$$(\psi^*_{x_i}, XY\psi_{x_i})=(X^*\psi^*_{x_i}, Y\psi_{x_i})=x_i\bar{y}=\bar{x}\bar{y}.$$

The same result holds when ψ is an eigenstate of Y.

In general, however, $\mathrm{Cov}(XY)$ according to Eq. (8') is complex. But

$$\mathrm{Cov}^*(XY)=\mathrm{Cov}(YX).$$

It is therefore indicated that Eq. (8') be replaced by the definition

$$\mathrm{Cov}(XY)=\left(\psi^*, \frac{XY+YX}{2}\psi\right)-(\psi^*, X\psi)(\psi^*, Y\psi). \tag{8}$$

We shall thus base our further inquiry upon this formulation, which leaves the conclusions concerning eigenstates unaltered.

A further interesting relation can be easily established. If, in one of the well-known forms of Schwarz' inequality, to wit

$$(f^*, f)(g^*, g)\geq\frac{1}{4}[(f^*, g)+(f, g^*)]^2$$

we take

$$f=(X-\bar{x})\psi,$$

$$g=(Y-\bar{y})\psi$$

we find at once

$$[\mathrm{Cov}(XY)]^2\leq\mathrm{Var}(X)\cdot\mathrm{Var}(Y). \tag{9}$$

In consequence one is permitted to define a *correlation coefficient* σ after the manner of random-variable theory:

$$\sigma(X, Y)=\mathrm{Cov}(XY)[\mathrm{Var}(X)\,\mathrm{Var}(Y)]^{-1/2}. \tag{10}$$

and the range of σ is the expected one, $-1\leq\sigma\leq 1$.

A few examples illustrating these relations seem to be in place.

a) The spin components of a single electron "anticommute". Hence, if X and Y are x- and y-components of the spin,

$$\mathrm{Cov}(XY)=-\bar{x}\bar{y}=(a^*b+ab^*)(ia^*b-iab^*)\frac{\hbar^2}{4},$$

provided the spin function is $\begin{pmatrix} a \\ b \end{pmatrix}$. For a state ψ which is an eigenstate of the z-component, either a or b is zero, and the covariance vanishes.

b) Less trivial, perhaps, is the case of a system with angular momentum $J=\hbar$, we now write X, Y, Z for the components of $\boldsymbol{J}/\hbar$. These operators have the matrix form:

$$X=\sqrt{\tfrac{1}{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y=\sqrt{\tfrac{1}{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad Z=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Their common eigenvalues are -1, 0 and 1 and their eigenvectors form the following list, to which reference will be made again in the next section. The

Eigenvalue	-1	0	1
$\psi_x=$	$\begin{pmatrix} \frac{1}{2} \\ -\sqrt{\frac{1}{2}} \\ \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} \sqrt{\frac{1}{2}} \\ 0 \\ -\sqrt{\frac{1}{2}} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} \\ \sqrt{\frac{1}{2}} \\ \frac{1}{2} \end{pmatrix}$
$\psi_y=$	$\begin{pmatrix} \frac{1}{2} \\ -\sqrt{\frac{1}{2}}i \\ -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} \sqrt{\frac{1}{2}} \\ 0 \\ \sqrt{\frac{1}{2}} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} \\ \sqrt{\frac{1}{2}}i \\ -\frac{1}{2} \end{pmatrix}$
$\psi_z=$	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

operator

$$\frac{1}{2}(XY+YX)=\frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

has eigenvalue -1, 0 and 1. Simple computation shows that, for a state

$$\psi=\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\mathrm{Cov}(XY)=-\frac{1}{2}(a^*c-ac^*)i+\frac{1}{2}[(a^*+c^*)b+(a+c)b^*]$$
$$\times[(a^*-c^*)b-(a-c)b^*]i.$$

Interestingly again, if ψ is an eigenvector of Z, the covariance vanishes. The same is true, of course, when ψ is one of the ψ_x or ψ_y in accordance with theorem a) following Eq. (8′).

c) Next, let us compute the covariance between position measurements at time $t=0$ and at time t. The operator X corresponding to a measurement of x at $t=0$ is simply $x\cdot$; at the later time it is

$$X_t \sim T^* xT,$$

where T is the time-development operator, $T=\exp\left(-i\int\frac{H}{\hbar}dt\right)$. For H we shall take the Hamiltonian of a free particle. In that case, since for any operators A and B

$$e^{A}Be^{-A}=B+[A, B]+\frac{1}{2!}[A, [A, B]]+\cdots$$

in the notation of commutator brackets, we have

$$X_t=e^{\frac{i}{\hbar}Ht}xe^{-\frac{i}{\hbar}Ht}=x+\frac{it}{\hbar}[H, X]=x+\frac{t}{m}P \tag{12}$$

where P is the linear momentum operator. In general T satisfies

$$T^{-1}=T^{*}$$

when it is written in the form of a differential operator; in matrix form it is unitary. If the eigenfunctions of an operator Q at $t=0$ are ψ_i, then the eigenfunctions of the same operator at time t are $T^{*}\psi_i$. To see this, assume the equation

$$Q\psi_i=q_i\psi_i$$

to be satisfied at $t=0$. Let $\psi_i{}^t$ be the solution of

$$(T^{*}QT)\psi_i{}^t=q_i{}^t\psi_i{}^t.$$

On multiplying by T from the left we obtain

$$Q(T\psi_i{}^t)=q_i{}^t(T\psi_i{}^t)$$

which means that $T\psi_i{}^t$ is an eigenfunction and $q_i{}^t$ an eigenvalue of Q. Hence $q_i{}^t$ must equal q_i; the eigenvalues cannot change in time. And since $T\psi_i{}^t=\psi_i$, we have $\psi^t=T^{*}\psi_i$ in view of Eq. (11). Evidently, $\psi_i{}^t$ is the ancestor of ψ_i at time $-t$ in accordance with the Schrödinger equation.

On using Eqs. (8) and (12) we obtain, upon expansion,

$$\mathrm{Cov}(XX_t)=\mathrm{Var}(X)+\frac{t}{m}\mathrm{Cov}(XP) \tag{13}$$

where X without subscript refers to $t=0$.

For ψ we assume a packet of the form

$$\psi=(\pi a^2)^{-1/4}\exp\left(ikx-\frac{x^2}{2a^2}\right) \tag{14}$$

at $t=0$, which transforms itself into

$$\psi_t=(\pi a^2)^{-1/4}\left(1+\frac{i\hbar t}{ma^2}\right)^{-1/2}\exp-\left[\frac{x^2-2ia^2kx+\frac{ia^2k^2\hbar t}{m}}{2\left(a^2+\frac{i\hbar}{m}t\right)}\right] \tag{15}$$

at time t. It is then easily seen that

$$\mathrm{Cov}(XP)=0 \tag{16}$$

so that

$$\mathrm{Cov}(XX_t)=\mathrm{Var}(X)=\frac{a^2}{2} \tag{17}$$

at all times.

The correlation coefficient defined in Eq. (10), however, depends on t. $\mathrm{Var}(X)$ has just been computed, and

$$\mathrm{Var}(X_t)=\mathrm{Var}\left(X+\frac{t}{m}P\right)=\mathrm{Var}(X)+\frac{t^2}{m^2}\mathrm{Var}(P)+\frac{2t}{m}\mathrm{Cov}(XP).$$

But $\mathrm{Var}(P)=\hbar^2/2a^2$, hence, in view of Eqs. (16) and (17),

$$\mathrm{Var}(X_t)=\frac{a^2}{2}\left[1+\left(\frac{\hbar t}{ma^2}\right)^2\right].$$

Therefore the correlation coefficient between a position measurement at time 0 and time t is

$$\sigma=\left[1+\left(\frac{\hbar t}{ma^2}\right)^2\right]^{-1/2}.$$

It remains near 1 if the initial extent of the wave packet, a, is large. The fact that it does not depend on k, the speed of the packet, is also reasonable, for σ is 1 for random variables which are connected by a linear relation, e.g. $x_t=x_0+vt$.

§ 3. Joint probabilities

Having shown that the covariance defined in Eq. (8) is not without attractive features, we next examine the joint probabilities it entails. As is known from statistics[9]

$$\mathrm{Cov}(XY)=\sum_{ij}P(x_iy_j)x_iy_j-\bar{x}\bar{y}. \tag{18}$$

Eq. (8) takes this form if we use the alternate expansions

$$\psi=\sum_i a_iu_i \quad \text{and} \quad \psi=\sum_j b_jv_j$$

in orthonormal sets u and v which are, respectively, eigenstates of X and Y. For then

$$(\psi^*, XY\psi)=\int X^*\sum_\lambda a_\lambda^*u_\lambda^*Y\sum_\mu b_\mu v_\mu d\tau=\sum_{ij}(a_i^*b_j\int u_i^*v_jd\tau)x_iy_j.$$

Comparison with Eqs. (18) and (8) allows a simple identification of P:

$$P(x_i, y_j; \psi) = R[a_i^* b_j \int u_i^* v_j d\tau] = R[(\psi^*, \psi_{x_i})(\psi, \psi_{y_j}^*)(\psi_{x_i}^*, \psi_{y_j})]. \qquad (19)$$

Because of the closure relation these joint probabilities satisfy Eqs. (1), (2) and (3). If ψ is either ψ_{x_i} or ψ_{y_j}, they also lead to Eq. (4), but they are not in general accord with the projection postulate in the narrow form here under consideration. Yet, as is clear from the preceding section, they introduce correlations between measurements. When X and Y commute

$$P(x_i, y_j; \psi) = P(x_i, \psi) \cdot \delta_{ij},$$

provided proper allowance is made for degeneracies. Most unfortunately, however, the defining relation (19) does not prevent the joint probabilities from being negative. This is a rather essential aspect connected with the present attempt to introduce correlations between measurements, and it is difficult to see how it can be avoided; its causes lie very deep, and their removal would require some radical changes in the axioms listed in the Introduction. For such reasons it seemed worthwhile to illustrate the case by computing a few examples.

a) Returning to the system with angular momentum $J=1$ to which discussion b) of § 2 was devoted, we label the eigenvalues -1, 0 and 1 by indices ρ and σ, allowing them the values -1, 0 and 1. The state ψ has again the form of a column vector with components a, b and c. For convenience we introduce the abbreviations

$$s = \sqrt{\frac{1}{2}}(a+c), \quad d = \sqrt{\frac{1}{2}}(a-c).$$

Now

$$P(x_i, y_j; \psi) \equiv P_{\rho\sigma} = R(\psi^+, u_\rho)(v_\sigma^+, \psi)(u_\rho^+, v_\sigma),$$

u_ρ and v_σ being the eigenstates ψ_x and ψ_y listed in 2b. One finds

$$(\psi^+, u_{-1}) = \sqrt{\frac{1}{2}}(s^* - b^*),$$

$$(\psi^+, u_0) = d^*,$$

$$(\psi^+, u_1) = \sqrt{\frac{1}{2}}(s^* + b^*),$$

$$(v_{-1}^+, \psi) = \sqrt{\frac{1}{2}}(d + ib),$$

$$(v_0^+, \psi) = s,$$

$$(v_1^+, \psi) = \sqrt{\frac{1}{2}}(d - ib).$$

The last factor of $P_{\rho\sigma}$ is

$$(u_\rho^+, v_\sigma) = \begin{array}{c|ccc} \rho \backslash \sigma & -1 & 0 & 1 \\ \hline -1 & +\frac{1}{2}i & \sqrt{\frac{1}{2}} & -\frac{1}{2}i \\ 0 & \sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{1}{2}} \\ 1 & -\frac{1}{2}i & \sqrt{\frac{1}{2}} & +\frac{1}{2}i \end{array}.$$

From these all joint probabilities can be compounded. In the case where ψ is an eigenstate of J_z (see § 2b for the form of these eigenstates!) one obtains

$$P_{\rho\sigma} = \begin{vmatrix} 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{vmatrix} \quad \text{if } J_z = -1 \text{ or } +1$$

and

$$P_{\rho\sigma} = \begin{vmatrix} \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{vmatrix} \quad \text{if } J_z = 0.$$

These are perfectly acceptable probability distributions whose meaning is clear and simple. For instance, if J_z is known *a priori* to be 1 or -1, there is no chance of finding J_x and J_y manifesting equal values in simultaneous measurements, nor can they exhibit opposite values. One can only obtain a zero value for one and $+1$ or -1 for the other. If J_z is 0, the situation is the reverse. Notably, negative probabilities do not occur.

Generally, however, the situation is quite different. If s, d and b (which must obey the normalization rule $|s|^2+|d|^2+|b|^2=1$) are taken to be, respectively, 0.6, 0.6 and 0.53, one calculates

$$P_{-1-1} = \frac{1}{4} R\{i(s^*-b^*)(d+ib)\} = -0.009.$$

b) Let x_i be the position x of a particle, y_j its simultaneously measured linear momentum, so that (*if* $p \equiv k\hbar$).

$$P(x,\,k\,;\,\psi)=R\int\delta(q_1 z)\,\psi^*(q_1)\,dq_1\int ce^{-ikq_2}\psi(q_2)\,dq_2\int\delta(q_3,x)\,c^*\,e^{ikq_3}\,dq_3$$

$$=|c|^2\,R\{\psi^*\,e^{ikx}\int\psi(q)\,e^{-ikq}\,dq\}.$$

Assume for ψ a wave packet of the form (14). Evaluation then leads to the expression

$$P(x,\,k;\,\psi)=\sqrt{2}\,|c|^2\cos(k-k_0)\,x\exp-\frac{1}{2}\left[\frac{x^2}{a^2}+(k-k_0)^2a^2\right]. \qquad (20)$$

The fact that P has an absolute maximum at $x=0$ and $k=k_0$ is satisfying, of course, as is the diffusive exponential decrease with x and k; but the oscillating factor prevents acceptance of the result.

c) As a last example, we discuss the joint probability, again in the simple case of a moving free particle, for position measurements at different times. Here we have

$$P(x_1,x_2^t;\,\psi)=R\{(\psi^*,\,\psi_{x_1})\,(\psi,\,\psi_{x_2}^{t*})\,(\psi_{x_1}^*\psi_{x_2}^t)\}, \qquad (21)$$

$$\psi_{x_1}=\delta(x,\,x_1),$$

$$\psi_{x_2}^t=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left[ik(x-x_2)+i\frac{\hbar t}{2m}k^2\right]=\frac{1-i}{2}\left(\frac{m}{\pi\hbar t}\right)^{1/2}\exp\left[-i\frac{m}{2\hbar t}(x-x_2)^2\right].$$

Insertion in Eq. (21) gives (with certain precautions when $t\to 0$)

$$P(x_1x_2^t,\,\psi)=R\left\{\frac{m}{ht}\psi^*(x_1)\exp\left[-i\frac{m}{2\hbar t}(x_1-x_2)^2\right]\right.$$

$$\left.\times\int\psi(x)\exp\left[i\frac{m}{2\hbar t}(x-x_2)^2\right]dx\right\}.$$

Here we use once more the form (14) for ψ. We carry out the integration and, to make the result less unwieldy, put $x_1=0$, obtaining

$$P(0,\,x_t\,;\,\psi)=R\left\{\frac{\sqrt{2}\,m}{ht}\left(1-\frac{ima^2}{\hbar t}\right)^{-1/2}\exp-\frac{1}{2}\left[\frac{a^2\left(k-\frac{mx}{\hbar t}\right)^2}{1-\frac{ima^2}{\hbar t}}\right]\right\}. \qquad (22)$$

Our choice of normalization has been such that p now represents a probability density (per unit range of x_1 and x_2).

Two limiting cases are of primary interest. For large t, i.e. if $\hbar t/m\gg a^2$,

$$P(0,x^t;\,\psi)\approx\sqrt{2}\,\frac{m}{ht}\exp\left[-\frac{a^2}{2}\left(k_0-\frac{mx}{\hbar t}\right)^2\right].$$

This is always positive and has a maximum at the classical position of the particle, $x=k_0\hbar/m\cdot t$.

In the opposite extreme, when $\hbar t/m \ll a^2$

$$P(0, x^t; \psi) \approx R\left\{\left(\frac{m}{\pi h a^2 t}\right)^{1/2} \exp\left[i\left(\frac{\pi}{4} - \alpha\right)\right]\right\} \tag{23}$$

where

$$\alpha = \left(k - \frac{mx}{\hbar t}\right)^2 \frac{\hbar t}{2m}$$

or

$$P(0, x^t; \psi) \propto \cos\alpha + \sin\alpha .$$

For small t, x can be chosen so that this quantity is negative. Closer inspection of Eq. (23) reveals, for finite k_0 and x, the satisfactory limit

$$\lim_{t\to 0} P(0, x; \psi) \propto \delta\left(x, \frac{k_0 h}{m} t\right).$$

In all preceding examples Eq. (19) was employed as the definition of joint probability. Although reasonable, this definition is not unique. Its lack of uniqueness arises from the circumstance that we have inferred its form from the single product moment, $(\psi^*, XX\psi)$, leaving all higher moments $(\psi^*, X^n Y^m \psi)$ with their possible variants $(\psi^*, X^{n-1} Y^m X\psi)$, etc., unspecified. We have therefore considered another possibility which is also compatible with Eq. (18) and has certain other advantages, namely

$$P(x, y; \psi) = (2\pi)^{-2} \int\int_{-\infty}^{\infty} \rho(\alpha, \beta) \exp[-i(\alpha x + \beta y)]\, d\alpha\, d\beta$$

with

$$\rho(\alpha, \beta) = (\psi^*, \exp[i(\alpha X + \beta Y)]\psi)$$

where in accordance with our previous notation X and Y are operators.[10] When the analysis is carried through for a case similar to example b in which X is the position and Y the momentum of a particle, but ψ is chosen to be a superpositian of two gaussian functions

$$\psi(x) = [2a\sqrt{\pi}\,(1 + e^{-b^2/a^2})]^{-1/2}[e^{-(x-b)^2/2a^2} + e^{-(x+b)^2/2a^2}],$$

one again obtains a result with a trigonometric factor which can cause P to be negative.

There is, however, a convincing argument which establishes the impossibility of introducing any sensible joint probability distribution that exhibits correlations.

The analysis of the possibility of introducing joint probabilities in quantum mechanics proceeds from the observation that a classical joint probability distribution is completely determined only if a certain set of its moments is specified. In the case of random variables which range over an infinite set of values,

specification of all the moments is necessary.

Let us now return to the previous example of angular momentum $\hbar$ and attempt to construct a joint probability distribution $P(x_i, y_i; \psi)$, which we now write for brevity in the form $f(x, y)$, where X and Y are taken to be $1/\hbar J_x$ and $1/\hbar J_y$. The eigenvalues of X and Y are then 1, 0, and -1. It follows from postulate 3 that we need to consider only $(f(1, \pm 1), f(-1, \pm 1), f(0, \pm 1), f(\pm 1, 0)$ and $f(0, 0)$ — i.e. the space of our random variables has only 9 points, so that only nine moments are needed to detemine $f(x, y)$. These moments are $\langle x\rangle$, $\langle y\rangle$, $\langle x^2\rangle$, $\langle y^2\rangle$, $\langle xy\rangle$, $\langle x^2y\rangle$, $\langle xy^2\rangle$, $\langle x^2y^2\rangle$, and the normalization $\langle 1\rangle$. The classical formulae

$$\langle x^n y^m\rangle = \sum_{xy} f(x, y)\, x^n y^m \quad m=0, 1, 2 \text{ and } n=0, 1, 2$$

for these nine moments can be inverted to give

$$\begin{aligned}
f(1, \pm 1) &= 1/4\,(\pm\langle xy\rangle \pm \langle x^2y\rangle + \langle xy^2\rangle + \langle x^2y^2\rangle),\\
f(-1, \pm 1) &= 1/4\,(\mp\langle xy\rangle \pm \langle x^2y\rangle - \langle xy^2\rangle + \langle x^2y^2\rangle),\\
f(0, \pm 1) &= 1/2\,(\pm\langle y\rangle + \langle y^2\rangle \mp \langle x^2y\rangle - \langle x^2y^2\rangle),\\
f(\pm 1, 0) &= 1/2\,(\pm\langle x\rangle + \langle x^2\rangle \mp \langle xy^2\rangle - \langle x^2y^2\rangle),\\
f(0, 0) &= 1 + \langle x^2y^2\rangle - \langle x^2\rangle - \langle y^2\rangle.
\end{aligned}$$

Our previous treatment left all moments with n or $m > 1$ undefined.

We now examine all possible quantum mechanical definitions of the moments. Moments of the form $\langle x^n\rangle$ or $\langle y^n\rangle$ are already specified as $(\psi^*, X^n\psi\rangle$ or $\langle\psi^*, Y^n\psi)$ by the axioms of quantum mechanics. The other moments ("cross moments") will be taken, in similar fashion, to be the expectation values of hermitean operators. These operators will be homogeneous polynomials in X and Y, and must reduce to X^nY^m for commuting operators. Furthermore, symmetry between X and Y must be maintained—i.e. $\langle x^ny^m\rangle$ must go over into $\langle x^my^n\rangle$ on interchange of X and Y. The following choices are then possible with α, β, γ, δ, and ε as arbitrary real parameters.

$$\begin{aligned}
\langle xy\rangle &= 1/2(\psi^*, XY+YX\psi),\\
\langle x^2y\rangle &= 1/(2\alpha+\beta)(\psi^*, |\alpha(X^2Y+YX^2)+\beta XYX|\psi),\\
\langle xy^2\rangle &= 1/(2\alpha+\beta)(\psi^*, |\alpha(XY^2+Y^2X)+\beta YXY|\psi),\\
\langle x^2y^2\rangle &= 1/(2\gamma+2\delta+2\varepsilon)(\psi^*, |\gamma(X^2Y^2+Y^2X^2)+\delta(XYXY+YXYX)\\
&\qquad +\varepsilon(XY^2X+YX^2Y)|\psi).
\end{aligned}$$

The question of interest is then the following: Do there exist values of these arbitrary parameters for which all of the $f(x, y)$ ($x=-1, 0, 1$ and $y=-1, 0, 1$) are positive for an arbitrary state function

$$\psi=\begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad ?$$

Let us assume that such values exist—in particular, that a satisfactory value of ε exists. In order to find what it should be, we calculate certain of the moments for the arbitrary state:

$$\langle 1\rangle=|a|^2+|b|^2+|c|^2=1, \text{ if } \psi \text{ is normalized}$$
$$\langle x^2\rangle=1/2(|b|^2+a^*c+ac^*+1),$$
$$\langle xy\rangle=-i/2(a^*c-ac^*),$$
$$\langle xy^2\rangle=\alpha/(2\alpha+\beta)\sqrt{2}\,[(a^*+c^*)b+(a+c)b^*],$$
$$\langle x^2y\rangle=-i\alpha/(2\alpha+\beta)\sqrt{2}\,[(a^*-c^*)b-(a-c)b^*],$$
$$\langle x^2y^2\rangle=1/2(\gamma+\delta+\varepsilon)[2\gamma|b|^2+\varepsilon(|a|^2+|c|^2)].$$

Now the quantities $f(1, 1)+f(1, -1)=1/2(\langle xy^2\rangle+\langle x^2y^2\rangle)$ and $f(1, 0)+f(-1, 0)=\langle x^2\rangle-\langle x^2y^2\rangle$ must be positive or zero for any ψ. In particular, for the state $a=-1/\sqrt{2}$, $b=0$, $c=1/\sqrt{2}$, $\langle x^2\rangle=\langle xy^2\rangle=0$ and $\langle x^2y^2\rangle$ must also be zero in order to make the two quantities positive or zero. But for this state, $\langle x^2y^2\rangle=\varepsilon/2(\gamma+\delta+\varepsilon)$, from which we conclude that ε must be zero. Having fixed the value of ε, we now compute $f(1, 1)$ for the state $a=\exp(i\pi/4)/\sqrt{2}$, $b=0$, $c=\exp(-i\pi/4)/\sqrt{2}$, for which $\langle xy^2\rangle=\langle x^2y\rangle=0$, $\langle x^2y^2\rangle=0$ (as a consequence of $\varepsilon=0$) and $\langle xy\rangle=-1/2$ to give $f(1, 1)=-1/8$. Thus it is not possible to find a satisfactory value of ε, from which we conclude that one cannot define a non-negative $f(x, y)$ which exhibits correlations.

The conclusion that joint probabilities of the usual type cannot be formulated in quantum mechanics was reached by De Broglie,[11] whose reasoning was based, however, entirely on Eq. (5). He regards this equation as fundamental and inevitable.

There arises at this point a rather subtle distinction which is worth noting. The inevitable equation, which relates joint probabilities to so-called conditional probabilities, is this:

$$P(x_i, y_j; \psi)=P(x_i)P'(x_i, y_j) \qquad (5')$$

where the conditional probability P' is the probability that y_j will be observed when x_i is known to be present or, on our specific condition, P' is the relative frequency of y_j in a subclass of measurements all of which yield x_i. It is *not* necessarily the same as $W(x_iy_j)$, which represents the probability for the occurrence of y_j when x_i is *known to occur with certainty*, that is, when ψ is an eigenstate of x_i. The projection postulate is the statement which says that $P'=W$. De Broglie's argument therefore embraces this postulate; instead of rejecting it, he perhaps redefines joint probabilities in an unusual way.

If, as we suggest, there are no correlations between measurements, $P'(x_i y_j)$ is simply $P(y_j)$ in accordance with Eq. (6), and Eq. (5′) above is satisfied without artificiality, although it is now trivial.

§ 4. Conclusions

There appears to be no reasonable criterion for ruling out those instances in which joint probabilities are negative. Hence it is necessary to reject *in toto* both our definition of the covariance, Eq. (8), and its consequence, such as Eq. (19). The axioms stated in the Introduction do not seem to tolerate this extension. We are thus led back with evident cogency to the proposition which the projection postulate endeavored to avoid, namely to Eq. (6) with its implication that measurements of non-commuting observables are in general uncorrelated. For if (6) is used in Eq. (18), $\mathrm{Cov}(XY)=0$.

By this return, however, nothing of value in quantum mechanics is surrendered. For a certain kind of correlation is automatically carried in the state function itself and becomes manifest in the temporal changes of ψ. The structure of the axioms is such that the question as to the connection between a measurement at time t_1 and at another time t_2 need not be asked. Each independent factor of the definition (6) alone provides the answer to the proper question: What is the probability that X shall have the value x_i at time t if the state is $\psi(t)$? It is, of course, $|(\psi^*(t), \psi_{x_i})|^2$. Employing once more the example of a wave packet, whose form is Eq. (14) at $t=0$, while $\psi(x, t)$ is given by Eq. (15), we find for the probability of encountering the particle at x, t

$$|(x, t)|^2=\left\{(\pi a^2)\left[1+\left(\frac{\hbar t}{ma^2}\right)^2\right]\right\}^{-1/2}\exp-\left[\frac{\left(x-\frac{k\hbar t}{m}\right)^2}{a^2+\frac{\hbar^2 t^2}{m^2 a^2}}\right]. \qquad (24)$$

This result has all the qualities needed for the description of the motion of particles: It reduces to classical propagation with speed $k\hbar/m$ if either m or a is large; it exhibits quantum mechanical diffusion through the dependence of its half-width on t. And it clearly makes no concession to the occurrence of a measurement. If a measurement is made with minimal disturbance, the presumption is that it can be repeated at a later time, yielding a result in accordance with Eq. (24); if it is destructive, repreparation of the state ψ at time 0 must be understood. For example, nothing more than Eq. (24) is needed to explain the production of ion tracks.

There is in fact a bonus in this more modest interpretation. For it preserves reference to the prepared state throughout the process of measurement. If a measurement did produce an eigenstate, the memory of the state before measurement would be lost. An exact measurement of x made upon a state represented by a plane wave would yield a δ-function, and so would a measure-

ment upon a spherical wave. Yet we know that the probability for a second measurement at a later time and a different place will depend on the type of wave that existed before the first measurement was made. This fact is correctly rendered by Eq. (6) and its corollary, Eq. (24), but is denied by the projection postulate.

Earlier, attention was called to von Neumann's three stages of causality. Our conclusion indicates that his threefold division does violence to quantum mechanics and that, insofar as it is tenable at all, we are closer to the first than to the second stage. The logical inconsistency of using a probability theory in which a single observation determines a distribution function is thus removed.

The version of the projection postulate we were forced to deny is the one most widely featured, but probably not the one most generally believed. Thoughtful physicists, when reflecting upon the measurement act, find themselves committed to a statement which is not postulational and is far milder than von Neumann's axiom; it is this. Many types of measurement (not all) involve a *selection* of systems from an original ensemble, and by proper subsequent manipulation one such separated set can then be isolated. This set may represent an eigenstate of the measured observable. One might even wish to argue—against all reasonable practice—that this specification *defines* a quantum mechanical measurement. Even if this argument is accepted there is still no reduction of the wave packet in the sense that the original wave function is suddenly and acausally changed to another form; for the new state, if any, refers to a different physical ensemble for which the old state had no meaning. A simple analogue is the transition from a game involving two dice to one with a single die, for which the spread of probabilities is smaller. This change is not properly discussed as a sudden collapse of probabilities, nor is it puzzling.

A fuller theory of measurement in which all these questions are discussed will be published elsewhere. It describes the measuring act as a conversion from a pure case to a mixture, in von Neumann's sense. This accounts for the so-called "cancelling of phases" said to occur during measurement. The classical probabilities present in the mixture can then, if it is desired, be altered to form new states. Except for the concept of a mixture of states, this complete theory requires no postulate additional to those listed in § 1.

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2) See, for example, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., 1955).

3) H. Margenau, Phys. Rev. **49** (1936), 240.
4) J. L. McKnight, Phil. of Sci. **24** (1957), 321; **25** (1958), 209;
L. Durand, Phil. of Sci. **27** (1960), 115.
5) An acceptable modification, which in essence denies the postulate, is formulated at the end of this article.
6) H. Margenau, Phil. of Sci. **25** (1958), 23; reprinted in C. W. Churchman and P. Ratoosh, *Measurement: Definitions and Theories* (John Wiley and Sons, 1959).
7) This set has not been shown to be consistent or complete in a strict logical sense, a goal which at present eludes our competence. Nor has an attempt ever been made to define the primitive terms or the rules of operation to be employed in the use of the system. Loosely, the latter are the rules of mathematics. The axiomatic system is intuitively felt to be complete, and this feeling is enhanced by the realization that no inconsistency has yet been encountered among its consequences.
8) The abbreviations, $\bar{p}$ for "expectation of P", and the scalar product form (ϕ^*, P, ϕ) will be employed throughout this paper.
9) The terminology here adopted follows W. Feller, *An Introduction to Probability Theory and its Applications* (John Wiley and Sons, 1957).
10) If X and Y denote positron and momentum operators, the present $P(x, y; \phi)$ becomes Wigner's distribution function. See E. Wigner, Phys. Rev. **40** (1932), 749.
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Theory of Impurity Band Conduction in Semiconductors

——*An Approach to Random Lattice Problem*——

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The impurity band problem in semiconductors is formulated in terms of the Green function, the calculation of which has been carried out by a graphical method. Making use of the hydrogen model for the impurity state, the level density of the impurity band as well as the conductivity are calculated for various concentrations of impurities, the results being in qualitative agreement with observations. The effects of correlation and compensation are not taken into account, since the main interest in this paper consists in the randomness of the impurity array.

§ 1. Introduction

Low temperature anomalies in electrical resistance and Hall effect were observed in SiC by Busch and Labhart[1] and in Ge by Hung and Gliessman,[2] who ascribed them to the existence of conduction through the impurity states. Since then a number of experimental and theoretical works have been carried out on the various problems involved in the phenomena.

From the theoretical point of view, we have to consider (1) the electron correlation problem,[3] (2) the possibility of absence of diffusion under the spatially fluctuating potential,[4] (3) the role of lattice vibration or distortion in assisting or hindering the motion of carriers,[5] and (4) the problem of random distribution of impurities, including the justification of the impurity band model upon which the anomalies were explained phenomenologically with great success, at least in the early stage.[2),6]

In real phenomena of impurity conduction, these effects, more or less independent in themselves, seem to play roles correlating with each other in a complicated way. According to experimental results,[7] however, it seems rather safe to assume that at "high concentration", from 10^{17} to 10^{18} impurities per cc in n-type Ge, the carriers move in the so-called impurity band, without serious effect of correlation as far as the main feature is concerned. Both the electrical

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resistivity and Hall coefficient increase as the specimen is compensated, and this tendency is more remarkable for the former.[7] This suggests that the compensation has two effects in this concentration range, namely the decrease in the number of carriers on the one hand and the increase in the scattering probability of carriers on the other hand, probably due to the resulting charged centers (minority impurities). The fact that resistivity and Hall coefficient are almost independent of temperature in uncompensated specimens seems to show that the carriers are in rather mobile band-like states, and almost degenerate in this quasi-band, and further that the scattering is mainly due to the randomness in the spatial distribution of impurities because there are no charged centers in these uncompensated specimens.

At "low concentration" ($<10^{16}$/cc in Ge), compensation is essential for the conduction to take place, and the electron correlation and the potential fluctuation resulting from compensation seem to play important roles. In this case, the carriers will hop from site to site with the assistance of phonons.[5),8)] At "very high concentration" ($>10^{18}$/cc in n-type Ge), the Fermi energy seems to merge into the conduction band, in view of the fact[7] that the characteristic peak disappears in the Hall coefficient-temperature curve. One must then assume that the conduction takes place essentially in the conduction band, although the compensation may reduce the Fermi energy to such an extent that the characteristic nature of the impurity band appears again.

We are here concerned with the "high concentration" range, with no compensation, and investigate mainly the problem of random lattice. Most of the problems in transport phenomena in metals and semiconductors have been formulated so far first by introducing perfectly mobile states (energy band under periodic lattice), and then by taking into account the effects of various kinds of disturbance (lattice vibration or impurities) based on simple perturbation approximation. Some other phenomena have been dealt with by starting from the localized states and taking account of the effect of thermal disturbance which gives rise to the hopping of the carriers. A common feature in these two extreme cases is that the states prepared in the starting points are more or less good eigenstates and the transition between these states is a well-defined and well-separated elementary process in quantum mechanical sense.

Such feature is perfectly absent in the present problem. First of all, an eigenstate in random lattice has neither definite momentum nor definite localization. Even within a small range of eigen-energy, the degree of localization differs quite irregularly from state to state, and it does not seem very useful or meaningful to investigate the individual eigenstates. The level density, however, is expected to be a continuous function of energy, forming the so-called impurity band, and may possibly be observed more or less directly, for instance, by measuring magnetic susceptibility. In the second place, there is no disturbance causing the scattering of carriers other than the random distribution of

impurities itself which is already taken into account in the formation of impurity band. An alternative approach getting out of this embarrassing situation may be to describe the motion of carriers as diffusion from site to site. However, a transfer of an electron between two sites is not a separable elementary process and has some correlation with preceding and succeeding transfers, as is inferred from the existence of Hall effect. Thus we should devise a method which enables us to deal with this inseparable problem in the random lattice.

As for the level density for random lattice, James and Ginzbarg[6] investigated the case of one-dimensional random lattice composed of two kinds of atoms with a fixed concentration, while Lax and Phillips[9] dealt with the random distribution of δ-function type potentials on a continuous one-dimensional space for a number of concentrations. In particular, the latter authors compared their numerical results calculated by making use of a high speed computer with those obtained through various approximations. For high density the optical model seems to provide a very good approximation in the main, while the low energy tail of the level density curve can be approximately described by taking into account the local density fluctuation. For low density, the impurity band become split off from the continuum, and the tail parts of the former can be approximated by the pair theory. Kurosawa[10] treated the impurity band problem in three dimensions with tight binding model. Making use of analogy with lattice vibration in random lattice, he inferred that the low energy levels may have approximate wave numbers in spite of the absence of translational invariance, and calculated the mobility of electrons assuming the Rayleigh type scattering caused by the local density fluctuation. Aigrain[11] made use of Dyson's method in calculating the level density. Kasuya[12] made another approach to the impurity band conduction, by making use of phase randomness of wave functions due to the random distribution of impurities, and by determining the amplitude on the basis of perturbation theory and normalization condition. This method seems to be essentially a kind of damping theory.

In this paper we propose an approach to this problem of random lattice which utilizes the Green function formalism. In calculating the Green function we develop a graphical method with which one can improve the approximation to any desired order. The method is to sum up selectively the graphs appearing in the perturbational expansion, retaining all the graphs which become important at high densities. The results obtained are, therefore, essentially high density expansion, which rather seems fairly good for the range of concentrations we are considering here. The level density is simply given as the trace of the Green function, while the electrical conductivity can be expressed by the product of two Green functions connecting impurity sites. As for the latter, general formula for the transport coefficients derived by Kubo and Nakano[13] has been made use of, because it does not necessitate the separation of the procedure into the calculation on elementary process and on kinetic transport equation.

§ 2. Formulation of the problem in terms of the Geen function

In the concentration range of the so-called impurity band conduction, which corresponds to $10^{17}\sim10^{18}$ impurities per cc in the case of n-type Ge, the Fermi energy for the electron is well inside the impurity band, although the high energy side of the impurity band may merge into the conduction band. Neglecting the influence of conduction band as well as that of excited states of the impurity, we adopt the tight binding approximation with the ground state of the impurity. Let $V_{mn}=V(|R_m-R_n|)$ be the energy integral for the transfer of an electron from the m-th site to the n-th site, then the Hamiltonian of the system can be simply written as

$$H=\sum_{m\neq n}\sum V_{mn}a_m^* a_n, \tag{2·1}$$

where a_m^* and a_m represent the creation and annihilation operators of an electron at the m-th site. The impurities are assumed to be distributed completely at random. The ground state energy of an isolated impurity is taken as the origin of the energy. When there is compensation, the charged minority impurities give rise to the spatially fluctuating potential, and we must take into account the diagonal part $\sum_m V_m a_m^* a_m$, where V_m's differ from site to site. Assuming no compensation, we confine ourselves to the case of $V_m=0$ for all sites. The generalization to the case of $V_m\neq0$ is not difficult. We also neglect the correlation as well as electron spin. The effect of the latter will partly be taken into account later simply by doubling the occupation numbers.

Since (2·1) is of quadratic form, it can be diagonalized as

$$H=\sum_{\mu}E_\mu A_\mu^* A_\mu \tag{2·2}$$

by orthogonal transformation

$$a_m=\sum_{\mu}c_{m\mu}A_\mu, \tag{2·3}$$

where A_μ^* and A_μ are the creation and annihilation operators for the μ-th eigenstate of a single particle. Electric current operator in this system is written, by definition, as

$$\boldsymbol{J}=ie/\hbar[H,\ \sum_m \boldsymbol{R}_m a_m^* a_m]=ie/\hbar\sum_{m\neq n}\sum \boldsymbol{R}_{mn}V_{mn}a_m^* a_n=\sum_{\mu}\sum_{\nu}\boldsymbol{J}_{\mu\nu}A_\mu^* A_\nu, \tag{2·4}$$

where

$$\boldsymbol{R}_{mn}=\boldsymbol{R}_m-\boldsymbol{R}_n \quad\text{and}\quad \boldsymbol{J}_{\mu\nu}=ie/\hbar\sum_{m\neq n}\sum \boldsymbol{R}_{mn}V_{mn}c_{m\mu}^* c_{n\nu}. \tag{2·5}$$

According to Kubo[13] the static conductivity tensor is given by

$$\sigma=\lim_{s\to+0}\int_0^\infty dt\int_0^\beta d\lambda\exp(-st)\langle \boldsymbol{J}(-i\hbar\lambda)\,\boldsymbol{J}(t)\rangle, \tag{2·6}$$

where $\langle\cdots\rangle$ means the quantum statistical average under no external field, and $\boldsymbol{J}(t)$ is the Heisenberg representation of the operator $\boldsymbol{J}$. Making use of the relation

$$A_\mu^*(t)=\exp(iHt/\hbar)\,A_\mu^*\exp(-iHt/\hbar)=A_\mu^*\exp(+iE_\mu t/\hbar),\ \text{etc.,} \tag{2·7}$$

and

$$\langle A_\alpha^* A_\beta A_\gamma^* A_\delta\rangle=\delta_{\alpha\delta}\delta_{\beta\gamma}\langle N_\alpha\rangle\langle 1-N_\beta\rangle+\delta_{\alpha\beta}\delta_{\gamma\delta}\langle N_\alpha\rangle\langle N_\gamma\rangle \tag{2·8}$$

where

$$\langle N_\mu\rangle=[\exp[\beta(E_\mu-E_F)]-1]^{-1}=f(E_\mu) \tag{2·9}$$

with Fermi energy E_F and reciprocal temperature β, and assuming

$$\langle\boldsymbol{J}\rangle=\sum_\mu \boldsymbol{J}_{\mu\mu}\langle N_\mu\rangle=0 \tag{2·10}$$

which is obvious in the present case, we can rewrite (2·6) as

$$\sigma=\pi\hbar\sum_\mu\sum_\nu \boldsymbol{J}_{\mu\nu}\boldsymbol{J}_{\nu\mu}\frac{e^{\beta(E_\mu-E_\nu)}-1}{E_\mu-E_\nu}f(E_\mu)\{1-f(E_\nu)\}\left[\delta(E_\mu-E_\nu)+\frac{i}{\pi}\,\mathcal{P}\,\frac{1}{E_\mu-E_\nu}\right]. \tag{2·11}$$

The second term in square brackets contributes nothing, since the product outside the brackets is not altered when μ and ν are interchanged. (2·11) is then written as

$$\sigma=\pi\hbar\int dE\sum_\mu\sum_\nu \boldsymbol{J}_{\mu\nu}\boldsymbol{J}_{\nu\mu}\delta(E-E_\mu)\,\delta(E-E_\nu)[-df(E)/dE] \tag{2·12}$$

where we have made use of (2·9).

In order to rewrite (2·12) in the original representation, it is convenient to introduce the Green functions defined by

$$G_{mn}^{(\pm)}(E)=\left\langle 0\left|a_m\frac{1}{E-H\pm i\varepsilon}a_n^*\right|0\right\rangle=\sum_\mu\frac{1}{E-E_\mu\pm i\varepsilon}c_{m\mu}c_{n\mu}^*\quad(\varepsilon\to+0), \tag{2·13}$$

$$G_{mn}(E)=(i/2\pi)[G_{mn}^{(+)}(E)-G_{mn}^{(-)}(E)]=\sum_\mu\delta(E-E_\mu)\,c_{m\mu}c_{n\mu}^* \tag{2·14}$$

where $|0\rangle$ is the vacuum state. (2·12) now reduces to

$$\sigma=-(\pi e^2/\hbar)\int dE\,\boldsymbol{R}_{mn}V_{mn}G_{nl}(E)\,\boldsymbol{R}_{lk}V_{lk}G_{km}(E)[-df(E)/dE]. \tag{2·15}$$

On the other hand, level density is given by

$$D(E)=\sum_\mu\delta(E-E_\mu)=\sum_m G_{mm}(E) \tag{2·16}$$

with the use of the orthonormality relations:

$$\sum_\mu c_{m\mu}c_{n\mu}^*=\delta_{mn}. \tag{2·17}$$

If N impurities are contained in a unit volume, we have

$$\int D(E)\,dE=N \tag{2·18}$$

levels as a whole. Taking into account the electron spin, and assuming that compensation degree is given by K (that is, there are NK minority impurities each of which takes one carrier from the impurity band of majority centers, $0<K<1$), one can determine the Fermi energy from the equation

$$2\int f(E)\,D(E)\,dE=N(1-K). \tag{2·19}$$

Owing to the random distribution of impurities, the conductivity tensor should be isotropic. Again taking into account the spin degeneracy, we have the scalar conductivity

$$\sigma=2(\pi e^2/3\hbar)\int dE\,\Xi(E)[-df(E)/dE], \tag{2·20}$$

$$\Xi(E)=-\sum_m\sum_n\sum_l\sum_k \boldsymbol{R}_{mn}V_{mn}G_{nl}(E)\cdot\boldsymbol{R}_{lk}V_{lk}G_{km}(E), \tag{2·21}$$

where $\boldsymbol{R}_{mn}\cdot\boldsymbol{R}_{lk}$ should be understood as scalar product.

§ 3. Graphical calculation of the Green function

Since the level density and conductivity have been given by (2·16) and (2·15) or (2·20), the problem is now to calculate the Green function. Expanding the resolvent in (2·13) formally, we have

$$\frac{1}{z_\pm-H}=\frac{1}{z_\pm}+\frac{1}{z_\pm}H\frac{1}{z_\pm-H}=\sum_{\nu=0}^{\infty}z_\pm^{-(\nu+1)}H^\nu,\quad z_\pm=E\pm i\varepsilon. \tag{3·1}$$

Inserting this into (2·1), we get

$$z_\pm G_{mn}^{(\pm)}(E)=\delta_{mn}+\frac{1}{z_\pm}V_{mn}+\sum_{\nu=1}^{\infty}\frac{1}{z_\pm^{\nu+1}}\sum_{l_1}\sum_{l_2}\cdots\sum_{l_\nu}V_{ml_1}V_{l_1l_2}\cdots V_{l_{\nu-1}l_\nu}V_{l_\nu n} \tag{3·2}$$

where summation on each l_i is to be extended over all impurity sites. We can represent each term in the right-hand side of (3·2) graphically as shown in Fig. 1. Contrary to the usual Feynman graphs, for each impurity site denoted by a (vertex) point a factor $1/z$ (propagator) is associated, and for each transfer, say from the n-th site to the m-th site, denoted by a bonding line which joins n and m a factor V_{mn} (transition matrix) is associated. The Green function G_{mn} is therefore graphically represented by a sum over all the graphs which represent various possible "journeys" from the n-th site to the m-th site.

As the impurities are distributed completely at random, we can replace (3·2) by the ensemble average, by making the positions of all intermediate impurity sites l_i's run over the whole space with equal probability. In effect, we may

replace the summation over intermediate impurity sites by integral

$$\sum_{l} \to N \int dR_l$$

and we then have

$$\underset{\sim}{z}_{\pm} G_{mn}^{(\pm)}(E) = \delta_{mn} + \frac{V_{mn}}{\underset{\sim}{z}} + \sum_{\nu=1}^{\infty} \sum_{\mu=1}^{\nu} \frac{N^{\mu}}{\underset{\sim}{z}^{\nu+1}} \int dR_1 \int dR_2 \cdots \int dR_{\mu}$$

$$\times V(R_{mp_1})\, V(R_{p_1 p_2}) \cdots V(R_{p_\nu n}). \qquad (3\cdot3)$$

where μ is the total number of intermediate sites (white points in Fig. 1) and $\nu+1$ is the number of bonding lines appearing in a typical graph. Topological interconnection of points within a graph is characterized by a set of indices $(p_1, p_2, \cdots, p_\nu)$. For given μ and ν, we have to sum over all those possible sets of $(p_1 \cdots p_\nu)$ which correspond to the graphs shown inside each section of Fig. 1.

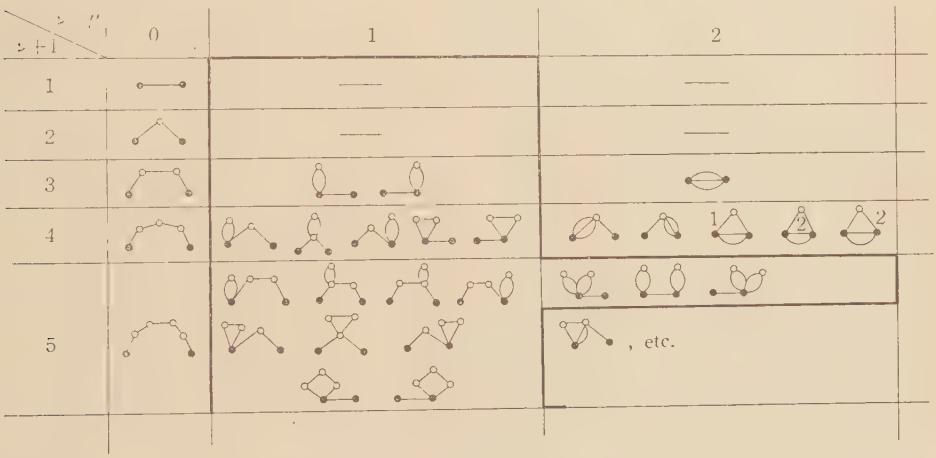


Fig. 1. Graphical representation of the typical terms in (3·2). The black points represent fixed sites and the white points represent intermediate sites which are to be averaged with respect to their random distribution. $\nu+1$ is equal to the total number of bonding lines. μ is the total number of white points. All the graphs inside the sections enclosed by the solid line are summed up in the approximation described in the text.

For a given value of z, those terms with smaller $\nu-\mu$ values make larger contributions when the concentration N is large, as is seen from (3·3). That is, *when one wants to get the result which is accurate at high concentrations, the leading principle in selecting the graphs is to sum up those graphs with smaller $\nu-\mu$ values as many as possible.*

First we calculate the diagonal element $\overline{G_{mm}^{(\pm)}(E)}$ which is obviously independent of the position R_m. Graphical representation of $\overline{G_{mm}^{(\pm)}}$ is given in Fig. 2. It is more convenient to consider a quantity $\xi^{(\pm)}$ defined by

$$\xi^{(\pm)} = z_{\pm} \overline{G_{mm}^{(\pm)}(E)}. \qquad (3\cdot4)$$

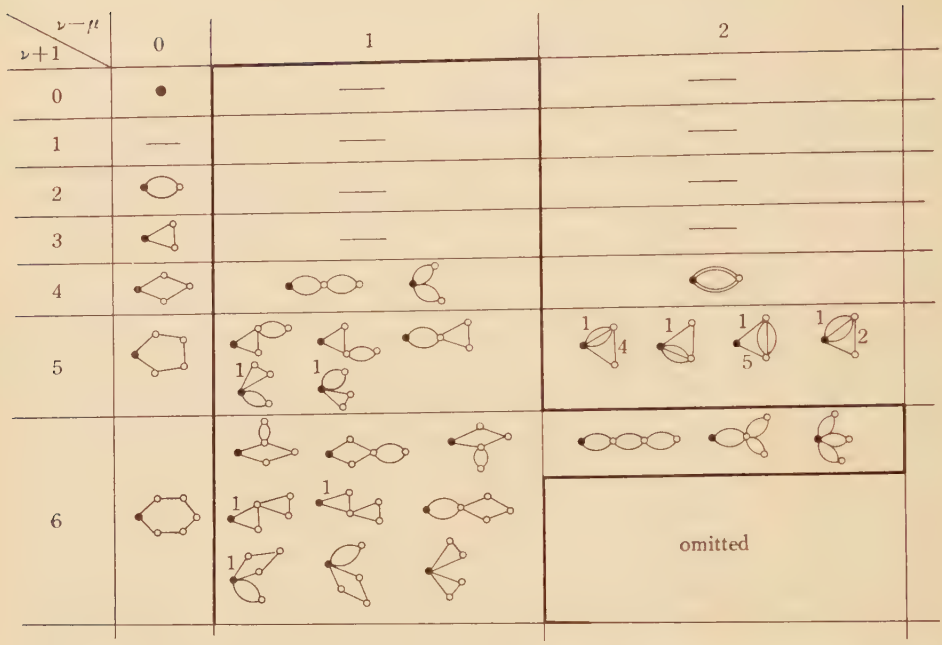


Fig. 2. Graphical representation of $G^{(\pm)}_{mm}(E)$. The meaning of the notation is the same as in Fig. 1.

From Fig. 2 we note that $\hat{\xi}^{(\pm)}$ equals unity plus sum of graphs which represent various journeys starting from and terminating at the site m. We can classify these graphs for $\hat{\xi}$ according to the number of returns to the home site m on the way. (See Fig. 3a.) Following the leading principle stated above, we take only those graphs in which different journeys have no common intermediate sites with one another. This amounts to putting

$$\hat{\xi} \doteqdot 1+\eta+\eta^2+\cdots=\frac{1}{1-\eta}, \tag{3·5}$$

where η represents a sum of all "irreducible" journeys which never pass the home site m on the way. The irreducible η itself can be calculated in the same manner, by taking only those graphs in which the main route has no common intermediate sites but from each intermediate site one can make any times of sub-journeys. (See Fig. 3b.) Neglecting the overlaps between these sub-journeys, but extending this procedure infinitely, we see that for each intermediate site a factor $\hat{\xi}$ has to be associated as the result of the contributions from all the sub-journeys starting from and terminating at that site. Thus we can write down η as

$$\eta=\frac{N\hat{\xi}}{z^2}\int V(\boldsymbol{R}_1)\,V(-\boldsymbol{R}_1)\,d\boldsymbol{R}_1+\frac{(N\hat{\xi})^2}{z^3}\iint V(\boldsymbol{R}_1)\,V(\boldsymbol{R}_2)\,V(-\boldsymbol{R}_1-\boldsymbol{R}_2)\,d\boldsymbol{R}_1\,d\boldsymbol{R}_2+\cdots$$

$$= \frac{1}{8\pi^3} \frac{N\tilde{\xi}}{z^2} \int \frac{v(\boldsymbol{k})^2 d\boldsymbol{k}}{1-(N\tilde{\xi}/z)v(\boldsymbol{k})} \tag{3·6}$$

where $v(\boldsymbol{k})$ is defined by

$$v(\boldsymbol{k}) = \int V(\boldsymbol{R}) \exp(-i\boldsymbol{k}\boldsymbol{R})\, d\boldsymbol{R}. \tag{3·7}$$

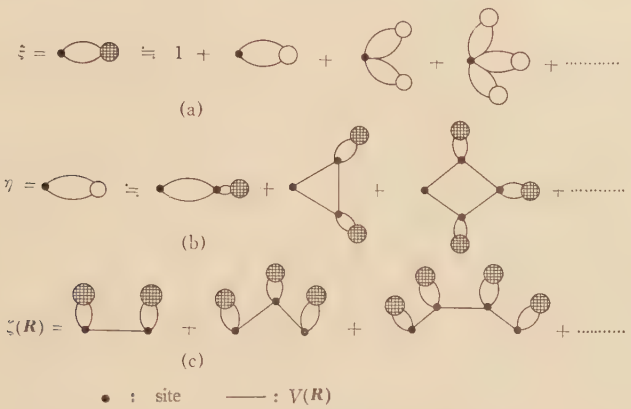


Fig. 3. Graphical representation of $\tilde{\xi}$, η, and ζ (R).

Combining (3·6) with (3·5), we get the equation

$$\eta = 1 - \frac{1}{\tilde{\xi}} = -\frac{V(0)}{z} - \frac{1}{8\pi^3} \int \frac{d\boldsymbol{k}}{N\tilde{\xi} - zv(\boldsymbol{k})^{-1}}. \tag{3·8}$$

In calculating the non-diagonal element $G_{mn}^{(\pm)}(E)$ $(m \neq n)$ which is obviously a function of $\boldsymbol{R}_{mn}$, we put

$$\zeta^{(\pm)}(\boldsymbol{R}_{mn}) = z_{\perp} \overline{G_{mn}^{(\pm)}(E)} \tag{3·9}$$

and carry out calculation similar to (3·6), following the prescription shown in Fig. 3c. This amounts to

$$\zeta(\boldsymbol{R}) = \frac{\tilde{\xi}^2}{z} V(\boldsymbol{R}) + \frac{N\tilde{\xi}^3}{z^2} \int V(\boldsymbol{R}_1)\, V(\boldsymbol{R}-\boldsymbol{R}_1)\, d\boldsymbol{R}_1$$
$$+ \frac{N^2\tilde{\xi}^4}{z^3} \iint V(\boldsymbol{R}_1)\, V(\boldsymbol{R}_2)\, V(\boldsymbol{R}-\boldsymbol{R}_1-\boldsymbol{R}_2)\, d\boldsymbol{R}_1 d\boldsymbol{R}_2 + \cdots \tag{3·10}$$

or

$$\zeta(\boldsymbol{R}) = -\frac{1}{8\pi^3} \tilde{\xi}^2 \int \frac{\exp(i\boldsymbol{k}\boldsymbol{R})\, d\boldsymbol{k}}{N\tilde{\xi} - zv(\boldsymbol{k})^{-1}}. \tag{3·11}$$

Solving (3·8) and (3·11), we get $\tilde{\xi}$, η and $\zeta(R)$ as functions of energy z. It is evident that in deriving these equations on the basis of the prescription shown in Fig. 3, we have taken into account all of those graphs in Fig. 1 and Fig. 2

for which $\nu-\mu=0$ or $\nu-\mu=1$, and even a part of the graphs with $\nu-\mu\geqq 2$, as are enclosed by the solid line in the figures. (3·8) and (3·11) are therefore exact in the limit of high concentration.

It might seem that the expansion (3·3) as well as (3·1) converge very slowly, or even diverge for small values of z, and necessarily that (3·8) and (3·11) are not valid. However, this is not the case because by rewriting (3·8) as

$$z-N(N\tilde{\xi}/z)^{-1}=-V(0)-\frac{1}{8\pi^3}\int\frac{d\boldsymbol{k}}{(N\tilde{\xi}/z)-v(\boldsymbol{k})^{-1}}, \qquad (3\cdot 12)$$

we see that $(N\tilde{\xi}/z)$ is finite when $|z|\to 0$. Therefore, the expansions (3·6) and (3·10), which are essentially power series in $(N\tilde{\xi}/z)$, will not particularly show bad behaviours in the neighborhood of $|z|=0$. That is, by rearranging the terms in apparently divergent series (3·3) by the graphical method shown in Fig. 3, we obtained a maybe convergent series (3·6) and (3·10), as far as the z-dependence is concerned. Inclusion of $\tilde{\xi}$ or sub-journeys in Fig. 3 is essential in removing this divergence.

Another problem is the limit $N\to\infty$. In this case $z/\tilde{\xi}$ remains finite as is obvious from (3·12), but (3·6) and (3·10) may diverge for large values of N. We expect, however, that (3·8) and (3·11), which are analytic continuation of (3·6) and (3·10), are valid even for large N, since we get reasonable result by solving (3·8) and (3·11), as will be shown in later sections.

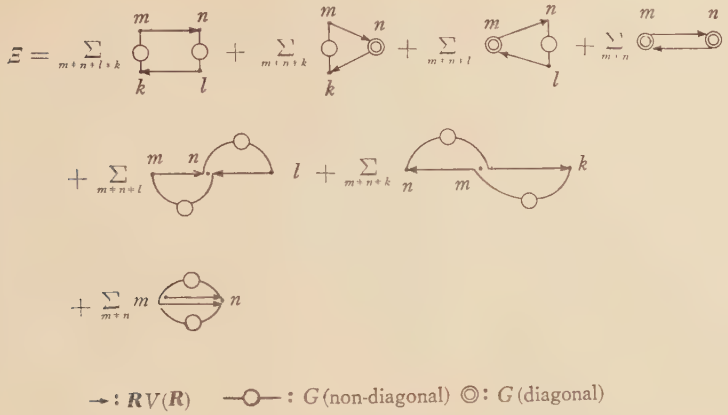
§ 4. Level density and conductivity

It is now straightforward to write down the level density. Making use of (2·16), (2·14) and (3·4), we have

$$D(E)=\sum_m G_{mm}(E)=NG_{mm}(E)=\frac{i}{2\pi}\frac{N(\tilde{\xi}^{(+)}-\tilde{\xi}^{(-)})}{E} \qquad (4\cdot 1)$$

where $\tilde{\xi}^{(\pm)}$ are solutions of (3·8) for $z_\pm=E\pm i\varepsilon(\varepsilon\to+0)$. The existence of imaginary part in $\tilde{\xi}^{(\pm)}$ corresponds to finite density of continuous spectrum. As we shall see below, this also corresponds to the existence of transport. In the second equation of (4·1) we have replaced the sum over m for fixed distribution of impurities by N times the ensemble average of a typical term. If we take a macroscopic volume V, the most probable relative error committed by this replacement is of the order of $(NV)^{-1/2}$, which is quite small.

As for the conductivity, we have only to calculate (2·21), which can be divided into seven terms as shown in Fig. 4. Let us take a typical term under the first summation of Fig. 4, and write down G_{nl} and G_{km} as sums of graphs. Generally the graphs may have any number of sites common to both G's. Moreover G_{nl} may contain m and k, while G_{km} may contain n and l. If we



→ : $RV(R)$ —○— : G(non-diagonal) ◎ : G(diagonal)

Fig. 4. Graphical representation of seven terms in (2·21).

neglect such multiply connected graphs which become less important at high concentrations, we can replace the ensemble average of product by the product of ensemble average

$$\overline{G_{nl}G_{km}}=\overline{G_{nl}}\,\overline{G_{km}}. \tag{4·2}$$

If we follow the same procedure in all the other terms in Fig. 4, we see that the fifth and sixth terms vanish, because the integrals split into two factors, each containing one vector multiplied by direction-independent factor. Defining

$$g(\boldsymbol{k})=\int\overline{G_{mn}(E)}\,\exp(-i\boldsymbol{k}\boldsymbol{R}_{mn})\,d\boldsymbol{R}_{mn} \tag{4·3}$$

and making use of the relation

$$\frac{\partial v(\boldsymbol{k})}{\partial \boldsymbol{k}}=-i\int\boldsymbol{R}V(\boldsymbol{R})\exp(-i\boldsymbol{k}\boldsymbol{R})\,d\boldsymbol{R},$$

we can rewrite the sum from the first to the fourth terms in Fig. 4 as

$$\begin{aligned}\Xi_1&=-\frac{1}{8\pi^3}\int\left\{\frac{\partial v(\boldsymbol{k})}{\partial \boldsymbol{k}}\right\}^2\left\{N\overline{G_{mm}}+N^2g(\boldsymbol{k})\right\}^2 d\boldsymbol{k}\\&=-\frac{1}{8\pi^3}\cdot\frac{1}{4\pi^2}\int\left\{\frac{1}{v(\boldsymbol{k})}\frac{\partial v(\boldsymbol{k})}{\partial \boldsymbol{k}}\right\}^2\left[\frac{N\xi^{(+)}}{N\xi^{(+)}-z_+v(\boldsymbol{k})^{-1}}-\frac{N\xi^{(-)}}{N\xi^{(-)}-z_-v(\boldsymbol{k})^{-1}}\right]^2 d\boldsymbol{k},\end{aligned} \tag{4·4}$$

where we have made use of (2·14), (3·4), (4·3), (3·9) and (3·11). The last term in Fig. 4 is simply written down as

$$\Xi_2=\frac{1}{4\pi^2}\int\boldsymbol{R}^2V(\boldsymbol{R})^2d\boldsymbol{R}\left[\frac{1}{8\pi^3}\int d\boldsymbol{k}\,e^{i\boldsymbol{k}\boldsymbol{R}}\left\{\frac{\xi^{(+)}}{z_+}\frac{N\xi^{(+)}}{N\xi^{(+)}-z_+v(\boldsymbol{k})^{-1}}\right.\right.$$

$$-\frac{\xi^{(-)}}{\approx_-}\quad\frac{N\xi^{(-)}}{N\xi^{(-)}-\approx_- v(\boldsymbol{k})^{-1}}\Bigg\}\Bigg]^2. \tag{4·5}$$

§ 5. Calculation by hydrogenic model

We now apply the results obtained in the preceding sections to the impurity band conduction in semiconductors. Taking the simplest model for the impurity state, we assume hydrogen-like 1s-state wave function:

$$\phi_m(\boldsymbol{r})=(\alpha^3/\pi)^{1/2}\exp(-\alpha|\boldsymbol{r}-\boldsymbol{R}_m|). \tag{5·1}$$

The transfer energy is given by

$$V_{mn}=\int\phi_m(\boldsymbol{r})\frac{-e^2}{\kappa_0|\boldsymbol{r}-\boldsymbol{R}_m|}\phi_n(\boldsymbol{r})\,d\boldsymbol{r}=-V_0(1+\alpha R_{mn})\,e^{-\alpha R_{mn}} \tag{5·2}$$

where

$$V_0=e^2\alpha/\kappa_0=2\varepsilon_0 \tag{5·3}$$

is twice the ionization energy ε_0 of the 1s-state. κ_0 is dielectric constant and α^{-1} is the radius of the 1s-orbit. Making use of the Fourier component (3·7):

$$v(k)=-V_0\frac{32\pi\alpha^3}{(\alpha^2+k^2)^3}, \tag{5·4}$$

we can put (3·8) and (3·11) into the forms

$$(1-w_\pm)+\frac{w_\pm}{\xi^{(\pm)}}=\frac{8}{\pi}\int_{-\infty}^{\infty}\frac{t^2\,dt}{(t^2+1)^3+(p\xi^{(\pm)}/w_\pm)}, \tag{5·5}$$

$$\zeta^{(\pm)}(R)=-\frac{\xi^{(\pm)}}{w_\pm}\frac{8}{\pi ir}\int_{-\infty}^{\infty}\frac{t\exp(i\boldsymbol{t}\boldsymbol{r})\,d\boldsymbol{t}}{(t^2+1)^3+(p\xi^{(\pm)}/w_\pm)} \tag{5·6}$$

where we have defined dimensionless concentration, energy and distance by

$$p=32\pi N\alpha^{-3}, \tag{5·7}$$

$$w_\pm=\frac{E\pm i\varepsilon}{V_0}, \tag{5·8}$$

$$r=R\alpha. \tag{5·9}$$

When the concentration p is very large, we can put $t^2+1\approx t^2$ in the denominator of the integrand in (5·5). By putting $t^3=s$, the integration is straightforwardly carried out, and we get a quadratic equation for ξ, the roots being given by

$$\xi^{(\pm)}=-\frac{1}{1-w}+\frac{32}{9}\frac{w}{p(1-w)^2}\mp\frac{8i}{3}\frac{w\sqrt{p(1-w)-16/9}}{p(1-w)^2}. \quad (5\cdot10)$$

From this we immediately get the level density:

$$D(E)=\frac{\alpha^3}{V_0}\frac{1}{12\pi^2}\frac{\sqrt{p(1-w)-16/9}}{(1-w)^2}. \quad (5\cdot11)$$

As is shown by the broken line in Fig. 6 for the case of $p=8$, (5·11) is of strongly asymmetric shape with low energy tail extending to infinity but with cutoff on the high energy side at

$$w_2=1-\frac{16}{9}p^{-1}. \quad (5\cdot12)$$

These features are to be expected. It is easy to show that eigenvalues of the matrix (V_{mn}) with $-V_0\leqq V_{mn}\leqq 0\,(m\neq n)$ and $V_{mm}=0$, never exceed V_0, and if all V_{mn} are equal to $-V_1$, say, the eigenvalues are $-(N-1)V_1$ (non-degenerate) and $+V_1$ ($(N-1)$-fold degenerate). The curves in Fig. 6 are somewhat reminiscent of the latter characteristic, smoothed out by statistical distribution. In spite of its approximate nature, (5·11) satisfies the normalization condition (2·8) exactly.

Although (5·11) presents a tolerable approximation in its main portion, we see that for large negative values of w

$$|p\xi/w|\gg 1 \quad \text{only if } |w|\ll 2p^{1/3}, \quad (5\cdot13)$$

that is, for too large negative values of w, the approximation $t^2+1\approx t^2$ in (5·5) is not good. In fact, if we solve (5·5) exactly, the level density curve has low energy cutoff, too, although the tail is rather long for high concentrations, as we shall see below. Now (5·5) itself is already an approximation. If we could take into account all the graphs, the result should give low energy tail extending to infinity, because clusters of any number of impurities with large mutual interaction $V_{mn}\approx -V_0$ are possible. The probability for such large clusters to appear, however, is quite small, and the true tail should fall, with decreasing energy, much more rapidly than (5·11) predicts. Moreover, as for the conductivity, the approximation $t^2+1\approx t^2$ in (5·5) and (5·6) leads to contradictory results that $\sigma\to\infty$ as compensation tends to unity, that is, as the number of carriers tends to zero.

For the reasons stated above it is necessary to solve (5·5) exactly. The numerical calculation has been carried out by making use of the electronic computer, JEIDAC 101. The outline of the procedure is as follows. Put

$$X=-\frac{p}{w}\xi, \quad (5\cdot14)$$

then (5·5) can be written as

$$w=1-\frac{p}{X}-J(X) \tag{5·15}$$

where

$$J(X)=\frac{8i}{3}\sum_{j=1}^{3}\frac{t_j}{(1+t_j^2)^2}, \tag{5·16}$$

t_j being the three roots of the algebraic equation

$$(1+t^2)^3=X \tag{5·17}$$

with positive imaginary parts. For the real interval $-\infty<X<1$, the right-hand side of (5·15) is as shown in Fig. 5. As is evident from this figure, for w in the range $w_1<w<w_2$ there is no real root X of (5·15). This is nothing but the energy range in which eigenvalues are distributed continuously. Solving for t_j or X in this range, we can calculate the level density by taking the imaginary part of X:

$$D(E)=\frac{\alpha^3}{32\pi^2V_0}\,\mathrm{Im}(X). \tag{5·18}$$

The Fermi energy $E_F=V_0w_F$ is determined at absolute zero of temperature by (2·19) or

$$\int_{w_1}^{w_F}\frac{1}{\pi p}\,\mathrm{Im}(X)\,dw=\frac{1}{2}\,(1-K) \tag{5·19}$$

for given concentration p and compensation K. At high concentration in which we are mainly interested, Ξ_2 can be neglected compared to Ξ_1 (see Fig. 4). Then assuming absolute zero of temperature, we have for conductivity

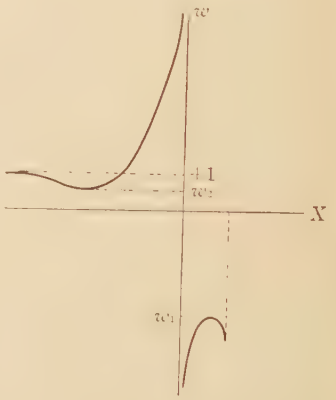


Fig. 5. The curve $w=1-p/X-J(X)$ as a function of X.

$$\sigma=2\frac{e^2\alpha}{\hbar}L \tag{5·20}$$

$$L=\frac{1}{4\pi^2}\sum_{j=1}^{3}\mathrm{Im}\left[\frac{t_j(3+7t_j^2)}{3(1+t_j^2)}-\frac{2t_j^3(1+t_j^2)^2}{i\,\mathrm{Im}(X)}\right] \tag{5·21}$$

where the use of (2·20), (4·4), (5·4), (5·14) and (5·17) have been made. It is obvious from (4·4) that conductivity vanishes at $w_F=w_1$ and $w_F=w_2$ just as the level density does.

In Fig. 6 we show the results of calculation. The level density as a function of energy w, and the conductivity as a function of Fermi energy w_F, are plotted by full and dotted lines respectively, for various values of concentration p. In order to show how the Fermi energy and conductivity vary as the compensation changes, we attached the values of K along the latter curves. Although we

cannot prove analytically that the root of (5·15) satisfies the normalization condition

$$\int_{w_1}^{w_2} \frac{1}{\pi p} \operatorname{Im}(X)\, dw = 1, \tag{5·22}$$

the result of numerical calculation shows that (5·22) is satisfied to within 1% even for the lowest concentration $p=0.25$, for which our approximation (3·8) is expected to be the poorest.

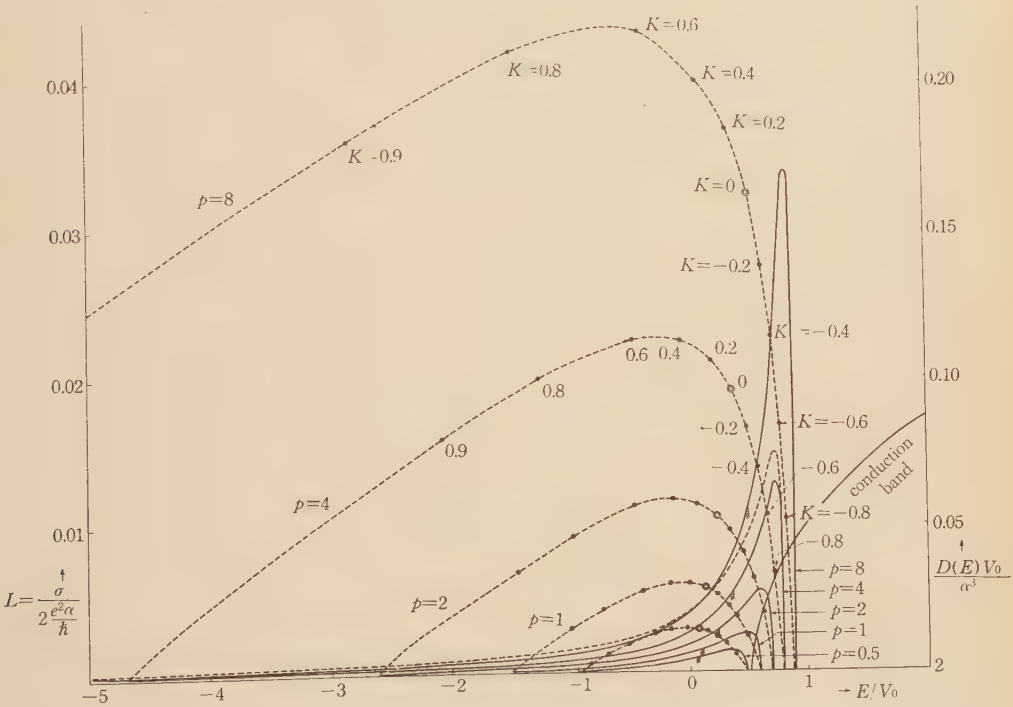


Fig. 6. The level density as a function of energy (full line, right scale) and the conductivity as a function of Fermi energy (dotted line, left scale), for various values of concentration p. The broken line is the level density curve calculated from (5·11) for $p=8$.

The numerical values attached to the conductivity curves represent the degree of compensation K.

§ 6. Discussions and comparison with experiments

As has been remarked by a number of authors, the optical model is expected to present a good approximation at high concentration. In our case the optical model corresponds to taking

$$E(\boldsymbol{k}) \approx \sum_n V(\boldsymbol{R}_{mn}) \exp(i\boldsymbol{k}\boldsymbol{R}_{mn}) \approx N\int V(\boldsymbol{R}) \exp(i\boldsymbol{k}\boldsymbol{R})\, d\boldsymbol{R} = -V_0 p \left(1+\frac{k^2}{\alpha^2}\right)^{-3} \tag{6·1}$$

for the approximate energy band as a function of wave vector, where it does not matter whether the impurities form a periodic lattice or random lattice. This approximation is valid only for small wave number k. The electron propagating with small k is not influenced seriously by short range randomness of impurity array.

The approximation which we have developed in the present graphical method goes one step forward from the optical model, because we have taken into account the effect corresponding to shift and broadening of levels by introducing complex number $\tilde{\xi}$ at each site in writing (3·6). The low energy edge $w_1=E_1/V_0$ of the impurity band calculated by solving (5·14) should be deeper than, but should approach in the limit $N\to\infty$ the value $w_1(\text{opt.})=E(0)/V_0=-p$ calculated based on the optical approximation (6·1). This is in fact the case, as is shown in Fig. 7.

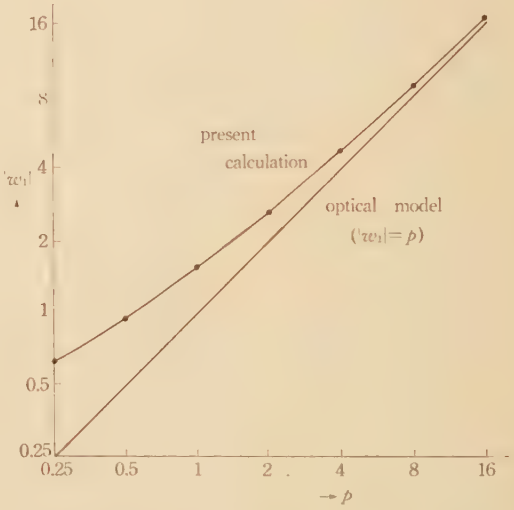


Fig. 7. The low energy band edge $|w_1|$ as a function of concentration p. The corresponding values for optical model is represented by the straight line.

As for the level density, (6·1) gives

$$D(E)=\frac{\alpha^3}{V_0}\frac{1}{12\pi^2}p^{1/2}(-w)^{-3/2}\times\left[1-\left(-\frac{w}{p}\right)^{1/3}\right]^{1/2}, \qquad (6\cdot2)$$

which almost coincides with (5·11) in the energy range

$$1\ll|w|\ll p \quad (w<0).$$

It is interesting to note that introduction of complex factor $\tilde{\xi}$ for each site is analogous to taking into account the shift and broadening for each level of continuous spectrum in the conventional damping theory. It should be emphasized, however, that our Hamiltonian (2·1) has no diagonal singularity in the sense that Van Hove[11] defined in his work on damping theory. As soon as we improve the approximation made in § 3, by taking into account more multiply connected graphs such that the sub-journeys starting from different sites include common sites, the correspondence to the conventional damping theory seems to become obscured.

Turning to the conductivity, we note in Fig. 6 a remarkable feature that although the level density is quite small at low energy, the corresponding conductivity (as a function of Fermi energy) is comparatively large in spite of the small number of carriers. In other words, the low energy electron has much larger mobility than the high energy electron does. This is consistent with the

above-mentioned fact that the low energy electron is not seriously influenced by the random array of impurities. Kurosawa[10] estimated the mean free path l of an electron in the impurity band by assuming the Rayleigh type scattering, obtaining $l \propto k^{-4}$.

Since we have neglected the spacial fluctuation of potential, let us compare the calculated conductivity with the observed value for the case of no compensation. The calculated concentration dependence of resistivity ($\rho=\sigma^{-1}$) is shown in Fig. 8. The dependence is slightly weaker than $\rho \propto N^{-1}$. This is in qualitative agreement with the observations in n- and p-type Ge for the impurity band range of concentration. As for the absolute value, the observed value of resistivity in Sb-doped Ge with concentration $N=2.7\times10^{17}$/cc is given by $\rho=3.2\times10^{-2}$ ohm cm at very low temperatures where ρ is almost independent of temperature.[7] If we take $\alpha=65$Å, this concentration corresponds to $p=7.45$. We can read $L^{-1}\sim33$ from Fig. 7, which gives $\rho=4.4\times10^{-2}$ ohm cm according to (5·20), in satisfactory agreement with the observation, in view of the approximations we have made and the simple model we have taken.

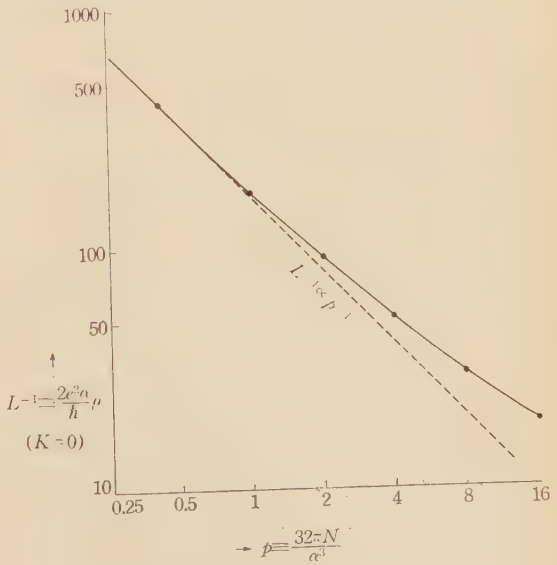


Fig. 8. The electrical resistivity ρ as function of concentration p.

As for the Fermi energy, it gets to the bottom of the conduction band at $p\sim8$ (See Fig. 6) or at $N\sim2.9\times10^{17}$/cc, if we put $\alpha=65$Å. On the other hand, the peak in the Hall coefficient versus temperature curve seems to exist up to 10^{18}/cc according to the observation. This discrepancy may be partly due to the existence of higher excited state of the impurity, but may be mostly due to the fourfold degeneracy of conduction band minima. The latter has the effect of making level density four times larger, depressing the Fermi energy. A part of this effect, however, will be canceled by the Coulomb repulsion between the electrons in the same site but belonging to different valleys or to different spin states which effectively raises the Fermi energy. On the other hand, our calculation is based upon the tight binding model which does not take into account the interaction between the impurity band and the conduction band. It is hoped to improve these points as well as to take into account the effect of potential fluctuation.

Our hope is simply that the method proposed in this paper may present a

new approach to the random lattice problem, as well as a typical example of treating non-Boltzmann (or non-Markovian) transport problem. The graphical method seems especially useful in that one could get as accurate result as one wishes, if one does not mind the accompanying labor. The optical model could not be a starting point to the random lattice problem in the same degree as the band model can be to the real periodic crystals with imperfection, because the deviation of random lattice from the optical model cannot be defined properly, except for the approximate procedure of taking into account the local fluctuation of concentration, which is inevitably a concept of macroscopic nature.

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Non-Static Effects in the Nucleon-Nucleon Interaction*

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A numerical method was used to solve the momentum space integral equation for the scattering amplitude. The complete OPEP phase shifts were computed and compared with the static OPEP phases. The non-static part of OPEP was found to be insignificant for $^1D_2$ at moderate energies.

§ 1. Introduction

It is well known that the nucleon-nucleon interaction is non-local in co-ordinate space. The usual procedure has been to make approximations which render the interaction local, hence easily calculable. Although phenomenological non-local potentials have occasionally been used,[1] no convincing estimate has been made of all of the non-static effects of the interactions given by field theory. In this paper the non-static one-pion-exchange potential is treated in momentum space, thus including the effects of both the static and non-static parts.

§ 2. Method and accuracy

The scattering amplitude, $f_l(p)$, for the uncoupled states is given by the singular integral equation :[2]

$$f_l(p)=f_{l,B}(p)+\mathrm{P}\int_0^\infty \frac{s^2 ds V_l(s,p) f_l(s)}{2E_k-2E_s}, \tag{2·1}$$

where $f_{l,B}(p)$ is a Born approximation amplitude and is given by

$$f_{l,B}(p)=\frac{kE_k}{2} V_l(k,p). \tag{2·2}$$

E_k is half the energy of either nucleon in the center-of-mass system, $k=\sqrt{E_k^2-M^2}$ and $V_l(s,p)$ is the angular-momentum-decomposed momentum-space interaction.

Since the equation for $f_l(p)$ is doubly singular (it has a singularity with the range of integration due to the denominator, and the range of integration extends to infinity) it was decided to employ a successive approximation method.

* Work supported in part by the U. S. Atomic Energy Commission.

$f_{l,B}(p)$ was used as a first order approximation to $f_l(p)$ in the integrand in Eq. (2·1), and a second approximation, $f_{l,2B}(p)$, was obtained. This procedure was continued until $(f_{l,nB}(k)-f_{l,(n-1)B}(k))/f_{l,nB}(k)<10^{-5}$. Then from the relation for the uncoupled states:[2]

$$\tan\delta_l=-\pi f_l(k), \tag{2·3}$$

phase shifts were calculated at various energies from 37 to 310 Mev. Two approximations were necessary:

Approximation 1) The upper limit for the (numerical) integrations was taken to be $15\mu c$, where μ is the mass of the pion. A test run showed the contribution from greater than $15\mu c$ to be less than one part in 10^5.

Approximation 2) In the neighborhood of the singularity, the non-singular part of the integrand was approximated by the straight line $s^2V_l(s,p)f_l(s)=As+B$ and the denominator was expanded in a Taylor series about the singularity. Then this principal value integral was performed analytically in the neighborhood of the singularity. The contribution to the total integral from this small neighborhood, $\sim.2\mu c$, was sufficiently small for the error introduced to be assumed negligible.

The accuracy and convergence of the method was tested using a *local* Yukawa potential of range $1/\mu$ and strength $\approx\pm 20$ Mev at a lab energy of $\approx$70 Mev. Figure 1 shows the successive Born approximations compared with the exact solutions for the two runs. The latter were obtained by numerical integration of the Schrödinger equation in coordinate space.

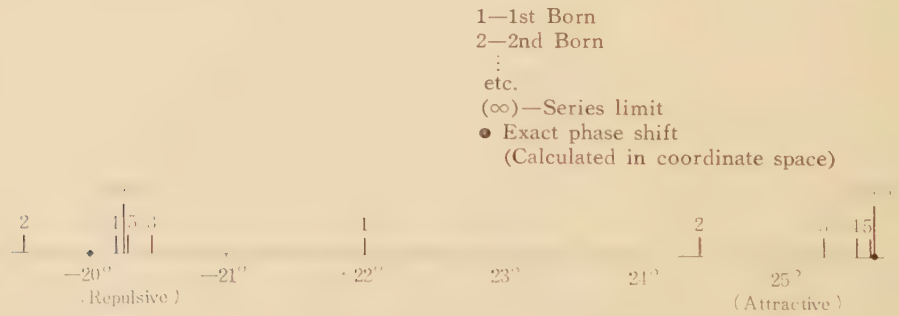

Fig. 1. Born phase shifts calculated in momentum space for Yukawa potential.

§ 3. Results for OPEP

The $^1D_2$ and $^1G_4$ states were investigated for the non-static pseudoscalar one-pion potential.[3] Figure 2 shows the scattering amplitude, $f_l(p)$, for $^1D_2$ at 310 Mev. The successive approximations to the phase shift for this case were 2.45°, 2.46°, 2.47°.

In the table the complete phases are compared with those calculated from the local (static) coordinate space potential. As can be seen, the contribution

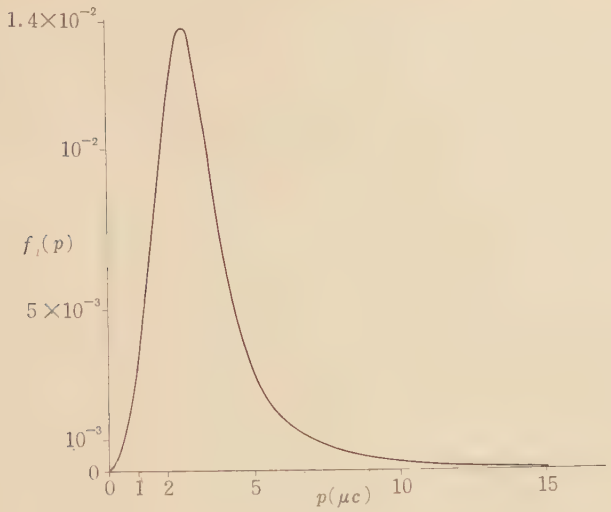

Fig. 2.

from the non-static parts of the one-pion-exchange potential is slight (in all cases $<10\%$ of the one-pion value). Therefore, for energies up to 310 Mev, OPEP can be accurately represented by a local potential in coordinate space.

	Lab energy (Mev)	Coordinate space (static) OPEP	Momentum space (non-static) OPEP	
			First Born	Complete
$^1D_2$	37.3	.89°	.87°	.88°
	95.	1.80°	1.72°	1.74°
	310.	2.75°	2.45°	2.47°
$^1G_4$	95.	.37°	.36°	.36°

Hoshizaki and Machida,[3] by making a q/M expansion,[4] have transformed the OPEP to a coordinate space potential which includes explicitly non-static terms whose signs depend upon whether pseudoscalar or pseudovector coupling is used. They investigated one of the non-static terms, the quadratic spin orbit term, which for PV coupling changes the $^1D_2$ phase shift at 310 Mev in the direction found necessary by Hamada.[5] PS coupling would give a similar shift but in the wrong direction. For the $^1D_2$ state at 310 Mev, the addition of the quadratic spin-orbit term alone to the static OPEP changes the phase shift by 3°. From our results, however, the *entire* pseudoscalar non-static change is only $\sim$.3°. Preliminary results of a similar calculation by J. Goto seem to be in agreement with our work.[6]

It appears, then, that the other non-static terms largely cancel the effect

of the quadratic spin-orbit potential in the pseudoscalar OPEP, or that the q/M expansion is not accurate.

The calculations reported here were carried out in the Computation Center at The Pennsylvania State University and the A.E.C. Computation Center at New York University.

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Persistent Ring Currents in an Ideal Bose Gas

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The production and persistence of ring currents is investigated in a simple model, consisting of Bose-Einstein particles contained in a ring-shaped volume; the particles do not interact with each other, but interact with randomly distributed scattering centres, and carry a test charge through which they respond to external fields. Except for the Bose-Einstein statistics, this model is identical with the model of Kohn and Luttinger, for which a normal, non-zero resistivity has been shown to exist. Furthermore, our model is known to give a qualitatively correct insight into the Meissner effect in actual superconductors, where the "bosons" are attractively correlated electron pairs.

We are able to elucidate the mechanism by which ring currents are generated, and to prove that these ring currents, within our model, have infinite lifetime. There are similarities to the ideas of London, but also significant differences. Our model has no energy gap, and thus our explanation of persistent ring currents is qualitatively dissimilar to the two-fluid model of Bardeen, in which the energy gap is essential. The model is a simple example of a strictly non-ergodic system, and therefore has some interest for the foundations of non-equilibrium statistical mechanics.

The model predicts a new, experimentally observable effect: If the ring is maintained between the poles of the magnet, and cooled well below the transition temperature, small ring currents are generated during this cooling. The effect of these ring currents is to force the flux through the ring very close to an integral number of flux "units" introduced by London. This approximate quantization of flux is destroyed when the ring is moved out of the external field.

§ 1. Introduction

The two most striking properties of superconductors are the magnetic field expulsion (Meissner effect) and the existence of persistent currents in superconducting rings. Although persistent ring currents were originally interpreted in terms of an infinite electrical conductivity for the medium, London[1] showed that the data on the magnetic field expulsion are inconsistent with such an interpretation, and suggested that the ring currents are essentially diamagnetic in origin.

A simple thermodynamic argument[1] shows that the state in which the ring carries a current which maintains a flux through the ring, has higher free energy than the state in which both current and field are zero. Thus the

persistent current state is not the state of thermodynamic equilibrium, but rather a long-lived, perhaps infinitely-long-lived, metastable state. Experimentally, there is no evidence for a finite lifetime of these ring currents; currents have been kept going for periods of over a year without observable decay.

London[1] suggested a set of phenomenological equations relating the density of supercurrent, $\boldsymbol{j}_s$, to applied electric and magnetic fields, namely

$$\text{curl}\,(\Lambda \boldsymbol{j}_s) = -\boldsymbol{H}/c \tag{1·1}$$

and

$$\frac{\partial(\Lambda \boldsymbol{j}_s)}{\partial t} = \boldsymbol{E}, \tag{1·2}$$

where Λ is a material constant of the medium. The first London equation, (1·1), suffices for the discussion of the Meissner effect, which is an equilibrium phenomenon. The persistent ring currents, which are non-equilibrium phenomena, require also the second London equation, (1·2). Starting from these two equations, London proved a conservation law for the "fluxoid" through a superconducting ring, namely the quantity

$$\Phi_C = c\oint_C \Lambda \boldsymbol{j}_s \cdot d\boldsymbol{s} + \iint_S \boldsymbol{H}\cdot \boldsymbol{n}\, dS. \tag{1·3}$$

Here C is a closed curve drawn inside the ring, and S is a surface bounded by the curve C, so that the second term of (1·3) is simply the magnetic flux enclosed by the curve C. London showed that his basic equations (1·1), (1·2) imply that Φ_C is zero for any closed curve C which is located entirely in the interior of the superconductor and which does not embrace a hole. If the curve C does embrace a hole, then the value of Φ_C is a constant in time and does not depend on the shape of the curve C. That is, the fluxoid is not a property of the specific curve C, but rather of the topology of this curve.

London then draws the following conclusion: "The persistent currents are to be considered *metastable* under *macroscopic* conditions in the same sense as a picture is stable when it is hung on the wall by a nail. A supply of a *finite* amount of energy is required; for instance, the ring has to be heated above the transition temperature or a strong magnetic field has to be applied on a section of it in order to set free the energy of the current and of the magnetic field which is locked in".

Although the analogy to a picture hanging on a wall is appealing, it encounters difficulties. The flux, or the fluxoid,* is a continuous variable in this picture, and it is not obvious from a microscopic point of view why Φ_C should

* In a superconducting ring of thickness in excess of the London penetration depth, one can choose a curve C well inside the superconductor. Then the first term of (1·3) vanishes, and the fluxoid Φ_C is identical with the flux through the enlarged (by the penetration depth) hole.

be unable to decrease slowly, in a continuous fashion, even though the macroscopic London equations do not allow this. The analogy seems to be more close if the nail holding the picture against the wall is slanted downwards rather than upwards, thereby allowing the picture to slide down slowly.

This would be quite different, of course, if the fluxoid were quantized; and London suggests that this might be so, the quantum of the flux being

$$\Phi_1 = hc/e. \qquad (1\cdot4)$$

Here $h=2\pi\hbar$ is Planck's constant, and e is the charge of the "superelectrons"; if one interprets this as the charge of a correlated electron pair, i.e. two electronic charges, the flux unit is approximately 2×10^{-7} gauss-cm^2. If the flux is quantized, then it becomes immediately obvious why it fails to decrease continuously; it can only decrease in quantum jumps, and there is no occasion for such jumps.

Appealing as this explanation appears at first sight, it is nevertheless untenable. London's argument for quantization of flux is invalid (see § 2), and recent experiments[2] show that the flux through a superconducting ring is not in fact quantized, at least not with quanta of the order of magnitude (1·4).

A much simpler line of argument for persistence of the ring current bases itself directly on the Meissner field expulsion, without use of special phenomenological equations. This argument runs as follows: Suppose there is some flux through the hole in the ring; this state can be achieved by cooling the ring through the transition temperature, while it is kept between the poles of a magnet. Now pull out the (already superconducting) ring; a ring current establishes itself by ordinary electromagnetic induction. In a normal conductor, this initial ring current would die down quickly, due to Ohmic resistance, and the flux through the ring would decrease to zero. But in a superconductor, the lines of flux are unable to pass through the material of the ring (Meissner effect!), and therefore the flux must remain constant. But, the only way in which the flux can remain constant is that the circulating current in the ring remains constant also. Q.E.D.

This argument is disproved by a recent, very important, experiment by Garwin.[3] Garwin uses extremely thin rings, of thickness much less than the London penetration depth. If the absence of penetration of the field is essential for the persistence of ring currents, then Garwin's ring currents should die down quickly; in fact they do no such thing.

More recently, two lines of argument have been put forth as a result of recent microscopic theories of superconductivity. Bogoliubov et al.[4] suggest the picture of the "displaced Fermi sphere". Instead of coupling, by means of their canonical transformation, single-electron states with momenta k and $-k$, one could equally well couple momenta k and $-k+q$. The state so obtained would be substantially stable, and would certainly carry current.

However, the smallest value of the momentum q is of order $\hbar/L$ where L is the circumference of the ring. The corresponding current per electron is $e\hbar/ML^2$, and if there are N electrons in the material, the ring current is quantized with a quantum value

$$i_1 = Ne\hbar/ML^2. \tag{1·5}$$

P.W. Anderson[5] first pointed out that this "quantum" has an insultingly macroscopic size; using $L=1$ cm, $N=10^{22}$ electrons, we get $i_1 \cong 5.5\times10^{12}$ eus $=1800$ amperes! This would be the *minimum* current possible with a displaced Fermi sphere. In normal metals, only a tiny fraction of the electrons is responsible for the observed current. In the displaced Fermi sphere state, all electrons contribute, hence this very large quantum.

The second recent line of argument is Bardeen's[6] two-fluid model. This model is constructed in strict analogy to Landau's theory of liquid helium.[7] The crucial quantity here is the energy gap; persistence of the supercurrent is attributed to the impossibility of creating elementary excitations which would slow down this current. We shall show, in this paper, that persistent ring currents can be created, and maintained, in the complete absence of an energy gap. Furthermore, the model we use, the ideal Bose-Einstein gas, has been shown[8] to yield a Meissner effect as well, and it has also been shown[9] that the Meissner effect in actual superconductors can be understood in precise analogy to the Bose-Einstein gas model. Thus, it is likely that our model calculation of persistent currents is meaningful also for actual superconductors, in the sense that it will be possible to construct a theory of persistent currents in actual superconductors in close analogy to this model. However, even in the absence of this full demonstration, the present model calculation proves for the persistent currents what we already know for the Meissner effect: *The energy gap in actual superconductors is not an essential factor in understanding persistent ring currents.* For if it were, no model lacking an energy gap could give persistent ring currents.

Let us now describe our model; discussion of the results from this model is best deferred to the last section. We consider a ring shaped volume, containing a macroscopic number of scattering centres, distributed randomly throughout the ring. This volume is filled with an ideal Bose-Einstein gas, i.e. the particles interact with the scattering centres but not with each other. Except for the Bose-Einstein statistics, this model is identical with the model of Kohn and Luttinger,[10] who proved explicitly that the model yields a normal, non-zero resistivity. Furthermore, they showed (in their Appendix F) that the quantum statistics makes no difference to their results. Of course, this statement refers to the "normal" state of the Bose-Einstein gas, and a separate investigation is needed below the Bose-Einstein condensation temperature. It is this separate investigation which is carried out in the present paper, and which shows that the model yields persistent ring currents.

§ 2. Quantum states in a ring without impurities

Consider a ring-shaped volume V, which need not be a figure of revolution, but merely must have the topology of a ring. Let there exist a magnetic field $\boldsymbol{B}$ with div $\boldsymbol{B}=0$, so that

$$\boldsymbol{B}=\operatorname{curl}\boldsymbol{A}. \tag{2·1}$$

It is convenient to consider the vector potential $\boldsymbol{A}$ as a sum of two terms.

$$\boldsymbol{A}(x,\,t)=A_0(t)\,\boldsymbol{v}_0(x)+\boldsymbol{A}_1(x,\,t) \tag{2·2}$$

where the vector function $\boldsymbol{v}_0(x)$ is defined as follows:

$$\operatorname{curl}\boldsymbol{v}_0=0, \tag{2·3a}$$

$$\operatorname{div}\boldsymbol{v}_0=0, \tag{2·3b}$$

$$\boldsymbol{v}_0\cdot\boldsymbol{n}=0 \quad \text{on the boundary of the ring,} \tag{2·3c}$$

$$\int(\boldsymbol{v}_0)^2 d^3x=1. \tag{2·3d}$$

The integration in (2·3d) goes over the volume V occupied by the ring. If this volume has the topology of a ring, i.e. encloses exactly one hole, then there exists one and only one vector field $\boldsymbol{v}_0$ satisfying the conditions (2·3). The decomposition (2·2) is then made unique by the requirement

$$\int\boldsymbol{v}_0(x)\cdot\boldsymbol{A}_1(x,\,t)\,d^3x=0. \tag{2·4}$$

This also gives the relation

$$A_0(t)=\int\boldsymbol{v}_0(x)\cdot\boldsymbol{A}(x,\,t)\,d^3x. \tag{2·5}$$

Although the vector potential $\boldsymbol{A}$ allows a gauge transformation,

$$\boldsymbol{A}'=\boldsymbol{A}+\operatorname{grad} f, \tag{2·6}$$

the coefficient A_0 is gauge-invariant. To prove this, we observe that

$$\int_V\boldsymbol{v}_0(x)\cdot\operatorname{grad}(f)\,d^3x=\int_\Sigma f\boldsymbol{v}_0\cdot\boldsymbol{n}\,dS-\int_V f\operatorname{div}\boldsymbol{v}_0\,d^3x=0\,,$$

where the first term on the right vanishes by (2·3c), the second by (2·3b). Though gauge invariant, the vector potential $A_0\boldsymbol{v}_0$ produces zero magnetic field $\boldsymbol{B}$ in the material of the ring, by (2·3a). The physical meaning of this is simply that this part of the vector potential is associated with the flux through the hole in the ring, not with any magnetic field in the volume occupied by the ring itself.

If one makes a decomposition of the current density $\boldsymbol{j}(x,\,t)$ analogous to (2·2), i.e.

$$\boldsymbol{j}(x, t)=j_0(t)\,\boldsymbol{v}_0(x)+\boldsymbol{j}_1(x, t) \tag{2·7}$$

with

$$j_0(t)=\int \boldsymbol{v}_0(x)\cdot\boldsymbol{j}(x, t)\,d^3x \quad \text{and} \quad \int \boldsymbol{v}_0(x)\cdot\boldsymbol{j}_1(x, t)\,d^3x=0, \tag{2·8}$$

then the current $i(t)$ circulating in the ring is entirely detemined by j_0, i.e. the current density $\boldsymbol{j}_1(x, t)$ gives zero when integrated across a full cross section of the ring.[11] Thus the vector field $\boldsymbol{v}_0(x)$ is the crucial field for any discussion of ring currents, and we shall henceforth ignore the parts $\boldsymbol{A}_1(x, t)$ and $\boldsymbol{j}_1(x, t)$ altogether. To the extent that the phenomenological equations for superconductors are *linear*, which is certainly true of the London equations, this does *not* represent an extra assumption, but merely a convenience for calculative purposes.

The one-particle Hamiltonian for this ring, in the absence of impurities, is simply

$$H_0=(\boldsymbol{p}-e\boldsymbol{A}/c)^2/2M. \tag{2·9}$$

Although, in general, the operators

$$\boldsymbol{\pi}=\boldsymbol{p}-e\boldsymbol{A}/c \tag{2·10}$$

satisfy commutation relations different from those of the components of $\boldsymbol{p}$, this is not so for the special case

$$\boldsymbol{A}(x, t)=A_0(t)\,\boldsymbol{v}_0(x) \tag{2·11}$$

which we shall assume henceforth. For then we get directly

$$[\pi_x, \pi_y]=[p_x, p_y]=0. \tag{2·12}$$

Thus, as far as commutation rules are concerned, H_0 of (2·9) is the same as it would be in the absence of the vector potential. However, the Hilbert space on which H_0 operates does, in general, depend on the vector potential. That is, we shall now show that the condition that the wave function $\psi(x, t)$ is single-valued, does not commute with the S-transformation required to go from the operators $\boldsymbol{\pi}$ back to the operators $\boldsymbol{p}$, except under special circumstances. Let us construct the S-transformation explicitly: Since curl $\boldsymbol{v}_0=0$, there exists a function $f(x)$, defined but multiple-valued over the volume of the ring, such that

$$\boldsymbol{v}_0(x)=-\mathrm{grad} f(x). \tag{2·13}$$

If one goes once around the ring, the value of f increases by the amount

$$F=\oint_C \boldsymbol{v}_0(x)\cdot d\boldsymbol{s}. \tag{2·14}$$

Since curl $\boldsymbol{v}_0=0$, this line integral F is independent of the detailed path C,

and depends only on the topology of the path. We now define the operator S to mean multiplication by $(e/\hbar c)A_0 f(x)$, i.e.

$$\exp(iS)\,\psi(x)=\exp\left[\frac{ie}{\hbar c}A_0 f(x)\right]\psi(x). \tag{2·15}$$

Then a simple calculation gives the identity

$$\mathrm{e}^{-iS}\boldsymbol{\pi}\,\mathrm{e}^{iS}=\boldsymbol{p}. \tag{2·16}$$

Although the transformation (2·15) formally looks like a gauge transformation, it is nothing of the sort; we already know that A_0 is a gauge invariant. We call the S-transformation (2·15) a "pseudo-gauge" transformation.

If $\psi(x)$ is single-valued over the ring, then $\exp(iS)\psi(x)$ is in general multiple-valued; it becomes multiplied by a phase factor $\exp(ieA_0F/\hbar c)$ if one goes once around the ring. Thus the pseudo-gauge transformation leads, in general, to unacceptable functions, i.e. to functions outside the Hilbert space of single-valued functions. An exception occurs only if the phase factor is in fact unity, i.e. if

$$\frac{eA_0F}{\hbar c}=n\cdot 2\pi \tag{2·17}$$

where n is an integer. This is a condition on the flux through the ring: a simple calculation, using (2·11) for the vector potential in the volume occupied by the ring,* gives for the flux

$$\Phi=\int \boldsymbol{B}\cdot\boldsymbol{n}\,dS=\int A_0\boldsymbol{v}_0\cdot d\boldsymbol{s}=A_0F.$$

Thus the condition (2·17) becomes

$$\Phi=n\frac{2\pi\hbar c}{e}=n\Phi_1. \tag{2·18}$$

This is the "flux quantization" condition of London. However, this is not really a quantization condition, in the sense that no solutions of the Schroedinger equation exist for other values of the flux. Rather, (2·18) is the condition that the solutions for flux Φ can be derived from the solutions for zero flux by means of the pseudo-gauge transformation (2·15). Since an S-transformation preserves the eigenvalue spectrum of an operator, we conclude that, for these flux values, the energy spectrum is the same as for zero flux. With other flux values, solutions of the Schroedinger equation most definitely exist, but the energy spectrum is different from the one for zero flux, and hence no mere similarity transformation suffices to generate these eigenfunctions, starting from

* This form for the vector potential applies only inside the material of the ring. The vector potential in the rest of the space, in particular in the hole, has non-zero curl. It is defined uniquely, however, by the condition that it joins smoothly at the surface of the ring.

the zero-flux eigenfunctions. London's argument for flux quantization depends upon considering (2·15) as an ordinary gauge transformation, and is therefore invalid. These matters have been discussed extensively in recent times, in connection with the work of Bohm et al.[12] Since the full energy spectrum of the system, for all flux values, will be essential for our discussion of persistent currents, we may consider the persistent currents in superconducting rings as a particularly striking example of Bohm-Aharanoff effects.

We conclude this section by obtaining explicit solutions of the Schroedinger equation, for all flux values, for a ring of precisely cylindrical shape. The ring has inner radius a, outer radius b, and height L. The vector field $\boldsymbol{v}_0(x)$ is then of the form

$$\boldsymbol{v}_0(x)=\frac{K}{r}\boldsymbol{e}_\varphi \tag{2·19}$$

where $\boldsymbol{e}_\varphi$ is a unit vector along the direction of increasing φ (we use cylindrical coordinates r, φ, z), and the normalization constant K is

$$K=[2\pi L \ln(b/a)]^{-1/2}. \tag{2·20}$$

It is convenient to introduce the ratio of the actual flux to the London quantum Φ_1, i.e.

$$\mu=\Phi/\Phi_1=e\Phi/2\pi\hbar c=eKA_0/\hbar c. \tag{2·21}$$

With this notation, the Hamiltonian (2·9) becomes

$$H_0=-\frac{\hbar^2}{2M}\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}-\left(-i\frac{\partial}{\partial\varphi}-\mu\right)^2\right]. \tag{2·22}$$

The single-valued eigenfunctions of this operator are of the form

$$\psi(r, \varphi, z)=R(r)\,Z(z)\exp(im\varphi) \tag{2·23}$$

where m is an integer; $Z(z)$ is a sine function:

$$Z(z)=\sin\left(\frac{n_1\pi z}{L}\right) \tag{2·24}$$

with n_1 integral; and $R(r)$ satisfies the Bessel equation

$$R''+r^{-1}R'+\left[k^2-\frac{(m-\mu)^2}{r^2}\right]R=0. \tag{2·25}$$

The eigenvalues k are determined by the condition that $R(a)=R(b)=0$, and in terms of these eigenvalues, the energy is given by

$$E=\frac{\hbar^2}{2M}[k^2+(n_1\pi/L)^2]. \tag{2·26}$$

We have written the solutions in such detail primarily to show that there is no difficulty about their existence, i.e. there is no need for quantization of flux.

If the ring is "thin", i.e. $b-a \ll a$, one can get an approximation to the value of k by neglecting the variation of r across the ring. This procedure gives

$$k^2 \simeq \left(\frac{n_2\pi}{b-a}\right)^2 + \frac{(m-\mu)^2}{ab} - \frac{1}{4ab} \qquad (2\cdot27)$$

where n_2 is integral. The constant term at the end is of no interest; dropping it, the eigenvalue spectrum becomes

$$E \simeq \frac{\hbar^2}{2M}\left[\frac{(m-\mu)^2}{ab} + \left(\frac{n_1\pi}{L}\right)^2 + \left(\frac{n_2\pi}{b-a}\right)^2\right]. \qquad (2\cdot28)$$

The ground state for a given value of μ (of the flux) has the following quantum numbers: $n_1=n_2=1$, and m is the integer closest to μ. If both the width of the ring, $b-a$, and the height L of the ring, are much smaller than its mean radius $\rho=(ba)^{1/2}$, then the low-lying excited states all have $n_1=n_2=1$ and differ from each other only through their m-values. Under those circumstances, which we shall assume to hold henceforth,* the energy spectrum is shown schematically in Fig. 1. We see how the spectrum is reproduced for integral values of μ, as deduced earlier from the pseudo-gauge transformation. We also note that there are many crossings of energy levels. These crossings occur for integral and half-integral values of μ. The possibility of levels crossing each other is connected with the invariance of the ring under rotations about its axis. This invariance property is destroyed by impurity scattering centres, and the resulting energy spectrum will be studied in the next section. We expect, and shall find, that levels do *not* cross

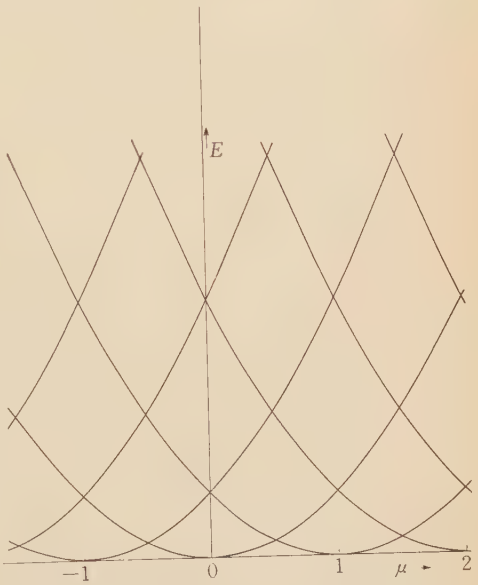

Fig. 1. Energy levels of a single particle in a perfectly cylindrical ring without impurity scattering centres; the abscissa is the flux through the hole in the ring, expressed in the "flux units" of London. Note the periodicity of the diagram, associated with the "pseudo-gauge transformation". Note also the many crossings of energy levels; these crossings are associated with the rotational symmetry of the ring.

* We do this for the sake of simplicity. As we shall see later on, this condition is not required to understand supercurrents in ordinary rings; for supercurrents in Garwin rings, the condition is usually satisfied. We do not expect any qualitative change if the condition is violated.

each other if there are impurity scattering centres.

Before closing this section, let us write down the current per particle. The expectation value of the quantity j_0 in (2·7) is

$$j_0 = \frac{e}{M}\int \psi^* \boldsymbol{v}_0(x)\cdot \boldsymbol{p}\psi\, d^3x - \frac{e^2}{Mc}A_0\int \psi^*(\boldsymbol{v}_0)^2\psi\, d^3x. \qquad (2\cdot29)$$

A simple calculation shows that the two terms combine to give

$$j_0 = (e\hbar K/M)(m-\mu)\int |\psi|^2/r^2\, d^3x. \qquad (2\cdot30)$$

For our thin ring, the integral can be replaced approximately by $1/\rho^2 = 1/ab$. Using this approximation, and replacing μ by its value (2·21), the quantity j_0 becomes

$$j_0 = -(e^2K/2\pi Mc\rho^2)(\Phi - m\Phi_1). \qquad (2\cdot31)$$

The current per particle in the ring, i, is given by

$$i = j_0/2\pi K = -\frac{e^2}{(2\pi\rho)^2 Mc}(\Phi - m\Phi_1). \qquad (2\cdot32)$$

We note that an increase in the flux Φ through the ring always produces a decrease in the current i, no matter what the sign of the charge e on the particles. In this sense the current (2·32) is diamagnetic.

However, this diamagnetism is rather peculiar. In normal diamagnets, the effects of the Ampère "ring currents" is to oppose the applied field; the magnetic field inside the material is closer to zero than it would be in the absence of the material. Here, there is no magnetic field *in the material* to start with, i.e. the magnetic effect is due to flux through the hole in the ring, not to any magnetic field in the volume of the ring. Furthermore, the tendency of the current i as given by (2·32) is *not* to decrease this flux to zero, but rather to move it closer to an integral number of flux quantum units.

If this is a Bose-Einstein system in its ground state, there are N particles contributing to the current; furthermore, the difference $\Phi - m\Phi_1$ is at most $(1/2)\Phi_1$ in magnitude. Let us estimate the order of magnitude of the current under these circumstances. Using two electronic charges for e and a radius ρ of the order of 1 cm, we get the estimate

$$I = Ni \sim 10^{-20} N \text{ amperes}. \qquad (2\cdot33)$$

If the ring is an ordinary, macroscopic ring, of total volume of order 1 cm^3, this is a current of some 100 amperes. Thus even a slight departure of the flux Φ from an integral number of flux units results in truly macroscopic currents, and there is no need to worry about crossing of levels — the currents become too large for superconductivity before we ever get to that stage.

On the other hand, the estimate is different for the rings used by Garwin.[3]

The volume of those rings is of order $10^{-6}\,\mathrm{cm}^3$, the radius ρ is of order $10^{-1}\,\mathrm{cm}$, and the current (2·33) is then of order 10^{-2} amperes. If the current is itself to act as the source of the flux (as it does once the ring is pulled out from between the poles of the magnet), currents of this order of magnitude are insufficient. This indicates the fundamental importance of Garwin's experiments for the theory of persistent ring currents.

§ 3. Quantum states in a ring with impurities

Although the results of the preceding section are simple and exact, they are not sufficient for a discussion of persistent ring currents. In the absence of any scattering, and with a ring of cylindrically symmetrical shape, there is no mechanism by which particles can be slowed down. Thus, a current, once started, persists indefinitely in any case. *In order to be convincing, a theory of persistent ring currents must start from a model which yields a non-zero resistivity under "normal" conditions.*

We therefore add to our model a set of S scattering centres, located at positions $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_S$. These positions are assumed to be random, and we shall, later on, estimate certain matrix elements by averaging over these random locations. The Hamiltonian H is now

$$H = H_0 + \sum_{i=1}^{S} v(\boldsymbol{x} - \boldsymbol{x}_i) \tag{3·1}$$

where H_0 is given by (2·9). Since each particle interacts with all the scattering centres, but not with any other particle of the gas, this is still an ideal Bose-Einstein gas, with the usual Bose-Einstein distribution of occupation numbers. The only effect of the scattering centres is to alter the energy levels and corresponding wave functions.

The model defined by (3·1) is precisely the one discussed in great detail in the fundamental paper of Kohn and Luttinger.[1] In this paper, they show that the model gives rise to an ordinary, finite resistivity if the particles obey Boltzmann statistics. Furthermore, they show, in their appendix F, that the introduction of quantum statistics (either Fermi-Dirac or Bose-Einstein) makes no qualitative difference.

In their calculation, see their formulas (40) and (41), Kohn and Luttinger find that averaging over the random locations of scattering centres is *not* enough to give a meaningful ensemble average; that is, the fluctuations of typical matrix elements about their "average" values are comparable to the average values themselves. In order to get quantities with small fluctuations, they are forced to average also over groups of neighbouring quantum states, their formula (43). This procedure is completely all right for the normal state of a Bose-Einstein system. It fails, however, in the condensed state: For then one quantum state, the single-particle ground state, has a macroscopic occupation number, and can-

not be averaged in with the other quantum states. Thus we may expect, and will indeed find, that the ordinary resistivity found by Kohn and Luttinger does not apply to the temperature region in which the Bose gas is condensed.

In order to discuss the energy levels and wave functions of the Hamiltonian (3·1), we start as follows. Consider (3·1) for the special case of zero flux, $\mu=0$, and let E_0 and Ψ_0 be the energy and wave function, respectively, of the ground state of this zero-flux Hamiltonian:

$$[H_0(0)+\sum_{i=1}^{S} v(\boldsymbol{x}-\boldsymbol{x}_i)]\Psi_0=E_0\Psi_0. \tag{3·2}$$

Although the function Ψ_0 would be quite hard to find explicitly, we need only certain simple qualitative properties of it. Ψ_0 is a *real* function, because the Hamiltonian allows time-reversal in the absence of flux. Ψ_0 has no nodes, since it is the ground state wave function. Qualitatively, Ψ_0 is a constant of the order of magnitude $V^{-1/2}$, except in the neighbourhood of scattering centres. We shall assume that the scattering centres are repulsive (to avoid the possibilities of bound states about particular scattering centres or groups of them); Ψ_0 then has "dips" in the neighbourhood of the scattering centres. If the scattering centres are spaced well apart, the "dips" are all similar to each other. These simple properties of Ψ_0 will suffice for our needs.

We now construct a set of wave functions Ψ_m as follows:

$$\Psi_m=\exp(im\varphi)\Psi_0. \tag{3·3}$$

This set is neither complete nor orthogonal. In order to make it complete, we would have to premultiply Ψ_0 by a complete set of functions in the ring, including functions with oscillatory behaviour in the radial and z-directions. However, the low-lying states of the Hamiltonian (3·1) are predominantly superpositions of states (3·3), since functions with oscillatory behaviour in the radial and z-directions have appreciably higher kinetic energies, and are therefore not admixed to any significant extent. Furthermore, such oscillatory behaviour is associated with particle motion in the corresponding directions, and is therefore of no interest for currents circulating around the ring.

The functions (3·3) are normalized to unity, since Ψ_0 is so; they are furthermore nearly orthogonal. Let us estimate the overlap integral

$$\beta_{m-m'}=\int \Psi_m^*\Psi_{m'}d^3x=\int \exp[-i(m-m')\varphi]\Psi_0^2 d^3x. \tag{3·4}$$

According to the preceding discussion, the function Ψ_0^2 has the qualitative form

$$\Psi_0^2\cong C[1-\sum_{i=1}^{S} g(\boldsymbol{x}-\boldsymbol{x}_i)] \tag{3·5}$$

where C is a constant of order of magnitude V^{-1}, and the function $g(x-x_i)$ describes the "dip" in the neighbourhood of the ith scattering centre. If we

substitute (3·5) in (3·4), the constant term integrates to zero. Defining the function $G(r_i, z_i)$ by

$$G(r_i, z_i) = \int g(\boldsymbol{x}-\boldsymbol{x}_i) \exp[-i(m-m')(\varphi-\varphi_i)] d^3x, \tag{3·6}$$

the sum in (3·5) gives rise to the estimate

$$\beta_{m-m'} \cong -C \sum_{i=1}^{S} \exp[-i(m-m')\varphi_i] G(r_i, z_i). \tag{3·7}$$

To estimate the order of magnitude of β, we now average over random locations $\boldsymbol{x}_i$ of the scattering centres. This average is clearly zero for $m \neq m'$. We therefore consider, not β itself, but its absolute square. A straightforward calculation, using the assumption of statistical independence of the locations of different scattering centres, yields

$$\langle |\beta_{m-m'}|^2 \rangle_{Ave} \cong C^2 S \langle G^2(r_i, z_i) \rangle_{Ave}. \tag{3·8}$$

Now C^2 is of order of magnitude V^{-2}, and the integral G is essentially the volume "cut out" by the scattering centre. Since a zero-energy wave function approaches its asymptotic value only very slowly, like $1-a/r$ where a is the scattering length, the average volume cut out by each scattering centre is comparable to the volume per scattering centre, V/S. We therefore get

$$\langle |\beta_{m-m'}|^2 \rangle_{Ave} \sim V^{-2} S (V/S)^2 = S^{-1}. \tag{3·9}$$

The root-mean square value of the overlap integral is therefore of order $S^{-1/2}$. If we assume an average distance of the order of 10^{-6} cm between scattering centres, just as an example, and take an ordinary ring with volume V of order 1 cm^3, this gives $S \sim 10^{18}$ and overlap integrals of the order of 10^{-9}. Small as these are, we shall see that they cannot be neglected altogether. For a "Garwin ring" of volume 10^{-6} cm^3, we get $S \sim 10^{12}$ and overlap integrals β of order 10^{-6}.

The choice of origin of the angle coordinate φ is arbitrary, of course. Inspection of the defining formula (3·4) shows that a redefinition of φ has the effect of multiplying $\beta_{m-m'}$ by a phase factor $\exp[-i(m-m')\gamma]$ where γ is the angle between the original and the modified origins of φ. By a suitable choice of the origin of φ we can therefore ensure that, for a particular value of $m-m'$, the integral $\beta_{m-m'}$ is *real*. This will prove useful later.

Next, we need the matrix elements of the Hamiltonian (3·1) for the functions Ψ_m. The expression for $H_{mm'}$ is easily written down. It can then be simplified greatly by making use of (3·2), and some integrations by parts. The result is

$$H_{mm'} = \int \Psi_m^* H \Psi_{m'} d^3x = E_0 \delta_{mm'} + (\hbar^2/2M)(m-\mu)(m'-\mu) J_{m-m'} \tag{3·10}$$

where

$$J_{m-m'}=\int r^{-2}\exp[-i(m-m')\varphi]\Psi_0^2\,d^3x. \tag{3·11}$$

Except for the factor r^{-2}, this is the same integral as ${}_i\beta$, (3·4), and the estimates for it are completely similar. For diagonal elements, $m=m'$, $J=J_0\sim 1/\rho^2$ where $\rho=(ab)^{1/2}$ is the mean radius of the ring. For off-diagonal elements, the absolute value of J is smaller by a factor of the order of $S^{-1/2}$. The straightforward averages of J and ${}_i\beta$, over random locations of scattering centres, both vanish. It is *not* permissible, however, to assume that $J_{m-m'}\simeq {}_i\beta_{m-m'}/\rho^2$. On the contrary, the arguments of Kohn and Luttinger show that these quantities are sensitive to details of the distribution of scattering centres, particularly as regards their complex phases. It is fortunate, therefore, that we shall not need to make any special assumption about either one of $J_{m-m'}$ or ${}_i\beta_{m-m'}$. All we shall require is the general order of magnitude of these two quantities.

In most cases, a single function Ψ_m is quite sufficient as an approximation to a true eigenfunction of H. The energy is then approximately equal to the diagonal element H_{mm}, i.e. to

$$H_{mm}=E_0+(\hbar^2/2M_i\rho^2)(m-\mu)^2. \tag{3·12}$$

This is just the result for the ring without scattering centres, modified by a constant zero-point energy due to the scattering centres.

This approximation fails, however, when two levels m_1, m_2 are degenerate or nearly degenerate. Degeneracy between m_1 and m_2 occurs at a flux value given by

$$\mu=\mu_0\equiv\frac{1}{2}(m_1+m_2). \tag{3·13}$$

In the neighbourhood of this flux value, the eigenfunctions of H are close to neither one of Ψ_{m_1} or Ψ_{m_2}, but take the form

$$\Psi\simeq a_1\Psi_{m_1}+a_2\Psi_{m_2} \tag{3·14}$$

where a_1 and a_2 have comparable size. Other functions Ψ_m are not admixed appreciably. The equation determining the a_i and the energy E is $Ha=E\mathcal{N}a$, where both H and $\mathcal{N}$ are two-by-two matrices. We use the notation

$$\alpha=(2M_i\rho^2/\hbar^2)J_{m_1-m_2} \tag{3·15}$$

and we drop the subscript on the overlap integral ${}_i\beta$. The equation then becomes

$$E_0\begin{pmatrix}a_1\\a_2\end{pmatrix}+(\hbar^2/2M_i\rho^2)\begin{pmatrix}(m_1-\mu)^2 & \alpha(m_1-\mu)(m_2-\mu)\\ \alpha^*(m_1-\mu)(m_2-\mu) & (m_2-\mu)^2\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}$$
$$=E\begin{pmatrix}1 & \beta\\ \beta & 1\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}. \tag{3·16}$$

Here we have made use of the freedom in the choice of origin of the φ-coordinate so as to make the overlap integral β real.

The discussion of this equation is straightforward but slightly tedious. It is done most conveniently by noting that the matrix equation

$$Ha=E\mathcal{N}a \tag{3·17}$$

can be rewritten as

$$\mathcal{N}^{-1/2}H\mathcal{N}^{-1/2}b=Eb \quad \text{with } b=\mathcal{N}^{1/2}a. \tag{3·18}$$

We shall merely quote results. We introduce the following quantities:

$$x=\frac{1}{2}[(1+\beta)^{-1/2}+(1-\beta)^{-1/2}]\cong 1 \tag{3·19a}$$

$$y=\frac{1}{2}[(1+\beta)^{-1/2}-(1-\beta)^{-1/2}]\cong -\frac{1}{2}\beta \tag{3·19b}$$

$$\sin\tilde{\xi}=y(x^2+y^2)^{-1/2} \qquad \cos\tilde{\xi}=x(x^2+y^2)^{-1/2} \qquad \tilde{\xi}\cong -\frac{1}{2}\beta \tag{3·19c}$$

$$X=\frac{1}{2}(m_1-\mu)^2+\frac{1}{2}(m_2-\mu)^2+\frac{1}{2}(m_1-\mu)(m_2-\mu) \times(\alpha+\alpha^*)\sin(2\tilde{\xi}) \tag{3·19d}$$

$$Y=\frac{1}{2}[(m_1-\mu)^2-(m_2-\mu)^2]\cos(2\tilde{\xi}) \tag{3·19e}$$

$$Z=\frac{1}{2}[(m_1-\mu)^2+(m_2-\mu)^2]\sin(2\tilde{\xi})+(m_1-\mu)(m_2-\mu) \times(\alpha\cos^2\tilde{\xi}+\alpha^*\sin^2\tilde{\xi}). \tag{3·19f}$$

With this notation, the eigenvalues E of (3·16) are given by

$$E=(1-\beta^2)^{-1}\{E_0+(\hbar^2/2M\rho^2)[X\pm(Y^2+Z^*Z)^{1/2}]\}. \tag{3·20}$$

We now discuss this result. The region of applicability of (3·20) is for μ near μ_0, Eq. (3·13). In this region, we have

$$m_1-\mu\cong-(m_2-\mu). \tag{3·21}$$

Thus Y passes through zero, whereas X and Z remain substantially constant over the small range of μ-values (flux-values) in question here. It then follows directly from (3·20) that *the two levels no longer cross each other*. Rather, the minimum energy difference between the upper and lower level is given by

$$(E_+-E_-)_{min}=(1-\beta^2)^{-1}(\hbar^2/M\rho^2)(Z^*Z)^{1/2} \cong(\hbar^2/4M\rho^2)(m_1-m_2)^2|\alpha+\beta| \tag{3·22}$$

where we have neglected second-order quantities in α and β for the estimate. The extent of the transition region is determined by the condition $Y^2=Z^*Z$; estimating the same way, this means

$$|\mu-\mu_0| \sim \frac{1}{4}(m_1-m_2)|\alpha+\beta|\,. \tag{3·23}$$

Outside this transition region, the two energy values obtained from (3·20) are nearly equal to the simpler estimates (3·12) for $m=m_1$ and $m=m_2$, respectively.

According to our earlier order-of-magnitude estimates for the integrals β and J, this means that the transition region is small compared with 1/2 (the spacing between successive degeneracy values of the flux parameter μ), provided that m_1-m_2 is less than 10^8 for the ordinary ring, less than 10^5 for the Garwin ring. This is adequate for our needs, as we shall see.

A schematic picture of the energy levels in the presence of scattering centres is given in Fig. 2 (the size of the transition regions is vastly exaggerated in this Figure). There are no longer any level crossings.

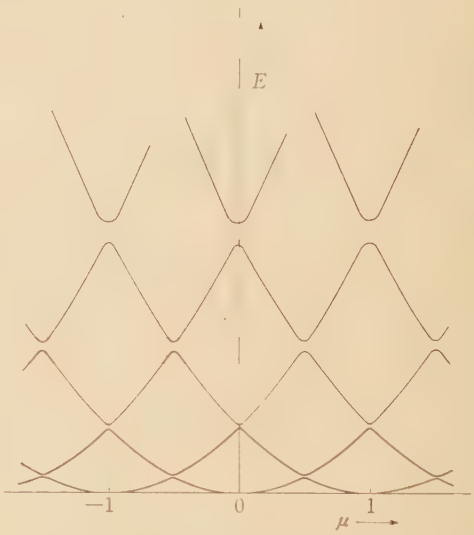

Fig. 2. Energy levels of a single particle in a cylindrical ring, in the presence of impurity scattering centres. The periodicity of the levels as a function of the flux through the hole (expressed in flux units) is preserved. But levels do not cross any longer, since the scattering centres have destroyed the rotational invariance of the problem. The size of the "transition regions" is exaggerated enormously on this Figure.

We shall be interested in the current I as a function of the flux Φ, or of the flux parameter $\mu=\Phi/\Phi_1$, when the system is in its ground state. *In the absence of impurities*, this relation is given by (2·32), for any one state m. However, the state m is the ground state only for a very limited range of flux values, namely $\Phi-m\Phi_1$ less than $(1/2)\Phi_1$ in magnitude. Thereafter, the energy of state m crosses one of the other energies (of states $m-1$ or $m+1$) and state m becomes an excited state. *In the presence of impurities*, there are no crossings. Rather, whenever μ is close to half-integral, we are in a transition region, in which the wave function changes from being mostly Ψ_m to being mostly Ψ_{m-1} or Ψ_{m+1}, and in which the ground state energy passes through a maximum. A straightforward calculation gives the identity for j_0, (2·29):

$$j_0=-c\frac{\partial E}{\partial A_0}=-\frac{eK}{\hbar}\frac{\partial E}{\partial \mu} \tag{3·24}$$

where the constant K is defined by (2·20). The current per particle, i, is given by (2·32), hence

$$i = -\frac{e}{2\pi\hbar}\frac{\partial E}{\partial \mu}. \qquad (3\cdot25)$$

For half-integral values of μ, in the middle of the transition region, the ground state energy has a maximum, and thus the current i is zero. More generally, the current is, except for a factor, just the derivative of the ground state energy with respect to the flux parameter μ, and therefore has the "sawtooth" behaviour shown in Fig. 3. *In the presence of impurities, the ground state current is no longer "diamagnetic" in character.* Rather, the current associated with the ground state wave function is limited from above and below, and shows narrow "paramagnetic" regions between adjacent, wide "diamagnetic" regions.

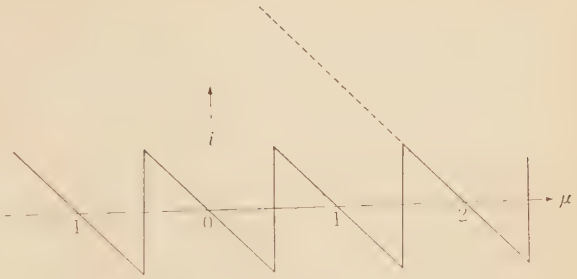

Fig. 3. The ring current due to a single particle in the ground state, as a function of the flux (expressed in flux units). This current is limited above and below. Big ring currents in "Garwin" rings can be built up only if the system does *not* stay in the ground state, i.e., if the "transition regions" are traversed "suddenly" rather than "adiabatically".

§ 4. Persistent currents in the limit of zero temperature

Let us start by showing the relation between the results obtained so far and the second London equation (1·2). We make a Fourier decomposition of the vector potential $\boldsymbol{A}(x, t)$ and of the current density $\boldsymbol{j}(x, t)$, according to

$$\boldsymbol{A}(x, t) = \sum_k A_k(t)\,\boldsymbol{v}_k(x) \qquad (4\cdot1)$$

and

$$\boldsymbol{j}(x, t) = \sum_k j_k(t)\,\boldsymbol{v}_k(x) \qquad (4\cdot2)$$

where $\boldsymbol{v}_k(x)$ is the complete set of vector eigenfunctions defined in reference 11). The function $\boldsymbol{v}_0(x)$ defined by (2·3) is the leading member of the set $\boldsymbol{v}_k(x)$. We write the second London equation in the form

$$\Lambda\frac{\partial j_s}{\partial t} = \boldsymbol{E} = -\operatorname{grad}\varphi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}. \qquad (4\cdot3)$$

We then take the scalar product of both sides with $\boldsymbol{v}_0(x)$, and integrate over the volume of the ring. The term—grad φ gives zero after integration (see the discussion after Eq. (2·6)), and we obtain

$$\Lambda\frac{dj_0}{dt} = -\frac{1}{c}\frac{dA_0}{dt}. \qquad (4\cdot4)$$

Although (4·4) is only one Fourier component of (1·2), *this Fourier component alone is of relevance for persistent ring currents.* As is shown in reference

11), all other terms of the expansion (4·2) give zero when integrated over a cross section of the ring, i.e. they do not lead to any charge transport around the ring. Since the London equations are *linear*, we can consider each term of the Fourier decomposition by itself, and we see that only this one term need to interest us here.

If there are no scattering centres in the ring, the current component j_0 is given by (2·31). If we differentiate both sides of this expression with respect to time, we obtain

$$\frac{dj_0}{dt} = -\frac{e^2}{Mc}\frac{K^2}{\rho^2}\frac{dA_0}{dt} \simeq -\frac{e^2}{Mc}\frac{1}{V}\frac{dA_0}{dt} \tag{4·5}$$

where we have made use of the relation $b-a \ll a$ in order to replace ρ^2/K^2 by the volume V of the ring. This is the current per particle; if the Bose gas is condensed, there are N_0 particles in this same quantum state, and Eq. (4·5) becomes identical with (4·4), with London's Λ taking the form

$$\frac{1}{\Lambda} = \frac{N_0}{V}\frac{e^2}{M}. \tag{4·6}$$

Except for notation, this is precisely London's own estimate (Eq. (18), page 60, of reference 1)).

Although this derivation of the London equation is highly suggestive, it is insufficient for a number of reasons.

(1) Our Eq. (2·31) relates j_0 to the flux. We must determine under which conditions we are allowed to differentiate both sides with respect to time —after all, (2·31) is derived on the assumption of a given, time-independent flux through the ring!

(2) Even if the time differentiation is permitted, some very relevant information is lost by this process. (2·31) is much more specific in its predictions than (4·5); in particular, the flux "quantum" Φ_1 appears in (2·31), but not in (4·5).

(3) Eq. (2·31) was derived on the basis of a ring of perfect cylindrical symmetry, containing no scattering centres. Under those conditions, there is no mechanism for stopping a ring current, and we obtain persistent currents from any model at all! As we have seen in § 3, Eq. (2·31) is modified significantly if there are impurity scattering centres in the ring, and this requires discussion.

We can give a hasty (and, as it will turn out, wrong!) answer to query number 1, as follows: If the change in flux is carried out very slowly compared to all relevant quantum frequencies $(E_i - E_j)/\hbar$, then the wave function changes "adiabatically"; in particular, if the system condenses initially into its ground state, changes in the applied flux will result in making the system go to the ground state appropriate to the altered flux, i.e. Eq. (2·31) will remain valid. We can then differentiate both sides with respect to time, and

(4·5) results. This argument is suspect because, in the absence of scattering centres, there are degeneracies between the quantum energies at integral and half-integral values of the flux parameter $\mu=\Phi/\Phi_1$. At a degeneracy point, the quantum transition frequency vanishes, and no physical process can be carried out more slowly than that! Also, as London[1] has emphasised, what we really want is not that the system should stay, adiabatically, in its quantum ground state, but rather that the system should retain the same wave function; in London's terminology, the wave function is supposed to be "stiff".

Since the degeneracy points are clearly going to be important, and since these degeneracies are destroyed by the scattering centres, a much fuller discussion is required, and this will now be done.

Rather than deducing further phenomenological equations, let us simply follow along the experimental procedure used for establishing ring currents in superconductors, and let us see what our model says about the happenings at each stage of that procedure. The procedure consists of three parts:

(1) The ring is placed, at room temperature, between the poles of a magnet, and we wait until thermal equilibrium is achieved.

(2) The ring is then cooled to a very low temperature, well below the superconducting transition temperature. In this section, we shall simplify matters by assuming the ring is cooled all the way down to absolute zero; extension to low but finite temperatures will be carried out in § 5.

(3) The external magnetic field is then removed, for example, by pulling the magnet away from the ring. It is observed that the flux threading the ring remains unaltered, the flux now being maintained by a persistent ring current.

Above the transition temperature, the Bose gas has a normal, non-zero resistivity.[10] Thus any ring current induced initially dies down quickly, and we may start our consideration from a state in which the gas carries no current. As we cool down below the Bose-Einstein condensation temperature, a temperature-dependent, macroscopic number of particles $N_0=N_0(T)$ settles into the single-particle ground state of the system. Since the system is between the poles of a magnet, the ground state in question is the lowest state in Fig. 2, at the particular value of $\mu=\Phi/\Phi_1$ appropriate to the external flux $\Phi=\Phi_e$ set up by the magnet. It should be noted that μ is a very large number in practice, since the flux unit Φ_1 is only 2×10^{-7} gauss-cm^2.

For simplicity of the discussion, let us assume that this initial value of μ is far enough removed from half-integral values, so that we are in a "diamagnetic" region. The current I associated with the condensed bosons is then given by $I=N_0(T)i$, with i determined by (2·32), that is,

$$I(T)=-\frac{e^2N_0(T)}{(2\pi\rho)^2Mc}(\Phi-m\Phi_1) \qquad (4\cdot7)$$

where m is the integer closest to $\mu_e=\Phi_e/\Phi_1$. This current, in turn, sets up flux lines through the hole in the ring.* Let $\mathcal{L}$ be the self-inductance of the ring, then the flux is given by

$$\Phi=c\mathcal{L}I(T)+\Phi_e. \tag{4·8}$$

We substitute (4·7) into (4·8). Introducing the notation

$$q=c\mathcal{L}\frac{e^2}{(2\pi\rho)^2Mc}=\frac{e^2}{(2\pi\rho)^2M}\mathcal{L}, \tag{4·9}$$

the resulting equation yields, for the flux Φ,

$$\Phi-m\Phi_1=\frac{\Phi_e-m\Phi_1}{1+qN_0(T)}. \tag{4·10}$$

Thus, as we cool down and $N_0(T)$ increases to the total number $N=N_0(0)$, the actual flux Φ through the ring approaches more closely an integral number of flux units.

Let us estimate orders of magnitude. The self-inductance of a torus-shaped ring with radius ρ, and thickness $2r$, is given by[13]

$$\mathcal{L}=(4\pi\rho/c^2)[\ln(8\rho/r)-7/4]. \tag{4·11}$$

Our ring is not quite of this shape, since it has rectangular rather than circular cross section, but the formula (4·11) should suffice for an order-of-magnitude estimate. If we put $\rho\sim1$ cm, and $\rho/r\sim3$, we get $\mathcal{L}\sim2\times10^{-20}$; using two electronic charges for "e" and two electronic masses for "M", the quantity q is estimated to be

$$q\sim3\times10^{-13}. \tag{4·12}$$

In an "ordinary" ring, of volume $V\sim1$ cm^3, the total number of particles is of order 10^{22}. Thus qN_0 is of order 10^9, and the flux Φ very rapidly becomes extremely close to a flux unit, no matter what the initial flux Φ_e set up by the magnet. In a "Garwin" ring, with $\rho\sim10^{-1}$ cm and $V\sim10^{-6}$ cm^3, q is of order

* Strictly speaking, we are departing from our independent-particle model at this stage, in the sense that the flux lines set up by each particle now influence the motion of all the other particles. We treat this interaction self-consistently, in the Hartree-Fock sense. That is, the flux set up by all the particles is inserted into the single-particle Hamiltonian as a c-number parameter. In an exact treatment, this term is really a sum of two-particle interaction terms, the interaction being the magnetic dipole-dipole interaction between two moving charges. Needless to say, the resulting problem is insoluble.

In a fully self-consistent treatment, the electrostatic interaction between these charged particles should also be included, to the same approximation. We have not done so primarily in order to simplify the discussion. This work is now under way, and will be reported later on. Preliminary estimates suggest the following: (1) The self-consistent treatment of the Coulomb interaction does give rise to important changes in some of our estimates, (2) these changes are such as to improve the theory, i.e. they make it easier to set up and maintain persistent ring currents, and (3) there is no change in the prediction of approximate flux quantization, upon cooling.

3×10^{-12}, and qN_0 is still of order 10^4, so the same conclusion applies.

This is a theoretical result capable of direct experimental verification: *As long as the ring remains stationary in the field, and is cooled down well below the transition point, the final flux through the ring should be very nearly quantized.* As far as we are aware, no other theory of superconductivity gives rise to this prediction, and it is to be hoped that the experiment will prove feasible from the practical point of view. It should be noted that details of the model, e.g. the density of scattering centres, do not enter into this effect.

Having cooled the ring down to, ideally, the absolute zero of temperature, the next step consists in removing the external magnet from the neighbourhood of the ring. This means that the quantity Φ_e, the external flux, is slowly decreased to zero. If Eq. (4·10) remains valid during this process, it gives, for the final flux,

$$\Phi=[1-(1+qN_0)^{-1}]m\Phi_1. \tag{4·13}$$

Let us determine to what extent $\mu=\Phi/\Phi_1$ deviates from m. (4·13) gives

$$\mu-m=-\frac{m}{1+qN_0}. \tag{4·14}$$

In an "ordinary" ring with a flux of order 1 gauss-cm^2 through its hole, we have $m\sim10^7$ and $qN_0\sim10^9$. Thus, even after the external magnet has been removed, the flux is still reasonably close to an integral number of flux units. In particular, the flux parameter μ does not differ from the integer m by as much as 1/2. We are nowhere near the "transition" regions in Fig. 2. The basic equation (2·32) for the current per particle is thus valid throughout the process, and our results are consistent. Let us estimate the quantum-mechanical transition frequencies $\omega_{ij}=(E_i-E_j)/\hbar$ for this regime. The lowest such frequency is obtained by letting E_i be the first excited state, E_j the ground state. According to (3·12), this energy difference, for μ near some integer m, is equal to $\hbar^2/2M_i\rho^2$, so that $\omega\sim\hbar/2M_i\rho^2\sim1$ sec^{-1}. This is neither particularly slow nor particularly fast, compared to macroscopic times. However, there is no need to invoke a special "adiabatic" behaviour at all! Over the region of μ-values in question, the ground state *wave function* is substantially unaltered, namely it is equal to Ψ_m, although the ground state *energy* changes proportionately to $(\mu-m)^2$. Thus, no matter whether the flux change is "slow" (so that the new wave function is the ground state wave function for the new flux) or "sudden" (so that the new wave function is equal to the old wave function), the result is the same!

The situation is very different with "Garwin" rings. For the same flux of order 1 Gauss-cm^2, we have $m\sim10^7$, as before, but $qN_0\sim10^4$. Thus the deviation $\mu-m$, Eq. (4·14), is of order 10^3, and we must most definitely consider what happens in the "transition" regions of Figs. 2 and 3.

The time development of the wave function of the system is given by

$$H(\mu)\Psi = i\hbar \frac{\partial \Psi}{\partial t} \tag{4·15a}$$

with

$$\mu = \mu(t). \tag{4·15b}$$

For the moment, we restrict ourselves to the idealized case of a system cooled all the way down to absolute zero, so that, at time $t=0$, the system is in the ground state wave function appropriate to the flux parameter $\mu(0)$. Since the system is an independent particle system, with Bose statistics, the ground state wave function of the N-particle system is just the product of single-particle ground state wave functions, and it follows from (4·15a) that, at all later times as well, the N-particle wave function is a product of N identical factors, one for each particle. Thus, it is sufficient to calculate with a one-particle system, and multiply by N afterwards. The finite-temperature case, where this procedure fails, will be considered in § 5.

The time-development of a wave function as a result of an externally varied parameter μ in the Hamiltonian is a well-known problem, e.g. see Schiff.[14] We introduce eigenfunctions of $H(\mu)$ for given μ by

$$H(\mu)\,\varphi_s(x;\,\mu) = E_s(\mu)\,\varphi_s(x;\,\mu) \tag{4·15}$$

and time-dependent phases $\chi_s(t)$ by

$$\chi_s(t) = (i/\hbar)\int_0^t E_s[\mu(t')]\,dt'. \tag{4·16}$$

The time-dependent single-particle wave function is written as a linear superposition:

$$\psi(x,\,t) = \sum_s c_s(t)\,\varphi_s[x;\,\mu(t)]\exp[-\chi_s(t)]. \tag{4·17}$$

The time-development of the coefficients $c_s(t)$ is then determined by the equation

$$\frac{dc_s}{dt} = -\frac{d\mu}{dt}\sum_{s'} U_{ss'}c_{s'} \tag{4·18}$$

where

$$U_{ss'} = \exp[\chi_s(t)-\chi_{s'}(t)]\int \varphi_s^*(x;\,\mu)\,\frac{\partial \varphi_{s'}(x;\,\mu)}{\partial \mu}\,d^3x \tag{4·19}$$

where the integral is to be evaluated for $\mu=\mu(t)$.

Let us follow the time-development of the wave function as the external flux Φ_e is decreased. We shall let the index s increase with the single-particle energy E_s, so that $s=0$ is the ground state, $s=1$ the first excited state, and so

on. At time $t=0$, the system is condensed in the ground state appropriate to the flux Φ given by (4·10), i.e. μ is very nearly equal to an integer m, and the wave function is

$$[\psi(x, t)]_{t=0}=\varphi_0(x; \mu(0)) \cong \Psi_m \tag{4·20}$$

where Ψ_m is defined by (3·3). As the external flux is now decreased, the actual flux through the ring also decreases, albeit much more slowly. Thus $d\mu/dt$ in (4·18) is negative. For some time, μ is still larger than the value appropriate to the first "transition" region ($\mu \cong m-1/2$). During that period, the ground state wave function $\varphi_0(x; \mu)$ is nearly independent of μ, and remains equal to Ψ_m. Thus all matrix elements $U_{ss'}$ with $s=0$ vanish,* and $c_0(t)=c_0(t=0)=1$.

We now enter the transition region, $\mu \cong m-1/2$. In this region the wave functions φ_0 and φ_1 are linear combinations of the functions Ψ_m and Ψ_{m-1}, with coefficients determined by the secular equation (3·16). Fortunately, there is no need to discuss these coefficients in detail; all we require is the end result of this change-over. For μ slightly *less* than $m-1/2$, on the other side of the transition region, the ground state wave function is essentially equal to Ψ_{m-1}, whereas the wave function φ_1 of the first excited state is essentially equal to Ψ_m.

The net effect of passing through the transition region depends on whether the rate of change $d\mu/dt$ is "adiabatic" or "sudden".[13] If the flux parameter μ changes very slowly, the system stays in the ground state, i.e. the wave function changes slowly from its initial form Ψ_m to its final form Ψ_{m-1}. The current produced by the gas then follows the ground state curve in Fig. 3, and we do *not* build up a large circulating current. On the other hand, if the flux parameter μ changes rapidly enough so that the change is "sudden", the wave function remains unchanged in form, i.e. equal to Ψ_m, and the result is that we have moved onto another branch of the energy level diagram; the system is in the first excited state $\varphi_1=\Psi_m$ rather than in the ground state $\varphi_0=\Psi_{m-1}$. Under those conditions, the relation between current and flux remains governed by (2·32), and we have made the first step towards building up a large circulating current.

Thus, if the flux change is "sudden", we can get past the transition regions and build up the required circulating currents. What, then, is the time scale against which we must compare the experimental times? The relevant frequency is the quantum mechanical transition frequency $(E_+-E_-)/\hbar$, where the

* The overlap integral $\int \varphi_s^*(x; \mu)\, \varphi_{s'}(x; \mu)\, d^3x=\delta_{ss'}$ for all values of μ. Differentiating this equation with respect to μ, we conclude that the $\partial/\partial\mu$ in (4·19) can be shifted to the factor φ_s^*, provided we also change the sign in front of the integral. Thus our statement in the text can be deduced from properties of the ground state wave function, for which our approximations are reasonable, and does not depend on properties of wave functions of highly excited states, for which our approximations become questionable.

energy difference is estimated by (3·22) with $m_1-m_2=1$. Inserting $\rho\sim10^{-1}$ cm and $|\alpha+\beta|\sim S^{-1/2}\sim10^{-6}$, we get a time scale of the order of 25 years! Thus, the transition regions are only an apparent, not a real, difficulty.

So far, we have estimated for the *first* transition region encountered by the system, i.e. $m_1-m_2=1$. As the value of the flux parameter decreases, so does $m_1-m_2=2(m-\mu)$; this difference enters into the energy difference estimate (3·22), and hence into the time scale. However, the discussion of the behaviour of our model becomes extremely complex if the flux change is not obviously "sudden". The equations governing the time development of the system are non-linear, and more work is required. For the moment, all we can say is that at least the first few transition regions present no difficulty at ordinary time scales. Even with $m_1-m_2\sim10^3$ (see (4·14)), our estimated time scale is still of the order of weeks, considerably slower than the usual times for removal of the ring from the magnet.*

Finally, let us ask what is the lifetime of the circulating ring current? With an "ordinary" ring, of volume $V\sim1$ cm^3, the flux parameter μ remains close to the integral value m, see (4·14) and the discussion there, and consequently the wave function of the condensed particles remains equal to Ψ_m. Since this is a correct eigenfunction of the Hamiltonian for the final flux parameter μ in question, there is no reason for any subsequent change in the wave function. The same statement holds true for Garwin rings, provided that the transition regions are traversed in a "sudden" fashion. *Within our model, the circulating current in a ring has a strictly infinite lifetime.* So far, however, this result has been obtained from an idealized treatment in which the system is assumed to be at the absolute zero of temperature precisely. Since the lifetime of the ring current could, *a priori*, be temperature-dependent, we must remove this limitation on our proof, before we can discuss the result itself.

§ 5. Persistent currents at finite temperatures

At a finite temperature, the system is described by a density matrix. We "prepare" the system by placing the ring between the poles of a magnet, at room temperature, and then cooling the system to some final temperature, well below the Bose-Einstein condensation point. We shall *assume* that enough time has elapsed, and other necessary conditions have been fulfilled, so that full thermodynamic equilibrium has been achieved at this final temperature, at time $t=0$.

Let $H_N(0)$ be the Hamiltonian of the N-particle system at this time, and let $\Psi_\alpha(0)$ be the eigenfunctions of this Hamiltonian. Then the density matrix $U(0)$ at $t=0$ can be written as

* Furthermore, one possible (and, according to our preliminary estimates, likely) effect of including the Coulomb forces into the self-consistent treatment, is to increase these time scales appreciably.

$$U(0)=\sum_{\alpha}\pi_{\alpha}P[\Psi_{\alpha}(0)] \tag{5·1}$$

where $P(\Psi_{\alpha})$ is the projection operator onto the wave function Ψ_{α}, and the π_{α} are probabilities:

$$0\leq\pi_{\alpha}\leq 1 \quad \text{and} \quad \sum_{\alpha}\pi_{\alpha}=1. \tag{5·2}$$

The probabilities π_{α} are proportional to $\exp(-E_{\alpha}/kT)$.

By assumption, we are dealing with an independent particle system, so that the Hamiltonian has the form

$$H_N=\sum_{i=1}^{N}H(i) \tag{5·3}$$

where each single-particle Hamiltonian $H(i)$ may still be time-dependent, through a time-dependent flux, say. Let us introduce, at time $t=0$, the eigenfunctions $\psi_s(x, 0)$ of the single-particle Hamiltonian. The index α in (5·1) now stands for a "configuration", i.e. a set of occupation numbers $\{n_0, n_1, n_2, \cdots\}$ with

$$\sum_{s}n_s=N. \tag{5·4}$$

The wave function $\Psi_{\alpha}(0)$ is the usual symmetrized product function, formed from n_0 factors $\psi_0(x, 0)$, n_1 factors $\psi_1(x, 0)$, etc. We now observe that the time dependent Schroedinger equation for the N-particle system,

$$H_N\Psi_{\alpha}=i\hbar\frac{\partial\Psi_{\alpha}}{\partial t}, \tag{5·5}$$

can be solved exactly, by defining $\psi_s(x, t)$ to be the solution of the one-particle Schroedinger equation

$$H(i)\,\psi_s(x_i, t)=i\hbar\frac{\partial\psi_s(x_i, t)}{\partial t} \tag{5·6}$$

and then constructing $\Psi_{\alpha}(t)$ from these functions $\psi_s(x_i, t)$ by the *same* method, with the *same* occupation numbers, as was used to construct the original $\Psi_{\alpha}(0)$ from the $\psi_s(x_i, 0)$. Furthermore, with $\Psi_{\alpha}(t)$ defined in this way, the time-dependent density matrix is given by

$$U(t)=\sum_{\alpha}\pi_{\alpha}P[\Psi_{\alpha}(t)] \tag{5·7}$$

with *time-independent* probabilities π_{α}. The change of basis functions from $\Psi_{\alpha}(0)$ to $\Psi_{\alpha}(t)$ is a similarity transformation, according to (5·5); it is obviously the "natural" way to describe the system.

The operator for the current is a sum of single-particle operators. Therefore, by well-known methods, the expectation value of this operator at time t assumes the form, for any one wave function $\Psi_{\alpha}(t)$,

$$\int\Psi_{\alpha}^{*}\,j^{(N)}\Psi_{\alpha}\,d^{3N}x=\sum_{s}n_s(\alpha)\,j_s(t) \tag{5·8}$$

where $j_s(t)$ is the expectation value of the current in the single-particle state $\psi_s(x, t)$;

$$j_s(t) = \int \psi_s^*(x, t)\, j^{(1)} \psi_s(x, t)\, d^3x \tag{5·9}$$

and $n_s(\alpha)$ is the occupation number of single-particle state s in the configuration α. Note that there are no cross terms between states s and s' in Eq. (5·8).

If we now consider the ensemble as a whole, the current at time t is given by

$$j(t) = \sum_\alpha \pi_\alpha \sum_s n_s(\alpha)\, j_s(t) = \sum_s \bar{n}_s j_s(t) \tag{5·10}$$

where $\bar{n}_s$ is the average occupation number of state s:

$$\bar{n}_s = \sum_\alpha \pi_\alpha n_s(\alpha). \tag{5·11}$$

We now observe the following:

(1) By our method of construction, the average occupation numbers $\bar{n}_s$ are constants of motion.

(2) For temperatures below the transition temperature of the Bose gas, the ground state term $\bar{n}_0 j_0(t)$ must be considered separately from all the others; at temperatures well below the transition temperature (but still finite!) this one term *dominates* the sum (5·10).

(3) Our previous calculation is precisely a calculation of this one term.

This chain of reasoning establishes that the assumption of strictly absolute zero temperature, made in § 4, is not in fact essential. The result is unchanged if the temperature of the system is finite but well below the Bose-Einstein transition temperature.* *This conclusion applies in particular to the infinite lifetime of the supercurrent.*

§ 6. Discussion

We now summarise our results, and discuss their significance.

* *A priori*, a calculation based on the ground state wave function of an N-particle system is valid only at unreasonably low temperatures, namely NkT less than the first excitation energy of the system; i. e. the upper limit on the temperature T is of order $1/N$. Our argument proves that the actual limit of validity on the temperature T is much higher, and independent of N.

The argument depends on the three-dimensional nature of the Bose-gas; a two-dimensional Bose gas does not condense. There then arise the questions:

(1) Is a layer of matter of thickness 10^{-6} cm to be considered a two-dimensional or a three-dimensional Bose gas of electron pairs? (We note that the internal size of each pair is the Pippard coherence length, of order 10^{-4} cm).

(ii) If the correct analogy in that case is to a two-dimensional Bose gas, are persistent supercurrents set up in such a ring? The answer to the second question is *not* obvious; the calculation of R. M. May, Phys. Rev., **115** (1959), 254, has shown that the two-dimensional Bose gas gives rise to a field expulsion law which would be experimentally almost indistinguishable from a true Meissner effect!

First, we emphasize that our results are for a mathematical model, and therefore may or may not be applicable to actual superconductors. The Bose-Einstein model has been shown to have close analogies to actual superconductors, especially as regards the Meissner effect; but there are also significant differences, of which the most important is presumably the absence of an energy gap from our model. We shall take the optimistic point of view, that the energy gap will be as unimportant for persistent ring currents as it has been shown to be for the Meissner effect. But estimates of transition times, which entered the discussion in § 4, may well be changed fundamentally by the energy gap. Presumably, though, this change would be in the right direction, i.e. to lengthen our already quite large time scales.

Second, we observe that the flux parameters μ which enter our estimates are very large numbers, of order 10^7. Since the flux is proportional to the crucial component A_0 of the vector potential, *it is not sufficient to restrict oneself to terms linear in the vector potential.* This means that the Kubo formalism[15] is inapplicable to this problem. A modified form of the Kubo formalism, where the expansion is in the *change* in A_0 as the ring is removed from the magnet, is more realistic; but even that fails for Garwin rings, where the *change* in μ carries us across a number of "transition" regions. It should be noted, however, that the Kubo formalism does show, very directly, that the condensed Bosons contribute a significant current; this is in fact how the present investigation started.

Third, our theory is *not* identical with the London phenomenology. In particular, the flux quantum Φ_1 enters explicitly, and we predict a new experimentally observable effect, namely that the flux through the ring should be very nearly quantized as long as the ring is not moved, but merely cooled. If the Bose gas analogy is good for actual superconductors, this prediction should apply to them.

Fourth, we obtain a strictly infinite lifetime for the ring current, once it has been set up. Although this prediction is not contrary to any observation so far, we consider it not a success, but a failure of the theory. The state of the ring in which there is a circulating current which maintains a flux through the ring, is a state of higher free energy than the state with no current and no flux. Thus, it is not the thermodynamic equilibrium state, and a complete theory should allow us to estimate the time of relaxation to full thermodynamic equilibrium. If the theory gives an infinite time, this means merely that the correct relaxation mechanism has been omitted from the initial assumptions of the theory.

True as this is, the omission in question is by no means just a trivial error. The infinite lifetime for our supercurrent is connected directly with the present limitations on the foundations of statistical mechanics.[16] The model used by us is the *same* model which gives a finite conductivity above the Bose-Einstein

transition temperature.[10] But, whereas at high temperatures the "coarse-graining" associated with the measurement of the macroscopic current is sufficient to hide the fundamental difficulties of the Kohn-Luttinger theory, it is insufficient to hide them below the condensation point. The difficulty is the absence of any mechanism by which the "fine-grained entropy"

$$S = -k \operatorname{Tr}(U \ln U) \tag{6·1}$$

can increase with time. Relaxation to equilibrium means the attainment of a state of higher entropy, without a change in the energy of the system (we are considering adiabatic conditions). No such relaxation can in fact take place in this model. The assumption that the scattering centres are located at random is irrelevant here: Wherever the scattering centres may be in any one sample, the time-development of the wave functions of the system is completely causal. The probabilities π_α in § 5 are constant, and hence so is the entropy.

It is for this very reason that the circulating ring currents are so interesting theoretically. For here we have an example of a strictly non-ergodic system, when considered along the usual lines. It is therefore at least possible that the actual lifetimes will be determined by the effects neglected in the usual theory, and discussed in reference 15), that is, a stochastic interaction with the "rest of the world".*

The author is grateful to Drs. P.W. Anderson, L.M. Delves, and R. Garwin for interesting and useful discussions, and to Bell Telephone Laboratories for a most fruitful and stimulating visit, in the course of which many of these ideas were conceived.

Additional Note:

After completion of this manuscript, we received preprints from Drs. Deaver and Fairbank, reporting the experimental observation of the flux quantization upon cooling, in full agreement with our model; i.e. the flux quantum corresponds to *two* electronic changes. We also received a preprint from Drs. Byers and Yang containing theoretical considerations related to this experiment. There are similarities to the present work, but there are also significant differences. In particular, Byers and Yang consider only the thermodynamic equilibrium state, not time-dependent processes, and they do not have a detailed microscopic model such as we have proposed here.

* Another, and much less interesting, possibility is that the lifetimes are determined by the formation of three-electron and higher complexes in actual superconductors, complexes which are ignored by the quasi-chemical equilibrium theory of superconductivity which restricts itself to single electrons in chemical equilibrium with electron pairs.

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Letters to the Editor

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On a Fixation of Gauge and True Observables of Yang-Mills Field

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The present author has recently obtained the gravitational interaction potential between two point particles by separating the transverse-traceless parts of metric tensor and requiring the consistency of the coordinate condition[1] in the analogous way to the electromagnetic field. Since one may consider that Yang-Mills's $\boldsymbol{b}$-field[2] seems to have a character intermediate between the gravitational and electromagnetic fields, it will be interesting to investigate the problem of the fixation of the gauge and the true observables of the Yang-Mills field.

As the $\boldsymbol{b}$-field was originally introduced in order to preserve the invariance of the Lagrangian of nucleon field under the generalized isotopic gauge transformation, we shall start with the following Lagrangian density:

$$L=-(1/4)\boldsymbol{f}_{\mu\nu}\boldsymbol{f}_{\mu\nu} -\bar{\psi}\gamma_\mu(\partial_\mu-i\epsilon\boldsymbol{\tau}\cdot\boldsymbol{b}_\mu)\psi-m\bar{\psi}\psi \quad (1)$$

where $\boldsymbol{f}_{\mu\nu}=\partial_\nu\boldsymbol{b}_\mu-\partial_\mu\boldsymbol{b}_\nu-2\epsilon\boldsymbol{b}_\mu\times\boldsymbol{b}_\nu$ and $\boldsymbol{b}_\mu$ is transformed as

$$\boldsymbol{b}_\mu\to\boldsymbol{b}_\mu'=S(\epsilon,\boldsymbol{\Lambda})(\boldsymbol{b}_\mu+f(-\epsilon,\boldsymbol{\Lambda}))\partial_\mu\boldsymbol{\Lambda} \quad (2)$$

in which we have used Umezawa-Kamefuchi's $S(\epsilon,\boldsymbol{\Lambda})$ and $f(\epsilon,\boldsymbol{\Lambda})$.[3] The canonical momentum $\boldsymbol{\Pi}_\mu=\partial L/\partial(\partial_4\boldsymbol{b}_\mu)$ and the Hamiltonian are defined by the usual method. We have the following constraint equations:

$$\boldsymbol{\Pi}_4=0, \quad (3)$$

$$\partial_j\boldsymbol{\Pi}_j+2\epsilon(\boldsymbol{b}_j\times\boldsymbol{\Pi}_j)+\epsilon\boldsymbol{J}_4=0, \quad (4)$$

where $\boldsymbol{J}_\mu=i\bar{\psi}\boldsymbol{\tau}\gamma_\mu\psi$. They constitute a generator of infinitesimal transformation of (2). The other field equations are

$$\partial_4\boldsymbol{b}_k=-\boldsymbol{\Pi}_k+\partial_k\boldsymbol{b}_4+2\epsilon(\boldsymbol{b}_k\times\boldsymbol{b}_4), \quad (5)$$

$$\partial_4\boldsymbol{\Pi}_k=\partial_j\boldsymbol{f}_{kj}-2\epsilon\boldsymbol{f}_{kj}\times\boldsymbol{b}_j+\epsilon\boldsymbol{J}_k +2\epsilon\boldsymbol{\Pi}_k\times\boldsymbol{b}_4. \quad (6)$$

In contrast to the case of electrodynamics, where one can take the transverse potential (which is gauge invariant) as a true observable,[4] it is difficult to find such variables that are commutable with constraints (3) and (4). (Though we can obtain them with the aid of the approximation method given by Newman-Bergmann for the gravitational field,[5] infinitely many observables are necessary.) We shall then notice the following fact. The transformation (2) can be de-

composed into two steps:

$$\boldsymbol{b}_\mu \to \boldsymbol{b}_\mu + \partial_\mu \boldsymbol{\Lambda} \to G^{-1}(\boldsymbol{b}_\mu + \partial_\mu \boldsymbol{\Lambda})G \quad (7)$$

where G is a unitary transformation whose explicit expression is given by Umezawa-Kamefuchi.[3] Now we shall define the true variables as the variables which are left invariant under the first transformation of (7). Therefore, the problem of defining true observables $\boldsymbol{b}_i{}^T$ is the same as that in the electromagnetic case. We define it by $\boldsymbol{b}_i{}^T = \boldsymbol{b}_i - \partial_i \boldsymbol{\lambda}$, $\boldsymbol{\lambda} = (1/\Delta)\partial_j \boldsymbol{b}_j$.

Imposing the gauge condition (which corresponds to the Coulomb gauge in the case of electromagnetic field)

$$\Delta \boldsymbol{\lambda} = \partial_j \boldsymbol{b}_j \approx 0 \quad (8)$$

and requiring its consistency with Eq. (5), we shall determine $\boldsymbol{b}_4$ conjugate to $\boldsymbol{\Pi}_4 (=0)$. The consistency condition between Eqs. (5) and (8) is

$$-\Delta \boldsymbol{b}_4 \approx \epsilon \boldsymbol{J}_4 + 2\epsilon(\boldsymbol{b}_k{}^T \times \boldsymbol{\Pi}_k{}^T) + 4\epsilon \boldsymbol{b}_k{}^T \times \partial_k \boldsymbol{b}_4 + 4\epsilon^2 \boldsymbol{b}_k{}^T \times (\boldsymbol{b}_k{}^T \times \boldsymbol{b}_4) \quad (9)$$

where $\boldsymbol{\Pi}_k{}^T = -\partial_4 \boldsymbol{b}_k{}^T$, which satisfies the relations $[\Pi_i{}^{Ta}(x), b_j{}^{Tb}(x')] = \delta_{ab} \times [\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{x}') - (1/4\pi)\partial_i \partial_j' \{(\boldsymbol{x}-\boldsymbol{x}')^2\}^{-1/2}]$.

By taking into account Eqs. (4) and (9), the Hamiltonian can be written as

$$\boldsymbol{H} = \int d\boldsymbol{x} [-(1/2)\boldsymbol{\Pi}_k{}^T \boldsymbol{\Pi}_k{}^T + (1/4)\boldsymbol{f}_{ij}{}^T \boldsymbol{f}_{ij}{}^T - \epsilon \boldsymbol{J}_i \boldsymbol{b}_i{}^T - (1/2)\epsilon \boldsymbol{J}_4 \boldsymbol{b}_4 - \epsilon(\boldsymbol{\Pi}_k{}^T \times \boldsymbol{b}_k{}^T) \times \boldsymbol{b}_4 + H_N] \quad (10)$$

where $\boldsymbol{b}_4$ is a solution of Eq. (9) and H_N represents the Hamiltonian density for free nucleon field. Since we cannot directly solve the non-linear equation (9) for $\boldsymbol{b}_4$, we solve the simultaneous equations for $\boldsymbol{b}_4$, $\boldsymbol{b}_i{}^T$ and ψ by expanding $\boldsymbol{b}_4$, etc., in a power series of the parameter ϵ, $\boldsymbol{b}_4 = \Sigma \epsilon^j \boldsymbol{b}_4{}^{(j)}$, etc., and making use of a perturbation iteration method as done in the gravitational case.[1] The lowest order solution of $\boldsymbol{b}_4$ is

$$\boldsymbol{b}_4^{(1)} = (1/4\pi) \int d\boldsymbol{x}' \{(\boldsymbol{x}-\boldsymbol{x}')^2\}^{-1/2} \times \{\boldsymbol{J}_4^{(0)}(x') + 2(\boldsymbol{b}_k^{T(0)}(x') \times \boldsymbol{\Pi}_k^{T(0)}(x'))\},$$

and the static interaction potential of the lowest order is obtained from (10),

$$H' = -(\epsilon^2/8\pi) \iint d\boldsymbol{x}\, d\boldsymbol{x}' \{(\boldsymbol{x}-\boldsymbol{x}')^2\}^{-1/2} \{\boldsymbol{J}_4^{(0)}(x)\boldsymbol{J}_4^{(0)}(x') - 4(\boldsymbol{b}_k^{T(0)}(x) \times \boldsymbol{\Pi}_k^{T(0)}(x)) \times (\boldsymbol{b}_k^{T(0)}(x') \times \boldsymbol{\Pi}_k^{T(0)}(x'))\}.$$

In conclusion, we note that our transverse field $\boldsymbol{b}_i{}^T$ is not the true observable in its original sense, unlike the electromagnetic case, since $\boldsymbol{b}_i{}^T$ is not invariant under the generalized isotopic gauge transformation but suffer the second step transformation of (7). The similar feature occurs also in the usual formalism of electrodynamics where ψ (which suffers gauge transformation of the first kind) is not commutable with a constraint div $\boldsymbol{E} + ie\bar{\psi}\gamma_4\psi = 0$. The further discussions concerning the true observables will be made on another occasion.

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A Note on the Non-leptonic Decay Modes of the Ξ-Hyperon

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A pole approximation calculation for the decay mode $\Xi \to \Lambda + \pi$ is presented. It is suggested that the curious pattern of the Σ-decay asymmetries arise perhaps from the fact that α_Ξ and α_Λ have opposite signs. Finally the K_1^0-K_2^0 mass difference is calculated in the pole approximation.

Recently, experimental results on the magnitude of the asymmetry parameter for the decay mode $\Xi^- \to \Lambda^0 + \pi^-$ have become available.[1] We present below a calculation using the pole approximation for the decay following Feldman, Matthews and Salam.[2] A possible bearing of the fact that the asymmetry parameters in Ξ- and Λ-decays have opposite signs[1] on the Σ-decay asymmetries will also be discussed. Finally we discuss the K_1^0-K_2^0 mass difference in the pole approximation.

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Fig. 1(a).

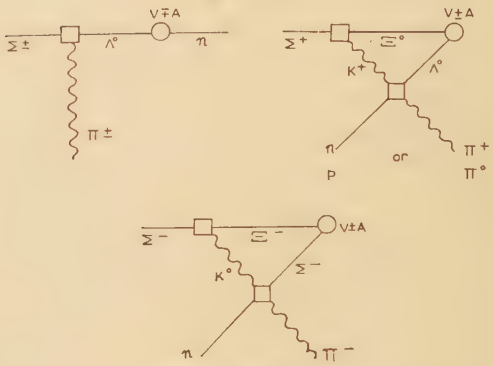

Fig. 1(b).

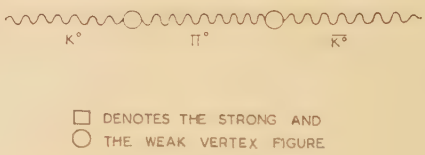

Fig. 1(c).

The diagram of interest for the Ξ-decay is shown in Fig. 1(a). The diagram with the K-meson pole makes a comparatively small contribution and hence is neglected in order to reduce the number of unknowns. G_1 denotes the renormalized Ξ-Ξ-π coupling constant while $a+b\gamma_5$ denotes the Ξ-Λ vertex. The decay asymmetry parameter $\alpha_\Xi = \pm 0.69$[1] yields, after a standard calculation, $a/b = \mp 0.58$ or ∓ 4.42. Using the value of the Ξ lifetime $\tau = 1.28 \times 10^{-10}$ secs.,[1] we then find $G_1^2/4\pi \cdot (b/m_\pi)^2 = 0.24 \times 10^{-11}$ for $|a/b| = 0.58$ and 0.31×10^{-12} for $|a/b| = 4.42$. The experimental results also indicate that α_Ξ and α_Λ have opposite

signs. However, the sign of α_Λ is uncertain, different groups of workers obtaining contradictory results[3] so that the sign of a/b is uncertain. Note that the values of these parameters for the mode $\Xi^0 \to \Lambda^0 + \pi^0$ follow from the above because of the $|\Delta I| = 1/2$ rule.

It is possible that the observed pattern of the Σ-decay asymmetries[4] arises due to a parity clash between diagrams involving Λ and those involving Ξ. Fig. 1 (b) indicates the relevant diagrams. For the Λ-intermediate state, the pole diagram has been drawn and is assumed to be dominant. For Ξ-intermediate states, pole diagrams are not possible and those next in order of simplicity are taken to be the important ones. The $\Xi^- \text{-} \Sigma^-$ vertex can arise from $\Xi^- \to \Lambda^0 + \pi^- \to \Sigma^-$. It is clear that for suitable values of the parameters, these three diagrams can yield parity-conserving $\Sigma^+ \to n + \pi^+$ and $\Sigma^- \to n + \pi^-$ decay modes and parity-violating $\Sigma^+ \to p + \pi^0$ decay mode, the proton from Σ^+-decay and the Λ from Ξ- decay having helicities of the same sign. Such a procedure would however imply that relations like $R(\Sigma^+ \to p + \pi^0) \approx 2R(\Lambda^0 \to p + \pi^-)$[5] are the nature of dynamical accidents. The validity of the approximation is also very uncertain.

In this connection, we would like to remark that the pole approximation can be used to calculate the $K_1^0 \text{-} K_2^0$ mass difference. We consider Fig. 1 (c). Taking the $K_1^0 \text{-} K_2^0$ mass difference δm to be $\sim 10^{-5}$ev,[6] we find $(c/m_\pi^2)^2 \sim 0.3 \times 10^{-11}$, where c is the coupling constant for the weak K-π vertex. A precise value for δm would allow us to determine c more accurately (provided, of course, the approximation is valid). Similar calculations can be performed, for instance, for the Σ^0 or Λ^0 self-energy due to transitions of the type $\Sigma^0 \to \Lambda^0 \to \Sigma^0$.

We are indebted to Professor Alladi Ramakrishnan and Mr. N. R. Ranganathan for stimulating discussions. We also wish to thank the Atomic Energy Commission for providing us with Fellowships.

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Phonon-Phonon Interaction in Liquid Helium II

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Despite the recent developments in the theories of liquid helium and interacting Boson systems, the treatments of interactions between excitations in liquid helium have been either phenomenological[1] or unsatisfactory.[2] On the other hand, Feynman[3] succeeded in deriving spectrum of excitations without going into full details of the ground state wave function, requiring the knowledge of only a few distribution functions. Here we propose to use his theory for determining interactions between excitations in liquid helium. In particular, we describe here the interaction between phonons.

According to Feynman, the wave functions Ψ_1 and Ψ_2 of the two states having phonons with wave vectors $\boldsymbol{k}_1, \boldsymbol{k}_2, \cdots \boldsymbol{k}_l$ and $\boldsymbol{k}_1', \boldsymbol{k}_2', \cdots \boldsymbol{k}_n'$, respectively, can be written as follows, Φ being the ground state wave function,

$$\Psi_1 = F_1\Phi, \quad \Psi_2 = F_2\Phi \tag{1}$$

where

$$F_1 \equiv q_{k_1} q_{k_2} \cdots q_{kl}, \quad F_2 \equiv q_{k_1'} q_{k_2'} \cdots q_{k_n'} \tag{2}$$

with $q_k \equiv \sum_i e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i}$.

The interaction between phonons is given by the matrix element of the Hamiltonian between these states, namely

$$H_{1;2} = \frac{\langle F_1^* H F_2\rangle - E_2\langle F_1^* F_2\rangle}{\langle F_1^* F_1\rangle^{1/2}\langle F_2^* F_2\rangle^{1/2}} \tag{3}^*$$

where the angular brackets imply expectation values in the ground state and E_2 denotes the energy of non-interacting phonons in the state 2. The second term in the numerator arises from the non-orthogonality of the phonon states. Calculating the energy from the ground state, the first term of the numerator of Eq. (3) can be transformed to yield the following,

$$\frac{\hbar^2}{2m}\sum_i \langle \boldsymbol{\nabla}_i F_1^* \cdot \boldsymbol{\nabla}_i F_2\rangle. \tag{4}$$

After a little manipulation, this expression can be brought to the following form,

$$\frac{\hbar^2}{2m}\sum_{r=1}^{l}\sum_{s=1}^{n}(\boldsymbol{k}_r\cdot\boldsymbol{k}_s')\langle q_{-k_2}\cdots q_{-k_{r-1}} q_{k_s'-k_r} \times q_{-k_{r+1}}\cdots q_{-k_l} q_{k_1'}\cdots q_{k'_{s-1}} q_{k'_{s+1}}\cdots q_{k'_n}\rangle. \tag{5}$$

For long wavelengths of the involved phonons, the denominator of Eq. (3) can be transformed by factorizing it in the product of the quantities of the form $\langle q_k q_{-k}\rangle$ and using the well-known relation $\langle q_k q_{-k}\rangle = N(\hbar k/2mc)$, c being the sound velocity.

Thus we obtain the following final form for the matrix element,

$$\left(\frac{2mc}{\hbar N}\right)^{(l+n)/2}\frac{\hbar^2}{2m}\frac{\sum_{r=1}^{l}\sum_{s=1}^{n}(\boldsymbol{k}_r'\boldsymbol{k}_s')\langle q_{-k_1}\cdots q_{-k_{r-1}} q_{k_s'-k_r} q_{-k_{r+1}}\cdots q_{-k_l} q_{k_1'}\cdots q_{k'_{s-1}} q_{k'_{s+1}}\cdots q_{k_n'}\rangle}{(k_1k_2\cdots k_lk_1'k_2'\cdots k_n')^{1/2}} - \hbar c(k_1'+k_2'+\cdots+k_n')\left(\frac{2mc}{\hbar N}\right)^{(l+n)/2}\frac{\langle q_{-k_1} q_{-k_2}\cdots q_{-k_l} q_{k_1'} q_{k_2'}\cdots q_{k_n'}\rangle}{(k_1k_2\cdots k_lk_1'k_2'\cdots k_n')^{1/2}}. \tag{6}$$

* Actually, for those matrix elements which connect the states with different energies, there is a certain ambiguity arising from the diagonal part of the Hamiltonian in connection with the non-orthogonality of phonon states. Therefore, the matrix elements we obtain is meaningful only on the same energy shell. This point will be discussed in a full note. The author is grateful to Drs. K. Yamazaki and H. Yamamoto for illuminating discussions on this point.

For calculation of the quantities of the form $\langle q_k q_{k'} \cdots \rangle$, a quantum hydrodynamical consideration may be used for long wavelengths. Eq. (3) in the subsequent article was obtained in this way.

As has been noted by Feynman, the different phonon states having the same total momentum are not orthogonal in general, and there appears to be a difficulty in calculating the transition probability in this representation. However, the situation is similar to that encountered in Dyson's theory of spin wave interaction.[4] If one considers the scattering process as the scattering between phonon wave packets, the non-orthogonality of the states will have no effects on the motion of the wave packets, at least for the first order process. Therefore, we have simply ignored it here.

In the subsequent article, we shall apply the result obtained here to calculating the sound attenuation in liquid helium. We hope to treat the interactions involving rotons in addition to phonons along a similar line.

The author wishes to express his sincere thanks to Professor T. Usui, Dr. H. Mori, and Dr. T. Tsuneto for very helpful discussions.

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Sound Attenuation in Liquid Helium II at Very Low Temperatures

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Attenuation of sound in liquid helium has been extensively studied by Khalatnikov[1] on the basis of Landau's quantum hydrodynamics with considerable success. However, below about 1°K, there is a discrepancy between his theory and the recent measurements.[2] Namely, the theory predicts almost on attenuation in this temperature region, whereas the considerable amount of attenuation was observed. Here we discuss the attenuation at temperatures low enough so that rotons can be ignored and the phonon mean free path is much greater than the wavelength of sound. In this case the sound wave is represented by an assembly of sound wave quanta and the attenuation can be visualized as the decrease of the number of the sound wave quanta caused by the collisions with thermal phonons. Among possible processes contributing to the attenuation, the three phonon process have been ignored heretofore due to the reason that energy and momentum cannot be conserved simultaneously because of the dispersion of phonons. However, here we argue that this process can contribute to the attenuation provided that the inverse of the life-

time of average thermal phonons is comparable to or greater than the degree of violation of the law of energy conservation for this pocess. For the confluence process, in which a sound quantum collides with a thermal phonon to yield a thermal phonon, this latter quantity is

$$\hbar\Delta\omega=c\hbar k_s-\frac{d\epsilon(k_t)}{dk_t}k_s=3c\gamma\hbar^3k_t^2k_s \quad (1)$$

where t and s refer to thermal phonons and sound quanta respectively, and the phonon spectrum, $\epsilon(k)$, was put in the form

$$\epsilon(k)=c[\hbar k-\gamma(\hbar k)^3]. \quad (2)$$

For $\gamma=10^{37}$ *c.g.s.*, $c\hbar k_t=k_BT$, $T=0.5°K$, $ck_s=2\pi\times10^7$ sec^{-1}, we obtain $\Delta\omega/2\pi c\sim$ 0.1 cm^{-1}. This is comparable to the observed attenuation coefficient, or the inverse of the mean free path of the sound quanta. If we assume that the lifetime of a phonon decreases with its wave number, we can conclude that the uncertainty of energy of a thermal phonon associated with this lifetime exceeds the degree of violation of the law of energy conservation in the three phonon process.

Thus, we have calculated the attenuation coefficient resulting from the three phonon processes. The relevant matrix element was obtained employing the result of the preceding article with a quantum hydrodynamical consideration for calculating the density correlation. For $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ in the same direction, the result is

$$H_{k_1,k_2;\,k_1+k_2}=H_{k_1+k_2;k_1,k_2}=\frac{1}{2\sqrt{V}}\left(\frac{c\hbar^3}{2\rho}\right)^{1/2}$$
$$\times\sqrt{k_1k_2(k_1+k_2)}\left(3+\frac{\rho^2}{c^2}\frac{dc^2/\rho}{d\rho}\right), \quad (3)$$

ρ being the liquid density.

The perturbation calculation, in which the dispersion of the phonon spectrum is neglected, gives the following result for the attenuation coefficient at 11.8 Mc/sec, where the dominant contribution comes only from the confluence process,

$$\alpha=3.24\frac{\omega(k_BT)^4}{c^6\pi\hbar^3\rho}\left(1+\frac{\rho}{c}\frac{dc}{d\rho^2}\right)^2$$
$$=12.2\times T^4\ \text{cm}^{-1} \quad (4)$$

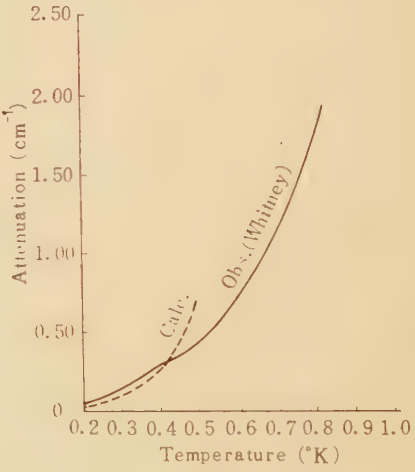

Fig. 1.

where we have chosen: $\rho=0.145$ gr cm^{-3}, $c=2.37\times10^4$ cm/sec and $dc/d\rho=4.3\times10^5$ *c.g.s.* This result is shown in Fig. 1 along with the observed attenuation coefficient. The agreement is satisfactory below about 0.4°K, considering the fact that no adjustable parameter is involved in the calculation. At higher temperatures, the calculated attenuation coefficient rises too rapidly. This is probably due to the fact that the phonon mean free path begins to be comparable to the sound wavelength and our simple picture of the attenuation breaks down.

Much of the ideas of the present work emerged from the very stimulating conversations with Professor T. Usui, Dr. H. Mori, and Dr. T. Tsuneto.

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Analysis of $(n, n'\gamma)$ Cross Section for Nuclei $A\sim100$

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The $(n, n'\gamma)$ cross section near threshold gives information concerning the optical potential which is complementary to the elastic scattering, if the spin-parity of the excited state is known. The reaction proceeds mainly via compound nucleus. With the penetrability $T_i^J(E)$ calculated from the optical potential with spin-orbit force, the cross section becomes, generalizing the Hauser-Feshbach theory,[1]

$$\frac{d\sigma(i,i';i'')}{d\Omega_\gamma}=\frac{\lambda^2}{8(2i+1)}\sum_{L=even}\sum_{J,\pi}\times\frac{(2J+1)}{\sum_{(J,\pi)}T_{l''}^{J''}(E'')}(-)^{i+i''}\sum_{l,j}B_{jJ}^{iL}T_l^j(E)\times\sum_{l',j'}C_{j'J}^{i'L}T_{l'}^{j'}(E')\sum_{\substack{\Lambda p\\ \Lambda' p'}}A_{\Lambda\Lambda'}^{ii'i''L}a_\Lambda^p a_{\Lambda'}^{p'}\times P_L(\cos\theta),$$

where

$$A_{\Lambda\Lambda'}^{ii'i''L}=\sqrt{(2\Lambda+1)(2\Lambda'+1)}\times(\Lambda\Lambda'1-1|L0)\sqrt{2i'+1}\times W(i'\Lambda i'\Lambda';i''L),$$

$$B_{jJ}^{iL}=(2j+1)(jj\,1/2-1/2|L0)\times\sqrt{2J+1}\;W(jJjJ;iL),$$

$$C_{j'J}^{i'L}=(-)^{j'-1/2}\sqrt{(2J+1)(2i'+1)}\times W(i'Ji'J;j'L),$$

and i and i' are the spins of the ground and excited states respectively. γ-decay leads to a state with spin i''. J, π and Λ, p are the spin-parities of the compound state and the γ-ray respectively. a_Λ^p is the relative transition amplitude of γ-ray normalized to unity. The sum in the denominator is to be done for all channels consistent with J, π.

For the potential, we adopted the following form.

$$V\cdot f(\rho)+iW\cdot g(\rho)+v\frac{r_0}{R_0+\rho}h(\rho)a_l^j$$

where

$\rho=r-R_0$, $R_0=r_0A^{1/3}-b$,

$r_0=1.25$ fermi, $W=4$ Mev and 7Mev,

$b=2.52$ fermi, $v=6$ Mev,

$V=50$ Mev,

$$a_l^j\begin{cases}=l & \text{for } j=l+1/2\\ =-(l+1) & \text{for } j=l-1/2,\end{cases}$$

$$f(\rho)\begin{cases}=1 \text{ if } \rho-c\leq0 & c=0.52171 \text{ fermi}\\ =\exp\left[-\left(\frac{\rho-c}{l'}\right)^2\right] \text{ if } \rho-c>0 & l'=2.40 \text{ fermi},\end{cases}$$

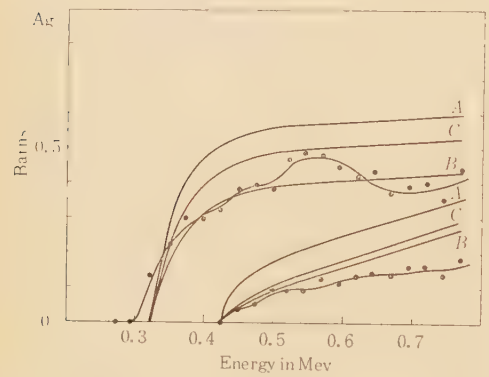
$$g(\rho)=\exp\left[-\left(\frac{\rho-b}{l''}\right)^2\right]$$

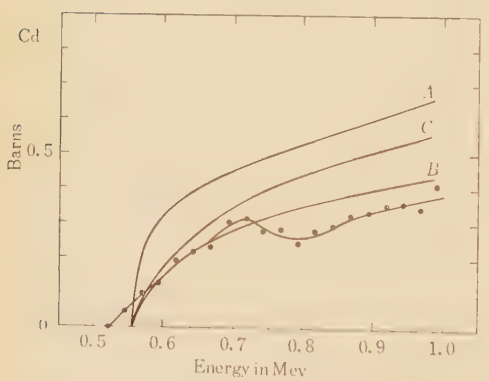
for all ρ, $l''=0.98$ fermi,

$$h(\rho)=0 \text{ if } \rho-c\leq 0$$

$$=1/2\left(\frac{\rho-c}{l'}\right)\exp\left[-\left(\frac{\rho-c}{l'}\right)^2+1/2\right] \text{ if } \rho-c>0.$$

This potential is almost the same as the one in reference 4).





$4\pi(d\sigma(90°)/d\Omega_\gamma)$ versus incident neutron energy.
A: $W=7$ Mev, B: $W=4$ Mev, C: $W=$ {4 Mev for even-waves, 7 Mev for odd-waves.

An experiment was carried out for Ag and Cd using the ringgeometry,[2] details of which will be published separately.[3] γ-ray was detected at around 70°~110°. The results are given in the Figure and are compared with the surface absorption optical potential.[4] The angular distribution of γ-ray for Ag ($i=i''=1/2$, $i'=3/2$ or $5/2$) is nearly isotropic but that for Cd ($i=i''=0$, $i''=2$) shows a marked minimum at 90°($4\pi d\sigma(90°)/d\Omega_\gamma$ is 0.7 times the integrated cross section). This is understandable since near the threshold the process with outgoing s-wave is predominant and in this case the z-component of $M(E2)$ can take all possible values for Ag but only 0 and ± 1 for Cd.

From the Figure it is seen that W, the maximum of the imaginary part at the surface, should be about 4 Mev or less which is fairly small compared to the value obtained from the elastic scattering at 4 Mev for various nuclei, $W=7$Mev,[4] but is consistent with the zero-energy strength functions which are very small here. According to a theory,[5] W for odd-waves will be larger than that for even-waves, but experimentally this is not the case. However, there are uncertainties in regarding the real part of the potential for the p-state[6] (derived from the p-state strength functions) and since T_l^j for the p-state is very sensitive on the real part here (as the giant resonance is near) more extensive analysis is required for the discussion of the difference in W for odd- and even-waves. Further, the coupling with the vibrational mode may be important for the odd waves since they are close to the giant resonances.[7]

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A Note on the Dielectric Constant and Causality

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There are two quite different ways to define the dielectric constant for solids on the basis of many electron theory. One of them is to define it in terms of the response to an external electric field.[1] The other is to define it in terms of the modified Coulomb interaction.[2] In what follows the dielectric constant defined by the former way will be referred as the "external" dielectric constant and that defined by the latter as the "internal" dielectric constant. The aim of the present letter is to elucidate the relation between the "external" and "dynamical" dielectric constants.

According to Izuyama,[3] the external dielectric constant of a solid, $\epsilon_{ex}(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega)$, is given in a form of reciprocal matrix by

$$\epsilon_{ex}(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega) = \delta(\boldsymbol{k}_1+\boldsymbol{k}_2) + v(\boldsymbol{k}_2)\chi(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega), \quad (1)$$

where $v(\boldsymbol{k}_1) = 4\pi e^2/k_1^2$ is the Fourier transform of the instantaneous Coulomb interaction $v(x_1-x_2) = e^2/|\boldsymbol{r}_1-\boldsymbol{r}_2| \cdot \delta(t_1-t_2)$ and $\chi(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega)$ is given by

$$\chi(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega) = \lim_{\epsilon\to 0}(-i)\int_0^\infty dt e^{-i\omega t-|\epsilon|t} \times\langle[\rho(\boldsymbol{k}_1, t), \rho(\boldsymbol{k}_2)]\rangle. \quad (2)$$

In Eq. (2), $\langle\mathfrak{A}\rangle$ stands for the average of a quantity $\mathfrak{A}$ over the canonical ensemble of the system and $\rho(\boldsymbol{k}, t)$ is the Heisenberg representation of $\rho(\boldsymbol{k})$ which is the Fourier component of the density operator $\rho(\boldsymbol{x}) = \psi^+(\boldsymbol{x})\psi(\boldsymbol{x})$, where $\psi(\boldsymbol{x})$, is the quantized wave function of the electron cloud. Here it is to be noted that $\chi(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega)$ has non-zero value when and only when $\boldsymbol{k}_2 = \boldsymbol{k}_1+\boldsymbol{q}$, where $\boldsymbol{q}$ is one of the reciprocal lattice vectors, because of the periodic structure of the crystal lattice.

In a many-electron system, the instantaneous Coulomb interaction between electrons is modified by the polarization of the medium. Hubbard[4] has shown that the modified Coulomb interaction between space-time points x_1 and x_2, $\mathcal{V}(x_1, x_2)$, satisfies the integral equation

$$\mathcal{V}(x_1, x_2) = v(x_1-x_2) + \int v(x_1-x') \overline{\mathcal{W}}(x', x'')v(x''-x_2)d^4x'd^4x'', \quad (3)$$

where $\overline{\mathcal{W}}(x', x'')$ represents contribution of all polarization diagrams (not proper in general). The Fourier

transform of Eq. (3) gives us

$$\tilde{v}(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega) = v(\boldsymbol{k}_1)\delta(\boldsymbol{k}_1+\boldsymbol{k}_2) + v(\boldsymbol{k}_1)\tilde{\mathscr{W}}(\boldsymbol{k}_1, \boldsymbol{k}_2\, \omega)v(\boldsymbol{k}_2). \quad (4)$$

The internal dielectric constant $\epsilon_{in}(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega)$ is defined, according to Pratt,[2] by

$$\tilde{v}(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega) = \epsilon_{in}^{-1}(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega)v(\boldsymbol{k}_1). \quad (5)$$

Using standard methods in field theory,[5] it can be shown that

$$\tilde{\mathscr{W}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega) = \lim_{\epsilon\to 0} i\int_{-\infty}^{\infty} dt e^{-i\omega t-|\epsilon t|} \times\langle 0|\mathrm{T}\rho(\boldsymbol{k}_1, t)\rho(k_2)|0\rangle, \quad (6')$$

where T is Dyson's chronological operator and $|0\rangle$ is the exact ground state of the system, namely the lowest eigenstate of the total Hamiltonian H. Eq. (6') is for the absolute zero of temperature. At a finite temperature Eq. (6') should be replaced by

$$\mathscr{W}(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega) = \lim_{\epsilon\to 0} i\int_{-\infty}^{\infty} dt e^{-i\omega t-|\epsilon t|} \times\langle(\mathrm{T}\rho(\boldsymbol{k}_1, t)\rho(\boldsymbol{k}_2))\rangle. \quad (6)$$

After some mathematical manipulation, one can show that $\tilde{\mathscr{W}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega)$ is rewritten as

$$\tilde{\mathscr{W}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega) = \tilde{\mathscr{W}}_1(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega) + \tilde{\mathscr{W}}_2(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega), \quad (7)$$

where

$$\tilde{\mathscr{W}}_1(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega) = -\sum_n\sum_m \frac{\mathrm{P}}{\omega_{nm}+\omega} \times e^{\beta\Omega}(e^{-\beta E_n} - e^{-\beta E_m}) \times\langle n|\rho(\boldsymbol{k}_1)|m\rangle\langle m|\rho(\boldsymbol{k}_2)|n\rangle \quad (8)$$

and

$$\tilde{\mathscr{W}}_2(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega) = -i\pi\sum_n\sum_m(1+e^{-\beta\omega})e^{\beta(\Omega-E_n)} \times\delta(E_n-E_m+\omega)\langle n|\rho(\boldsymbol{k}_1)|m\rangle \times\langle m|\rho(\boldsymbol{k}_2)|n\rangle. \quad (9)$$

In the above, $H|n\rangle = E_n|n\rangle$, $\hbar\omega_{nm} = E_n - E_m$, and

$$e^{-\beta\Omega} = \mathrm{Tr}e^{-\beta H} = \sum_n e^{-\beta E_n}.$$

It is also shown that the following relation holds between $\chi(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega)$ and $\tilde{\mathscr{W}}(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega)$:

$$\chi(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega) = \chi_1(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega) + \chi_2(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega), \quad (10)$$

$$\chi_1(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega) = \tilde{\mathscr{W}}_1(\boldsymbol{k}, \boldsymbol{k}_2, \omega), \quad (11)$$

and

$$\chi_2(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega) = \tanh(-\beta\omega/2) \times\tilde{\mathscr{W}}_2(\boldsymbol{k}_1, \boldsymbol{k}_2, \omega), \quad (12)$$

where, in the limit that $T=0°\mathrm{K}$, i.e. $\beta=\infty$, the factor $\tanh(-\beta\omega/2)$ is reduced to the so-called step-function $\eta(-\omega)$, which is equal to 1 for $\omega<0$ and equal to -1 for $\omega>0$. When $\boldsymbol{k}_1=-\boldsymbol{k}_2$, as will be seen easily, χ_1 and $\tilde{\mathscr{W}}_1$ are real, and χ_2 and $\tilde{\mathscr{W}}_2$ pure imaginary.

It has been done here to elucidate the relation between the "external" dielectric constant and the "internal" one. Now it is easy to see that the "external" dielectric constant, ϵ_{ex}, satisfies the Kramers-Kronig dispersion relation, i.e. the requirement of causality, but the "internal" one, ϵ_{in}, *does not.*

As is seen from the definition mentioned above, the external dielectric constant plays a role in the response to an external applied field. On the other hand, the internal dielectric constant plays a role in theories of excitons and impurity levels where the modified coulomb interaction between an electron and a hole or between an

electron and an impurity charge is essential.

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Three-Pion Resonance and High Energy Phenomena

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August 30, 1961

It seems that there is some evidence for three-pion resonance in the experiment for $\bar{p}+p\rightarrow 2\pi^{+}+2\pi^{-}+\pi^{0}$ reaction.[1] According to the experimental results, the isotopic spin of the resonant state ω seems to be equal to 0 and its mass is about (790 ± 20) Mev.* By this information we are motivated to study the problems for the ω. When we denote the three pions of which the ω consists as π_1, π_2 and π_3, the isotopic spin of π_1-π_2 system must be equal to 1. Therefore the orbital angular momentum l in π_1-π_2 system must be odd owing to the Bose statistics. As the most possible case we may consider $l=1$. Then we are inclined to regard the ω as a resonant state in ρ-π_3 system,* where ρ indicates the resonant state in π-π system with the mass 750 Mev, isotopic spin $I=1$ and spin $j=1$ which has been established in the recent experiment.[2] If the ω corresponds to the neutral vector meson which has been predicted by Nambu,[3] the orbital angular momentum l' in ρ-π_3 system must be equal to 1. It is of interest for us to derive such a resonant state from the current field theory, but it may be a difficult problem because the mass of ω is much smaller than the sum of those of ρ and π_3 in spite of the existence of the centrifugal barrier.**

Recently Hasegawa[4] has shown that the phenomena of jet shower in extreme high energy region can be explained in terms of the effects of the fictitious H^* quantum which may be a bound state in baryon-antibaryon system. According to his analysis, the mass of H^* seems to be about $2M$, where M is the nucleon mass. There may exist several kinds of H^*. In this case the mass of $2M$ should be interpreted as the mean value of the masses of these quanta.*** Basing on the experimental results for the existence of ρ and ω, and introducing the H^* quanta belonging to the bound states in nucleon-

* The predictions of the ω have been made by Nambu, Chew and Gell-Mann.

* The π_3 may be a π^0 for an instant and also may be a π^+ or π^- for another instant, because the width of ω seems to be much narrower than that of ρ.

** We heard that Sudarshan pointed out $m_\rho < m_\omega < m_\rho + m_\pi$.

*** We are indebted to Dr. S. Hasegawa who has kindly discussed with us.

antinucleon system, that is, H_0^* and H_1^* with isotopic spin 0 and 1 respectively, for the reaction of antiproton annihilation we should like to suggest the following model as an attempt to establish the relation between the description of extreme high energy phenomena and that of phenomena in Bev region or low energy region. The large multiplicity of pions in antiproton annihilation at rest may be due to the effects of ρ and ω.

$$\bar{p}+p\rightarrow H_1^*\rightarrow\rho+\omega\rightarrow(\pi^++\pi^-)+(\pi^++\pi^-+\pi^0), \quad \text{(I)}^*$$

$$\bar{p}+p\rightarrow H_1^* \text{ or } H_0^*\rightarrow\rho+\rho\rightarrow(\pi^++\pi^-)+(\pi^++\pi^-) \text{ or } (\pi^++\pi^0)+(\pi^-+\pi^0), \quad \text{(II)}^*$$

$$\bar{p}+p\rightarrow H_0^*\rightarrow\omega+\omega\rightarrow(\pi^++\pi^-+\pi^0)+(\pi^++\pi^-+\pi^0). \quad \text{(III)}^*$$

Next let us make some discussion about K^--p collision. We have previously predicted the resonance phenomena for K^--p scattering in the neighborhood of 1.1 Bev/c which are due to the strong interaction in $I=0$, $d_{3/2}$-state.[6] These resonance phenomena due to $I=0$ state have been confirmed by the recent experiment for K^--p scattering.[7] In addition, recent measurements[8] on

$$K^-+p\rightarrow\Sigma+\pi+\pi+\pi, \quad \text{(IV)}$$

$$K^-+p\rightarrow\Lambda+\pi+\pi+\pi, \quad \text{(V)}$$

at 1.15 Bev/c have shown the existence of Y_0^*, but there seems to be no evidence for the strong π-Λ interaction (Y_1^*) in the reaction (V). We may consider the following reaction mechanism for these reactions at (1.1~1.4) Bev/c, since they correspond to the phenomena in the neighborhood of the threshold energy of $K^-+p\rightarrow Y+\omega$.

$$K^-+p\rightarrow Y+\omega\rightarrow Y+\pi+\pi+\pi. \quad \text{(VI)}$$

But such a model as this can be applied only to the reaction (V), because it is the $I=0$ state that reacts strongly and because the Y_0^* plays an important role in the process (IV). We should like to get the experimental data for the energy spectrum of Λ in the reaction (V).

Finally let us examine the effects of ω on π-N scattering at high energy. As was pointed out by Nambu,[3] the existence of ω may have something to do with the second or third resonance for π^--p scattering. There will be the following reaction mechanism for π^--p scattering at high energy.

$$\pi^-+p\rightarrow n+\omega\rightarrow N+m\pi \quad (m=1, 2, 3). \quad \text{(VII)}$$

It is needless to say that the isotopic spin $I=1/2$ state reacts strongly. In such an energy region that the second or third resonance takes place, we may safely suppose that the ω and neutron in the intermediate state will be in an s- or p-state relative to each other. The s- and p-states in ω-n system may correspond respectively to the $d_{3/2}$- and $f_{5/2}$-states in the initial π^--p system which are responsible respectively for the second and third resonances. Moreover, it may be expected that the existence of ω has also something to do with the fourth resonance for the isotopic spin $I=3/2$ state, because the

* The (ρ-ω), (ρ-ρ) and (ω-ω) pairs in these processes may be produced in part without the excitation of H^*. These correspond to the processes which have been suggested by Sakurai.[5]

reaction process $\pi+N\rightarrow N^*+\omega\rightarrow(N+\pi)+(\pi^++\pi^-+\pi^0)$ may play the important role in π-N scattering in the neighborhood of 1.35 Bev, where N^* indicates the(3/2, 3/2)-resonant state in π-N system.

The author would like to express his thanks to Dr. B. C. Maglić for his kind information on the experiment at Berkeley.

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Proton-Antiproton Collision and Isobaric States

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August 14, 1961

We previously tried to estimate the branching ratios for the reaction $p+\bar{p}\rightarrow K+\bar{K}+\pi+\pi$ under the assumption that this reaction takes place through the process $p+\bar{p}\rightarrow K^*+\bar{K}^*\rightarrow K+\bar{K}+\pi+\pi'$.* The result in the most practical case $I_{K^*}=1/2$ (I_{K^*} represents the isotopic spin of K^*) showed that it will be impossible to tell by the branching ratios at production whether the initial state is $I=0$ or $I=1$.** The final isotopic spin state $I=0$ (or $I=1$) is divided into the symmetric and antisymmetric parts with respect to the final two pions. The reaction amplitudes for these final substates were estimated previously with equal weight for simplicity.

In this note we should like to examine how the branching ratios are modified when either substate plays the most important role in the following reactions and to discuss which substate has a predominant effect on these reactions.

$$p+\bar{p}\rightarrow K^*+\bar{K}^*\rightarrow K+\bar{K}+\pi+\pi; \quad \text{(I)}$$
$$N^*+\bar{N}^*\rightarrow N+\bar{N}+\pi+\pi, \quad \text{(II)}$$
$$Y_1^*+\bar{Y}_1^*\rightarrow \Lambda+\bar{\Lambda}+\pi+\pi, \quad \text{(III)}$$
$$Y_0^*+\bar{Y}_0^*\rightarrow \Sigma+\bar{\Sigma}+\pi+\pi, \quad \text{(IV)}$$
$$Y_1^*+\bar{Y}_0^*\rightarrow \Lambda+\bar{\Sigma}+\pi+\pi, \quad \text{(V)}$$
$$Y_0^*+\bar{Y}_1^*\rightarrow \Sigma+\bar{\Lambda}+\pi+\pi, \quad \text{(VI)}$$

where K^*, N^*, Y_1^* and Y_0^* indicate the isobaric states in π-K, π-N, π-Λ and π-Σ systems respectively, and their isotopic spins seem to be equal to 1/2, 3/2, 1 and 0, respectively.[3),4)] The

* According to the recent experiments, the observed $K\pi$ (or $\bar{K}\pi$) mass distributions for this reaction seem to be consistent with both K^* and $\bar{K}^*$ resonances.[2)]

** The author thanks Dr. J. Ballam for his helpful comment on this point.

branching ratios for reactions (I) and (II) are shown in Tables I and II. (S) and (A) in the tables represent respectively the symmetric and antisymmetric states for the final two pions.

The symmetric and antisymmetric states for reaction (III) correspond to the final isotopic spin states $I=0$ and $I=1$ respectively, and the following results can easily be obtained.

$$(\Lambda\bar{\Lambda}\pi^+\pi^-):(\Lambda\bar{\Lambda}\pi^0\pi^0)=1:0 \text{ for } I=1 \text{ state},$$
$$=2:1 \text{ for } I=0 \text{ state}.$$

In our model, only the $I=0$ state contributes to reaction (IV) and there are nine possible states belonging to the final $(\Sigma\bar{\Sigma}\pi\pi)$. If the reaction amplitudes for the symmetric and antisymmetric states are estimated with equal weight, we can make a prediction such that each final state will be realized with equal probability. The branching ratios of reaction (IV) into the final states except for $(\Sigma^+\bar{\Sigma}^-\pi^-\pi^-)$, $(\Sigma^0\bar{\Sigma}^0\pi^0\pi^0)$ and $(\Sigma^-\bar{\Sigma}^+\pi^+\pi^+)$ do not depend on whether the symmetric state reacts strongly or the antisymmetric state reacts strongly. Because the z-component of the initial isotopic spin is equal to zero, the Λ or $\bar{\Lambda}$ in reaction (V) or (VI) must be accompanied by π^0, so long as reaction (V) or (VI) takes place through the process $p+\bar{p}\to Y_1^*+\bar{Y}_0^*\to\Lambda+\bar{\Sigma}+\pi+\pi$ or $p+\bar{p}\to Y_0^*+\bar{Y}_1^*\to\Sigma+\bar{\Lambda}+\pi+\pi$. Therefore we have only to consider the possible three states $(\pi^0\pi^+)$, $(\pi^0\pi^0)$ and $(\pi^0\pi^-)$ with respect to the final pions. The final states for these reactions can be specified by the charge states of the final two pions. The branching ratios both for reaction (V) and for reaction (VI) are given as follows.

$$(\pi^0\pi^+):(\pi^0\pi^0):(\pi^0\pi^-)=1/6:1/3:1/6$$
for the symmetric state,
$$=1/6:0:1/6$$
for the antisymmetric state.

Next we should like to discuss which state of the symmetric and antisymmetric ones plays the most important role in reactions (I) ~ (VI). The final configurations $K+\pi+\bar{K}+\pi$ may

Table I. Branching ratios for reaction (I) in the case where each isotopic spin state mentioned in this table reacts strongly.

reactions	branching ratios			
	$I=1$		$I=0$	
	(S)	(A)	(S)	(A)
$p+\bar{p}\to K^++K^-+\pi^++\pi^-$	1/9	1/9	1/9	1/9
$K^++K^-+\pi^0+\pi^0$	1/18	0	1/18	0
$K^++\bar{K}^0+\pi^-+\pi^0$	2/9	0	0	2/9
$K^0+\bar{K}^0+\pi^0+\pi^0$	1/18	0	1/18	0
$K^0+\bar{K}^0+\pi^++\pi^-$	1/9	1/9	1/9	1/9
$K^0+K^-+\pi^++\pi^0$	2/9	0	0	2/9

Table II. Branching ratios for reaction (II) in the case where each isotopic spin state mentioned in this table reacts strongly.

reactions	branching ratios			
	$I=1$		$I=0$	
	(S)	(A)	(S)	(A)
$p+\bar{p}\to p+\bar{p}+\pi^+ +\pi^-$	8/45	5/18	2/9	1/18
$n+\bar{n}+\pi^+ +\pi^-$	8/45	5/18	2/9	1/18
$p+\bar{p}+\pi^0+\pi^0$	1/45	0	1/9	0
$n+\bar{n}+\pi^0+\pi^0$	1/45	0	1/9	0
$p+\bar{n}+\pi^- +\pi^0$	1/45	0	0	1/9
$n+\bar{p}+\pi^+ +\pi^0$	1/45	0	0	1/9

generally be specified by giving l, l' and L, where l (or l') indicates the orbital angular momentum in K-π system (or $\overline{K}$-π system) which corresponds to the K^*(or $\overline{K}^*$) spin and parity, and L indicates the orbital angular momentum of the center-of-mass for K and π with respect to that for $\overline{K}$ and π. The final configurations for reactions (II)$\sim$(VI) may also be specified similarly. Let us now study reaction (I) of proton annihilation at rest and reactions (II)$\sim$(VI) in the case where the available energies in the final states are very low. Then we may safely suppose that the Y^* and $\overline{Y}^*$ will be in an S-state relative to each other. Dalitz and Miller[5] tried to explain the experimental results for $K^- +p\to\Lambda+\pi+\pi$ reaction by taking into account the interference effects arising from the requirement of Bose statistics for the final pions. Their discussions are based mainly on the analysis of experimental data for the following configurations. (i) The two pions have the same c.m. momentum. (ii) The two pions have equal and opposite momenta in the c.m. system. Of course the configuration (i) results from the reaction amplitude for the symmetric state of isotopic spin for the final pions. Now let us pay attention to the configuration (ii) in our case. When $l+l'+L$ is even (odd), the symmetric (antisymmetric) state plays the important role in these reactions. Since $l=l'$ for the reactions (I)$\sim$(IV), we may say as follows. (1) If the initial p-$\bar{p}$ system is in an S-state it may be expected that the antisymmetric state has the large effect on reaction (I).* (2) In reactions (II)$\sim$(IV), the symmetric states react strongly. If Y_0^* and Y_1^* are the isobaric states with the same orbital angular momentum in Y-π system, the same conclusion as (2) may also be obtained with regard to reactions (V) and (VI).

* It must be noted that the parity of antiproton is odd.

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Errata

Parametric Integral Formulas and Analytic Properties in Perturbation Theory

Noboru Nakanishi

Prog. Theor. Phys. Suppl. No. **18** (1961), 1

p. 16	Eq. (4.8)	$\sum_{\sigma\in\sigma(i)}=\partial U/\partial x_i$	$\sum_{\sigma\in\sigma(i)}U_\sigma=\partial U/\partial x_i$
p. 19	8th line from bottom	x_{ij}	X_{ij}
p. 20	19th line	$C_i\cup(G-H)\neq\phi$	$C_i\cap(G-H)\neq\phi$
	22th line	$C_j\cup(G-(HC_i))\neq\phi$	$C_j\cap(G-(HC_i))\neq\phi$
p. 41	Eq. (10·29)	$(m_iy_i-m_jq_j)^2$	$(m_iy_i-m_jy_j)^2$
p. 58	Eq. (14·30)	$G(s)$	$G(\omega)$
p. 60	15th line from bottom	It it	It is

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CONTENTS

Page

Nuclear Elastic Scattering of Photons near the Particle Threshold Energy (I) ..... Tsutomu TŌHEI, Masumi SUGAWARA, Shigeki MORI and Motoharu KIMURA 1657

Some New Activities Produced by Fast Neutron Bombardments ..... Kasuke TAKAHASHI, Tokihiro KUROYANAGI, Haruo YUTA, Kyûya KOTAJIMA, Kunio NAGATANI and Haruhiko MORINAGA 1664

Weak Gamma Rays in the Decay of La^{140} ..... Hidekuni TAKEKOSHI, Naomoto SHIKAZONO and Poh-Kun TSENG 1674

The Magnetic Investigation of Cu-Butyrate Monohydrate and Cu-Stearate ..... Takashi WATANABE 1677

Velocity of Slowly Moving Dislocation ..... Tomoyuki TAKEUCHI 1682

A Note on the Theory of Barium Titanate ..... Hidetosi TAKAHASI 1685

Internal Magnetic Field in Rare Earth Metals ..... Jun KONDO 1690

Quenched-in Vacancy in Solid Solution Alloys ..... Yoshikazu HAMAGUCHI 1692

Some Experimental Studies on the Parasitic Ferromagnetism of αFe_2O_3 ..... Akira TASAKI and Shūichi IIDA 1697

Determination of the Extension of Linear Polymer Molecule in the Solution by the Light Scattering ..... Yasuhiro MIYAKE 1703

Three-Level Maser Action in Gas II. Experimental Study on Formic Acid ..... Tatsuo YAJIMA 1709

Plasma Oscillations in External Fields ..... Masao SUMI 1718

Characteristics of an Ammonia Beam Maser ..... Koichi SHIMODA 1728

Optical Study of Aniline Hydrobromide Single Crystal ..... Tsuruji IWAI 1739

On the *R* Centers in Lithium Fluoride ..... Akizo OKUDA 1746

Impulsive Motion of a Circular Cylinder in a Viscous Fluid at Small Reynolds Numbers ..... Sinzi KUWABARA 1762

Study of Temperature Variation of Ultrasonic Velocity and Adiabatic Compressibility in Some Liquids Part 1 ..... P. R. K. L. PADMINI and B. Ramachamdra RAO 1770

SHORT NOTES

Scintillation from the Solid Xenon ..... Sigeo TAKEMOTO, Takeo HASEGAWA and Saburo HOMMA 1777

Proton Magnetic Resonance and the Polymorphism of Triglycerides ..... Haruo NAKAJIMA 1778

Observation of the Tracks of Fission Fragments in Molybdenite ..... Kazuhiko IZUI and F. Eiichi FUJITA 1779

On the Extraordinary X-ray Diffraction Intensities of Quenched TiC Powder ..... Shiro FUJISHIRO 1780

The Flow of a Viscous Fluid past a Porous Flat Plate ..... Mitutosi KAWAGUTI 1781

Observations on the Etch Patterns in Deformed KCl Crystals .. Masayoshi SAKAMOTO 1782

Preparation of Gallium Phosphide Crystals and Their Etch Pattern ..... Sigeru IIZIMA and Makoto KIKUCHI 1783

Electroluminescence in Gallium Phosphide Single Crystals ..... Sigeru IIZIMA and Makoto KIKUCHI 1784

Crystal Distortion of FeO on Cooling past its Néel Temperature ..... Yoshiko SUZUKI and Eiichi ASADA 1785

Note on Fe^{3+} ions in Spinel Structure ..... Yoshikazu SUGIURA 1786

Surface Field Effect and Noise in Ge Grain Boundary ..... Yoshio INUISHI, Ken KOTANI and Ichiro TSURUMI 1787

Semiconducting Properties of $SnSe_2$ and $GeSe_2$ ..... Sizuo ASANABE 1789

Temperature Dependence of Nuclear Resonance Absorption Line Width in Dy^{161} .. Poh-Kun TSENG, Naomoto SHIKAZONO, Hidekuni TAKEKOSHI and Tokio SHOJI 1790

Cubic Contribution to EPR of Mn^{++} in $MgCl_2$ ..... Hiroaki MATSUMOTO, Kuniya FUKUDA, Masumi TAKESHIMA and Yoichi UCHIDA 1791
An Evidence for the Existence of a Diffusion Layer in the Melting Process of Semiconductors ..... Shoji TAUCHI 1792
The Dynamic Bulk and Shear Viscosities of Polyvinylchloride ..... Ryusuke KONO 1793
Reflection of a Weak Shock Wave in a Perfectly Conducting Gas ..... Shigeki MORIOKA 1794
Some Extra Lines in Paramagnetic Resonance Spectrum of Heavily Cr^{3+} Doped $K_3Co(CN)_6$ ..... Takeshi MITSUMA 1796
Electrical Properties of TlSe ..... Ryuichi MORI and Kimio HASHIMOTO 1797

Errata

Cleavage Fracture in Single Crystals of Silicon Iron ..... Ryūkichi HONDA 1776

JOURNAL OF THE PHYSICAL SOCIETY OF JAPAN

Volume 16, Number 10, October 1961

CONTENTS

Page

Angular Distributions of Photo-protons from Li, Al, S, Ni, and Cu ..... Masayoshi MASUDA 1801
Photoprotons from Silicon and Phosphorus ..... Katsufusa SHODA, Shoichi SHIINA, Keisuke KOBAYASHI, Ken ABE and Motoharu KIMURA 1807
A Linear Electron Accelerator as the Injector of 1 BeV Synchrotron ..... Tetsuji NISHIKAWA, Jiro TANAKA, Akira MIYAHARA and Hiroo KUMAGAI 1817
"Gamma Rays from the Proton Bombardment of Sulphur and the O^+ Second Excited State of ^{32}S" ..... Yasuo HIRAO 1828
Reaction $O^{16}(\gamma, p)N^{15}$ by 20.5 MeV Polarized Bremsstrahlung ..... Katsufusa SHODA 1841
Angular Distribution for the $N^{14}(d, p)N^{15}$ Ground State Reaction ..... Susumu MORITA, Toshiyuki ISHIMATSU, Tetsuji CHÔ, Yutaka NAKAJIMA, Nawoyuki KAWAI, Tôru MURATA and Yûki HACHIYA 1849
(α, p) Reactions near $Z=26$ ..... Itaru NONAKA, Hisashi YAMAGUCHI, Takashi MIKUMO, Iwao UMEDA, Tatsuro TABATA and Saburo HITAKA 1853
Resonant Scattering of Photons in Mg and Si ..... Masumi SUGAWARA 1857
Effect of Impurity-Core on Carrier Mobility in Heavily Doped Germanium ..... P. CSAVINSZKY 1865
Sample Sandwiched Plastic Scintillators for Activation Measurements by Use of Fast Neutrons ..... Masakatsu SAKISAKA, Bunzaburō SAEKI, Michio TOMITA and Fumio FUKUZAWA 1869
On the New Magnetic Phase in Manganese-Aluminium-Copper System ..... Ichiro TSUBOYA 1875
Magnetic and Electric Properties of Chalcopyrite ..... Teruo TERANISHI 1881
Effect of the Substructure upon Magnetic Properties of Iron-Silicon Single Crystals ..... Yoji NAKAMURA 1888
Dislocation Images in Pure Iron Observed by Transmission Electron Microscopy ..... Hiroshi FUJITA and Zenji NISHIYAMA 1893
The Thermoelectric Power of the Gamma-Brasses ..... Seiichiro NOGUCHI 1909
Magnetic Anisotropy of Evaporated Films. I. Iron Films Formed in Magnetic Field ..... Minoru TAKAHASHI, Denjiro WATANABE, Tetsuzo SASAGAWA, Tatsuo KŌNO and Shiro OGAWA 1913
The Effect of Heat Treatment on the Minority Carrier Lifetime in Silicon ..... Toshihumi ASAKAWA 1923
Magnetic Anisotropy of $KCoF_3$ and $KNiF_3$ ..... Kazuyoshi HIRAKAWA, Takasu HASHIMOTO and Kinshiro HIRAKAWA 1934
Electron Paramagnetic Resonance for Metals with Two Conduction Bands ..... Tsuyoshi MURAO 1940
Theory of Cyclotron Resonance in Anisotropic Metals ..... Hideo SUZUKI, Masumi HATTORI and Kanetaka ARIYAMA 1947
The Shielding of a Fixed Charge in a Metal ..... Shozo SHINOHARA 1963

Galvanomagnetic Effects in Bismuth Selenide Bi_2Se_3 ..... Kimio HASHIMOTO 1970
Hall Effect in *N*- and *P*-type Germanium at 24 Gc ..... Naozo WATANABE 1979
Structure of Thin Layers of Some F.C.C. Metals Deposited on Oriented Ag, Pd and Ni Films.....Shunji SHIRAI, Yoshiichi FUKUDA and Masayoshi NOMURA 1989
X-ray and Magnetic Studies of the Manganese-Germanium System....Tetuo OHOYAMA 1995
Friction and Deformation of Polymers.....Kyuichiro TANAKA 2003
Self-Absorption of Spectral Line ..... Toshiharu TAKO 2016
Structure and Properties of *A*-Centers in KCl .....
.....Kazuko KOJIMA, Noriko NISHIMAKI and Tadanobu KOJIMA 2033
Microscope Observations of Crystals Appearing in the Process of Cooling Aqueous Solutions of Alcohols.....Kenji YAMAJI 2045
High Temperature Determination by a New Gas Thermometer with Constant Bulb Temperature. I. Measurimg Methods ..... Jiro OISHI 2056

SHORT NOTES

The Specific Heat of $Ni(NH_3)_6Cl_2$ and $Ni(NH_3)_6Br_2$ at Low Temperatures.....
..... Kazutoshi UKEI and Eizo KANDA 2061
The Relaxation Phenomena in Polyelectrolytes Solutions..... Noboru NISHIDA 2062
Observation of Optical Maser Oscillation in Ruby..Kanji KUBOTA and Kazuo HAYASHI 2063
Frequency Dependence of Ferromagnetic Crystalline Anisotropy of $Ni_xFe_{3-x}O_4$ ....
..... Seiji USAMI, Nahonori MIYATA and Zenya FUNATOGAWA 2064
Nuclear Magnetic Resonance of Fe^{57} in Fine Powdered Iron .....
..... Shizuko OGAWA and Setsu MORIMOTO 2065
Stress Annealing Effect in Cobalt .....Takashi SAMBONGI and Tadayasu MITUI 2066
Heavily Deffused Si pn-Junction ..... Masakatsu TOKUNAGA and Katsufusa SHONO 2067
Phonon Spectrum of Germanium-Silicon Alloys.....
..... R. BRAUNSTEIN, A. R. MOORE and F. HERMAN 2068
Helium Discharge Chamber.....
.....Shūji FUKUI, Satio HAYAKAWA, Ryoichi KAJIKAWA and Ken KIKUCHI 2069
Electron Microscopic Observation of the Aggregates of Vacancies in Germanium ..
....Kenzo TANAKA, Michihiko MANNAMI, Kunihide IZUMI and Hiroko NIWA 2070
Theory of Non-Linear Responses of Many Variable Systems .....
.....Koji OKANO and Osamu NAKADA 2071
Several Experiments on the Memory Effect in αFe_2O_3.....
.....Masako IWATA, Takao IWATA and Mikio YAMAMOTO 2072
Photoconduction of γ-ray Irradiated KCl Single Crystals at Higher Electric Field ..
..... Yoshio INUISHI and Yung-Kwun SUNG 2074
On the Coupling between Optical Lattice Vibrations and Carrier Plasma Oscillations in Polar Semiconductors .....Isaaki YOKOTA 2075
Electron Spin Resonance of the *M* center in Potassium Chloride .....
.....Hiroshi OHKURA and Kazuo MURASE 2076
Ultrasonic Attenuation in Al and Al-Cu Alloys at Low Temperature.....
..... Kazuo KAMIGAKI and Tokutaro HIRONE 2077

Errata

Magnetic Properties of Iron Phosphide Fe_2P .....Shū CHIBA 2078
Local Description of the Flow behind a Steady, Two-Dimensional, Curved Shock Wave.....Zirō HASIMOTO and Shigeki MORIOKA 2078
Measurement of Anomalous Absorption Coefficients of Electrons for MgO Crystals
.....Kazutake KOHRA and Hiroshi WATANABE 2078
Announcement of Proceedings of the International Conference on Cosmic Rays and the Earth Storm..... 2079
Announcement of Proceedings of the International Conference of Magnetism and Crystallography..... 2080

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Partial Equilibrium Model for Nuclear Reactions

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A new model is proposed for nuclear reactions in order to investigate the "intermediate resonances" detected experimentally in the intermediate energy region. The basic assumption is that only several nucleons of the target nucleus are excited at the compound system. Calculations are made in terms of the concept of the partial-equilibrium-compound-nucleus and the direct interaction developed by Austern-Butler-McManus. Numerical results for the level distances of the "intermediate resonances" agree with experiment. The number of excited nucleons in the compound system is about six and depends slightly on the excitation energy or the mass number of target nucleus (in this energy region). Further it is pointed out that outgoing channels couple with each other to form a *gross channel* corresponding to the "intermediate level" of the residual nucleus. The average distances between the *gross channels* are about 300 kev.

§ 1. Introduction

The concept of the compound nucleus has succeeded in explaining low energy nuclear reactions.[1] On the other hand, the idea of direct interaction[2] has been useful for high energy nuclear reactions. The angular distribution is usually explained by the theory of direct interactions, for example, pick up or stripping process,[3] while the energy dependence is calculated by the use of the statistical model[4] in the intermediate energy region.

Recently the excitation curves and the angular distributions of protons inelastically scattered from various nuclei have been investigated experimentally by many authors.[5)~20)] Sharp energy dependences have been observed in these excitation curves (typical examples are shown in Figs. 1~4). There is no exception under the following circumstances:

$$\begin{aligned} &A > 20, \\ &\varepsilon = 15 \sim 30 \text{ Mev}, \\ &\Delta E < 100 \text{ kev}. \end{aligned} \tag{1}$$

Here A, ε, and ΔE are the mass number of target nucleus, excitation energy, and energy resolution of the experiments, respectively. These sharp energy dependences of the excitation curves seem to be a general feature of intermediate energy nuclear reactions, since they are independent of incident particle, target

nucleus, and reaction type. The simple direct interaction theory cannot explain these energy dependences because they fluctuate so rapidly, while the compound nucleus model produces too sharp resonances. The energy dependence of the angular distributions cannot also be explained by these usual theories. A new model[1] is introduced in this paper to explain these sharp energy dependences of the excitation curves at intermediate energies.

In this model these peaks are assumed to be some kind of resonances, where the level widths and the level distances are about 100 kev and 400 kev, respectively, which are about 10^3 times as large as those predicted by compound nucleus model, and are one tenth of the giant resonance in the optical model. We refer to these resonances as "*intermediate resonance*" in the following. To make the concept of the intermediate resonances clear, we roughly estimate a time τ during which the interaction takes place. For the compound nucleus τ_{com} is given by the reciprocal of level width Γ; for slow neutron resonance

$$\tau_{com}=\hbar/\Gamma\sim 10^{-16}\ \text{sec.}$$

For the direct interaction process τ_{dir} is about the transit time of the incident particle through the target nucleus; for the medium weight nuclei

$$\tau_{dir}=R/v\sim 10^{-22}\ \text{sec},$$

where R and v are the radius of target nucleus and the velocity of incident particle in the target nucleus, respectively. In the case of the intermediate resonance, τ_μ is given by

$$\tau_\mu=\hbar/\Gamma_\mu\sim 10^{-20}\ \text{sec.}$$

Then we obtain

$$\tau_{com}\gg\tau_\mu\gg\tau_{dir}$$

which shows that the reaction ends before statistical equilibrium is achieved. Namely, the reaction proceeds through a certain partial equilibrium whose lifetime τ_μ is between τ_{com} of the compound nucleus and τ_{dir} of the direct process.

This suggests that a certain number of nucleons in the target nucleus strongly interact with the incident particle; the others produce the averaged potential, only through which they interact with the incident particle.

As is well known, the excitation energy is assumed to be shared among all nucleons of the target nucleus in the formation of the compound nucleus at low energies. On the other hand, only one nucleon of target nucleus strongly interacts with the incident nucleon in the impulse approximation at high energy. That is, the excitation energy given by the incident particle is shared among a smaller number of nucleons in the target nucleus as the incident energy increases. In the intermediate energy region, the incident particle should interact strongly with a certain number of nucleons in the target nucleus while it interacts with the other nucleons only through an average potential.

These considerations lead to the assumption that *the reaction proceeds through a certain partial equilibrium* in the intermediate energy region. Let us divide the nucleons of the target nucleus into two groups: the inert core nucleons and the outer nucleons. *The partial equilibrium means that only outer nucleons interact strongly with the incident particle, while the inert core remains in its ground state but produces an average potential for the incident particle.*

When the incident particle once enters into the target nucleus, the *compound states* are formed. The *compound states* do not mean the states of the well-known compound nucleus but the compound states of the incident particle and outer nucleons only.

A general formalism in our model is developed on two basic assumptions: the *partial equilibrium* assumption and the *compound state* assumption. In § 2, the formalism of nucleon-nucleus scattering and the differential cross sections are derived. Some numerical results are obtained and compared with the experiments in § 3 and §4. The physical meaning of this model and some theoretical predictions are discussed in § 5.

§ 2. Partial equilibrium model in the intermediate energy region

In this section the formalism of nucleon-nucleus scattering is derived for the intermediate energy nuclear reaction based on the idea explained in § 1.

For simplicity only the nucleon-nucleus scattering is treated in this section. The following treatment can be easily extended for more complicated case, e.g. (p, α) reactions. The Hamiltonian H of the system is generally written as*

$$H=T_p+V_{pr}+H_r+V_{pc}+V_{rc}+H_c, \tag{2}$$

where T_p is the kinetic energy of the incident nucleon, V_{pr}, V_{pc}, and V_{rc} are the interactions between the incident nucleon and the outer nucleons, between the incident nucleon and the inert core nucleons, and between the inert core nucleons and the outer nucleons, respectively.

The inert core nucleons are considered to interact with the incident nucleon only through an average potential. Therefore the interation V_{pc} can be replaced by an average interaction U_p which contains only the coordinates of the incident nucleon $\mathbf{r}_p$:

$$V_{pc}\rightarrow U_p(\mathbf{r}_p). \tag{3}$$

On the other hand, the basic assumption of the partial equilibrium reduces not only the interaction V_{pc} to U_p but also the interaction V_{rc} to an average inter-

* The outer nucleons and the inert core are specified hereafter by the suffices r and c, respectively.

action U_r since the inert core does not change its state during the reaction,

$$V_{rc} \to U_r(\mathbf{r}_r), \tag{4}$$

where U_r contains only the coordinates of the outer nucleons.

Our system may be described by the model Hamiltonian H_M which is written

$$H_M = T_p + V_{pr} + H_r + U_p + U_r + H_c. \tag{5}$$

The wave function Ψ of the system is then written as follows:

$$\Psi = \chi_o \Phi \tag{6}$$

where Φ is the wave function of the outer nucleons defined by

$$\begin{aligned} &H_g \Phi = E\Phi, \\ &H_g = H_M - H_c \\ &\quad = T_p + V_{pr} + H_r + U_p + U_r. \end{aligned} \tag{7}$$

Here H_g and E are the Hamiltonian and the total energy of the system which consists only of the outer nucleons and the incident nucleon, and the inert core is expressed by the wave function χ_o which remains in its ground state. As is easily seen from Eq. (6), Ψ is a state of the resonating group[23] whose constituents are the outer nucleons and the inert core.

Owing to the compound state assumption it is convenient to separate V_{pr} into two parts, i.e. the outside interaction V_{pr}^{e} and the inside interaction V_{pr}^{i} defined by

$$\begin{aligned} V_{pr}^{i} &= \begin{cases} V_{pr} & (r_p < R) \\ 0 & (r_p \geqq R), \end{cases} \\ V_{pr}^{e} &= \begin{cases} 0 & (r_p < R) \\ V_{pr} & (r_p \geqq R), \end{cases} \end{aligned} \tag{8}$$

where r_p is the radial coordinate of the incident nucleon in the center of mass system, and R is the interaction radius at which the incident nucleon and the target nucleus form the *compound state*.* The Hamiltonian of the unperturbed system is written as

$$H_o^M = H_g - V_{pr}. \tag{9a}$$

The channel wave function ϕ_t satisfies the equation

$$H_o^M \phi_t = E_t^c \phi_t \tag{9b}$$

where E_t^c is the channel energy. The transition matrix T_{ba} can be separated into two parts,

* R can be chosen arbitrary in principle. However, to obtain a well defined reduced width it is most convenient to choose the same as the nuclear radius of the target nucleus.

$$T_{ba}\equiv T_{ba}^{(1)}+T_{ba}^{(2)}, \tag{10}$$

where suffices a and b denote the entrance and the outgoing channels, respectively. The first term in Eq. (10) contains only V_{pr}^{i} and is given by[24]

$$T_{ba}^{(1)}=\langle\phi_b|V_{pr}^{i}[1+(E+i\varepsilon'-H_o^M-V_{pr}^{i})^{-1}V_{pr}^{i}]|\phi_a\rangle, \tag{11}$$

where ε' is a small positive number. The matrix $T_{ba}^{(2)}$ is written as

$$\begin{aligned}T_{ba}^{(2)}=&\langle[1+(E-i\varepsilon'-H_o-V_{pr}^{i})^{-1}V_{pr}^{i}]\phi_b|V_{pr}^{e}|\\&\times[1+(E+i\varepsilon'-H)^{-1}V_{pr}^{e}]\cdot[1+(E+i\varepsilon'-H_o-V_{pr}^{i})^{-1}\cdot V_{pr}^{i}]\phi_a\rangle.\end{aligned} \tag{12}$$

It is convenient to introduce a complete set of the internal wave functions, since it was assumed in § 1 that the *compound states* are formed as soon as the incident nucleon enters into the target nucleus. The wave functions Λ_μ of the *compound state* satisfy the wave equation

$$H_g^{i}\Lambda_\mu=E_\mu\Lambda_\mu \tag{13a}$$

where

$$H_g^{i}=T_p+V_{pr}^{i}+H_r+U_p+U_r. \tag{13b}$$

The boundary conditions given by Teichman-Wigner[1] are taken as

$$\int_S\{(\mathrm{grad}\,\Lambda_\mu)_n+b_t\Lambda_\mu\}\,\phi_t\,dS=0, \tag{14}$$

where S is the surface of the intermediate system, n denotes the normal component of the surface, and b_t is a real constant. The set of functions ϕ_t is complete on the nuclear surface. Using the complete set of states $|\Lambda_\mu\rangle$, we obtain

$$T_{ba}^{(1)}=\langle\phi_b|V_{pr}^{i}|\phi_a\rangle+\sum_\mu\frac{\langle\phi_b|V_{pr}^{i}|\Lambda_\mu\rangle\langle\Lambda_\mu|V_{pr}^{i}|\phi_a\rangle}{E-E_\mu}. \tag{15}$$

This is easily rewritten by means of the reduced widths $\theta_{\mu a}$ and $\theta_{\mu b}$ (see the Appendix, Eq. (A1)) as

$$T_{ba}^{(1)}=\sum_\mu\sum_{l=|J-s|}^{J+s}\sum_{l'=|J-s'|}^{J+s'}(-2i)\frac{\theta_{\mu a}\theta_{\mu b}}{E-\varepsilon_\mu}A(a,b,\mu,\theta,\Phi), \tag{16}$$

where (θ,Φ) are the angles between the incident and the outgoing nucleons, and ε_μ is the complex eigenvalue of the intermediate system. The factor A is given by (see the Appendix, Eq. (A4))

$$\begin{aligned}A(a,b,\mu,\theta,\Phi)=&\left(\frac{\hbar^2}{km}\right)\left(\frac{k\hbar}{k'}\right)^{1/2}\pi^{1/2}i^{l-l'}(2l+1)^{1/2}\sum_{M=-J}^{J}\sum_{\nu'=-l'}^{l'}\\&\times(ls0m_s|l_sJM)\cdot(l's'\nu'm_{s'}|l's'JM)\,Y_{l'\nu'}(\theta,\Phi),\end{aligned} \tag{17}$$

where J, M, s, m_s, l, and ν are the total angular momentum, its component, the

channel spin, its component, the angular momentum of the incident wave, and its component, respectively. The quantities of the outgoing channels are discriminated by the prime (′).

It is to be noted that $T_{ba}^{(1)}$ and $T_{ba}^{(2)}$ are coherent in our model.* Thus using Eqs. (10), (15), and (17) we obtain the differential cross section for the scattering of nucleons by the nucleus (cf. the Appendix, Eq. (A3)) as

$$\frac{d\sigma(a,b)}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left(\frac{k'}{k}\right) (2I_a+1)^{-1} \cdot |T_{ba}^{(2)} + \sum_{\mu} \sum_{l} \sum_{l'} (-2i) \frac{\theta_{\mu a}\theta_{\mu b}}{E-\varepsilon_{\mu}} A(a,b,\mu,\theta,\Phi)|^2, \tag{18}$$

where k and k' are the wave numbers of the incident and the scattered nucleons, m is the nucleon mass and I_a is the channel spin of the initial state.

The second term in the last factor of Eq. (18) is not the one due to the formation of the usual compound nucleus, but is due to that of the *compound state* consisting of the incident nucleon and the outer nucleons only. Therefore the resonances would be very well spaced even in the intermediate energy region. On the other hand, $T_{ba}^{(2)}$, i.e. the direct interaction term is known to be also appreciable in the intermediate energy nuclear reactions. Thus the angular distribution of the inelastic scattering would be very complicated in general since these two terms interfere each other.

In practical cases in which the contribution of a few levels is dominant in the second term of Eq. (18) and the distorted wave Born approximation[25] is used for the first term, Eq. (18) may be approximated as follows,**

$$\frac{d\sigma(a,b)}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left(\frac{k'}{k}\right) (2I_a+1)^{-1} \cdot |\langle\phi_b|V_{pr}^{c}|\phi_a\rangle + \sum_{\mu}{}' \sum_{l=|J-s|}^{J+s} \sum_{l'=|J-s'|}^{J+s'} (-2i) \frac{\theta_{\mu a}\theta_{\mu b}}{E-\varepsilon_{\mu}} A(a,b,\mu,\theta,\Phi)|^2 \tag{19}$$

where the summation $\sum_{\mu}{}'$ is taken over a few neighbouring levels.

Eq. (19) consists of two terms: One is the resonance term just like that appears in Breit-Wigner's many level formula[1] and the other is the surface direct

* The condition under which the intermediate levels are detected experimentally is that

$$D_{\mu} > \Gamma_{\mu} > \Delta E$$

where D_{μ} is the level distance between the intermediate levels. In other words, the time duration of the incident and the outgoing wave packet ($\geq \hbar/\Delta E = 10^{-20}$ sec) is longer than the half lives of our *compound states* ($\hbar/\Gamma_{\mu} = 6\times10^{-21}$ sec) and the reaction time of the direct process given in §1 ($\sim 10^{-22}$ sec). Thus the waves emitted through the *compound state* and through the direct process should be coherent in our model, namely $T_{ba}^{(1)}$ and $T_{ba}^{(2)}$ interfere each other.

** The cross section Eq. (19) should be applied at lower energy. At higher energy, the first term in Eq. (15) becomes more important. Then, we should replace the first term in Eq. (19) by $\langle\phi_b|V_{pr}|\phi_a\rangle$ (see the Appendix, Eq. (A5)).

interaction term such as that given by Austern et al.[2] When the former contribution dominates the latter contribution the angular distribution changes its shape with the energy abruptly. Conversely, when the latter is dominant, the angular distribution shows a characteristic diffraction pattern and changes its shape with the energy very slowly.

§ 3. Intermediate levels

It was assumed in § 2 that the sharp energy dependences of the excitation curve were due to the resonances caused by the interaction between the incident particle and the outer nucleons. In this section, we are going to calculate the level distance D_μ by using H_0 in Eq. (7) and Weisskopf's semi-empirical level density formula[26] under this assumption.

Under the *partial equilibrium* assumption the nucleons in the *compound state* should spread over all the volume of the target nucleus since many collisions may happen in the compound system to attain the thermal equilibrium. Among the terms of Eq. (7) U_p and U_r may be considered to work to determine this size of the interaction volume, and the interaction would be described by

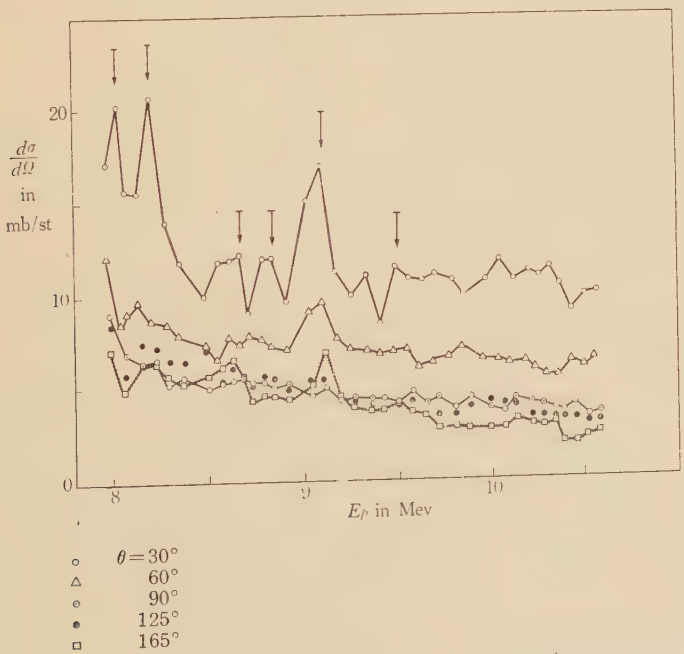

Fig. 1. Excitation curve of the $Ni^{58}(p, p')Ni^{58*}$ reaction.[5] The intermediate resonance peaks are shown by the arrows. The symbol * shows the first excited state. The angular distributions are also measured at several energies and show a typical diffraction pattern at high energy ($E_p \sim 12$ Mev), and in low energy region they become gradually isotropic.

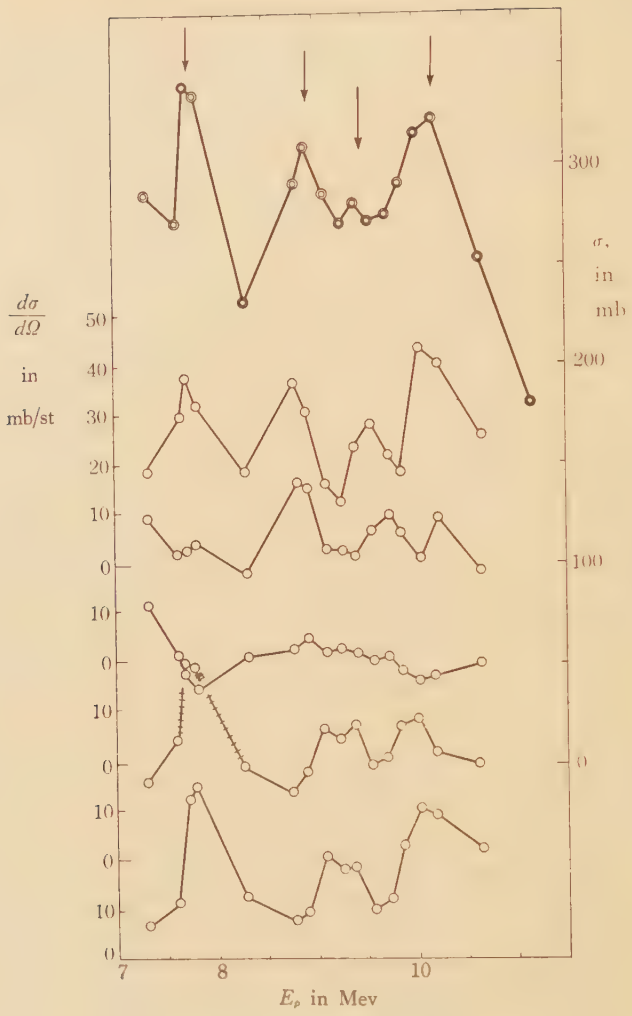

Fig. 2. The excitation curve for $Mg^{24}(p, p')Mg^{24*}$. The total cross section is estimated from the angular distribution measured at 17 angles. The arrows show the intermediate resonances observed in the total cross section.

the other terms.* Therefore in calculating $D_\mu(\varepsilon)$ we can replace H_g of Eq. (7) by

$$H_g' = H_g - U_p - U_r \qquad (20)$$
$$= H_r + V_{pr} + T_p.$$

Eq. (20) is the Hamiltonian for the scattering system which consists of the

* The average interactions U_p and U_r cause the giant resonance for the scattering. The intermediate resonances couple each other to form the giant resonance, and when these intermediate resonances are averaged by large energy scale the giant resonance only remains.

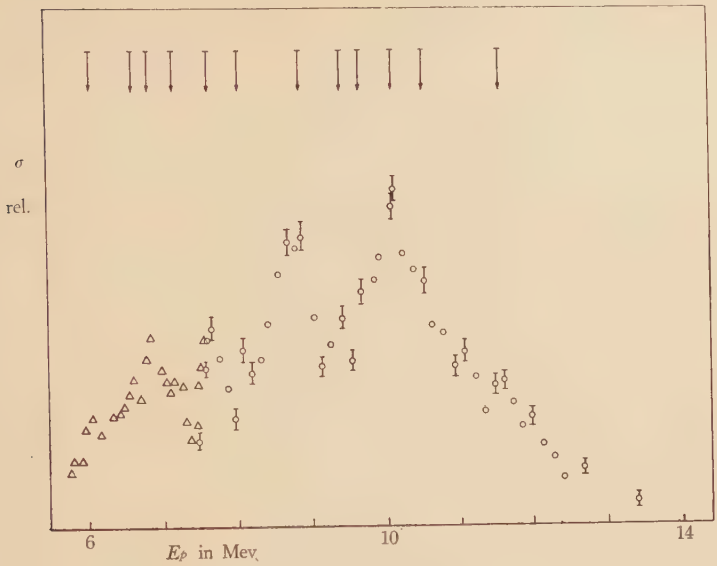

Fig. 3. Excitation curve of $Al^{27}(p, \gamma_0+\gamma_1)Mg^{28*}$. Transitions to the ground and the first excited states of Mg^{28} are mixed in it. The intermediate resonances are shown by the arrows.

incident particle and the outer nucleons. The interaction radius must be taken as same as that of the whole nucleus because of the assumption that the incident particle and the outer nucleons spread over all the volume of the nucleus.

The level density D_μ^0 of the above scattering system is calculated by using the Weisskopf semi-empirical level density formula which gives satisfactory results for the level densities of slow neutron scattering. That is,

$$D_\mu^0(\varepsilon_0) = \boldsymbol{c}^{-1} \exp\{-2\sqrt{\boldsymbol{a}\varepsilon_0}\}, \tag{21}$$

where ε_0 is the excitation energy of the compound system and $\boldsymbol{a}$ and $\boldsymbol{c}$ are parameters depending on the mass number A. It is to be noted that the interaction radius of the system in Eq. (20) is larger than that of the usual compound nucleus which has the particle number same as in our scattering system but has the normal density. As is well known, the level distance of the single particle model is proportional to R^{-2}. We assume this relation holds also for the transition from the usual compound nucleus in the above sense to our compound system which consists of the same number of nucleons but has larger interaction radius (cf. Fig. 8). Then we easily obtain*

$$\varepsilon_0 = \left(\frac{A+n}{N+n}\right)^{2/3} \varepsilon \tag{22}$$

* This does not correspond to the treatment for the ideal gas, but to that for the nucleon gas having some correlation among them.

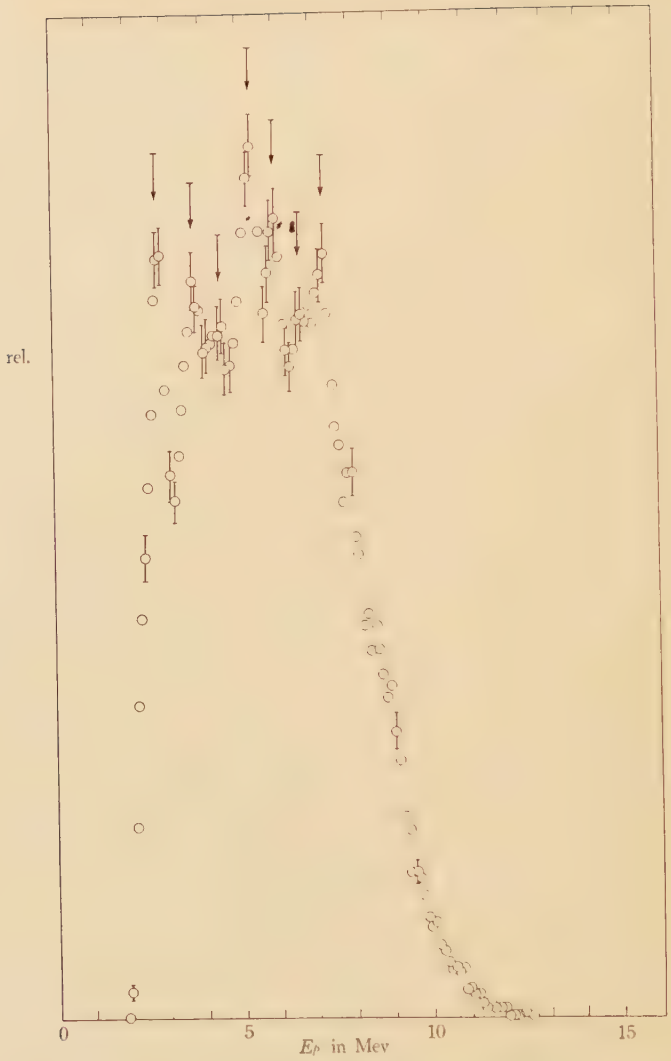

Fig. 4. The energy spectrum for $Si^{28}(\gamma, p)Al^{27}$. The ordinate is the relative numbers of protons. Bremsstrahlung γ-ray with 24 Mev maximum energy are used. Arrows show the gross channels.

and

$$D_\mu(\varepsilon) = \left(\frac{N+n}{A+n} \right)^{2/3} D_\mu{}^n(\varepsilon_a), \tag{23}$$

where n is the nucleon number of the incident particle, ε is the excitation energy of the compound system, and N and A are the number of outer nucleons and the mass number of target nucleus, respectively.

Next we have to determine the parametric values of $\boldsymbol{a}$ and $\boldsymbol{c}$ from known

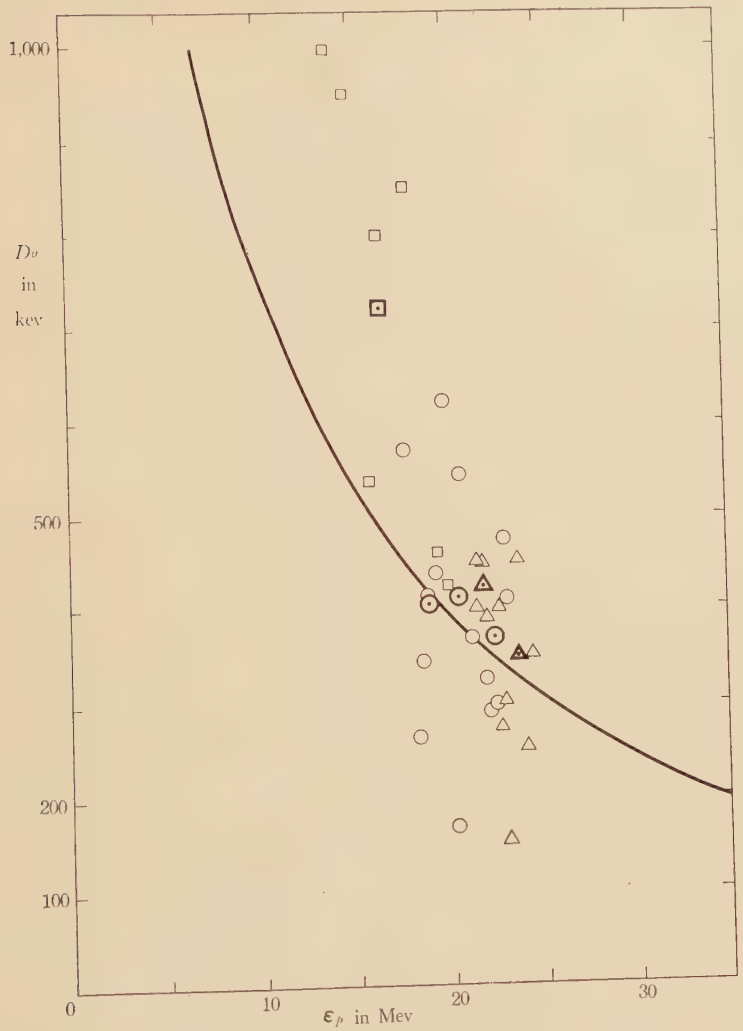

Fig. 5. The excitation energy dependence of the intermediate level distance D_μ for the Si^{28} compound nucleus. The theoretical curve is calculated from Eq. (23) for $(N+n)=6$ and $(A+n)=28$. The circles, triangles, and squares show the experimental level distances between each intermediate levels measured from the reactions $Al^{27}(p, \gamma)Si^{28*}$, $Al^{27}(p, \alpha)Mg^{24}$ and $Al^{27}(p, \alpha)Mg^{24*}$, and $Si^{28}(\gamma, p)Al^{27}$. The bold faced symbols show the averaged values.

data on the resonances in actual light nuclei. To do this we must first note the possible mass number dependence of these parameters and thus we need to know their values at least for two nuclei with different mass numbers. We first take the data on a nucleus with $A=27$,[26] and then equate the right-hand side of Eq. (21) to the mean value of the energy of the first excited state in

several light nuclei.*) The values of a and c thus fixed are

$$a=6.11\cdot(N+n)^2\cdot 10^{-4}\ \text{Mev}^{-1}, \qquad (23')$$

$$\log c=-1.20+0.0334(N+n)\ ;\ c \text{ in Mev}^{-1}.$$

Fig. 5 shows how Eq. (23) depends on the excitation energy in the case of $(A+n)=28$. The mean values of experimental level distances D_μ^{exp} lie on the theoretical curve corresponding to $(N+n)=6$. This figure also shows that the model can be applied up to about $\varepsilon=25$ Mev for the compound nucleus of $(A+n)=28$. On the other hand the excitation curves of $Ni^{58}(p, p')Ni^{58*}$, $Ni^{60}(p, p')Ni^{60*}$, $Mg^{24}(p, p')Mg^{24*}$, and $Ne^{20}(p, p')Ne^{20*}$ show that our model can be applied up to about 20 Mev (cf. Fig. 1, Fig. 2, and Table I). The mass number

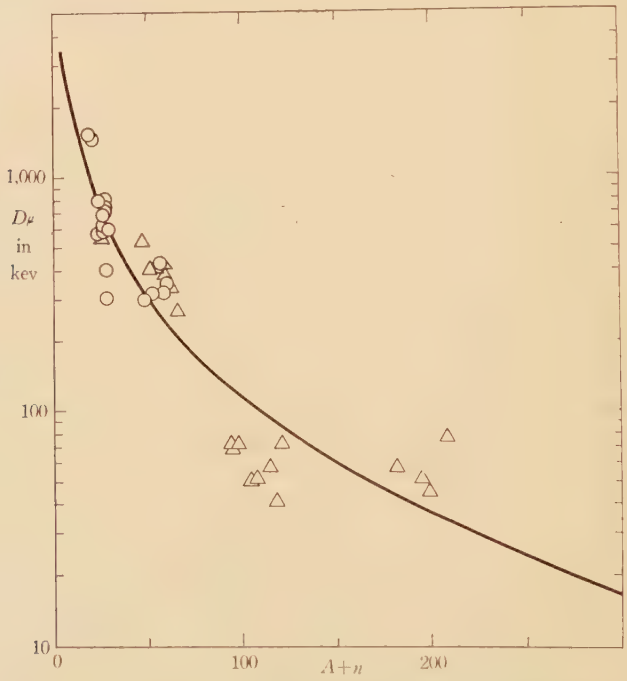

Fig. 6. The mass number dependence of the intermediate level distances D_μ and D_t. The theoretical curve is calculated from Eq. (23) for $(N+n)=6$ and $\varepsilon=12$ Mev. The circles and the triangles show the experimental points derived from the excitation curves and the energy spectrum, respectively. They are also shown in Tables I and II.

*) The parameters are determined by fixing p and q, of the equation,

$$\log D=p+(N+n)q$$

at $A=27$ and at light nuclei. Data on D at high excitation energy is not used since the energy resolusion is not reliable.

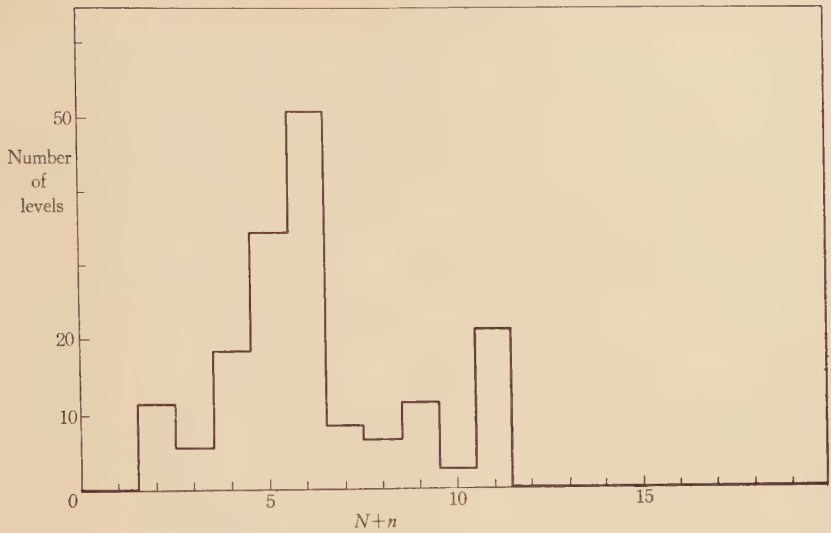

Fig. 7. The number of times each outer nucleon number $(N+n)$ appears, calculated from the experimental values D_μ^{exp} by means of Eq. (23).

Table I. The intermediate levels measured in excitation curves. The notations Q, B, and ε are the Q-value of the reaction, the binding energy of proton, and the mean excitation energy of the compound nucleus, respectively. D_μ^{exp} and D_μ^{theor} are the level distances measured experimentally and predicted theoretically, respectively. The number of the intermediate levels observed by experiments is shown in the eighth column. The * and ** show the first excited state and the mixture of the first and the ground state, respectively. Those without * means the ground state.

Type	Q (Mev)	B (Mev)	Energy range	ε (Mev)	D_μ^{exp} (kev)	D_μ^{theor} (kev)	No. of Peaks	Reference
$F^{19}(p, \alpha_0)O^{16}$	8.12	12.82	12.9~14.1	26.1	700	460	2	15)
$Ne^{20}(p, p')Ne^{20*}$	−1.63	2.0	7.4~15.5	13.5	1,200	830	6	16)
$Mg^{24}(p, p')Mg^{24*}$	−1.37	2.56	3.5~ 7.0	8.5	730	960	7	11)
			7.3~11.0	11.6	780	760	4	6)
$Al^{27}(p, \alpha)Mg^{24}$	1.60	11.57	10.5~14.5	23.6	350	320	7	9)
			9.5~11.0	21.8	420	360	3	12), 13)
$Al^{27}(p, \alpha)Mg^{24*}$	0.23	11.57	9.5~11.0	21.8	430	360	3	9)
			10.5~14.5	23.0	290	340	5	12), 13)
$Al^{27}(p, \gamma)Si^{28**}$	11.57	11.57	6.0~12.0	19.0	400	410	15	7), 17)
				20.6	410	380		7), 17)
				22.5	370	340		7), 17)
$Si^{28}(p, p')Si^{28*}$	−1.78	4.22	8.0~ 9.5	13.0	360	580	3	14)
			11.0~12.0	15.9	230	470	3	14)
$Ti^{48}(p, p')Ti^{48*}$	−0.99	6.34	4.0~ 6.5	11.1	310	340	9	11)
$Cr^{52}(p, p')Cr^{52*}$	−1.44	4.32	4.0~ 6.5	10.1	370	340	6	11)
$Fe^{56}(p, p')Fe^{56*}$	−0.85	6.29	4.0~ 6.5	12.2	410	250	5	11)
$Ni^{58}(p, p')Ni^{58*}$	−1.45	3.42	8.0~11.0	12.6	300	230	8	5), 10)
$Ni^{60}(p, p')Ni^{60*}$	−1.35	4.79	7.0~ 9.0	12.8	290	210	7	5), 10)

Table II. The gross channels measured in energy spectrum. The notations Q and ε are the Q-value of the reaction and the mean excitation energy of the residual nucleus, respectively. D_t^{exp} is the mean energy distance between the gross channels measured experimentally in the energy spectrum. D_t^{theor} is the theoretical prediction. The energy resolution of these experiments is less than 100 kev.

Type	Energy range (Mev)	ε (Mev)	D_t^{exp} (kev)	D_t^{theor} (kev)	Peaks	Reference
$Al^{27}(\gamma, p)Mg^{26}$	E_p=2.0~12.0	10.0	670	730	7	18)
$Si^{28}(\gamma, p)Al^{27}$	2.0~12.0	7.6	720	960	7	8)
$P^{31}(\gamma, p)Si^{30}$	2.5~14.0	11.1	660	630	7	18)
$Ti(p, d)Ti$	E_d=9.5~14.6	6.8	850	520	7	19)
$V(p, d)V$	9.0~12.2	5.3	800	580	5	19)
$Mn^{55}(p, d)Mn^{54}$	8.5~11.8	5.4	820	530	5	19)
$Fe(p, d)Fe$	8.9~12.6	5.5	920	520	5	19)
$Co^{59}(p, d)Co^{58}$	9.0~13.8	5.6	800	480	7	19)
$Ni(p, d)Ni$	8.5~11.1	4.5	870	560	4	19)
$Cu(p, d)Cu$	7.9~11.2	4.3	820	530	5	19)
$Zn(p, d)Zn$	8.1~11.3	4.4	640	510	6	19)
$Zr(d, p)Zr$	Q=0 ~ 5.0	3.9	210	370	25	20)
$Nb^{93}(d, p)Nb^{94}$	2.2~ 4.9	2.9	250	450	12	20)
$Mo(d, p)Mo$	1.3~ 5.2	3.2	240	410	17	20)
$Rh^{103}(d, p)Rh^{104}$	2.9~ 4.5	2.7	200	420	10	20)
$Pd(d, p)Pd$	0.9~ 5.0	3.4	180	350	24	20)
$Cd(d, p)Cd$	0.7~ 4.5	3.7	190	300	21	20)
$In(d, p)In$	2.6~ 4.6	2.7	170	370	13	20)
$Sn(d, p)Sn$	0.5~ 4.9	3.6	240	290	19	20)
$Ta^{181}(d, p)Ta^{182}$	1.0~ 3.7	3.5	240	180	12	20)
$Pt(d, p)Pt$	0.6~ 4.1	3.4	230	170	16	20)
$Au^{197}(d, p)Au^{198}$	1.1~ 4.4	2.9	240	190	15	20)
$Pb(d, p)Pb$	0 ~ 4.5	3.6	330	150	16	20)

dependence of Eq. (23) is shown for ε=12 Mev in Fig. 6. The experimental points concentrate also on the theoretical curve corresponding to $(N+n)=6$. The points for $A \geqq 100$ are obtained from the energy spectrum of (d, p) and (p, d) reactions. Their excitation energies are so small that it cannot be affirmed that they are the intermediate resonances. From known data on $D_\mu(\varepsilon)$, $(N+n)$ is calculated by using Eq. (23), and the number of times that each $(N+n)$ appears is plotted in Fig. 7 against $(N+n)$. This figure also shows the predominance of the cases with $(N+n)=6$. This parameter is inclined to increase for $\varepsilon \geqq 7$ Mev and to decrease gradually for $\varepsilon \gtrsim 30$ Mev in (p, p') data shown by Table I. The number $(N+n)$ does not seem to depend on the mass number within the presently available experimental data, but its mass number dependence or rather the dependence on the shell filling may be ascertained when enough experimental data are given.

The following two points are to be noted. First, the data given in Tables I, II, Figs. 5, 6, and 7 contain almost all types of nuclear reactions, (p, p'), (p, α), (p, γ), (γ, p) (d, p), etc. This indicates the validity of the *compound state* assumption in the intermediate system since it shows that the intermediate resonance does not depend on the type of nuclear reactions. Second, experimental points concentrate on the theoretical curves of $(N+n)=6$ rather than distribute themselves at random. This shows that nuclear reactions in the intermediate energy region proceeds mainly through the step of the *partial equilibrium* in the *compound state* of the resonating group consisting of outer nucleons and of incident particle rather than passage of the compound nucleus formation as a whole or the direct interaction. It can be concluded that only several nucleons (about six nucleons) form the compound system which plays an important role in the nuclear reaction, the excitation energy of which varying from 10 Mev to 30 Mev.

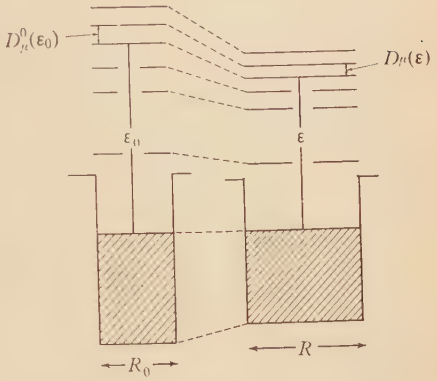

Fig. 8. The right-hand side shows the level scheme of our system and the left-hand side shows that of a nucleus of mass number $(N+n)$ but with the normal density. The levels of the former correspond one to one with those of the latter since it is assumed that the latter is formed by simply expanding the former.

§ 4. Gross channels

In the previous section we considered only the scattering of nucleons by nuclei, but it would be easy to understand that the same idea used there could as well be applied to other transmutation reactions. In either case, as the core is assumed to be inert, the number of possible channels for a fixed energy of the compound system would be much smaller than that expected in the case in which the whole nucleus is excited. Therefore, there would appear sharp bumps in the energy spectrum of the reaction products, the spacing of these bumps being again of the order of $D_\mu(\varepsilon)$. Actually, however, the core is not completely inert and thus the above sharp bumps would be broadened. In other words we get bunches of exit channels, which we shall call the "*gross channels*".

The channel wave function of the real system, ψ'_s, and that of our model, ϕ_t', are defined, respectively, by

$$H_o\psi'_s=\varepsilon_s^c\psi'_s, \qquad (24a)$$

$$H_o=H-V_{pr},$$

and

$$H_0^{M'}\phi_t'=E_t^{c'}\phi_t', \qquad (24b)$$

$$H_o^{M\prime}=H_M-V_{pr}.$$

Here ε_s^c and $E_t^{c\prime}$ are respective eigenenergies.

It is convenient to introduce here the complete set of wave functions defined inside the nuclear surface, since it is assumed that the intermediate system is expressed by our compound states which is also defined inside the nuclear surface. The complete set of the real system $|X_\lambda\rangle$, and that of our model $|\Lambda_\mu'\rangle$, are obtained as the solutions of the respective equations

$$HX_\lambda=\varepsilon_\lambda X_\lambda, \tag{25a}$$

and

$$H_M\Lambda_\mu'=E_\mu'\Lambda_\mu'. \tag{25b}$$

Here ε_λ, and E_μ' are the respective eigenvalues corresponding to the Teichman-Wigner boundary conditions:[1]

$$\int_S\{(\mathrm{grad}\,X_\lambda)_n+c_sX_\lambda\}\,\psi_s\,dS=0, \tag{26a}$$

and

$$\int_S\{(\mathrm{grad}\,\Lambda_\mu')_n+b_t'\Lambda_\mu'\}\,\phi_t'\,dS=0. \tag{26b}$$

S is the surface of the intermediate system, n is the suffix which shows the normal component of the surface, and c_s and b_t' are real constants. Then the reduced width of the real system, $\gamma_{\lambda s}$, and of our model, $\beta_{\mu t}$, are given by

$$\gamma_{\lambda s}=\left(\frac{\hbar^2}{2m}\right)^{1/2}\int_S\psi_sX_\lambda\,dS, \tag{27}$$

$$\beta_{\mu t}=\left(\frac{\hbar^2}{2m}\right)^{1/2}\int_S\phi_t'\Lambda_\mu'\,dS.$$

The reduced width $\gamma_{\lambda s}$ is expressed by replacing X_λ of Eq. (27) by that obtained from Eq. (26) as

$$\gamma_{\lambda s}=\sum_{\mu,t}\frac{\langle\phi_t'|(V_{rc}-U_r)|\psi_s\rangle\langle\Lambda_\mu'|(V_{rc}+V_{pr}-U_r-U_p)|X_\lambda\rangle}{(E_t^{c\prime}-\varepsilon_s^c)(E_\mu'-\varepsilon_\lambda)}\beta_{\mu t}. \tag{28}$$

This is the dispersion relation of the reduced width caused by the fluctuation of the potentials $(V_{pc}-U_p)$ and $(V_{rc}-U_r)$. It is to be noted that $\gamma_{\lambda s}$ disperses not only around the intermediate level μ but also around the channel t (cf. Fig. 9).

The strength function $s_\mu(E)$ becomes at $E=\varepsilon_\lambda$, ε_λ being the energy of an intermediate level, as

$$s_\mu(\varepsilon_\lambda)=\left\langle\frac{\gamma_{\lambda o}^2}{D_\lambda}\right\rangle_\mu, \tag{29}$$

where the symbol $\langle\ \rangle_\mu$ of the right-hand side shows that the average is taken over the states λ belonging to the intermediate level μ. The result after taking the average is*

$$s_\mu(\varepsilon_\lambda) = \frac{\beta_{\mu t}^2}{2\pi} \times \frac{W_1}{(E_\mu'-\varepsilon_\lambda)^2+1/4\cdot W_1^2}, \quad (30)$$

with

$$W_1^2 = C\langle \Lambda_\mu | (V_{pc}+V_{rc} -U_p-U_r)^2 | \Lambda_\mu \rangle. \quad (31)$$

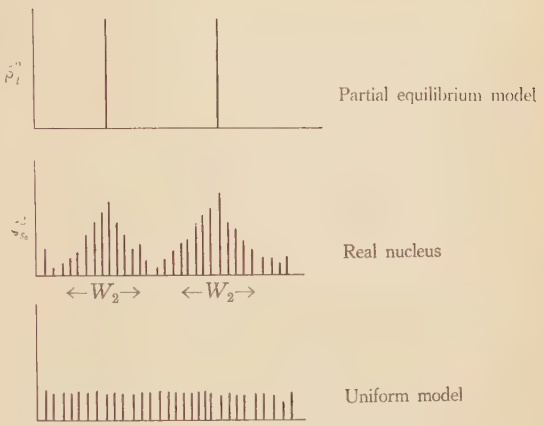

Fig. 9. The schematical explanation of the gross channel. The levels of the uniform model couple with the levels of our model to form the gross channel.

Eq. (30) shows that the intermediate γ-correlation[22] are to exist between the intermediate level and the compound nucleus levels.

Here we introduce the strength function s_t for the gross channel t. That is defined by

$$s_t(\varepsilon_s^c) = \left\langle \frac{\zeta_{so}^2}{D_s} \right\rangle_t, \quad (32)$$

where the symbol $\langle\ \rangle_t$ of the right-hand side shows that the average is taken over the channels s belonging to the gross channel t. The ζ_{so} is the reduced width for the s-channel which is mainly caused by the γ-decay of the excited residual nucleus and D_s is the energy distance between neighbouring s-channels. The channel s corresponds to one level λ' of the residual nucleus, and the gross channel t corresponds to one intermediate level μ' of the residual nucleus. Therefore, the function $s_t(\varepsilon_s^c)$ is equal to the strength function $s_\mu'(\varepsilon_\lambda')$ which might be observed in the elastic scattering of γ-ray by the residual nucleus. The energy ε_λ' is the excitation energy of the residual nucleus. Considering the dispersion relation Eq. (28) we obtain

$$s_t(\varepsilon_s^c) = s_\mu'(\varepsilon_\lambda') = \frac{\zeta_{so}^2}{2\pi} \frac{W_2}{(E_t^{c\prime}-\varepsilon_s^c)^2+1/4\cdot W_2^2} \quad (33)$$

with

$$W_2^2 = C\langle \phi_t' | (V_{rc}-U_r)^2 | \phi_t' \rangle. \quad (34)$$

* The reduced R-matrix is obtained by the same way as Thomas has developed (Phys. Rev. **97** (1955), 224). Then the strength function s_μ can be calculated by the Stieltjes transform of $\bar{R}$. The imaginary part W is treated more precisely by S. Hayakawa et al. (Prog. Theor. Phys. **21** (1959), 85) and C. Bloch (Nuclear Phys. **3** (1957), 137).

The right-hand side of Eq. (33) shows the gross structure of the channel s dispersing around the channel t. The function $s_t(\varepsilon_s^c)$ is smaller than $s_\mu(\varepsilon_\lambda)$ in general, since the reduced width ζ_{s0}^2 and the imaginary part W_2 are to be smaller than $\beta_{\mu t}^2$ and W_1.

These gross channels are to be observed when the compound system satisfies the circumstances (1) and when the excitation energy of the residual nucleus ε is larger than, say, 7 Mev so that the gross channel contains many real channels within its width.

The energy distances D_t between these channels can be calculated by using Eq. (23). In Figs. 5, 6, and Table II, the theoretical values of the energy distances D_t^{theor} are compared with the experimental values D_t^{exp} observed in (γ, p), (d, p), and (p, d) reactions. The agreement is quite good for $(N+n)=6$. However, in the relevant (d, p) and (p, d) reactions the excitation energy of the residual nucleus is so small (about 5 Mev) that each gross channel does not contain enough number of real channels. Also the gross channels which are observed in (γ, p) reactions are not very appropriate since the γ-rays used are not monochromatic. Thus it would have to be considered that the existence of the gross channels is not yet confirmed experimentally.

The width of the intermediate level Γ_μ and that of the gross channel Γ_t are given by

$$\Gamma_\mu = W_1 + \Gamma_\gamma + \Gamma_n \tag{34}$$

and

$$\Gamma_t = W_2 + \Gamma_\gamma, \tag{35}$$

where Γ_γ and Γ_n are the γ-ray width and the natural width for the outgoing channel. In order to form the stable partial equilibrium the *compound state* in our model must decay more rapidly to the outgoing channels than to the compound nucleus which excites the whole nucleus. In other words, the natural width Γ_n must be much larger than W_1. Comparing Eqs. (31) and (34) we obtain

$$\Gamma_n \gg W_1 > W_2. \tag{36}$$

This relation is rewritten as

$$\Gamma_\mu \sim \Gamma_n \gg \Gamma_t. \tag{37}$$

That is to say, the width of the intermediate level observed in the excitation curve is to be much larger than that of the gross channel which appears in the energy spectrum. From Eqs. (36) and (37) we can get an idea about the order of magnitudes of W_2 and Γ_n.

The natural width of s-wave resonance for the intermediate level Γ_n is proportional to the inverse of the nuclear radius R^{-1} since according to the result of the last section only six nucleons form the *compound system* regardless

of the target nucleus and the incident particle. Then we obtain

$$\Gamma'_{\mu} \sim \Gamma'_{n} \propto (A+n)^{1/3}.$$

This mass number dependence is weaker than that of the compound nucleus. The width Γ'_{μ} increases with the excitation energy more slowly in comparison with that of the compound nucleus model, since six nucleons are excited in our model, while all nucleons of the compound system are excited in the case of the compound nucleus model.

Finally it is to be noted that the excitation energy per nucleon in the *compound state* is larger than that in the compound nucleus for the same excitation energy. Therefore in our model the fraction of emission of high energy particles might be larger than that expected by the evaporation model. It is expected that the peak of the energy spectrum for particles nonelastically scattered by light nuclei shifts to higher energy by about 2~5 Mev compared with the evaporation model and is depressed by a factor of order 10.

§ 5. Conclusion and discussion

As the conclusion, under the circumstances shown in Eq. (1) we have the following results and the predictions:

i) The intermediate resonance is caused by the excitation of only several nucleons.

ii) About six nucleons are in partial equilibrium in the compound system and this number of nucleons depends slightly on the mass number of the target nucleus or the excitation energy.

iii) The gross channels caused because of the above mechanism of the reaction are to be measured in the energy spectrum.

iv) The level distance of the intermediate level and the channel distance of the gross channel are given by Eq. (23).

v) The imaginary parts of average potentials, W_1 and W_2, are smaller than about 200 kev. The level width of the intermediate level Γ'_{μ} will be larger than the width of the gross channel. The width Γ'_{μ} for s-wave resonance is proportional to $(A+n)^{-1/3}$ and it depends more weakly on the excitation energy than in the usual compound nucleus.

vi) The angular distribution of the nucleons scattered inelastically by various nuclei can be calculated by using Eq. (19). It should change its shape abruptly with the incident energy when the amplitudes which proceed through the intermediate levels are larger than those of the direct interactions. Otherwise it should change with energy smoothly.

vii) The peak of the energy spectra shifts to higher energy by about 2~5 Mev (for light nuclei, $A \sim 30$) and the peak height is much smaller than that expected from the evaporation model.

As is shown in Tables I and II the intermediate levels and the gross channels are observed in (p, p'), (p, α), (p, γ), (p, d), (γ, p), and (d, p) reactions. The excitation curves of the Si^{28}-compound nucleus formed by $Al^{27}(p, p')Al^{27*}$, $Al^{27}(p, \alpha)Mg^{24*}$, $Al^{27}(p, \gamma)Si^{28}$, and $Si^{28}(\gamma, p)Al^{27}$ reactions, are very similar to each other for the same excitation energy. This evidence suggests that the intermediate resonance is caused by a level of the compound nucleus. However, neither the spin nor the parity is yet measured experimentally. The gross channels are observed in the energy spectra of $Al^{27}(\gamma, p)$, $Si^{28}(\gamma, p)$, and $P^{31}(\gamma, p)$ reactions, but none of these experiments are very reliable since the γ-rays used are not monochromatic.

It is one of the important predictions that the intermediate resonances should be observed in the energy spectrum. This means that strong correlations are to exist among the reaction channels. To make these correlations clear will be a key for investigating the reaction mechanism, and experiments along this line are hoped to be performed.

The predictions of Eq. (23) agree reasonably well with the experiments for $(N+n)=6$. This number of $(N+n)$ is inclined to be larger in heavy unclei and at lower excitation energy, and it becomes smaller in light nuclei and at higher excitation energies. It depends weakly on the mass number of target nucleus and the excitation energy. However, its nuclear shell dependence or nuclear surface dependence might exist.

The width Γ_μ ranges from about 100 to 200 kev and this shows that the reaction mechanism is more complicated than the simple direct interaction process. The imaginary parts W_1 and W_2 can be determined by comparing Γ_μ with the width of the gross channels Γ_t. This will be a check for our model. The angular distribution of $Ni^{58}(p, p')Ni^{58*}$ is approximately independent of the incident energy, while that of $Mg^{24}(p, p')Mg^{24*}$, $Ne^{20}(p, p')Ne^{20*}$, and $Al^{27}(p, \alpha)Mg^{24}$ change abruptly with the incident energy. This evidence is not inconsistent with the predictions of our model since these differences could be attributed to the difference of the relative magnitudes of the first and second terms in the right-hand side of Eq. (19) in these two cases.

Acknowledgements

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Appendix

When the closed channels are eliminated in the matrix $T_{b\alpha}^{(1)}$ the collision

matrix U is easily obtained,[1]

$$U=\frac{1-iF/G}{1+iF/G}-2i\,\frac{\boldsymbol{\theta}_\mu\times\boldsymbol{\theta}_\mu}{E-\varepsilon_\mu} \tag{A1}$$

where

$$\begin{aligned}\varepsilon_\mu&=E_\mu-i(G\boldsymbol{\gamma}_\mu\cdot(1+iF/G)^{-1}G\boldsymbol{\gamma}_\mu),\\ \boldsymbol{\theta}_\mu&=(1+iF/G)^{-1}G\boldsymbol{\gamma}_\mu.\end{aligned} \tag{A2}$$

Here F and G are diagonal matrices with rows and columns which correspond only to open channels; their diagonal elements are the Coulomb wave functions F_a and G_a, respectively. The reduced width $\gamma_{\mu a}$ is also represented by vector form $\boldsymbol{\gamma}_\mu$.

The differential cross section for the scattering is given from Eq. (A1) and the matrix $T_{ba}^{(2)}$ [27]

$$\begin{aligned}\frac{d\sigma(a,b)}{d\Omega}=&\left(\frac{m}{2\pi\hbar^2}\right)^2\left(\frac{k'}{k}\right)(2I_a+1)^{-1}\cdot\Bigg|T_{ba}^{(2)}\\ &-\sum_\mu\sum_{l=|J-s|}^{J+s}\sum_{l'=|J-s'|}^{J+s'}\left(\frac{1-iF/G}{1+iF/G}-2i\,\frac{\boldsymbol{\theta}_\mu\times\boldsymbol{\theta}_\mu}{E-\varepsilon_\mu}\right)_{asl\,;\,bs'l'}\cdot A(a,b,\mu,\theta,\Phi)\Bigg|^2\end{aligned} \tag{A3}$$

where

$$\begin{aligned}A(a,b,\mu,\theta,\Phi)=&\left(\frac{mk}{\hbar^2}\right)^{-1}\left(\frac{k}{k'}\right)^{1/2}\pi^{1/2}i^{l-l'}(2l+1)^{1/2}\sum_{M=-J}^{J}\sum_{m'=-l'}^{l'}\\ &\times(ls0m_s|lsJM)\cdot(l's'\mu'm_s'|l's'JM)\,Y_{l\nu'}(\theta,\Phi).\end{aligned} \tag{A4}$$

Here J, M, s, m_s, l, and ν are the total angular momentum, its component, the channel spin, its component, the angular momentum of the incident nucleon, and its component, respectively. The quantities of the outgoing channels are shown by the prime ($'$). The average concerning the final states m_s is assumed to be already done. The symbol (|) is the Clebsch-Gordan coefficients.

At higher energies the first term in Eq. (15) becomes more important than the second term. Then there remain only a few level terms in Eq. (A1) which are calculated from the second term of Eq. (15) and the others rather play the part of the direct interaction. Then the cross section is approximately written as follows:

$$\begin{aligned}\frac{d\sigma(a,b)}{d\Omega}\cong&\left(\frac{m}{2\pi\hbar^2}\right)^2\left(\frac{k'}{k}\right)(2I_a+1)^{-1}\cdot\Big|T_{ba}^{(2)}+\langle\phi_b|V_{pr}^{i}|\phi_a\rangle\\ &-\sum_\mu{}'\sum_l\sum_{l'}\left(\frac{1-iF/G}{1+iF/G}-2i\,\frac{\boldsymbol{\theta}_\mu\times\boldsymbol{\theta}_\mu}{E-\varepsilon_\mu}\right)_{asl\,;\,bs'l'}\cdot A(a,b,\mu,\theta,\Phi)\Big|^2,\end{aligned} \tag{A5}$$

where $\sum'$ shows the summation of a few levels ε_μ neighbouring E.

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Relativistic Quantum Field Theory of Unstable Particle

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The relativistic theory of an unstable particle is formulated. The field operator of the unstable particle is expressed by the power series of the asymptotic field operator of its decay products with the coefficient of a-function. The recursion formula for a-function is given and solved in a simple model in comparison with one for the stable particle. Further a special attention is paid to the possibility to eliminate the field operator of unstable particle from the framework of theory.

§ 1. Introduction

Untill now, many authors[1)–10)] have attempted to formulate the theory of an unstable particle. As the results of these works, the essential features of the unstable particle were clarified at least in the well-known Lee model by virtue of the exact knowledge of its solution. However, this model has serious limitations with respect to its special form of interaction and to the relativistic invariance.

On the other hand, several authors[6),7),9),15)] have also tried to treat the relativistically invariant case. These works, however, seem to us not to be self-contained in the following point that they lack the means how to construct concretely the field operator of an unstable particle, which is rather implicitly assumed there to be already given.

It is the aim of the present paper to give the formalism to construct the field operator of an unstable particle and to get the self-contained and relativistically invariant theory.

In the recent axiomatic formalism of the relativistic quantum field theory, the asymptotic condition for field operators plays an essential role. Contrary to a stable particle, the asymptotic behavior of an unstable particle at the infinite past or future is not evident due to its unstable property itself. This is the main reason why it is difficult to construct the axiomatic formalism of an unstable particle.

In order to make the above situation clear, we will investigate the Lee model in the next section as a preliminary in the present work and show that the field operator of the unstable particle (V-particle) can be expressed by the asymptotic field operators only of its decay products (N- and θ-particles) and that in the asymptotic limit at the infinite past and future it vanishes in the sense of the

weak convergence. In the subsequent section, based on the above result of the Lee model we shall construct the relativistic theory of an unstable particle without explicit use of the asymptotic condition for the unstable particle. Further we shall pay our special attention also to the possibility to eliminate the field operator of an unstable particle from the framework of theory. This problem seems to have an actual importance in the case of interpretation of the resonance phenomena such as the well-known (3/2, 3/2) resonance in π-N scattering, π-π interaction or the recent new resonant states Y^*, etc.

§ 2. Field operators in the Lee model

In this section, we investigate the Lee model as a preliminary for the present work. From the beginning, one may easily imagine that the field operator of the unstable particle (ψ'_V) can be expressed by the asymptotic field operators only of its decay products (ψ'_N, ϕ_θ) from the well-known fact that one particle state of an unstable particle can not be an eigenstate of the total Hamiltonian and the scattering states of N-θ system form the complete set in the subspace with $n_V+n_\theta=n_V+n_N=1$. (n's are the number of the bare particles.) In the following, we shall briefly show that the power series expansion of $\psi'_V(t)$ in terms of ψ'^{out}_N and $\phi_\theta{}^{out}$ really begins from the product $\psi'^{out}_N \cdot \phi_\theta{}^{out}$.

Firstly, in order to investigate the time development of $\psi'_V(t)$, $\psi'_N(t)$ and $\phi_\theta(\boldsymbol{k}, t)$ (the momentum transform of $\phi_\theta(\boldsymbol{x}, t)$) in the Heisenberg representation, we expand them in the normal product of $\psi'_V(0)$, $\psi'_N(0)$ and $\phi_\theta(\boldsymbol{k}, 0)$,* which we briefly denote ψ'_V, ψ'_N and $\phi_\theta(\boldsymbol{k})$ hereafter. The calculation of the expansion coefficient is performed by remarking the selection rule in this model and the fact that the scattering states form the complete set of the subspace with $n_V+n_\theta=n_V+n_N=1$, when V-particle is unstable. Namely, by making use of the same notation as in reference 3),

$$\psi'_V(t) = \int d\boldsymbol{k} C(\boldsymbol{k}) C^*(\boldsymbol{k}) e^{-i(m_N+\omega_k)t} \psi'_V + \iint d\boldsymbol{k} d\boldsymbol{k}' C(\boldsymbol{k}) \chi^*(\boldsymbol{k}', \boldsymbol{k}) e^{-i(m_N+\omega_k)t} \psi'_N \phi_\theta(\boldsymbol{k}') + \text{“}\textit{higher order terms}\text{”}, \tag{2·1, a}$$

$$\psi'_N(t) = e^{-im_N t}\psi'_N + \int d\boldsymbol{p} d\boldsymbol{k} \chi(\boldsymbol{k}, \boldsymbol{p}) C^*(\boldsymbol{p}) e^{-i(m_N+\omega_p-\omega_k)t} \phi_\theta^\dagger(\boldsymbol{k}) \psi'_V + \int d\boldsymbol{p} d\boldsymbol{k} d\boldsymbol{k}' \{\chi(\boldsymbol{k}', \boldsymbol{p}) \chi^*(\boldsymbol{k}, \boldsymbol{p}) e^{-i(m_N+\omega_p-\omega_k)t} - \delta(\boldsymbol{k}-\boldsymbol{k}') e^{-im_N t}\} \phi_\theta^\dagger(\boldsymbol{k}') \psi'_N \phi_\theta(\boldsymbol{k}) + \text{“}\textit{higher order terms}\text{”}, \tag{2·1, b}$$

* For this purpose, we use the operator identity analogous to Eq. (4) in reference 11), i. e. $L=\sum \frac{1}{n!} \int dk_1 \cdots dk_n \langle \Omega | [\cdots [L, A^{in}(k_1)], \cdots], A^{in}(k_n)] | \Omega \rangle \epsilon(k_1) \delta(k_1^2+m^2) \cdots : A^{in}(k_1) \cdots A^{in}(k_n) :$, identifying L with $\psi_V(t)$, $\psi_N(t)$ or $\phi_\theta(\boldsymbol{k}, t)$ and $A^{in}(k)$ with the set of (ψ_V, ψ_N, $\phi_\theta(\boldsymbol{k})$) and their hermitian conjugate.

$$\phi_\theta(\boldsymbol{k}, t) = e^{-i\omega_k t}\phi_\theta(\boldsymbol{k}) + \int d\boldsymbol{p}\, e^{-i\omega_p t} C^*(\boldsymbol{p})\, \chi(\boldsymbol{k}, \boldsymbol{p})\, \psi_N^\dagger \psi_V$$
$$+ \int d\boldsymbol{p}\, d\boldsymbol{k}' \{\chi(\boldsymbol{k}, \boldsymbol{p})\, \chi^*(\boldsymbol{k}', \boldsymbol{p})\, e^{-i\omega_p t} - \delta(\boldsymbol{k}-\boldsymbol{k}')\, e^{-i\omega_k t}\} \psi_N^\dagger \psi_N \phi_\theta(\boldsymbol{k}')$$
$$+ \text{“}\textit{higher order terms}\text{”}, \qquad (2\cdot1, \text{c})$$

where

$$C(\boldsymbol{k}) = \frac{g}{\sqrt{2\omega_k}} G(\omega_k)\, S_V'^*(\omega_k + m_N), \qquad (2\cdot2)$$

$$\chi(\boldsymbol{k}, \boldsymbol{k}') = \delta(\boldsymbol{k}-\boldsymbol{k}') - gC(\boldsymbol{k}') \frac{G(\omega_k)}{\sqrt{2\omega_k}} (\omega_k - \omega_{k'} + i\epsilon)^{-1}, \qquad (2\cdot3)$$

$$S_V'(E) = \left\{E - m_0 + i\epsilon + g\int \frac{G^2(\omega_k)}{2\omega_k} \frac{d\boldsymbol{k}}{\omega_k - E + m_N - i\epsilon}\right\}^{-1}, \qquad (2\cdot4)$$

$$\omega_k = \sqrt{\boldsymbol{k}^2 + \mu^2}, \qquad (2\cdot5)$$

and m_0, m_N and μ denote the bare masses of V-, N- and θ-particles and further g and $G(\omega_k)$ mean the unrenormalized coupling constant and the cutoff factor, respectively. In Eqs. (2·1, a, b, c) “*higher order terms*” vanish in case of operation upon the subspace with $n_V + n_\theta = n_V + n_N = 1$.

Thus, the asymptotic operators for $\psi_N(t)$ and $\phi_\theta(\boldsymbol{k}, t)$:

$$\left.\begin{aligned} \psi_N^{out} &\equiv \lim_{t\to\infty} e^{im_N t}\psi_N(t), \\ \phi_\theta^{out}(\boldsymbol{k}) &\equiv \lim_{t\to\infty} e^{i\omega_k t}\phi_\theta(\boldsymbol{k}, t) \end{aligned}\right\} \qquad (2\cdot6)$$

are expressed as follows :

$$\psi_N^{out} = \psi_N + \int d\boldsymbol{k}\, C^*(\boldsymbol{k})\, \phi_\theta^\dagger(\boldsymbol{k})\, \psi_V$$
$$+ \int d\boldsymbol{k}\, d\boldsymbol{k}' \{\chi^*(\boldsymbol{k}, \boldsymbol{k}') - \delta(\boldsymbol{k}-\boldsymbol{k}')\} \phi_\theta^\dagger(\boldsymbol{k})\, \phi_\theta(\boldsymbol{k}')\, \psi_N + \cdots, \qquad (2\cdot7)$$

$$\phi_\theta^{out}(\boldsymbol{k}) = \phi_\theta(\boldsymbol{k}) + C^*(\boldsymbol{k})\, \psi_N^\dagger \psi_V$$
$$+ \int d\boldsymbol{k}' \{\chi^*(\boldsymbol{k}', \boldsymbol{k}) - \delta(\boldsymbol{k}-\boldsymbol{k}')\} \psi_N^\dagger \psi_N \phi_\theta(\boldsymbol{k}') + \cdots. \qquad (2\cdot8)$$

Now, let us consider the quantity $\psi_N^{out} \cdot \phi_\theta^{out}(\boldsymbol{k})$, which is expressed by Eqs. (2·7) and (2·8) as

$$\psi_N^{out} \cdot \phi_\theta^{out}(\boldsymbol{k}) = C^*(\boldsymbol{k})\, \psi_V + \int d\boldsymbol{k}'\, \chi^*(\boldsymbol{k}', \boldsymbol{k})\, \psi_N \phi_\theta(\boldsymbol{k}')$$
$$+ \text{“}\textit{higher order terms}\text{”}, \qquad (2\cdot9)$$

where “*higher order terms*” again vanish in case of operation upon the subspace mentioned above. Thus, comparing Eq. (2·9) with Eq. (2·1, a), we arrive at the final result that

$$\psi_V(t)=\int dk C(k)\, e^{-i(m_N+\omega_k)t}\psi_N^{out}\cdot\phi_\theta^{out}(k)+\text{“}\textit{higher order terms}\text{”}. \qquad (2\cdot10)$$

This expression also shows that $\psi_V(t)$ asymptotically vanishes at $t=\pm\infty$ in the sense of the weak convergence.

§ 3. Field operator of relativistic unstable particle and recursion formula of advanced functions

Taking into account the result of the previous section, we will investigate a relativistic unstable particle, where we consider as a simple model a neutral scalar particle which decays into two neutral scalar particles χ's. According to the results from the Lee model, we can reasonably assume the following condition for the unstable particle ϕ, in addition to the asymptotic condition for the stable particle χ. Namely we assume that (i) the whole Hilbert space is completely constructed by the true vacuum state multiplied only by any number of asymptotic field of χ; χ^{out} or χ^{in}. Then according to Glaser et al.,[11]

$$\phi(z)=\sum_{n=2}^{\infty}\frac{1}{n!}\int dx_1\cdots dx_n K_{1,2,\cdots,n}\, a(z\,;\,x_1,\,x_2,\,\cdots,\,x_n)$$
$$\times:\chi^0(x_1)\,\chi^0(x_2)\cdots\chi^0(x_n):. \qquad (3\cdot1)$$

Here χ^0 stands for χ^{out} and $K_{1,2,\cdots,n}=K_{x_1}K_{x_2}\cdots K_{x_n}$, where $K_x=(\Box_x-\mu^2)$ and μ denotes the mass of χ-particle. In the above expression it is assumed that (ii) the sum in the right-hand side begins with $n=2$. Since coefficient functions $a(z\,;\,x_1,\cdots,x_n)$'s are vacuum expectation values of the well-known advanced products of ϕ and χ, the latter assumption (ii) implies that $\langle 0|A[\phi(z)\,;\,\chi(x)]|0\rangle=0$.

Now, Eq. (3·1) means that $\phi(z)$ is completely determined if vacuum expectation values of the advanced products (we call it a-function below) are obtained. A-product, from its definition, satisfies the following identities:[11),12)]

$$A[\phi(x)\,;\,\phi(y)\,\phi(x_1)\cdots\phi(x_n)]-(x\leftrightarrow y)$$
$$=-i\sum_{comb.}[A[\phi(x)\,;\,\phi(x_1)\cdots\phi(x_k)],\ A[\phi(y)\,;\,\phi(x_{k+1})\cdots\phi(x_n)]] \qquad (3\cdot2)$$

and analogous equations where any number of ϕ's in the above equation are replaced by χ's. Then, the recursion formula for a-functions is obtained by inserting the complete set of states between the product of A's in the right-hand side of Eq. (3·2) and by taking the vacuum expectation value of the both sides. In this procedure, it should be noted that, according to the assumption i), the complete set of states is constructed only by χ^0's and ϕ does not explicitly contribute in it. Further, using the usual asymptotic condition for χ-field and assuming $a(x_1\,;\,x_2,\,x_3)=0$, we obtain the following equations in the approximation where intermediate states involve two χ^0's at most:

(A) $a(x_1; x_2, x_3, x_4) - (x_1 \leftrightarrow x_2)$

$$= \int du\, \Delta(x_1 - u)\, K_u a(x_2; x_3, x_4, u)$$

$$+ \frac{i}{2} \int du\, du'\, dv\, dv'\, K_{u,v} a(x_1; x_3, u, v) f(u-u', v-v')$$

$$\times K_{u',v'} a(x_2; x_4, u', v') - (x_1 \leftrightarrow x_2)$$

$$+ \int dz\, dz'\, K_z a(x_1; x_3, z)\, \Delta_\phi(z-z')\, K_{z'} a(x_2; x_4, z') - (x_1 \leftrightarrow x_2),$$

(B) $a(z; x_1, x_2) - (x_1 \leftrightarrow z)$

$$= \int du\, \Delta(x_1 - u)\, K_u a(z; x_2, u) + \frac{i}{2} \int du\, du'\, dv\, dv'\, K_{u,v} a(z; u, v)$$

$$\times f(u-u', v-v')\, K_{u',v'} a(x_1; x_2, u', v')$$

$$+ \int dz'\, \Delta_\phi(z-z')\, K_{z'} a(x_1; x_2, z'),$$

(C) $a(z; z') - (z \leftrightarrow z')$

$$= \frac{i}{2} \int du\, du'\, dv\, dv'\, K_{u,v} a(z; u, v) f(u-u', v-v')\, K_{u',v'} a(z'; u', v')$$

$$+ \Delta_\phi(z-z'),$$

etc.,

where $f(x, y) = f(y, x) = -f(-x, -y) = \Delta^{(+)}(x)\Delta^{(+)}(y) - \Delta^{(-)}(x)\Delta^{(-)}(y)$,* and z and z' appearing in a-functions imply the arguments of ϕ-field and $x_1, x_2, \cdots$ (or simply $1, 2, \cdots$), $u, v, \cdots$, those of χ-field, so far as this abbreviation does not give rise to any confusion.

Firstly, it is quite important to notice that terms surrounded by the dotted lines in the above equations do not occur in the present case, contrary to the case when ϕ-particle is stable. In the latter case they clearly correspond to the contribution from the intermediate state involving ϕ^{out}, and would offer the lowest order approximation terms for the left-hand side of the respective equations. Nevertheless, in the present case, this conventional perturbation method is not applicable in a simple way. However, we can show that the exact solution of the Lee model satisfies the above type of equations and it seems to justify our assumptions (i) and (ii).

Secondly, it should be noticed that the recursion formula is separated into two parts, one of which, such as Eq. (A), consists of equations concerning a-functions involving only χ-field, but the other part of which, such as Eqs. (B)

* Δ and Δ_ϕ denote the usual invariant function of χ- and ϕ-particles, respectively.

and (C) consists in equations concerning a-functions involving both ϕ- and χ-fields. This situation is easily understood from the derivation of recursion formula where only χ-field contributes to intermediate states. The former part, which we call χ-system, is obviously equivalent to the recursion formula derived from the assumptions that the system consists only in χ-field, excluding ϕ-field from the beginning and that $a(x_1\,;\,x_2,\,x_3)=0$. Since the solution obeying the both parts of recursion formula is, of course, the solution of χ-system, we find out that the introduction of the field operator for the unstable particle imposes the subsidiary condition upon the solution of χ-system. In § 6, basing on the result of § 4, we shall touch on this point more in detail.

§ 4. Relativistic solution

In this section we will seek for the solution of Eqs. (A), (B) and (C). At first, we note that the a-function, from its definition, has the following properties:

i) $a(z\,;\,x_1,\,x_2,\cdots x_n)$ is symmetric with respect to variables $x_1,\,x_2,\,\cdots,\,x_n$, (4·1)

ii) $a(x\,;\,y,\,z,\,\cdots)=0$ except to the case when x lies in the past for any of $y,\,z,\,\cdots$.* (4·2)

Let us assume that the vertex a-function $a(z\,;\,x_1,\,x_2)$ has the following form in a zero order approximation, corresponding to the interaction of the form $\lambda\phi(x)\chi^2(x)$ in the conventional Hamiltonian formalism:

$$\left.\begin{aligned} a(z\,;\,1,\,2)&=\lambda\int dz'\,a_\phi(z\,;\,z')\,\Delta_A(z'-1)\,\Delta_A(z'-2),\\ a(1\,;\,z,\,2)&=\lambda\int dz'\,a_\phi(z'\,;\,z)\,\Delta_A(1-z')\,\Delta_A(z'-2),\end{aligned}\right\} \tag{4·3}$$

where $\Delta_A(x)$ is a free advanced function of χ-field and so $K_x\Delta_A(x)=-\delta^4(x)$. Contrary to the usual perturbation method,[12] $a_\phi(z\,;\,z')$** is yet retained undetermined at the beginning. And the equation which determines $a_\phi(z\,;\,z')$ is obtained after the substitution of Eq. (4·3) into Eq. (C), namely

$$\begin{aligned} &a_\phi(z\,;\,z')-a_\phi(z'\,;\,z)\equiv\Delta_\phi'(z-z')\\ &\quad=\frac{i}{2}\,\lambda^2\int du\,du'\,a_\phi(z\,;\,u)\,f(u-u',\,u-u')\,a_\phi(z'\,;\,u'). \end{aligned} \tag{4·4}$$

In the next section, we shall give the solution of the above equation, while in this section we assume that the solution for $a_\phi(z\,;\,z')$ is already known.

* Here, $x,\,y,\cdots$ represent the arguments of either ϕ or χ without distinction.

** Hereafter, we shall use the notation $a_\phi(z\,;\,z')$ with an explicit use of the suffix for $a(z\,;\,z')$ of ϕ-field in order to avoid the confusion.

Substituting Eq. (4·3) in Eq. (B) and operating K_{x_1, x_2} on the both sides of Eq. (B), we obtain

$$\lambda[a_\phi(z\,;1)-a_\phi(2\,;z)]\delta^4(1-2)$$
$$=\frac{i}{2}\lambda\int du\,du'\,dv\,dv'\,a_\phi(z\,;u)\,f(u-u',\,v-v')\,K_{1,2,u',v'}a(1\,;2,\,u',\,v'). \tag{4·5}$$

Comparing Eq. (4·5) with Eq. (4·4), one gets

$$K_{1,2,3,4}a(1\,;2,\,3,\,4)=\lambda^2 a_\phi(1\,;3)\,\delta^4(1-2)\,\delta^4(3-4). \tag{4·6}$$

However, the right-hand side of the above expression does not satisfy the symmetry condition (4·1), so we are forced to symmetrize it as

$$K_{1,2,3,4}a(1\,;2,\,3,\,4)=\lambda^2\{a_\phi(1\,;3)\,\delta^4(1-2)\,\delta^4(3-4)$$
$$+a_\phi(1\,;2)\,\delta^4(1-3)\,\delta^4(2-4)+a_\phi(1\,;2)\,\delta^4(1-4)\,\delta^4(2-3)\}. \tag{4·7}$$

It will be shown later that Eq. (4·7) is just a correct solution in the zero order approximation by confirming that it really satisfies Eq. (A).

Secondly, let us consider the next order approximation. Now, Eqs. (4·3) and (4·7), of course, do not satisfy Eq. (B) owing to the last two terms in the right-hand side of Eq. (4·7) which appeared as the result of the symmetrization. However, this situation can be correctly improved by noting that the effect of these terms determines the next order correction to $a(z\,;x_1,\,x_2)$. According to this consideration, we put, instead of Eq. (4·3),

$$\left.\begin{aligned}K_{1,2}a(z\,;1,\,2)&=\lambda a_\phi(z\,;1)\,\delta^4(1-2)+\lambda^3 K_{1,2}a^{(1)}(z\,;1,\,2),\\ K_{1,2}a(1\,;z,\,2)&=\lambda a_\phi(1\,;z)\,\delta^4(1-2)+\lambda^3 K_{1,2}a^{(1)}(1\,;z,\,2),\end{aligned}\right\} \tag{4·8}$$

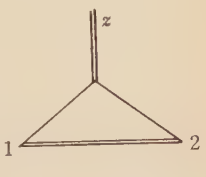
where $\lambda^3 a^{(1)}$'s denote the next order correction terms. Sustituting Eqs. (4·8) and (4·9) in (B), and taking Eq. (4·4) into consideration, we have

$$K_{1,2}\{a^{(1)}(z\,;1\,;2)-a^{(1)}(1\,;z,\,2)\}=ia_\phi(1\,;2)\int du\,a_\phi(z\,;u)\,f(u-1,\,u-2). \tag{4·9}$$

Hence, according to the boundary condition (4·2) for a-function, we get

$$K_{1,2}a^{(1)}(z\,;1,\,2)$$
$$=i\theta(1-z)\,a_\phi(1\,;2)\int du\,a_\phi(z\,;u)\,f(u-1,\,u-2), \tag{4·10}$$

which is schematically shown by Fig. 4·1 and seems to be the usual correction term to the vertex part. However, Eq. (4·10) again does not satisfy the symmetry condition



—— χ-propagation
═══ ϕ-propagation

Fig. 4·1. The diagram of $a^{(1)}(z\,;1,\,2)$.

(4·1). In order to get the symmetric solution, we must take into account the next higher order terms of Eq. (B) itself, which are obtained by inserting the three χ-particle state as the intermediate state into the expression $-i\langle 0|[A[\phi(z);\chi(2)],\chi(1)]|0\rangle$ in the left-hand side of Eq. (B). Denoting this term by $-i\langle 0|[A(z;2),\chi(1)]|0\rangle|_{(3)}$, it is calculated as

$$-iK_{1,2}\langle 0|[A(z;2),\chi(1)]|0\rangle|_{(3)}$$
$$=-i\frac{i^3}{3!}\int K_{2,u,v,w}a(z;2,u,v,w)\,g(u-u',v-v',w-w')$$
$$\times K_{1,u',v',w'}a(1;u',v',w')\,du\cdots dw', \tag{4·11}$$

where

$$g(x,y,z)=\Delta^{(+)}(x)\,\Delta^{(+)}(y)\,\Delta^{(+)}(z)+\Delta^{(-)}(x)\,\Delta^{(-)}(y)\,\Delta^{(-)}(z). \tag{4·12}$$

Now, $a(z;2,u,v,w)$ and $a(1;u',v',w')$ are considered to be of the order of λ^3 and λ^2 respectively, so the right-hand side of Eq. (4·11) as a whole, amounts to be of the order of λ^5 according to the conventional perturbation, counting the order of $a_\phi(z;z')$ as λ^0. However, contrary to this supposition, it turns out that some part of the right-hand side offers terms of the lower order of λ^3 than the above counting on account of Eq. (C) or (4·4), and that these terms are just necessary ones for the symmetrization of $a^{(1)}(z;1,2)$. As will be discussed in § 6 from a more general standpoint, this circumstance is quite important for the case of unstable particle, so we will precisely show the details of calculation.

In order to calculate $a(z;2,u,v,w)$ in the integrand of Eq. (4·11), let us again use the recursion formula in the lowest order approximation, namely

$$K_{2,u,v,w}a(z;2,u,v,w)-(z\leftrightarrow 2)$$
$$=\sum_{cycl.(u,v,w)}\int dx\,dx'\,K_{u,x}a(z;u,x)\,\Delta(x-x')\,K_{2,v,w,x'}a(2;v,w,x')$$
$$=\sum_{cycl.(u,v,w)}a_\phi(z;u)\,\Delta(u-2)\,a_\phi(2;v)\,\delta(v-w) \tag{4·13}$$

with the use of Eqs. (4·3) and (4·8). The substitution of $a(z;2,u,v,w)$ derived from Eq. (4·13) in Eq. (4·11) gives rise to the following term

$$-\frac{i}{2}\lambda^5\theta(2-z)\int a_\phi(z;u)\,\Delta(u-2)\,a_\phi(2;v)$$
$$\times g(u-1,v-v',v-v')\,a_\phi(1-v')\,du\,dv\,dv' \tag{4·14}$$

in addition to more complicated terms. Eq. (4·14) is shown schematically by Fig. 4 2 and is seen to involve a diagram of self-energy type. This self-energy part is further reduced to Δ_ϕ'-function by means of Eq. (4·4). In fact, by making use of relations:

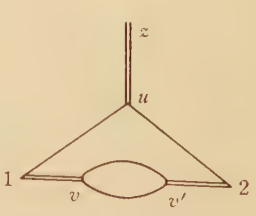


Fig. 4·2. The vertex correction involving a self energy part.

$$\Delta_\phi'^{(\pm)}(z-z') = \pm \frac{i}{2}\lambda^2 \int du\, du'\, a_\phi(z\,;\,u)\, \Delta^{(\pm)}(u-u')\, \Delta^{(\pm)}(u-u')\, a_\phi(z'\,;\,u'), \tag{4·15}$$

which are derived from Eq (4·4), we can rewrite Eq. (4·14) as

$$i\lambda^3 \theta(2-z) \int du\, a_\phi(z\,;\,u)\, \Delta(u-2) f_\phi(u-1,\, 2-1), \tag{4·16}$$

where

$$f_\phi(x,\, y) = \Delta^{(+)}(x)\, \Delta_\phi'^{(+)}(y) - \Delta^{(-)}(x)\, \Delta_\phi'^{(-)}(y). \tag{4·17}$$

Thus, the above term should be added to the right-hand side of Eq. (B), so Eq. (4·10) is replaced by

$$K_{1,2} a^{(1)}(z\,;\,1,\,2) = i\theta(1-z)\,\theta(2-1) \int du\, a_\phi(z\,;\,u) f(u-1,\, u-2)\, \Delta_\phi'(1-2)$$
$$+ i\theta(1-z)\,\theta(2-z) \int du\, a_\phi(z\,;\,u) f_\phi(u-1,\, 2-1)\, \Delta(u-2). \tag{4·18}$$

Although this expression is not yet explicitly symmetric with respect to 1 and 2, the direct calculation shows that its anti-symmetric part really vanishes and then the manifestly symmetric expression is given by

$$K_{1,2} a^{(1)}(z\,;\,1,\,2) = \frac{i}{2}\theta(1-z)\,\theta(2-z) \int du\, a_\phi(z\,;\,u) \{2\overline{\Delta_\phi'}(1-2) f(u-1,\, u-2)$$
$$+ \Delta(u-1) f_\phi(u-2,\, 1-2) + \Delta(u-2) f_\phi(u-1,\, 2-1)\}, \tag{4·19}$$

where

$$\overline{\Delta_\phi'}(z) = -\frac{1}{2}\epsilon(z_0)\, \Delta_\phi'(z).$$

Lastly, let us briefly refer to the next order correction to $a(1\,;\,2,\,3,\,4)$ i.e. Eq. (4·7). In a similar way to the case of $a(z\,;\,1,\,2)$, we put, instead of Eq. (4·7),

$$K_{1,2,3,4} a(1\,;\,2,\,3,\,4) = \lambda^2 \sum_{sym.(2,3,4)} a_\phi(1\,;\,2)\, \delta^4(1-3)\, \delta^4(2-4)$$
$$+ \lambda^4 K_{1,2,3,4} a^{(1)}(1\,;\,2,\,3,\,4), \tag{4·20}$$

where $\lambda^4 a^{(1)}(1\,;\,2,\,3,\,4)$ denotes the next order correction term. Sustituting Eq. (4·20) in Eq. (A), we obtain the equation which determines $a^{(1)}(1\,;\,2,\,3,\,4)$:

$$K_{1,2,3,4}\{a^{(1)}(1\,;\,2,\,3,\,4) - (1 \leftrightarrow 2)\} = i[\delta^4(1-3)\, a_\phi(2\,;\,4) \int du\, a_\phi(1\,;\,u) f(u-2,\, u-4)$$
$$- \delta^4(2-4)\, a_\phi(1\,;\,3) \int du\, a_\phi(2\,;\,u) f(u-1,\, u-4)$$
$$+ a_\phi(1\,;\,3)\, a_\phi(2\,;\,4)\, (f(1-2,\, 3-4) + f(1-4,\, 3-2)) - (1 \leftrightarrow 2)], \tag{4·21}$$

where we used Eq. (4·4) and reserved only the lowest order terms. The above calculation also guarantees that the expression (4·7) for $a(1;2,3,4)$ is a correct solution of Eq. (A) as the lowest order approximation, as previously referred. The expression for $a^{(1)}(1;2,3,4)$ derived from Eq. (4·21) also does not satisfy the symmetry condition. In order to obtain the correct solution, we have to take into consideration the next order approximation terms of Eq. (A) itself. Since the essential feature of calculation is same as in the case for Eq. (B) treated just before, we do not enter into more details.

Thus, in principle, we can obtain the solution for any a-functions, up to any order of approximation by succesively applying the above procedure. In § 6, we will argue the relation of the above solutions just derived for the unstable particle with the solution for the stable particle.

§ 5. Determination of $a_\phi(z;z')$

In this section we seek for the solution of Eq. (4·4), which was retained in the previous section. Taking Fourier's transform of Eq. (4·4) and after some elementary calculation, we have

$$a_\phi(p)-a_\phi(-p)=-2i\lambda^2|a_\phi(p)|^2\epsilon(p_0)\,\theta(-4\mu^2-p_\mu{}^2)\sqrt{1+4\mu^2/p_\mu{}^2}, \qquad (5\cdot1)$$

where $a_\phi(p)$ is the Fourier's transform of $a_\phi(z)$ and $p_\mu{}^2=\boldsymbol{p}^2-p_0{}^2$. Since $\phi(z)$ is an hermitian operator, so $a_\phi(z)$ is real and thus the left-hand side of Eq. (5·1) implies $2i\,\mathrm{Im}\,a(p)$. From Eq. (5·1) we get the following relation for $f_a(p)=a_\phi^{-1}(p)$:

$$\mathrm{Im}\,f_a(p)=\lambda^2\epsilon(p_0)\,\theta(-4\mu^2-p_\mu{}^2)\sqrt{1+4\mu^2/p_\mu{}^2}. \qquad (5\cdot2)$$

On the other hand, one finds from the condition (4·2) that $a_\phi(\boldsymbol{p},p_0)$ is an analytic function in the lower half on the complex p_0-plane. If we further assume that $a_\phi(\boldsymbol{p},p_0)$ has no zero-point in this domain, $f_a(\boldsymbol{p},p_0)$ also becomes an analytic function in the same domain. In this case, using the above analyticity of $f_a(\boldsymbol{p},p_0)$ and Eq. (5·2), we can derive the following equation concerning the real part of $f_a(\boldsymbol{p},p_0)$, i.e. $k(p_\mu{}^2)$, which is a function of $p_\mu{}^2$, by means of dispersion relation with two subtractions,

$$k(p_\mu{}^2)=k(-m^2)+(p_\mu{}^2+m^2)\,k'(-m^2) - \frac{1}{\pi}(p_\mu{}^2+m^2)^2\,\mathcal{P}\int_{4\mu^2}^{\infty}\frac{g(-\kappa^2)\,d\kappa^2}{(\kappa^2+p_\mu{}^2)(\kappa^2-m^2)^2}, \qquad (5\cdot3)$$

where $g(-\kappa^2)=\lambda^2\sqrt{1-4\mu^2/\kappa^2}\,\theta(\kappa^2-4\mu^2)$ and m denotes some real and positive parameter.* Hence, in order to obtain the complete form of $f_a(\boldsymbol{p},p_0)$, we

* In order to get a finite k-function, it is sufficient to take only one subtraction, which leads to the result that $k(p_\mu{}^2)\to$const. at $p_\mu{}^2\to\pm\infty$. However, we assume here that $k(p_\mu{}^2)\to O(p_\mu{}^2)$ in this limit. Of course, this assumption makes it necessary to take two subtractions and leads to the reasonable result in the weak coupling limit $\lambda\to0$, as will be shown later.

must introduce further conditions which determine $k(-m^2)$ and $k'(-m^2)$. These conditions just correspond to the asymptotic condition for a stable particle which determines the mass and the renormalization of the field operator of the stable particle.

According to the same consideration as in the case of the Lee model, two kinds of methods are conceivable to impose the above conditions. α) One of them characterizes the unstable particle by means of the resonance phenomena caused by this particle in χ-χ scattering and β) the other method, originally proposed by Peierls,[14] by means of the complex pole of the causal propagator in the momentum representation.

α) This method is briefly expressed by the following conditions,

$$k(-m^2)=0, \quad k'(-m^2)=1, \tag{5·4}$$

by identifying m with the mass of unstable particle. They are interpreted as conditions to determine the mass and the renormalization of the field operator of the unstable particle, respectively. Thus, from Eqs. (5·2), (5·3) and (5·4), one has

$$a_\phi^{-1}(p)=(p_\mu{}^2+m^2)\left\{1-\frac{p_\mu{}^2+m^2}{\pi}\mathcal{P}\int_{4\mu^2}^{\infty}\frac{g(-\kappa^2)\,d\kappa^2}{(\boldsymbol{p}^2-(p_0-i\epsilon)^2+\kappa^2)(\kappa^2-m^2)^2}\right\}. \tag{5·5}$$

The total cross-section for χ-χ scattering, which is easily calculated[11] through $a(1\,;\,2,\,3,\,4)$ given in the previous section, shows a resonance at $P_\mu{}^2=-m^2$ (P_μ: the total four momentum) and its width becomes as $\gamma=g(-m^2)/m=\lambda^2\left(1-\frac{4\mu^2}{m^2}\right)^{1/2}/m$ in the lowest order approximation. Further, Lehmann's spectral function $\rho(\kappa^2)$ is given by

$$\rho(\kappa^2)=-\frac{1}{\pi}\,\epsilon(p_0)\,\mathrm{Im}\,a_\phi(p)|_{p_\mu{}^2=-\kappa^2}=\frac{1}{\pi}\,\theta(\kappa^2-4\mu^2)\,g(-\kappa^2)$$

$$\times\left\{(m^2-\kappa^2)^2\left(1-\frac{m^2-\kappa^2}{\pi}\mathcal{P}\int_{4\mu^2}^{\infty}\frac{g(-\kappa'^2)\,d\kappa'^2}{(\kappa'^2-\kappa^2)(\kappa'^2-m^2)^2}\right)^2+g^2(-\kappa^2)\right\}^{-1}. \tag{5·6}$$

Eq. (5·6) shows that $\rho(\kappa^2)$ approaches to $\delta(\kappa^2-m^2)$ in the weak coupling limit $\lambda\to 0$, as remarked by Matthews and Salam.[6]

β) Firstly, instead of $f_a(p)$ $(=k(p_\mu{}^2)+i\epsilon(p_0)g(p_\mu{}^2))$ treated so far, let us introduce

$$f_c(p_\mu{}^2)=k(p_\mu{}^2)-ig(p_\mu{}^2), \tag{5·7}$$

which obviously denotes the inverse of Fourier's transform of the causal function. By making use of Eq. (5·3),

$$f_c(p_\mu{}^2)=f_c(-m^2)+(p_\mu{}^2+m^2)f_c'(-m^2)$$

$$-\frac{1}{\pi}(p_\mu{}^2+m^2)^2\int_{4\mu^2}^{\infty}\frac{g(-\kappa^2)\,d\kappa^2}{(\kappa^2+p_\mu{}^2-i\epsilon)(\kappa^2-m^2-i\epsilon)^2}. \tag{5·8}$$

This expression shows that $f_c(-z)$ is an analytic function in the complex z-plane with a branch cut C shown by Fig. 5·1. Now, on account of the analytic property of $g(-\kappa^2)$, we can perform the analytic continuation of $f_c(-z)$ from the upper- to the lower-half planes over the branch cut C by deforming the integration path C in Eq. (5·8) into C'. According to Peierls, we assert that $f_c(-z)$ has a zero-point at $z=\tilde{m}^2=(m_S-i\gamma_S/2)^2$ in the extended region through the above continuation, where $m_S(\fallingdotseq 2\mu)$ and $\gamma_S(\fallingdotseq 0)$ are interpreted as the mass and the lifetime of the unstable particle. Thus, setting m in Eq. (5·8) equal to m_S, we get

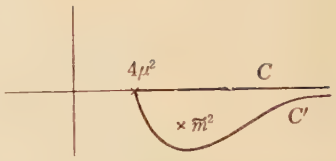


Fig. 5·1. Integration paths for $f_c(z)$.

$$k(-m_S^2)+(m_S^2-\tilde{m}^2)\,k'(-m_S^2)-2ig(-\tilde{m}^2)$$
$$-\frac{1}{\pi}(m_S^2-\tilde{m}^2)^2\,\mathcal{P}\int_{4\mu^2}^{\infty}\frac{g(-\kappa^2)\,d\kappa^2}{(\kappa^2-\tilde{m}^2)(\kappa^2-m_S^2)^2}=0, \qquad (5\cdot9)$$

where $\tilde{m}=m_S-i\gamma_S/2$. However, it turns out that Eq. (5·9) alone is insufficient to determine the unknown parameters $k(-m_S^2)$, $k'(-m_S^2)$ and γ_S in terms of λ, μ and m_S and thus one must further introduce another condition. As such a condition several possibilities are conceivable. Ida[15] has proposed the following one,

$$|f_c'(z)|_{z=-\tilde{m}^2}=1 \qquad (5\cdot10)$$

along the same line as conditions (5·4).

Another reasonable way is to impose a renormalization condition upon the spectral function $\rho(\kappa^2)$. As pointed out by Matthews and Salam,[6] $\rho(\kappa^2)$ is, in general, composed of a sharp peak $\rho_W(\kappa^2)$ due to the weak interaction and a diffuse spread $\rho_S(\kappa^2)$ due to the strong interaction, as shown schematically by Fig. 5.2a, in the case when the unstable particle paticipates in such two kinds of interactions. Then we assert, in general,

$$\int\rho_W(\kappa^2)\,d\kappa^2=1, \qquad (5\cdot11)$$

which guarantees that ρ_W tends to δ-function with the coefficient of unity when the weak interaction is switched off. However, the principle of separation of ρ_W from ρ_S, unfortunately, is not unambiguous except the above limit of switching off.[16]

Nevertheless, the discrepancy due to the above ambiguity is not appreciable in the sufficiently weak interaction. Further, in our simple model under consideration, $\rho_W(\kappa^2)$ is nothing but $\rho(\kappa^2)$ at least

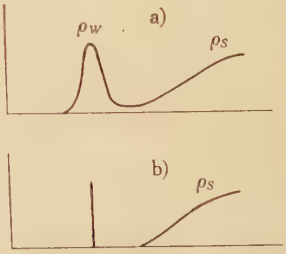


Fig. 5·2. The spectral function; a) The general case, b) The case when the weak interaction is switched off.

within the approximation adopted here. Although it is difficult to get an exact solution under the conditions of Eqs. (5·9) and (5·10) or Eqs. (5·9) and (5·11), either case leads to the same result in the weak coupling limit:

$$k(-m_S^2)=0, \quad k'(-m_S^2)=1,$$

and

$$\gamma_S=\lambda^2 \cdot 1-4\mu^2/m_S^2/m_S \qquad (5\cdot12)^*$$

in agreement with (5·8).

§ 6. Discussions and summary

In § 3, we have pointed out that, as the characteristic feature of the unstable particle distinct from the stable particle, the recursion formula of the advanced functions concerning the former particle lacks particular terms comparing with that concerning the latter particle. And this situation necessitated the complex procedure to solve it. Nevertheless, the result obtained in § 4 shows that the explicit form of solution is quite same as that for the stable particle only except that the spectral function of $a_\phi(z)$ or $\Delta_\phi'(z)$ which appears in this expression does not involve δ-function that is inherent in the stable particle. This agreement, however, is not accidental, but may be well understood by the following simple explanation.

As mentioned in § 3, the fact that the recursion formula lacks particular terms originates in that the unstable particle does not explicitly participate in the complete set of state vectors. For the sake of simplicity, let us consider any term in the recursion formula, where two χ-particles are inserted in the intermediate state (see Fig. 6·1a). If we follow the usual consideration of the perturbation theory, the order of this term concerning the coupling constant λ should be counted to be higher by λ^2 comparing with that in the case for the stable particle where the propagation of two χ-particles in the intermediate state is replaced by that of the single ϕ-particle, which is, of course, absent for the unstable particle. However, this expectation is wrong in our present case. In fact, the diagram in Fig. 6·1a involves a special diagram of such a self-energy type, as shown by Fig. 6.1b, where the end points of the propagation of two χ-particles come into a same vertex. On the other hand, Eq. (4·4) shows that such a self-energy

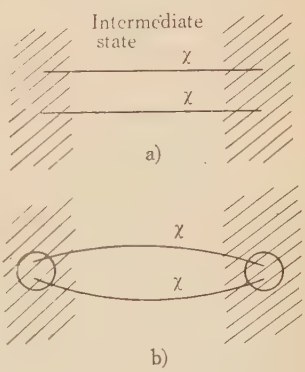


Fig. 6·1. The circle denotes the vertex part.

* In the derivation of this result it should be noticed that the imaginary part of the last term in the left-hand side of Eq. (5·9) offers a finite contribution, i.e. $g(-m_S^2)$ in the weak coupling limit.

part is just equal to Δ_ϕ', i.e. the propagation of a single ϕ-particle, contrary to the conventional expectation for the stable particle. Thus, extending the above consideration, it turns out that any term which involves n self-energy type diagrams should be regarded as terms with the lower order by λ^{2n} than the apparent order due to the conventional perturbation, counting the order of $a_\phi(z)$ as the zero order. And those terms effectively recover the missing terms in the recursion formula of the unstable particle, so that the resultant form of solution exactly coincides with that for the stable particle. Nevertheless, it seems still necessary to make more clear the counting method of the order of terms in the recursion formula, which was somewhat carelessly used so far. Now, the reason why the perturbational counting of the order is not adequate originates in the fact that it is wrong to regard the order of $a_\phi(z)$ as λ^0. From Eq. (4·4), one finds that any assignment of the definite order for $a_\phi(z)$ leads to the contradiction, because $a_\phi(z)$ obeys the special non-linear equation. However, from the result of § 5, we know that in the weak coupling limit $\lambda\to 0$, $a_\phi(z)$ tends to the usual free advanced function $\Delta_\phi^A(z)$ independent of λ. On the other hand, as seen from Eq. (5·6), the product of the above functions has the following limiting value, $\lambda^2 a_\phi^*(p_\mu) a_\phi(p_\mu)\sqrt{1+4\mu^2/p_\mu^2}\to\pi\delta(p_\mu^2+m^2)$. This implies that one has to count the order of any term to be the order of λ of this term as a whole, without any factorization, in the weak coupling limit $\lambda\to 0$. Thus, for instance, the order of the right-hand side of Eq. (4·4) should be counted as λ^0, contrary to the apparent order λ^2. In fact, it turns out that the approximation method in § 4 is really based on such a counting.

Now, let us turn our attention to the S-matrix concerning the unstable particle. We can really calculate the S-matrix for any processes by means of a-function where only stable particles participate in the initial and final states.[11] However, in order to deal with the process where unstable particles participate in the initial and/or final states, we must define one particle state for the unstable particle. If we follow the proposal by Ida,[16] such a state should be expressed as $\int\phi(x)h(x)d^4x$, where $h(x)$ is a space-time function associated with the observation (preparation or detection) of the unstable particle. Then, expressing $\phi(x)$ in the above state vector by the asymptotic field of χ through Eq. (3·1) and using the asymptotic condition for χ-field, we can calculate the S-matrix in terms of a-function, and $h(x)$ completely. Although in this article we restricted ourselves only to a simple model where both the production and the decay of the unstable particle are caused by the same interaction, the formalism presented here may be easily generalized to a more realistic case without any principal difficulty. For instance, let us consider π-meson, which decays into leptons and also strongly interacts with nucleons. In accordance with the present formulation, the field operator of π-meson may be represented by the asymptotic field operators of leptons and nucleons, if leptons and nucleons are stable. At the switching off of the weak interaction, the lepton components in

the field operator of π-meson will disappear, but give the contribution of δ-like part to the spectrum function of π-meson propagator, leading to the same result as π-meson being stable.

Finally, let us comment on the possibility to eliminate the field operator for the unstable particle, which was already pointed out in § 3. Now, if we regard the expression for $a(x_1\,;\,x_2,\,x_3,\,x_4)$, i.e. Eqs. (4·6) and (4·20) as the solution of only Eq. (A), that is, the χ-system, taking Eqs. (B) and (C) out of consideration, this expression enables us to interpret it as the result of a non-local interaction of the χ-system of the form $\lambda^2\int\chi^2(z)a_\phi(z\,;\,z')\chi^2(z')dz^4d^4z'$. Thus, in this case, the effect of an unstable particle is reduced to the problem of the structure of interaction of its decay products. The introduction of the local field operator for the unstable particle means only to impose the restriction, such as Eqs. (B) and (C), to the non-local form factor $a_\phi(z\,;\,z')$. Thus, from the standpoint of the formal axiomatic formalism presented here, any necessity to introduce the field operator for an unstable particle does not arise, unless at the same time other physical principle, such as a symmetry is taken into account; for instance, the introduction of field operator for the neutron has an essential meaning with respect to the isotopic symmetry in the strong interaction.

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Equilibrium Abundances of Stable Isotopes

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Equilibrium distribution calculated on the basis of the analogue of Saha's ionization formula is examined with the recent nuclear masses and abundance data. It is found that up to $A=92$, the abundance distribution can be well reproduced. The electron degeneracy which becomes important has also been considered for the region $A=64-92$. The equilibrium temperature is found to be 6.5×10^9 degrees. The neutrality is taken to have been satisfied by protons, neutrons, electrons and alpha-particles. The presence of positron is not considered. Though effects of the excited levels are neglected, the differences between the even and odd nuclei are taken into account.

§ 1. Introduction

Recently, Burbidge, Burbidge, Fowler and Hoyle[1] visualized some eight processes for the synthesis of elements in stars. Some sort of equilibrium was expected at a temperature above $\sim3\times10^9$ degrees when nuclear reactions of all sorts occur in profusion. Hoyle[2] calculated the equilibrium abundances of iron group elements and was successful in reproducing the abundance distribution for $A=48$–62. Burbidge et al.[1] re-examined the equilibrium abundances of this iron group elements with recent excited levels data. Hoyle and Fowler[3] examined the equilibrium distribution of elements inside the cores of type I and type II supernovae. At higher densities, they found that degeneracy effect was very important.

In the present note attempt is made to examine the equilibrium distribution of stable isotopes with recent nuclear masses[4] and abundance data.[5] The abandance distribution of elements has been modified since the days of V. M. Goldschmidt. The fixing of temperature here remains quite arbitrary. The abundances are calculated for different temperatures, and the temperature for which the best fit between the calculated and observed values is obtained has been considered as the required temperature. In the present paper, the whole region up to $A=92$ has been divided into two regions, the reason of which will be elaborated later on.

The neutrality condition is satisfied with protons, electrons and alpha-particles. Once the temperature is fixed, the concentrations of the above particles are determined for a given concentration of neutrons.

§ 2. Element abundances and nuclear data

Goldschmidt first compiled the relative abundances of nuclear species in the year 1937. Many authors in their different investigations utilized these abundances for comparing theory with observation. Goldschmidt's original table had been modified by Brown.[6] In his tabulation, Brown considered that the values of the relative abundances of elements in the visible universe would be derivable from the studies of the stellar spectra. There were difficulties in such consideration as was pointed out by Brown himself. His tabulation consisted of the meteoristic, solar and stellar data. Suess and Urey[5] modified the abundance table in the light of new analytical data. In the present paper this table has been used. Chandrasekhar and Henrich[7] used the atomic masses from Barkas.[8] Beskow, Treffenberg and Klein[9] used the binding energies tabulated by Mattauch and Flügge.[10] In the following we calculate the binding energies of the elements with the help of

$$Q(A, Z)=[M_n N+M_p Z-M(A, Z)]c^2 \tag{1}$$

$$A=N+Z$$

$$\text{AMU}=931.14 \text{ Mev},$$

where M_n, M_p are the masses of the neutron and proton respectively and $M(A, Z)$ is the mass of the nuclear species (A, Z). It is found that in most of the elements the differences in the values of binding energies tabulated by Mattauch and Flügge and that we have calculated are appreciable and they are found to affect the calculation of the abundances to a marked extent.

§ 3. Basic equations

We consider an equilibrium assembly where the following reactions take place[7]

$$e^-+e^+ \rightleftarrows \text{Gamma rays} \tag{2}$$

$$p+e^- \rightleftarrows n \tag{3}$$

$$X(A, Z)+\Delta p+\Delta n \rightleftarrows X'(A', Z') \tag{4}$$

where $X(A, Z)$ and $X'(A', Z')$ are nuclei of type (A, Z) and (A', Z') respectively and

$$\Delta p=Z'-Z$$
$$\Delta n=A'-(A+\Delta p). \tag{5}$$

The effect of neutrino is neglected.

At a temperature $\gtrsim 4\times 10^9$ degrees the relativistic mass variation of the electron has to be taken into account. Thus the relation between neutron,

proton and electron number densities for the reaction (3) is given by[11]

$$C_p C_{e^-}/C_n = 2\left(\frac{m_0 kT}{2\pi\hbar^2}\right)^{3/2}(1+15kT/(8m_0c^2)+\cdots)\,e^{-Q_0/kT} \tag{6}$$

where $Q_0=-0.78$ Mev and m_0 is the mass of an electron. C_p, C_{e^-} and C_n are the concentrations of proton, electron and neutron respectively. T is in degrees and the other terms have their usual meaning. Positrons being neglected from the neutrality condition (Eq. 7), an equation for electron and positron concentrations need not be considered.

The condition that the assembly is electrically neutral is

$$C_{e^-}=\sum_{i=2}^{\infty} Z_i C_i(A_i, Z_i)+C_p. \tag{7}$$

Neglecting other heavier nuclei except for He^4, the neutrality condition has been simplified to

$$C_{e^-}=C_p+2C_\alpha. \tag{8}$$

Then the equation giving the concentration of alpha particles in terms of proton and neutron number densities is[7]

$$C_p^2\cdot C_n^2/(2C_\alpha)=\left(\frac{2\pi MkT}{h^2}\right)^{9/2}\exp(-Q_\alpha/kT) \tag{9}$$

where M is the unit of atomic mass and Q_α, the binding energy of α-particle, is given by

$$Q_\alpha=(2M_p+2M_n-M_\alpha)c^2. \tag{9a}$$

Eqs. (6), (8) and (9) are solved (by trial and error) for any assigned neutron concentration and temperature, and consequently the concentrations of protons, alpha-particles and electrons are obtained. When the relativistic degeneracy of electrons is considered for the region of heavier nuclei, then instead of Eq. (6) we have[12]

$$C_n/C_p=\lambda(M_n/M_p)^{3/2}\exp(Q_0/kT) \tag{10}$$

where λ, the degeneracy parameter, is given by

$$\log_{10}\lambda\sim 0.214\,\frac{hc}{kT}\,C_e^{1/3} \tag{11}$$

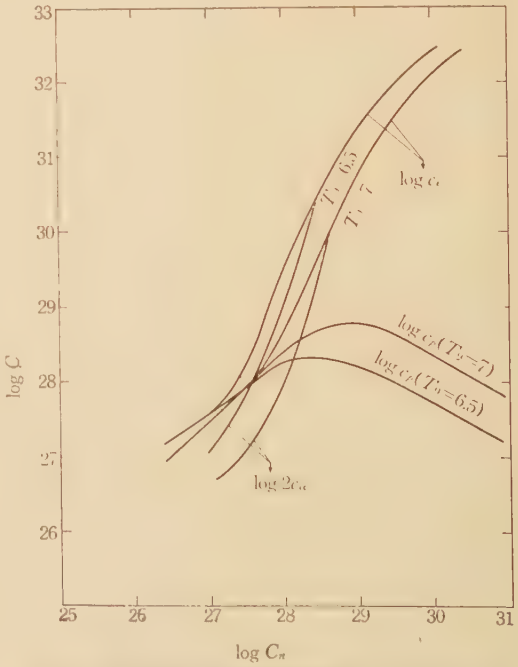

Fig. 1. Equilibrium concentrations of electrons, protons and alpha particles as a function of neutron concentrations at $T_9=6.5$ and $T_9=7.0$.

Now Eqs. (8), (9) and (10) are solved (by trial and error) for any assigned temperature and neutron concentration. The results for two different temperatures are shown in Fig. 1.

In order to fix up C_n and C_p values for further calculations of the abundances, we use the known concentrations of two nuclear species and put them in the expression[7]

$$\begin{aligned}\log C'(A', Z') &= \log C(A, Z) + \log\{w'(A', Z')/w(A, Z)\} \\ &+ 1.5 \log(A'/A) + \Delta p(\log C_p - 34.07 - 1.5 \log T_9) \\ &+ \Delta n(\log C_n - 34.07 - 1.5 \log T_9) + \frac{5.04}{T_9}(Q' - Q) \end{aligned} \tag{12}$$

where $w(A, Z)$ is the weight factor for the nucleus (A, Z), T_9 is in the unit of 10^9 degrees and Q_s are expressed in Mev.

For a particular temperature, the binding energies of two nuclear species and their abundances being known, the value of C_p and C_n can be fixed up. Once C_p and C_n are fixed, they can be used in the equation[1]

$$\begin{aligned}\log C(A, Z) &= \log w(A, Z) + 33.12 + 1.5 \log(AT_9) \\ &+ 5.04/T_9 \cdot Q(A, Z) + A(\log C_n - 34.07 - 1.5 \log T_9) + Z\theta \end{aligned} \tag{13}$$

where $\theta = \log C_p/C_n$ and $w = 2s+1$, s being the spin of the ground state of the nucleus. The numerical factor, which is slightly different from the previous ones is taken from Hoyle and Fowler.[3]

§ 4. Numerical calculations

Calculations for relative abundances are carried out for temperatures ranging from 4×10^9 to 8×10^9 degrees. It is found that at 6.5×10^9 degrees the best fit between the observed and calculated values is obtained. Taking the abundances of O^{16} and A^{36} from Suess and Urey,[5] we fix the following parameters:

$$\begin{aligned} T &= 6.5\times10^9 \text{ degrees} \\ \log C_n &= 27.42 \\ \log C_p &= 28.99 \\ \log C_\alpha &= 29.77. \end{aligned} \tag{14}$$

Therefore the material density is obtained to be $\rho = 2\times10^6$ g cm^{-3}. When these values are substituted, Eq. (13) reduces to the following simple form.

$$\log C(A, Z) = \log w + 1.5 \log A + 1.57Z + 34.34 + 0.775Q - 7.87A. \tag{15}$$

This equation has been used to calculate the abundances of elements as far as $A = 62$. Figure 2 shows the comparison between the calculated and the observed abundances in this region. To reduce the calculated abundances to the scale of observed abundances the observed abundances of O^{16} has been used.

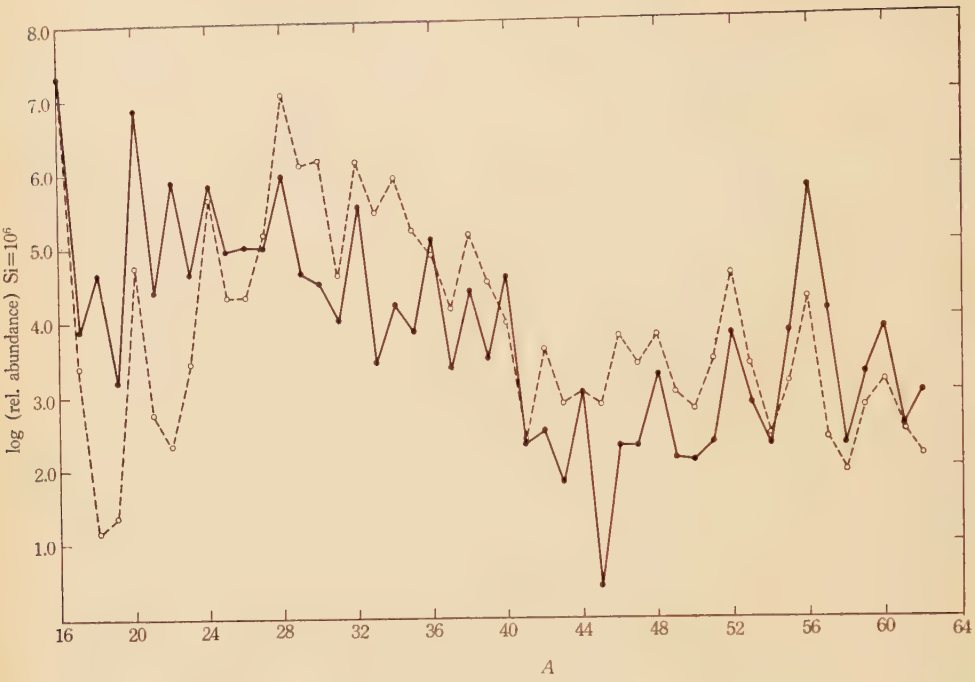

Fig. 2. Relative abundances as a function of atomic weight in the region A=16 to A=62. —●—●— observed abundance; —○—○— calculated abundance. $T=6.5\times10^9$ deg.; $\theta=1.57$. The observed data, normalized to Si=10^6, are those of Suess and Urey.[5]

If we continue our calculations a bit further, gross discrepancies occur. In this region beyond $A=64$, degeneracy becomes important. Therefore, in this region calculations are carried out taking into account the complete relativistically degenerate electron case. C_p and C_n values are fixed with the help of the observed abundances of Zn^{64} and Sr^{88} as it was done earlier. The best fit is obtained for the following parameters:

$$\begin{aligned} T&=6.5\times10^9 \text{ degrees}\\ \log C_n&=28.58\\ \log C_p&=28.32\\ \log C_\alpha&=30.45 \end{aligned} \tag{16}$$

and the density is found to be $\rho=2\times10^7$ g cm^{-3}. With these values Eq. (13) reduces to

$$\log C(A,Z)=\log w(A,Z)+1.5\log A+0.775Q -0.26Z+34.34-6.71A. \tag{17}$$

With the help of this equation, the abundances are calculated from $A=64$ to $A=92$. The calculated and the observed values of abundances are plotted in

Fig. 3. To reduce the calculated values to the scale of the observed abundances the observed abundances of Zn^{64} (relative to $Si=10^6$) has been used.

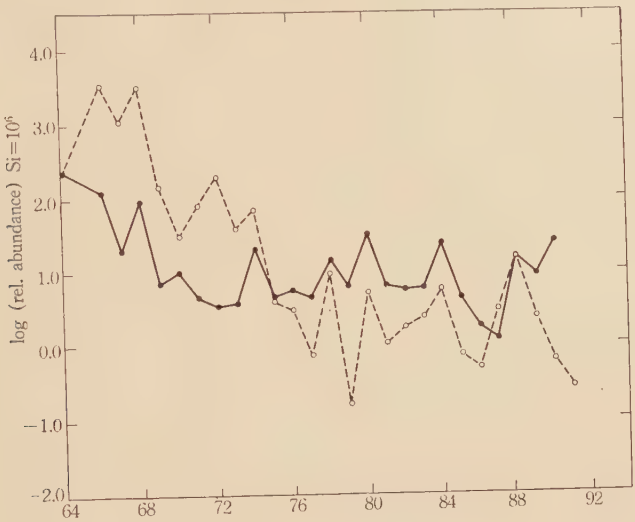

Fig. 3. Relative abundance as a function of atomic weight in the region A=64 to A=92. −●−●− Observed abundances; −○−−○−− calculated abundances. $T=6.5\times10^9$ deg. and $\theta=-0.26$. The observed data, normalized to $Si=10^6$, are those of Suess and Urey.

For known ground state spin values w is calculated and for the species where the ground state spin values of the odd nuclei are not available, we have taken $w=2$.

§ 5. Discussion and conclusion

The agreement between the calculated and observed abundances is fairly satisfactory, in view of the fact that there is already an uncertainty in observed values which may be as much as a factor five in some cases. However, for the region $A=64$–92, the fit is not very good. Therefore, a re-investigation of this region in the light of partial degeneracy may be important.

For the equilibrium process a reasonable fit is obtained for the temperature 6.5×10^9 degrees. It is not very certain whether this temperature can be available in the pre-supernova stage of the stellar evolution for a time long enough to acquire an equilibrium situation. But there is a probability that a temperature of this order can be reached in these stages. The re-examination of Hoyle's calculation by Burbidge et al. with new excited states of nuclei shows that the temperature required for the fit of the calculated and observed abundances is lower than the one required when the excited states are not considered. It is

therefore expected that such a consideration would lower the required temperature for a reasonable fit.

In the two regions considered, the densities are different, though the temperature is the same. When a stellar core shrinks the temperature does not rise considerably because the energy goes toward a mutual conversion of the nuclei; but all the same, the density increases. Such a consideration may explain the required change in density.

From this calculation it can be seen that for the range of $A > 100$, the equilibrium theory would require temperature and density much higher than can be expected at any stage of stellar evolution. Hence, if one believes in this hypothesis one has to look for some other processes for the formation of heavier elements as suggested by Burbidge et al. But higher density and temperature can easily be envisaged in big bang hypothesis. Thus it may be said that the results of this note are not quite decisive to decide in favour of one or the other but it seems that stellar hypothesis is not unworkable. Observations do seem to suggest that even if some percentage of elements were found at the beginning of the universe the abundances are being modified by the reactions in the interior of the stars. The theory of stellar structure is not yet developed enough to the point as to ascertain whether these densities and temperatures can be reached during the evolution of the stars.

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On the Plasma Oscillations in Metal Foils

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The dispersion relations, lifetimes of plasmons in metal foils and the probability of plasmon excitation by incident electrons are derived quantum mechanically. The results obtained are the generalizations of Ferrell's semi-classical theory.

§ 1. Introduction

Ritchie[1] has shown, using a dielectric theory, that for a thin metal foil the effect of the metal-vacuum boundary would be expected to cause a decrease in intensity of the plasma loss and the appearance of an additional lowered plasma loss at $\hbar\omega_p/\sqrt{2}$. A simple physical explanation of the oscillation of the frequency $\omega_p/\sqrt{2}$ as that due to a surface wave of the charge bound at the metal-vacuum interface has recently been given by Stern and Ferrell.[2] These authors have derived the dispersion relation of the "surface" plasmons

$$\omega=\omega_p/(1+\varepsilon)^{1/2}, \tag{1·1}$$

where ε is the dielectric constant of a dielectric layer. For a plasma bounded by vacuum one must set $\varepsilon=1$ and one obtains just Ritchie's result. But if ε is significantly greater than 1 for an actual oxide layer, the frequency of the "surface" plasmons is lowered. This prediction has recently been confirmed by experiments by Powell and Swan.[3]

In this paper it is shown quantum mechanically that there are two types of plasma oscillations, the "normal" oscillations of the frequency $\omega\sim\omega_p$ and the "tangential" oscillations of the frequency $\omega\sim\omega_p/\sqrt{2}$. In § 2 the dispersion relations, lifetimes of "normal" and "tangential" plasmons and the excitation rates of these plasmons by incident electrons are derived. The results obtained are the extensions of Ferrell's semi-classical theory. Our quantum mechanical treatment of electron interactions shows that Ferrell's explanation of the "surface" oscillation as that due to the surface wave of the charge bound at the plasma-vacuum interface should not be taken too literally. In the case of "tangential" oscillation the electric lines of force leak outside the metal foil, while in the case of "normal" oscillation the electric lines of force are confined inside

the foil.* Therefore the electric field within the foil is weaker in the former case than in the latter case, so that the frequency of "tangential" oscillation is lower than that of "normal" oscillation. When oxide layers are formed, the electric field outside the foil polarizes the oxide layers and the electric field induced by the polarizations of the oxide layers acts to reduce the electric field within the metal foil. Hence as the oxide layers are formed, the frequency of "tangential" oscillations is lowered, while that of "normal" oscillations remains unchanged. It will be shown in a subsequent paper that the interaction of electrons in a metal foil with electrons in dielectric layers modifies the Coulomb interaction between metal electrons, and this modification is effective only in the case of "tangential" oscillations.

§ 2. Metal foil bounded by vacuum

We consider a metal foil of thickness D bounded by vacuum. In the cartesian coordinate system we take the metal-vacuum interfaces at $z=\pm D/2$. We impose the periodic boundary conditions with period $L(\gg D)$ in x and y directions. Let us consider the Coulomb potential

$$v(\boldsymbol{r}-\boldsymbol{r}')=e^2/|\boldsymbol{r}-\boldsymbol{r}'|, \tag{2·1}$$

which can be expanded in the form

$$v(\boldsymbol{r}-\boldsymbol{r}')=\frac{1}{L^2}\sum_{k_x k_y} v(k_x, k_y, z-z')\exp[i\{k_x(x-x')+k_y(y-y')\}], \tag{2·2}$$

where $k_x=2\pi n_x/L$ and $k_y=2\pi n_y/L$, n_x and n_y being integers. Since $-D\leqq z-z'\leqq D$ and $v(k_x, k_y, -D)=v(k_x, k_y, D)$, we may expand $v(k_x, k_y, z-z')$ in a Fourier series with period $2D$. Then we get (Appendix I)

$$v(\boldsymbol{r}-\boldsymbol{r}')=\frac{1}{L^2 D}\sum_{\boldsymbol{k}} v(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}, \tag{2·3}$$

where

$$v(\boldsymbol{k})=(2\pi e^2/k^2)[1-\exp\{-(k_x^2+k_y^2)^{1/2}D-ik_z D\}], \quad (k_z=\pi n_z/D). \tag{2·4}$$

The Hamiltonian of our system is given by

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* The classical equation of motion of an electron is

$$m\ddot{\xi}=-eE,$$

where ξ is the displacement of the electron. From $D=\varepsilon E=E+4\pi P$, we get $E=-4\pi P/(1-\varepsilon)$. $\varepsilon(\omega)=1-\omega_p^2/\omega^2$ is the dielectric constant of the plasma. The equation of motion becomes

$$m\ddot{\xi}=-[4\pi ne^2/(1-\varepsilon)]\xi.$$

If $\varepsilon(\omega)+1=0$, the frequency is $\omega_p/\sqrt{2}$. In this case the electric displacement D is not zero, therefore the electric field outside the plasma is not zero. If $\varepsilon(\omega)=0$, the frequency is ω_p. In this case $D=0$, so that the electric field outside the plasma is zero.

$$H=\int \Psi^*(\boldsymbol{r})\left[-\frac{\hbar^2}{2m}\Delta+U(\boldsymbol{r})\right]\Psi(\boldsymbol{r})\,d\boldsymbol{r}$$
$$+\frac{1}{2}\iint \Psi^*(\boldsymbol{r})\,\Psi(\boldsymbol{r})\,v(\boldsymbol{r}-\boldsymbol{r}')\,\Psi^*(\boldsymbol{r}')\,\Psi(\boldsymbol{r}')\,d\boldsymbol{r}\,d\boldsymbol{r}'$$
$$-\frac{1}{2}\int \Psi^*(\boldsymbol{r})\,\Psi(\boldsymbol{r})\,v(0)\,d\boldsymbol{r}, \qquad (2\cdot5)$$

where $U(\boldsymbol{r})$ is the potential of ion cores (Fig. 1) plus a uniform background of negative charge, which is compensated by a uniform background of positive charge (Fig. 2) interacting with electrons. $\Psi(\boldsymbol{r})$ and $\Psi^*(\boldsymbol{r})$ satisfy the anticommutation relations

$$[\Psi(\boldsymbol{r}),\ \Psi^*(\boldsymbol{r}')]_+=\delta(\boldsymbol{r}-\boldsymbol{r}'),$$
$$[\Psi(\boldsymbol{r}),\ \Psi(\boldsymbol{r}')]_+=[\Psi^*(\boldsymbol{r}),\ \Psi^*(\boldsymbol{r}')]_+=0. \qquad (2\cdot6)$$

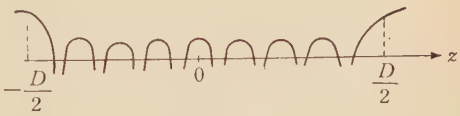

Fig. 1.

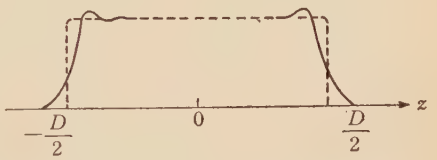

Fig. 2. The solid curve and the dashed line show the electron density and the uniform charge density respectively.

We expand $\Psi(\boldsymbol{r})$ and $\Psi^*(\boldsymbol{r})$ as follows,

$$\Psi(\boldsymbol{r})=\sum_{nq} b_{nq}\varphi_{nq}(\boldsymbol{r}),$$
$$\Psi^*(\boldsymbol{r})=\sum_{nq} b^*_{nq}\varphi^*_{nq}(\boldsymbol{r}), \qquad (2\cdot7)$$

where

$$-\frac{\hbar^2}{2m}\Delta\varphi_{nq}(\boldsymbol{r})+U(\boldsymbol{r})\,\varphi_{nq}(\boldsymbol{r})=\varepsilon_n(\boldsymbol{q})\,\varphi_{nq}(\boldsymbol{r}). \qquad (2\cdot8)$$

Inserting (2·7) into (2·5), we obtain

$$H=\sum_{nq}\varepsilon_n(\boldsymbol{q})\,b^*_{nq}b_{nq}+(1/2V)\sum_{\boldsymbol{k}} v(\boldsymbol{k})\,\rho_{\boldsymbol{k}}\rho_{-\boldsymbol{k}}-(N/2V)\sum_{\boldsymbol{k}} v(\boldsymbol{k}), \qquad (2\cdot9)$$

where $V=L^2D$ and $N=\sum_{nq} b^*_{nq}b_{nq}=\sum_{nq} N_{nq}$. The density fluctuation $\rho_{\boldsymbol{k}}$ is defined by

$$\rho_{\boldsymbol{k}}=\sum_{nq}\sum_{n'q'}\langle n'q'|e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}|nq\rangle\, b^*_{n'q'}b_{nq}, \qquad (2\cdot10)$$

where

$$\langle n'q'|e^{-i\boldsymbol{k}\boldsymbol{r}}|nq\rangle=\int \varphi^*_{n'q'}(\boldsymbol{r})\,e^{-i\boldsymbol{k}\boldsymbol{r}}\,\varphi_{nq}(\boldsymbol{r})\,d\boldsymbol{r}. \qquad (2\cdot11)$$

The dispersion relations and the lifetimes of plasmons can be derived by the same procedure as that of Nozieres and Pines.[4] We find (Appendix II)

$$1=\frac{2\pi e^2}{m\bar{V}}\left(1-\exp\{-(k_x^2+k_y^2)^{1/2}D-ik_zD\}\right)\sum_{nq}\sum_{n'q'}{}'\frac{f_{nq\,n'q'}(\boldsymbol{k})}{\omega^2-\omega^2_{n'q'\,nq}}, \qquad (2\cdot12)$$

$$\frac{1}{\tau}=\frac{\pi^2e^2}{mV}\left(1-\exp\{-(k_x^2+k_y^2)^{1/2}D-ik_zD\}\right)\sum_{nq}\sum_{n'q'}f_{nq\,n'q'}(\boldsymbol{k})\,\delta(\omega-\omega_{n'q'\,nq}), \tag{2·13}$$

where $\omega_{n'q'\,nq}=\{\varepsilon_{n'}(\boldsymbol{q}')-\varepsilon_n(\boldsymbol{q})\}/\hbar$. The longitudinal oscillator strength $f_{nq\,n'q'}(\boldsymbol{k})$ is defined by

$$f_{nq\,n'q'}(\boldsymbol{k})=(2m/\hbar^2k^2)\,\hbar\omega_{n'q'\,nq}|\langle n'q'|e^{-i\boldsymbol{k}\boldsymbol{r}}|nq\rangle|^2N_{nq}(1-N_{n'q'}). \tag{2·14}$$

We consider the following two cases.

Case A: $(k_x^2+k_y^2)^{1/2}D\ll1$

Since $1-\exp\{-(k_x^2+k_y^2)^{1/2}D-ik_zD\}\approx1-(-1)^{n_z}$, the dispersion relations for $n_z=\pm1,\ \pm3,\ \cdots$ become

$$1=\frac{4\pi e^2}{mV}\sum_{nq}\sum_{n'q'}\frac{f_{nq\,n'q'}(\boldsymbol{k})}{\omega^2-\omega_{n'q'\,nq}^2}. \tag{2·15}$$

(2·15) is the same as the dispersion relations of bulk plasmons. We shall call these plasmons "normal" plasmons, because in this case electrons oscillate back and forth across the foil from one surface to the other ("normal" oscillations). For $n_z=\pm2,\ \pm4,\ \cdots$ the dispersion relations become

$$1=\frac{2\pi e^2}{mV}(k_x^2+k_y^2)^{1/2}D\sum_{nq}\sum_{n'q'}\frac{f_{nq\,n'q'}(\boldsymbol{k})}{\omega^2-\omega_{n'q'\,nq}^2}. \tag{2·16}$$

We have not yet succeeded to determine whether (2·16) has stable solutions. Even if stable solutions exist, the excitation probability of these oscillations by incident electrons is small as compared with that of "normal" plasmons by a factor $(k_x^2+k_y^2)^{1/2}D(\ll1)$ as will be shown below. For $n_z=0$ the dispersion relations are also given by (2·16), but the oscillations are tangential. For an electron gas we get

$$\omega=(\omega_p/\sqrt{2})\,(k_x^2+k_y^2)^{1/4}D^{1/2}. \tag{2·17}$$

Since $\omega\propto k^{1/2}$, these oscillations are stable. However, it would be difficult to verify the existence of these oscillations experimentally.

Case B: $(k_c^2-k_z^2)^{1/2}>(k_x^2+k_y^2)^{1/2}\gtrsim\pi/D$*

The dispersion relations are given by

$$1=\frac{2\pi e^2}{mV}\sum_{nq}\sum_{n'q'}\frac{f_{nq\,n'q'}(\boldsymbol{k})}{\omega^2-\omega_{n'q'\,nq}^2}. \tag{2·18}$$

When $k_x,\ k_y\ll k_z$, electrons oscillate along a direction parallel to the surface of the foil ("tangential" oscillations). We shall call those plasmons which satisfy the dispersion relations (2·18) "tangential" plasmons, though generally oscil-

* $e^{-\pi}=0.043$.

lations are not tangential. In the case of an electron gas the dispersion relation for $k_z=0$ is given by

$$1=2\cdot\frac{2\pi e^2}{mV}\sum_{\substack{q\\|q|<k_0}}\frac{1}{\{\omega+\hbar(k_xq_x+k_yq_y)/m\}^2-\hbar^2(k_x^2+k_y^2)^2/4m^2}, \qquad (2\cdot19)$$

where the factor 2 is due to the electron spin and $\hbar k_0$ is the Fermi momentum. Replacing the summations by integrations, we find

$$\omega=\frac{\omega_p}{\sqrt{2}}\left\{1+\frac{6}{5}\frac{\hbar^2k_0^2}{m^2\omega_p^2}(k_x^2+k_y^2)+\cdots\right\}. \qquad (2\cdot20)$$

If, for a given k, the frequency of intraband transition $\omega_{n'q'nq}$ is on the average small compared with ω, then (2·15) and (2·18) may be written as

$$1=\frac{\omega_p^2}{\omega^2}+\sum_{\substack{n\neq n'\\nq\ n'q'}}\sum\frac{(4\pi e^2/mV)f_{nq\,n'q'}(k)}{\omega^2-\omega_{n'q'\,nq}^2}, \qquad (2\cdot15)'$$

$$1=\frac{\omega_p^2}{2\omega^2}+\sum_{\substack{n\neq n'\\nq\ n'q'}}\sum\frac{(2\pi e^2/mV)f_{nq\,n'q'}(\boldsymbol{k})}{\omega^2-\omega_{n'q'\,nq}^2} \qquad (2\cdot18)'$$

where $\omega_p^2=4\pi Ne^2/m^*V$. If interband parts are unimportant, the ratio of the frequencies of "normal" and "tangential" plasmons is nearly equal to $\sqrt{2}$. If an optical absorption band is present near ω_p (or $\omega_p/\sqrt{2}$) the frequency may be shifted upwards or downwards depending upon the relative positions of the absorption band and ω_p (or $\omega_p/\sqrt{2}$).[5]

The transverse dielectric constant is given by[6]

$$\varepsilon^{tr}(\omega_{k\mu})=1-\frac{4\pi e^2}{mV}\sum_{nq}\sum_{n'q'}\frac{f_{nq\,n'q'}^{tr}(\boldsymbol{k})}{\omega_{k\mu}^2-\omega_{n'q'\,nq}^2}. \qquad (2\cdot21)$$

For an isotropic solid the transverse oscillator strength is equal to the longitudinal one provided that the random phase approximation (RPA) is valid.[6] Hence within RPA $\varepsilon^{tr}=0$ and $\varepsilon^{tr}+1=0$ are just the dispersion relations (2·15) and (2·18) of "normal" and "tangential" plasmons respectively. Fröhlich and Pelzer[7] pointed out that $2nk/(n^2+k^2)^2$ should have a peak at the "normal" plasmon energy. Analogously, $nk/[(n^2-k^2+1)^2/4+n^2k^2]$ is expected to have a peak at the "tangential" plasmon energy, if plotted as a function of $h\nu$. Taft and Philipp[8] have measured the optical constants n and k for Ag. $\varepsilon^{tr}=n^2-k^2$ is zero at 3.75 ev which is in agreement with the value obtained by Steinmann.[9] Steinmann has observed the radiation of plasmons predicted by Ferrell.[10] Using the observed values of n and k, we find that $\varepsilon^{tr}+1=n^2-k^2+1$ is zero at about 3.4 ev, which is very close to the "normal" plasmon energy 3.75 ev. As is shown below, the width of the "normal" plasmon state is about 0.8 ev for Ag. Therefore the "normal" and "tangential" plasmon loss lines for Ag would not be clearly separated in energy loss experiments. In fact in energy loss experiments on Ag only a single loss line between 3.4 ev and 3.6 ev was observed.[11]

The lifetimes of "normal" and "tangential" plasmons are

$$\frac{1}{\tau_n}=4\pi\sigma(k,\omega_n), \tag{2·22}$$

$$\frac{1}{\tau_t}=2\pi\sigma(k,\omega_t), \tag{2·23}$$

where the longitudinal conductivity $\sigma(k,\omega)$ is

$$\sigma(k,\omega)=(\pi^2e^2/2mV)\sum_{nq}\sum_{n'q'}f_{nq\,n'q'}(\boldsymbol{k})\,\delta(\omega-\omega_{n'q'\,nq}). \tag{2·24}$$

If RPA is valid, the line widths may be written as

$$\frac{\hbar}{\tau_n}=2\hbar\omega_n n(\omega_n)\,k(\omega_n), \tag{2·25}$$

$$\frac{\hbar}{\tau_t}=\hbar\omega_t n(\omega_t)\,k(\omega_t). \tag{2·26}$$

For Ag $n(\omega_n)k(\omega_n)\sim0.1$, hence $\hbar/\tau_n\sim0.75$ ev. The observed values of n and k are not available at present except for silver and alkali metals.

The interaction Hamiltonian between incident electrons and plasmons is given by (Appendix II)

$$H'=\frac{D}{L}\sum_{k<k_c}\sum_{KK'}\left(\frac{\hbar\omega v(\boldsymbol{k})}{2V}\right)^{1/2}\delta(\Delta K_x-k_x)\,\delta(\Delta K_y-k_y)$$
$$\times\frac{\sin[(\Delta K_z-k_z)D/2]}{(\Delta K_z-k_z)D/2}\,(a_k^*+a_{-k})B_{K'}^*B_K, \tag{2·27}$$

where $\Delta\boldsymbol{K}=\boldsymbol{K}-\boldsymbol{K}'$. B_K, B_K^*, a_k and a_k^* are the destruction and creation operators of incident electrons and plasmons. It should be noted that (2·27) involves the factor $\sin[(\Delta K_z-k_z)D/2]/(\Delta K_z-k_z)D/2$ instead of $\delta(\Delta K_z-k_z)$. The momentum conservation holds in x and y directions, but the z-component of momentum is not conserved.* The momentum $\hbar(\Delta K_z-k_z)$ ought to be given to the lattice of the metal, or in other words it must be equal to the recoil momentum of the center of mass of the metal foil. From the standard perturbation theory, the rate of plasmon excitation is given by

* The conservation of z-component of momentum contradicts with the conservation of energy. If the energy-momentum conservation holds, we get

$$\hbar\omega-\hbar^2K^2/2m\quad\hbar^2(K-k)^2/2m=\hbar^2Kk_z/m-\hbar^2k^2/2m,$$

where we have used the fact that the direction of the momentum $\hbar K$ of the incident electron is along the z-axis. The second term of the above equation is small as compared with the first. Since $k_z=\pi n_z/D$, the right-hand side is equal to $\hbar^2Kn_z/mD$, which is not equal to $\hbar\omega$ except in the case of accidental coincidence. Fujimoto (unpublished) has recently developed a theory of electron diffraction, in which he has attributed the coherent interference of inelastically scattered waves in thin solid foils to the non-conservation of z-component of momentum.

$$(2\pi/\hbar)\,|(H')_{fi}|^2\rho_f=\frac{D}{L}\,\frac{\pi\hbar\omega v(\boldsymbol{k})mK}{\hbar^3}\left[\frac{\sin[(JK_z-k_z)D/2]}{(JK_z-k_z)D/2}\right]^2 d\Omega. \quad (2\cdot28)$$

Deviding (2·28) by $d\Omega_v/L$, we get the scattering coefficient

$$\mu(\theta, D)=\frac{D}{4\pi a_0}\,\frac{1+\exp\{-(k_x^2+k_y^2)^{1/2}D-ik_zD\}}{k^2/K^2}\,\frac{\hbar\omega}{mv^2}\left[\frac{\sin(JK_z-k_z)D/2}{(JK_z-k_z)D/2}\right]^2, \quad (2\cdot29)$$

where $v=\hbar K/m$ and $a_0=e^2m/\hbar^2$. Since $k_x^2+k_y^2\approx K^2\vartheta^2$, where ϑ is the angle of scattering (angle between $\boldsymbol{K}$ and $\boldsymbol{K}'$), the scattering coefficients for the excitation of "normal" and "tangential" plasmons are given by

$$\mu_n(\theta, D)=\frac{D}{2\pi a_0}\,\frac{1}{\theta^2+(k_z^2/K^2)}\,\frac{\hbar\omega_n}{mv^2}\left[\frac{\sin(JK_z-k_z)D/2}{(JK_z-k_z)D/2}\right]^2, \quad (2\cdot30)$$

$$\mu_t(\theta, D)=\frac{D}{4\pi a_0}\,\frac{1}{\theta^2+(k_z^2/K^2)}\,\frac{\hbar\omega_t}{mv^2}\left[\frac{\sin(JK_z-k_z)D/2}{(JK_z-k_z)D/2}\right]^2. \quad (2\cdot31)$$

Ferrell[10] has obtained the scattering coefficient for the "normal" plasmon excitation in thin metal foils by a semi-classical theory. His result is

$$\mu_n(\theta, D)=\frac{D}{2\pi a_0}\left[\frac{\hbar\omega_n/mv^2}{\theta^2+(JK_z/K)^2}\right]^2\left[\frac{\sin(JK_zD/2)}{JK_zD/2}\right]^2. \quad (2\cdot32)$$

Watanabe[12] has recently measured the angular distribution of electrons which suffered characteristic energy loss. The observed angular distribution agrees with that calculated by Ferrell[13] for a bulk electron gas. Ferrell's calculation[13] is based on the differential mean free path for a bulk electron gas

$$d(1/\lambda)=\left(\frac{d\Omega}{2\pi a_0}\right)\frac{\hbar\omega/mv^2}{\theta^2+(JK_z/K)^2}. \quad (2\cdot33)$$

(2·32) gives larger values at small angles and smaller values at large angles than those predicted by (2·30). Hence the angular distribution predicted by (2·32) seems to be in disagreement with Watanabe's experimental results. However, detailed comparison with experiments will be left to future investigations.

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Appendix I

For sufficiently large L

$$v(k_x, k_y, \zeta)=e^2\int_{-\infty}^{\infty}\int\frac{\exp\{-i(k_x\xi+k_y\eta)\}}{(\xi^2+\eta^2+\zeta^2)^{1/2}}\,d\xi\,d\eta$$

$$=e^2\int_0^\infty \frac{\rho\, d\rho}{(\rho^2+\zeta^2)^{1/2}}\int_0^{2\pi}\exp[-i(k_x^2+k_y^2)^{1/2}\rho\cos\varphi]\,d\varphi$$

$$=\frac{2\pi e^2}{(k_x^2+k_y^2)^{1/2}}\exp\{-(k_x^2+k_y^2)^{1/2}|\zeta|\}. \qquad (A\cdot 1)$$

Since $\int_{-D}^{D}\exp\{i(k_z-k_z')\zeta\}\,d\zeta=2D\delta_{k_zk_z'}$, we get

$$v(\boldsymbol{k})=\frac{1}{2}\int_{-D}^{D}v(k_x,\,k_y,\,\zeta)\exp\{-ik_z\zeta\}\,d\zeta$$

$$=\frac{2\pi e^2}{k^2}[1-\exp\{-(k_x^2+k_y^2)^{1/2}D-ik_zD\}], \qquad (A\cdot 2)$$

where $e^{ik_zD}=e^{-ik_zD}=(-1)^{n_z}$, so that $v(\boldsymbol{k})=v(-\boldsymbol{k})$. It is to be noted that in the limit of $L\to\infty$ and $D\to\infty$ we get

$$v(\boldsymbol{r}-\boldsymbol{r}')=\frac{1}{L^2D}\frac{L^2}{(2\pi)^2}\frac{D}{\pi}\int\frac{2\pi e^2}{k^2}e^{i\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')}\,d\boldsymbol{k}=\frac{1}{(2\pi)^3}\int\frac{4\pi e^2}{k^2}e^{i\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')}\,d\boldsymbol{k}.$$

Appendix II

The Hamiltonian of the system of incident electrons and a metal foil is given by

$$H=H_{metal}+\sum_K(\hbar^2K^2/2m)B_K^*B_K+H_{int}, \qquad (A\cdot 3)$$

where H_{metal} is given by (2·9). B_K^* and B_K are the creation and destruction operators of incident electrons respectively. The interaction Hamiltonian H_{int} between incident electrons and metal electrons is

$$H_{int}=\frac{1}{L^3}\sum_{KK'}\sum_{nqn'q'}\int\!\!\int_V e^{i(\boldsymbol{K}-\boldsymbol{K}')\boldsymbol{R}}\frac{e^2}{|\boldsymbol{r}-\boldsymbol{R}|}\varphi_{n'q'}^*(\boldsymbol{r})\,\varphi_{nq}(\boldsymbol{r})\,d\boldsymbol{r}\,d\boldsymbol{R}\,B_{K'}^*B_Kb_{n'q'}^*b_{nq}$$

$$=\frac{1}{L^3}\sum_k\sum_{KK'}v(\boldsymbol{k})\,\delta(\Delta K_x-k_x)\,\delta(\Delta K_y-k_y)\frac{\sin(\Delta K_z-k_z)D/2}{(\Delta K_z-k_z)D/2}\rho_{-k}B_{K'}^*B_K. \qquad (A\cdot 4)$$

We introduce the plasmons by adding certain terms to (A·3). The resultant extended Hamiltonian is given by

$$H_{ext}=\sum_{nq}\varepsilon_n(\boldsymbol{q})\,b_{nq}^*b_{nq}+\sum_K(\hbar^2K^2/2m)\,B_K^*B_K-(N/2V)\sum_k v(\boldsymbol{k})$$
$$+(1/2)\sum_{k>k_c}v(\boldsymbol{k})\,\rho_k\rho_{-k}$$
$$+(1/2)\sum_{k<k_c}\{(v(\boldsymbol{k})/V)^{1/2}\rho_k+p_k\}\{(v(\boldsymbol{k})/V)^{1/2}\rho_{-k}+p_{-k}\}+H_{int}. \qquad (A\cdot 5)$$

The subsidiary conditions are

$$p_k \Psi_{ext}=0 \quad (k<k_c). \tag{A·6}$$

We consider a canonical transformation generated by S_1 which is defined by

$$S_1=-\sum_{k<k_c}(1/\hbar)\,(v(\boldsymbol{k})/V)^{1/2}\rho_k q_k, \tag{A·7}$$

where $[p_k, q_{k'}]=-i\hbar\delta_{kk'}$. Introducing the creation and destruction operators of plasmons, a_k^* and a_k, by

$$\begin{aligned} q_k&=i(\hbar/2\omega)^{1/2}(a_{-k}-a_k^*),\\ p_k&=(\hbar\omega/2)^{1/2}(a_{-k}^*+a_k), \end{aligned} \tag{A·8}$$

we may write the new Hamiltonian as follows,

$$e^{-iS_1}H_{ext}e^{iS_1}=H_0+H_{s.r.}+H'+U+\Delta H_f+H_{int}, \tag{A·9}$$

where

$$H_0=\sum_{nq}\varepsilon_n(\boldsymbol{q})\,b_{nq}^*b_{nq}+\sum_K(\hbar^2K^2/2m)\,B_K^*B_K+\sum_{k<k_c}\hbar\omega(a_k^*a_k+1/2)$$
$$-(N/2V)\sum_{k<k_c}v(\boldsymbol{k}), \tag{A·10}$$

$$H_{s.r.}=(1/2)\sum_{k>k_c}v(\boldsymbol{k})\,\rho_k\rho_{-k}-(N/2V)\sum_{k>k_c}v(\boldsymbol{k}), \tag{A·11}$$

$$H'=\sum_{k<k_c}\sum_{nq}\sum_{n'q'}(v(\boldsymbol{k})/2\hbar\omega V)^{1/2}\langle n'q'|e^{-i\boldsymbol{k}\boldsymbol{r}}|nq\rangle\hbar\omega_{n'q'nq}b_{n'q'}^*b_{nq}(a_{-k}-a_k^*), \tag{A·12}$$

$$\Delta H_f=\sum_{k<k_c}\hbar(\omega_0^2(k)-\omega^2)/4\omega\cdot(a_ka_k^*+a_k^*a_k-a_ka_{-k}-a_k^*a_{-k}^*), \tag{A·13}$$

where $\omega_0^2(k)=k^2v(\boldsymbol{k})N/mV$. U is the "random phase" term which is neglected in RPA. Making use of the new subsidiary conditions

$$\Omega_k\Psi=\{a_{-k}^*+a_k-(2v(\boldsymbol{k})/\hbar\omega V)^{1/2}\rho_k\}\Psi=0 \quad (k<k_c), \tag{A·14}$$

H_{int} may be written as

$$H_{int}=\frac{1}{L^3}\sum_{k>k_c}\sum_{KK'}v(\boldsymbol{k})\,\delta(\Delta K_x-k_x)\,\delta(\Delta K_y-k_y)\,\frac{\sin(\Delta K_z-k_z)D/2}{(\Delta K_z-k_z)D/2}\,\rho_{-k}B_{K'}^*B_K$$
$$+\frac{D}{L}\sum_{k<k_c}\sum_{KK'}\left(\frac{\hbar\omega v(\boldsymbol{k})}{2V}\right)^{1/2}\delta(\Delta K_x-k_x)\,\delta(\Delta K_y-k_y)$$
$$\times\frac{\sin(\Delta K_z-k_z)D/2}{(\Delta K_z-k_z)D/2}\,(a_k^*+a_{-k})\,B_{K'}^*B_K. \tag{A·15}$$

In order to eliminate the virtual part of H' we consider the second canonical transformation generated by S_2 which is defined by

$$S_2=-iP\sum_{k<k_c}\sum_{nq}\sum_{n'q'}\left(\frac{v(\boldsymbol{k})}{2\hbar\omega V}\right)^{1/2}\langle n'q'|e^{-i\boldsymbol{k}\boldsymbol{r}}|nq\rangle\,\omega_{n'q'nq}\left[\frac{b_{n'q'}^*b_{nq}a_{-k}}{\omega-\omega_{n'q'nq}}+\frac{b_{n'q'}^*b_{nq}a_k^*}{\omega+\omega_{n'q'nq}}\right], \tag{A·16}$$

where P denotes the principal part. Using the relation

$$[H_0, iS_2]=-H' \text{ (virtual)}, \tag{A·17}$$

under RPA we obtain

$$\begin{aligned}(1/2)[[H_0, iS_2],\ iS_2]+[H', iS_2]&=(1/2)[H', iS_2]\\ &=H_{r.p.}+\text{terms involving } (a_k^* a_k+a_k a_k^*+a_k a_{-k}+a_k^* a_{-k}^*).\end{aligned} \tag{A·18}$$

Setting $\mathcal{H}_f$+the second term in (A·18)$=0$, we get the dispersion relation

$$1=\frac{k^2 v(\boldsymbol{k})}{mV}\sum_{nq}\sum_{n'q'}\frac{f_{nq\,n'q'}(\boldsymbol{k})}{\omega^2-\omega_{n'q'\,nq}^2}. \tag{A·19}$$

The part of H' which describes real transitions is not affected by the second transformation and contributes to the damping of plasmons. Standard perturbation theory gives (2·13). H_{int} is also not affected by the second transformation. Therefore the interaction Hamiltonian between incident electrons and plasmons is given by the second term in (A·15).

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Energy Spectrum and Determinant Method in Quantized Field Theory

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This is a continuation of the previous works on the energy spectrum and the phase shift. First, one's attention is called to the fact that, when multiple processes compete, a certain combination of phase shifts which is always real has elementary properties. As an illustration Levinson's theorem which has previously been generalized to the field theory is directly examined in a modified Lee model in which two channels compete. Next it is shown that the determinant quantity introduced in the foregoing paper has an analytic property corresponding to the structure of the energy spectrum. From this property one obtains a dispersion-like formula involving the stated phase shift, one by one for each eigenspace, which holds without any approximation on inelastic processes. An application of this property also to a statistical-dynamical system is discussed. Further a relation on the asymptotic behavior of the phase shift is suggested and a difference in whether a particle is elementary or composite is remarked.

§ 1. Introduction

In the previous papers, hereafter to be referred to as I[1] and II[2], we have pointed out several basic relations concerning the energy spectrum and the phase shifts. We want to consider in this paper some further points in this direction, putting stress on a connection between the energy spectrum and the determinant quantity which was introduced in II. In the following we want to summarize our main subjects.

One of the characteristic features in the quantized field theory is the appearance of multiple competing processes which complicates the matter. But in I, § 5, and II we have obtained, also in multi-channel case, identities which have formal resemblance with one-channel case. For instance, while the relation between the phase shift $\eta(\omega)$ and the interaction Hamiltonian H_1 in one-channel case is

$$d\eta/dg = -\pi k^{-1}(\Psi\, dH_1/dg\, \Psi)$$

(g is the coupling constant, k is a suitable normalization factor and Ψ is the eigenstate), it is replaced by

$$d\eta_j(\omega)/dg = -\pi k^{-1}(\omega)\, Sp[P_j\, dH_1/dg\, P_j]$$

in multi-channel case, where $\eta_j(\omega)$ is a certain combination of the phase shifts:

$$\eta_j(\omega) = \sum_a \eta_{ja}(\omega). \qquad (1\cdot1)$$

Here ω is the total energy, j denotes the j-th eigenspace of a commuting set of constants of motion corresponding to certain symmetries, $2i\eta_{ja}(\omega)$ is the logarithm of an eigenvalue of the S-matrix on an energy shell and P_j is the projection operator onto the j-th eigenspace. $\eta_j(\omega)$ is defined also by

$$e^{2i\eta_j(\omega)}=\det|P_jS(\omega)\,P_j|, \tag{1·2}*$$

where $S(\omega)$ is the S-matrix. $\eta_j(\omega)$ is always real. We have in I, § 5 (see (5·37)) obtained the field theoretical generalization of Levinson's theorm[3),4)] which is expressed in terms of $\eta_j(\omega)$ as

$$\eta_j(\infty)-\eta_j(\omega_0)=-(\mathrm{m}^j-\mathrm{m}_0{}^j)\pi, \tag{1·3}$$

where ω_0 is the threshold value of the total energy, and m^j and $\mathrm{m}_0{}^j$ are the numbers of discrete levels of the interacting and interaction-free system respectively belonging to the j-th space. In § 2 this relation is directly examined in a modified Lee model[5),6)] in which two channels compete, since the theorem in the case of multi-channels has been rather unfamiliar to us. One will see there that the completeness of state vectors is essential for the validity of this theorem.[4)]

It is shown that information on the energy spectrum of each eigenspace will be best expressed in terms of the analyticity of the determinant quantity in the complex plane of the energy variable. This determinant qnantity has been defined (see II, (2·10)), one by one for each eigenspace, by

$$\Delta^j(E)=\det|P_j\left\{1+\frac{1}{H_0-E}H_1\right\}P_j|, \tag{1·4}$$

where E is the complex energy variable. Hereafter ω shall represent the real and E the complex energy variable, and let us follow the conventional notation that

$$\Delta^j(\omega)=\Delta^j(E)\,]_{E=\omega+i\varepsilon}, \tag{1·5}$$

where ε is infinitesimal and positive. $\Delta^j(E)$ has the following fundamental properties :

1) By the nature of the determinant, $\Delta^j(E)$ is invariant under the following transformations :

$$\left.\begin{aligned}&\Delta^j(E)=\det|P_j\left\{1+U\frac{1}{H_0-E}H_1U^\dagger\right\}P_j|,\\&UU^\dagger=U^\dagger U=1,\quad [U,P_j]=0.\end{aligned}\right\} \tag{1·6}$$

2) As has been shown in II, Appendix III, it holds that

$$e^{2i\eta_j(\omega)}=\Delta^{j*}(\omega)/\Delta^j(\omega). \tag{1·7}$$

* As for a rigorous definition of "det" see Appendix IV.

3) $1/\Delta^j(E)$ is analytic on the complex E-plane except for the singularities:

a) a cut on the real axis corresponding to continuous levels,

b) poles on the real axis corresponding one to one to discrete levels of the interacting system in the j-th space.

The properties 2) and 3) are contained in the following one formula:

$$\Delta^j(E) = \prod_i \left(\frac{E-\omega_i}{E-\omega_0}\right) \cdot \exp\left\{-\frac{1}{\pi}\int_{\omega_0}^{\infty} d\omega \frac{\eta_j(\omega)}{\omega-E}\right\}, \tag{1·8}$$

where ω_i is the energy eigenvalue of the i-th bound state and ω_0 is the threshold value of the continuous energy region. This relation will be proved in § 4 and Appendix III. From this analyticity a dispersion-like formula for $\Delta^j(E)$ is obtained (§ 4).

4) $\Delta^j(E)$ is connected with the level density (being denoted by $\rho^j(\omega)$ in the place of $\Delta N^j(\omega)$) introduced in I, § 5, by (see II, (2·6))

$$\rho^j(\omega) = (1/2\pi i)\, d/d\omega \log(\Delta^{j*}(\omega)/\Delta^j(\omega)). \tag{1·9}$$

This reduces to (1·7) in the continuous energy region by virtue of the general connection between the level density and the phase shift (see II, (1·12)). In the discrete energy region, if the system is a statistical-dynamical one with large degrees of freedom, $\rho^j(\omega)$ represents just the level density e. g. of the electronic state of a solid. We shall in § 5 consider such a very simple example.

The other topics treated in this paper are as follows. We see in the example of § 1 that a difference in whether a particle is elementary or composite will give a difference in Levinson's relation.[7] In § 3, by virtue of the general connection between the phase shift and the interaction Hamiltonian (II, (1·4) or (1·6)) one obtains a somewhat general information on the asymptotic behavior of $\eta_j(\omega)$, a formula for the asymptotic phase shift in terms of potential in particle quantum mechanics and a relation between the asymptotic phase shift and the self energy in quantum field theory which will be illustrated in the Lee model. In § 4 it is suggested that some information on a new interaction which is responsible for a new production process will be obtained from a threshold behavior of $\eta_j(\omega)$ by an effective range-like formula which is obtained from the nature of $\Delta^j(E)$. In Appendix III a formula which expresses $\eta_j(\omega)$ in terms of S-matrix is given.

§ 2. Levinson's theorem in an example in which multi-channels compete

The relation between the phase shift and the number of bound states which had originally been given by Levinson[3] in the case of particle quantum mechanics has so far been not clear in the case of quantum field theory where multiple channels compete.[6),7] According to the scheme of I, § 5 (see I, (5·37))

Levinson's theorem in such a case holds in terms of our $\eta_j(\omega)$, yielding

$$\eta_j(\infty)-\eta_j(\omega_0)+\sum_l \Delta\eta_j(\omega_l)=-\pi(\mathrm{m}^j-\mathrm{m}_0^j), \tag{2·1}$$

where $\mathrm{m}^j(\mathrm{m}_0^j)$ is the number of discrete levels (at $g=0$) in the j-th eigenspace, ω_l are the energies of each production thresholds, and $\Delta\eta_j(\omega_l)$ are possible discontinuities in $\eta_j(\omega)$ at ω_l. One sees, however, that the discontinuity does not usually occur except in such cases in which a resonance energy coincides incidentally with the threshold, since, as will be seen in the example of § 2, the discontinuity due to a new production process in $\eta_j(\omega)$ has a factor k, the magnitude of the momentum of a newly produced particle which vanishes at the threshold. Hence we may usually put

$$\Delta\eta_j(\omega_l)=0. \tag{2·2}$$

For the purpose of examining the result (2·1) of the general theory, let us take up the modified Lee model which was introduced by Konishi and Ogimoto.[6] One will directly obtain (2·1) with $\mathrm{m}^j-\mathrm{m}_0^j=0$ (j: S-state). Thereby one will see that the completeness of state vectors is essential for its validity.[4]

Following Konishi and Ogimoto, consider the modified Lee model in which two kinds of θ-particles $\theta_\alpha(\alpha=1, 2)$ with masses $\mu_\alpha(\mu_2>\mu_1)$ and two kinds of V-particles $V_\lambda(\lambda=1, 2)$ with observed mass m which is identical with that of N-particle are interacting through the virtual processes

$$(N, \theta_\alpha)\rightleftarrows V_\lambda, \quad \alpha=1, 2\,;\ \lambda=1, 2. \tag{2·3}$$

We assume that no ghost appears. Hamiltonians are given by

$$H=H_0+H_1=H_0+\delta H_0+H_1-\delta H_0, \tag{2·4}$$

$$H_0+\delta H_0=m\sum_p \psi'_N{}^\dagger(p)\,\psi'_N(p)+m\sum_\lambda\sum_q \psi'^\dagger_{V(\lambda)}(q)\,\psi'_{V(\lambda)}(q)+\sum_\alpha\sum_k \omega_\alpha a_\alpha^\dagger(k)\,a_\alpha(k), \tag{2·5}$$

$$H_1=-\sum_\alpha\sum_\lambda\sum_{q=p+k} g_{\alpha\lambda}F_\alpha(\omega_\alpha)\{\psi'^\dagger_{V(\lambda)}(q)\,\psi'_N(p)\,a_\alpha(k)+\text{herm. conj.}\}, \tag{2·6}$$

$$\delta H_0=\sum_{\lambda,\mu}\sum_q \delta m_{\lambda\mu}\psi'^\dagger_{V(\lambda)}(q)\,\psi'_{V(\mu)}(q), \tag{2·7}$$

$$g_{\alpha\lambda}=g\,\epsilon_{\alpha\lambda}, \tag{2·8}$$

where the notations are in accordance with I, § 3, $g_{\alpha\lambda}$ are the unrenormalized coupling constants, ψ'_N and $\psi'_{V(\lambda)}$ are the unrenormalized operators and

$$\omega_\alpha=(\mu_\alpha^2+k^2)^{1/2}. \tag{2·9}$$

Only one channel $(N, \theta_1)\rightarrow(N, \theta_1)$ contributes to the (N,θ)-scattering in the energy region $\mu_1\leqq\omega<\mu_2$, and four channels $(N, \theta_\alpha)\rightarrow(N, \theta_\beta)$ $(\alpha, \beta=1, 2)$ in the region $\omega\geqq\mu_2$. Real V_λ-state is obtained as

$$\Psi_{V(\lambda)}=\sum_\mu \Phi_{V(\mu)}N_{\mu\lambda}+\sum_\alpha\sum_k \Phi_{k\alpha}(F_\alpha(\omega_\alpha)/\omega_\alpha\cdot\sum_\mu g_{\alpha\mu}N_{\mu\lambda}). \tag{2·10}$$

Let the (N, θ_α) in-state be

$$\Psi_{k\alpha}=\sum_{\mu}\Phi_{V(\mu)}\varphi_{\mu\alpha}(k)+\sum_{\beta}\sum_{k'}\Phi_{k'\beta}\varphi_{\beta\alpha}(k', k). \tag{2·11}$$

The completeness of $\{\Psi_{V(\lambda)}, \Psi_{k\alpha}\}$ in this eigenspace reads (as for the completeness in the Lee model, see Appendix II)

$$\sum_{\nu}N_{\lambda\nu}N_{\mu\nu}+\sum_{\alpha}\sum_{k}\varphi^{*}_{\lambda\alpha}(k)\varphi_{\mu\alpha}(k)=\delta_{\lambda\mu}. \tag{2·12}$$

From the relation between the phase shift and the interaction Hamiltonian (I, § 5, or II, § 1) it follows that (see Appendix I)

$$d/dg\cdot\eta(\omega)=-(4\omega/g)\sum_{\alpha}\sum_{\lambda}\sum_{k''}|\varphi_{\lambda\alpha}(k'')|^2\delta(\omega_\alpha''-\omega)\equiv-(4\omega/g)\Phi(\omega) \tag{2·13}$$

$$=-(16\pi/g)\sum_{\lambda}\sum_{\alpha}k_\alpha\omega_\alpha{}^2|\varphi_{\lambda\alpha}(k)|^2, \tag{2·14}$$

where $\eta(\omega)$ is defined by

$$e^{2i\eta(\omega)}=\det|S_{\alpha\beta}(\omega)|, \tag{2·15}$$

and

$$\varphi_{\lambda\alpha}(\omega)=0 \text{ for } \omega<\mu_\alpha, \tag{2·16}$$

$S_{\alpha\beta}(\omega)$ being the S-matrix element for the process $(N, \theta_\beta)\to(N, \theta_\alpha)$ and

$$S_{\alpha\beta}(\omega)=\delta_{\alpha\beta} \text{ for } (\alpha, \beta)\neq(1, 1) \text{ and } \mu_1\leqq\omega<\mu_2. \tag{2·17}$$

Taking the spur of both sides of (2·12)

$$\sum_{\lambda}\sum_{\nu}N_{\lambda\nu}^2+\int_{\mu_1}^{\infty}d\omega\,\Phi(\omega)=2. \tag{2·18}$$

Therefore, on account of $\Phi(\omega)\geqq0$, one obtains

$$0\leqq\int_{\mu_1}^{\infty}d\omega\,\Phi(\omega)<2. \tag{2·19}$$

The integral must be convergent so that

$$\Phi(\omega)\sim O(\omega^{-1-\alpha})\ (\alpha>0) \quad \text{for } \omega\sim\infty. \tag{2·20}$$

Substituting the above expression in (2·13) one obtains

$$d/dg\ \eta(\omega)\sim O(\omega^{-\alpha}) \qquad \text{for } \omega\sim\infty. \tag{2·21}$$

Integrating with respect to g,

$$\eta(\omega)\sim O(\omega^{-\alpha}) \qquad \text{for } \omega\sim\infty. \tag{2·22}$$

On the one hand, on inspecting (2·14) one sees that $d/dg\eta(\omega)$ is continuous at $\omega=\mu_2$ and

$$d/dg\ \eta(\omega)=0 \qquad \text{at } \omega=\mu_1, \tag{2·23}$$

therefore $\eta(\omega)$ is continuous at $\omega=\mu_2$ and

$$\eta(\omega)=0 \qquad \text{at } \omega=\mu_1. \tag{2·24}$$

(2·22) and (2·24) together constitute a special case of Levinson's theorem (2·1) with $m-m_0=0$ (j: S-state, and $m=m_0=1$).

As will be seen from the above deduction, (2·22), therefore Levinson's theorem, is the consequence of the completeness of state vectors.[4]

In the Lee model V-particle is interpreted as an elementary particle described by ψ_V. If V-particle were a bound system of interacting N- and θ-fields, Levinson's theorem would be satisfied with $m_0=0$ and $m=1$. Such a situation will be a distinction between an elementary and a compound particle.[7]

We have assumed in the above that $m_V=m_N=m$. The result of this section will hold in so far as

$$m_V<m_N+\mu_1. \tag{2·25}$$

In the case of

$$m_V>m_N+\mu_1,$$

there occurs a different circumstance that the vector space which describes the (N, θ) scattering is expanded by $\{\psi_N, a_\theta\}$ at $g=0$, but by $\{\psi_N, a_\theta, \psi_V\}$ for $g>0$, hence the Hilbert space is discontinuous at $g=0$. Therefore the additional fundamental assumption III′) in I, § 1, hence the result of § 5, are no more valid as it stands. In this case the formula (2·1) must be modified so as to take the discontinuity of the Hilbert space into consideration.

§ 3. Asymptotic behavior of the phase shift

We shall in this section obtain some information on the asymptotic behavior of the phase shift by virtue of the general connection between the phase shift and the interaction Hamiltonian (see I and II).

Consider first the scattering of a particle by a potential

$$V(r)=gv(r). \tag{3·1}$$

From the basic relation II, (1·9), it holds for continuous energy region

$$d/dg\ \eta_j(\omega)=-\pi k^{-1}(\omega)\,(\psi_{j\omega}v\psi_{j\omega}), \tag{3·2}$$

where

$$k(\omega)=(d\omega/dk)/k^2, \tag{3·3}$$

and $\psi_{j\omega}$ is the j-th partial wave, and for discrete levels

$$(\psi_{j\alpha}v\psi_{j\alpha})=d\omega_\alpha^{(\gamma)}/dg. \tag{3·4}$$

Therefore summing over all states in the j-th eigenspace, one obtains

$$\mathrm{Sp}[P_j v P_j] = \sum_\alpha d\omega_\alpha^{(j)}/dg - (1/\pi)\int_{\omega_0}^{\infty} d\omega\, d/dg\, \eta_j(\omega). \tag{3·5}$$

The left-hand side is independent of g, hence by comparing the values at $g=g$ and $g=0$ it follows

$$1/\pi\int_{\omega_0}^{\infty} d\omega\{d/dg\,\eta_j(\omega) - [d/dg\,\eta_j(\omega)]_{g=0}\} = \sum_\alpha d\omega_\alpha^{(j)}/dg. \tag{3·6}$$

Assume that $\omega_\alpha^{(j)}$ is differentiable with respect to g, then the integral of the left-hand side must converge. Hence

$$d/dg\,\eta_j(\omega) - [d/dg\,\eta_j(\omega)]_{g=0} \sim O(\omega^{-1-\alpha}) \quad (\alpha>0) \quad \text{for } \omega\sim\infty. \tag{3·7}$$

Meanwhile by (3·2)

$$[d/dg\,\eta_j(\omega)]_{g=0} = -\pi k^{-1}(\omega)\int_0^{\infty} dr\, r^2 v(r)\,(1/kr)\,(J_{j+1/2}(kr))^2, \tag{3·8}$$

where $J_a(x)$ is the a-th order Bessel function, and

$$r^2(1/kr)\,(J_{1/2}(kr))^2 = (2/\pi)\,(1/k^2)\,\sin^2 kr \sim 1/\pi k^2,$$
$$r^2(1/kr)\,(J_{j+1/2}(kr))^2 \sim 1/\pi k^2 \quad \text{for } \omega\sim\infty.$$

Hence

$$[d/dg\,\eta_j(\omega)]_{g=0} \sim -(d\omega/dk)^{-1}\int_0^{\infty} dr\, v(r) \quad \text{for } \omega\sim\infty, \tag{3·9}$$

therefore

$$d/dg\,\eta_j(\omega) \sim -(d\omega/dk)^{-1}\int_0^{\infty} dr\, v(r) \quad \text{for } \omega\sim\infty. \tag{3·10}$$

Integrating over g one finally obtains

$$\eta_j(\omega) \sim -(d\omega/dk)^{-1}\int_0^{\infty} dr\, V(r) \quad \text{for } \omega\sim\infty, \tag{3·11}$$

provided that $V(r)$ is integrable. It may be interesting to note that the right-hand side is independent of j. This result has something in common with a formula in a recent work of Malenka and Valk.[8]

Next consider the case of quantized field, supposing an interacting nucleon and meson fields if necessary. The Hamiltonian is

$$H = H_0 + H_1 = H_0 + \delta H_0 + H_1 - \delta H_0,$$
$$\delta H_0 \equiv \mathcal{H}_0^{(m)}\,\delta m + \mathcal{H}_0^{(\mu)}\,\delta\mu^2,$$
$$\mathcal{H} = dH/dg = \mathcal{H}_1 - \mathcal{H}_0^{(m)}\,d/dg\,\delta m - \mathcal{H}_0^{(\mu)}\,d/dg\,\delta\mu^2, \quad \mathcal{H}_1 = dH_1/dg. \tag{3·12}$$

If we assume that in the j-th eigenspace under consideration no composite particle participates in scattering and that the asymptotic condition holds, then similarly as in the preceding case

$$d/dg\ \eta_j(\omega) = -\pi k^{-1}(\omega) \sum_I (\mathfrak{F}_{I\omega} P_j \mathcal{H} P_j \mathfrak{F}_{I\omega}), \tag{3·13}$$

and

$$Sp[P_j \mathcal{H} P_j] = \mathrm{Sp}[P_j\{\mathcal{H}_1 - \mathcal{H}_0^{(m)}\ d/dg\ \delta m - \mathcal{H}_0^{(\mu)}\ d/dg\ \delta\mu^2\} P_j]$$
$$= \sum_\alpha d\omega_\alpha^{(j)}/dg - (1/\pi) \int_{\omega_0}^{\infty} d\omega\ d/dg\ \eta_j(\omega). \tag{3·14}$$

Note that

$$\mathrm{Sp}[P_j \mathcal{H}_1 P_j]$$

is independent of g, and usually the perturbation expansions of δm, $\delta\mu^2$ and $\eta_j(\omega)$ begin by second order terms. Therefore by comparing (3·14) at $g=g$ and $g=0$ it follows:

$$1/\pi \int_{\omega_0}^{\infty} d\omega\ d/dg\ \eta_j(\omega) = \sum_\alpha d\omega_\alpha^{(j)}/dg + C_j^{(m)}\ d/dg\ \delta m + C_j^{(\mu)}\ d/dg\ \delta\mu^2, \tag{3·15}$$

where

$$C_j^{(m)} = \mathrm{Sp}[P_j \mathcal{H}_0^{(m)} P_j],\quad C_j^{(\mu)} = \mathrm{Sp}[P_j \mathcal{H}_0^{(\mu)} P_j]. \tag{3·16}$$

Therefore one may expect that *the asymptotic behavior of* $d\ dg\eta_j(\omega)$ *is determined by the order of divergence of* δm *and* $\delta\mu^2$.

The relation (3·15) is directly examined in the Lee model. In Appendix II one sees

$$-\sum_k \omega|\varphi(k)|^2 = \delta m = (g/2)\ d/dg\ \delta m. \tag{3·17}$$

Substituting the above expression in (2·13) one obtains

$$1/\pi \int_{\mu 1}^{\infty} d\omega\ d/dg\ \eta(\omega) = -(4/\pi g) \int_{\mu_1}^{\infty} d\omega\, \omega\Phi(\omega) = (2/\pi)\ d/dg\ \delta m, \tag{3·18}$$

which is just (3·15) with $C^{(m)} = 2/\pi$.

§ 4. Analyticity of the determinant and an effective range formula at production threshold*

In this section we prove the form (1·8) for the $\Delta^j(E)$ and obtain a dis-

* After this paper had been written the author found similar results in the work of M. Baker[9] which is, however, restricted to the case where there are no multi-channels. It seems that the present method using the differentiation with respect to the coupling parameter is more compact and straightforward.

persion-like formula which follows from the analyticity. Further it is suggested that an effective range-like formula is extended for the phase shifts just above the production threshold.

As for the form (1·8) we shall here consider only the case of the scattering by a central force. The general case is proved in Appendix III. $\Delta^j(E)$ is transformed as follows:

$$\Delta^j(E)=\det\left|P_j\left\{1+\frac{1}{H_0-E}H_1\right\}P_j\right|$$

$$=\det_j|(H_0-E)^{-1}(H-E)|=\det_j^{-1}|(H_0-E)|\cdot\det_j|(H-E)|, \qquad (4\cdot1)$$

where det_j is the determinant defined on the j-th eigenspace. Suppose that the system is enclosed in a large spherical rigid wall with radious R, then the levels become discrete, so that

$$\begin{aligned}\det_j|(H_0-E)|&=\prod_i(\omega_i^0-E),\\ \det_j|(H-E)|&=\prod_\alpha(\omega_\alpha-E)\prod_i(\omega_i-E),\end{aligned} \qquad (4\cdot2)$$

where ω_i and ω_i^0 are the energies of the i-th normal mode with and without interaction, and ω_α is the energy of the α-th bound level, all belonging to the j-th eigenspace. Hence

$$\log\Delta^j(E)=\sum_\alpha\log(\omega_\alpha-E)+\sum_i\{\log(\omega_i-E)-\log(\omega_i^0-E)\}.$$

As R tends to infinity $\omega_i-\omega_i^0$ tend to zero, hence

$$\log(\omega_i-E)-\log(\omega_i^0-E)=\log\frac{\omega_i-E}{\omega_i^0-E}\approx\frac{\omega_i-\omega_i^0}{\omega_i-E}.$$

Meanwhile from the relation between the energy shift and the interaction Hamiltonian (see II, § 1)

$$\omega_i-\omega_i^0=-(1/R)(d\omega/dk)\eta_j(\omega_i),$$

and further the number of levels involved within $\omega\sim\omega+d\omega$ is

$$N_0(\omega)\,d\omega=(1/\pi)R(d\omega/dk)^{-1}.$$

Therefore

$$\sum_i\{\log(\omega_i-E)-\log(\omega_i^0-E)\}=-\frac{1}{\pi}\int_{\omega_0}^{\infty}d\omega\,\frac{\eta_j(\omega)}{\omega-E}, \qquad (4\cdot3)$$

so that one would be led to the result that $\Delta^j(E)$ is to be given by

$$\prod_\alpha(\omega_\alpha-E)\exp\left\{-\frac{1}{\pi}\int_{\omega_0}^{\infty}d\omega\frac{\eta_j(\omega)}{\omega-E}\right\}. \qquad (4\cdot4)$$

In the above, however, we have implicitly assumed that there always exists a

one to one correspondence between free and interacting levels. Such a correspondence, depending on the parameter g, breaks down at the threshold energy when a new bound state appears (see I, § 4 and § 5). The discrepancy will be settled by a requirement that $\Delta^j(E)$ should be continuous with respect to g. This requirement is satisfied by supplying a certain factor to (4·3), yielding

$$\Delta^j(E) = \prod_\alpha \left(\frac{E-\omega_\alpha}{E-\omega_0}\right) \exp\left\{-\frac{1}{\pi}\int_{\omega_0}^{\infty} d\omega \frac{\eta_j(\omega)}{\omega-E}\right\}, \tag{4·5}$$

which has the following zeros and singularities:

a) $\Delta^j(E)$ has a cut on the real axis ($\omega \geqq \omega_0$), discontinuity across the cut being characterized by

$$\Delta^{j*}(\omega)/\Delta^j(\omega) = e^{2i\eta_j(\omega)}. \tag{4·6}$$

b) $\Delta^j(E)$ has zero points at $E=\omega_\alpha$.

c) $\Delta^j(E)$ has a pole of m-th order at $E=\omega_0$, where m is the number of bound levels.

d) If

$$\int_0^\infty d\omega\, \eta_j(\omega)/\omega \quad \text{and} \quad \int_0^\infty d\omega\, \eta_j(\omega) \quad \text{are convergent} \tag{d_1}$$

or

$$\eta_j(\omega) \sim O(\omega^{-\alpha}) \ (\alpha > 0) \qquad \text{for} \quad \omega \sim \infty, \tag{d_2}$$

d_2) being weaker than d_1), then the exponent of the right-hand side of (4·5) tends to zero for $E \to \infty$, so that

$$\Delta^j(E) \to 1 \qquad \text{for} \quad E \to \infty. \tag{4·7}$$

For a non-relativistic scattering by a non-singular potential as well as for the Lee model, on account of (3·11) and (2·22) (with $d\omega/dk = k/m$) respectively, the condition d_2) is satisfied. For these cases satisfying d_1) or d_2), we obtain by Cauchy's theorem the relation

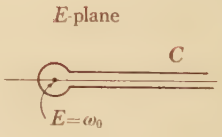

Fig. 1.

$$\Delta^j(E) = 1 + \frac{1}{2\pi i}\int_C dE' \frac{\Delta^j(E')}{E'-E}. \tag{4·8}$$

Also from (4·6)

$$\mathrm{Re}\,\Delta^j(\omega) = -\mathrm{Im}\,\Delta^j(\omega)\cdot\cot\eta_j(\omega). \tag{4·9}$$

Combining (4·8) and (4·9) one obtains the following dispersion-like formula:

$$-\mathrm{Im}\,\Delta^j(\omega)\,\cot\eta_j(\omega) = 1 + \frac{P}{\pi}\int_{\omega_0}^{\infty} d\omega' \frac{\mathrm{Im}\,\Delta^j(\omega')}{\omega'-\omega}. \tag{4·10}$$

Im $\Delta^{J}(\omega)$ may be compared with the source function in Chew and Low's effective range formula[10] and may be called "the effective interaction". There arises no trouble such as what arises when one wants to apply the unitarity to the dispersion formula involving the complex phase shift above the production threshold.[11] We have our formula for $\eta_J(\omega)$ one by one for each eigenspace without any such approximation as neglecting the inelastic processes.

By virture of our scheme it is suggested that, when a new production process is associated with a new interaction as is the case with the modified Lee model of § 2, information on the new interaction will be obtained from $\eta_J(\omega)$ just above the production threshold by the effective range-like formula. For compactness suppose that we are considering the modified Lee model of § 2, then by (4·5), since there is no bound state,

$$\Delta(E)=\det\left|1+\frac{1}{H_0-E}H_1\right|=\exp\left\{-\frac{1}{\pi}\int_{\mu_1}^{\infty}d\omega\frac{\eta(\omega)}{\omega-E}\right\}. \tag{4·11}$$

Let us decompose $\eta(\omega)$ as follows:

$$\eta(\omega)=\eta^{(1)}(\omega)+\eta^{(2)}(\omega), \tag{4·12}$$

where $\eta^{(1)}(\omega)$ *is differentiable at* $\omega=\mu_2$ *and* $\eta^{(2)}(\omega)=0$ *for* $\omega<\mu^2$, and factorize $\Delta(E)$ as follows,

$$\Delta(E)=\Delta^{(1)}(E)\cdot\Delta^{(2)}(E),\quad \Delta^{(\alpha)}(E)=\exp\left\{-\frac{1}{\pi}\int_{\mu_\alpha}^{\infty}d\omega\frac{\eta^{(\alpha)}(\omega)}{\omega-E}\right\}. \tag{4·13}$$

The effective range formula for $\omega\gtrsim\mu_1$ is written as

$$\left.\begin{aligned}&k\cot\eta^{(1)}(\omega)\simeq-1/a_1+r_1k^2/2,\\ &\text{where}\\ &\frac{1}{a_1}=\left\{1+\frac{P}{\pi}\int_{\mu_1}^{\infty}d\omega\frac{k\mathcal{J}_1(\omega)}{\omega-\mu_1}\right\}\Big/\mathcal{J}_1(\mu_1),\\ &r_1=-\frac{P}{\pi}\int_{\mu_1}^{\infty}d\omega\frac{k\mathcal{J}_1(\omega)}{(\omega-\mu_1)^2}\Big/\mu_1\mathcal{J}_1(\mu_1),\\ &\text{and}\\ &\operatorname{Im}\Delta^{(1)}(\omega)=k\mathcal{J}_1(\omega).\end{aligned}\right\} \tag{4·14}$$

We may write *also for* $\omega\gtrsim\mu_2$

$$\left.\begin{aligned}&k\cot\eta^{(2)}(\omega)\simeq-1/a_2+r_2k^2/2,\\ &|(4\cdot14)|_{1\to2},\end{aligned}\right\} \tag{4·15}$$

where a_2 and r_2 may be called the scattering length and the effective range for

the interaction responsible for the new production process. $\eta^{(2)}(\omega)$ is equal to $\eta(\omega)$ minus the value of $\eta(\omega)$ extrapolated from under the threshold: $\eta^{(2)}(\omega)=\eta(\omega)-\eta^{(1)}(\omega)$. $\mathrm{Im}\,\Delta^{(\alpha)}(\omega)$ are connected with the original total effective interaction $\mathrm{Im}\,\Delta(\omega)$ by

$$\mathrm{Im}\,\Delta(\omega)=-\mathrm{Im}\,\Delta^{(1)}(\omega)\cdot\mathrm{Im}\,\Delta^{(2)}(\omega)\{\cot\eta^{(1)}(\omega)+\cot\eta^{(2)}(\omega)\}. \tag{4·16}$$

§ 5. The level density of statistical-dynamical system

For a statistical-dynamical system with very large degrees of freedom, $\Delta^{j}(E)$ provides the level density of the system in terms of (1·9). As an illustration, consider an application of this relation to a very simple model which is composed of N fixed sources interacting with a scalar Fermion field $\psi(\boldsymbol{r})$ of mass m. $\psi(\boldsymbol{r})$ satisfies

$$\{\psi(\boldsymbol{r}),\psi^{*}(\boldsymbol{r}')\}=\delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{5·1}$$

and the Hamiltonians are given by

$$\begin{aligned} H&=H_0+H_1,\\ H_0&=(1/2m)\int d\boldsymbol{r}\,(\nabla\psi^{*}(\boldsymbol{r}))\,(\nabla\psi(\boldsymbol{r})),\\ H_1&=g\sum_{i=1}^{N}\Big(\int d\boldsymbol{r}\,v_i(\boldsymbol{r})\,\psi^{*}(\boldsymbol{r})\Big)\Big(\int d\boldsymbol{r}\,v_i(\boldsymbol{r})\,\psi(\boldsymbol{r})\Big), \end{aligned} \tag{5·2}$$

where

$$v_i(\boldsymbol{r})=v(\boldsymbol{r}-\boldsymbol{r}_i), \tag{5·3}$$

$\boldsymbol{r}_i$ being the position of the i-th source.

The determinant is expressed as (see II, § 2)

$$\Delta(E)=\det\left|1+\frac{1}{H_0-E}H_1\right|=\det|\delta_{ij}+gu(E,\boldsymbol{r}_{ij})|, \tag{5·4}$$

where

$$u(E,\boldsymbol{r}_{ij})=\sum_{k}\frac{v_i(\boldsymbol{k})\,v_j^{*}(\boldsymbol{k})}{k^2-E}=\sum_{k}\frac{|v(k)|^2}{k^2-E}e^{i\boldsymbol{k}\boldsymbol{r}_{ij}}, \tag{5·5}$$

$$\boldsymbol{r}_{ij}=\boldsymbol{r}_i-\boldsymbol{r}_j, \tag{5·6}$$

and $v(k)$ is the Fourier transform of $v(\boldsymbol{r})$.

First consider the case $N=1$ and assume that $\Delta(E)$ has one discrete level for this case, then

$$\Delta(E)_{N=1}=1+gu_0(E)=\frac{E-\omega_1}{E-\omega_0}\exp\left\{-\frac{1}{\pi}\int_{\omega_0}^{\infty}d\omega\,\frac{\eta(\omega)_{N=1}}{\omega-E}\right\}, \tag{5·7}$$

where ω_1 is the level energy, $\eta(\omega)_{N=1}$ is the phase shift of the scattering of the particle by a source, and

$$u_0(E) \equiv u_{ii}(E) = \sum_k \frac{|v(k)|^2}{k^2 - E}. \tag{5·8}$$

In the case of N sources the level will split into N different levels according to the interaction between the sources. Thus in the case of $N=2$, we have

$$\Delta(E)_{N=2} = (1+gu_0(E))^2 - u(E, \boldsymbol{r})\, u(E, -\boldsymbol{r}), \tag{5·9}$$

where $\boldsymbol{r}$ is the distance between two sources. From (5·7)

$$\Delta(E)_{N=2} \simeq c(E-\omega_1)^2 - u(E, \boldsymbol{r})\, u(E, -\boldsymbol{r}) \quad \text{for } E \simeq \omega_1, \tag{5·10}$$

where the coefficient c and the second term in the right-hand side is positive for real E. Therefore, if the second term has no zero in this neighbour, $\Delta(E)_{N=2}$ has two different zeros in this energy region.

Now consider the case of large N. One can easily see that the simple perturbation expansion of the determinant $\Delta(E)$ does not correctly represent the location of zero points. Accordingly, we prefer a different approximation. First we assume that the level structure becomes almost independent of N for sufficiently large N, which appears only as a multiplying factor in the expression for the level density, and that also the phase shifts $\eta(\omega)$ of particles scattered by a block of N sources becomes independent of N for sufficiently large N. Then we can put for sufficiently large N

$$\rho_N(\omega) = N d\sigma/d\omega, \quad \sigma(-\infty) = 0, \quad \int_{-\infty}^{\omega_0} d\omega\, \rho_N(\omega) = N = N\sigma(\omega_0), \tag{5·11}$$

and

$$\Delta(E) \equiv \Delta(E)_N = \exp\left\{-N\int_{-\infty}^{\omega_0} d\omega\, \frac{\sigma(\omega)}{\omega - E}\right\} \exp\left\{-\frac{1}{\pi}\int_{\omega_0}^{\infty} d\omega\, \frac{\eta(\omega)}{\omega - E}\right\}. \tag{5·12}$$

Expanding the determinant

$$\Delta(E)_{N+1} = \det|\delta_{ij} + g u(E, \boldsymbol{r}_{ij})| \tag{5·13}$$

with respect to the first row, we get

$$\Delta(E)_{N+1} = (1+gu_0(E))\, \Delta(E)_N + \sum_{l \neq 1} g u(E, \boldsymbol{r}_{1l})\, \Delta_{1l}(E)_N, \tag{5·14}$$

where $\Delta_{1l}(E)_N$ is the cofactor of the $(l, 1)$-element and is the N-th order determinant. Expanding $\Delta_{1l}(E)_N$ too, we get

$$\Delta_{1l}(E)_N = -g u(E, \boldsymbol{r}_{1l})\, \Delta(E)_{N-1} + O(g^2). \tag{5·15}$$

Retaining up to $O(g^2)$

$$\Delta(E)_{N+1} \simeq (1+gu_0(E))\, \Delta(E)_N - g^2 \sum_{l \neq 1} u(E, \boldsymbol{r}_{1l})\, u(E, \boldsymbol{r}_{1l})\, \Delta(E)_{N-1},$$

hence

$$\Delta(E)_{N+1}/\Delta(E)_N \simeq 1+gu_0(E)-g^2\sum_{l\neq 1} u(E, \boldsymbol{r}_{1l})\, u(E, \boldsymbol{r}_{1l})\cdot(\Delta(E)_{N-1}/\Delta(E)_N)$$
$$\simeq 1+gu_0(E)-g^2\sum_{l\neq 1} u(E, \boldsymbol{r}_{1l})\, u(E, \boldsymbol{r}_{1l}).$$

On account of (5·12) one obtains finally in the second order approximation

$$\rho_N(\omega)/N=\frac{1}{2\pi i}\frac{\partial}{\partial\omega}\log\frac{1+gu_0^*(\omega)-g^2\sum_{l\neq 1} u^*(\omega, \boldsymbol{r}_{1l})\, u^*(\omega, \boldsymbol{r}_{1l})}{1+gu_0(\omega)-g^2\sum_{l\neq 1} u(\omega, \boldsymbol{r}_{1l})\, u(\omega, \boldsymbol{r}_{1l})}, \quad (5\cdot 16)$$

from which the "*Zustand-Summe*" $Z(\beta)$, $\beta=1/kT$ of the system is obtained as

$$Z(\beta)=\mathrm{Sp}[e^{-H\beta}]=\int_{-\infty}^{\omega_0} d\omega\,\rho_N(\omega)\, f(\omega, \beta), \quad (5\cdot 17)$$

where Sp is the spur defined on the Hilbert space (i. e. also on the configuration space of quantum number 0 and 1), and $f(\omega, \beta)$ is the Fermi-Dirac distribution function:

$$f(\omega, \beta)=1/(e^{\alpha+\omega\beta}+1). \quad (5\cdot 18)$$

Although the present model is a very crude one, it suggests a general approach which makes it possible to deal directly with the level density of the system.

Appendix I

Modified Lee model

The Hamiltonians have been given in § 2. From the eigenvalue equation

$$(H-m)\Psi_{V(\lambda)}=0, \quad (\mathrm{AI}\cdot 1)$$

one gets (2·9) and

$$\delta m_{\mu\lambda}=-\sum_{\alpha}\sum_{k} g_{\alpha\mu}g_{\alpha\lambda}F_\alpha^{\,2}(\omega_\alpha)/\omega_\alpha. \quad (\mathrm{AI}\cdot 2)$$

For the in-(N, θ_α) scattering solution which satisfies

$$(H-m-\omega)\Psi_{k\alpha}=0, \quad (\mathrm{AI}\cdot 3)$$

the coefficients (see (2·11)) are given by

$$\varphi_{\beta\alpha}(k', k)=\delta_{k'k}\delta_{\beta\alpha}+\sum_{\lambda}\frac{g_{\beta\lambda}F_\beta(\omega')\,\varphi_{\lambda\alpha}(k)}{\omega_\beta'-\omega_\alpha-i\varepsilon}, \quad (\mathrm{AI}\cdot 4)$$

and

$$\varphi_{\lambda\alpha}(k)=-\sum_{\mu} A^{-1}_{\lambda\mu}(\omega_\alpha)\, g_{\alpha\mu}F_\alpha(\omega_\alpha)/\omega_\alpha, \quad (\mathrm{AI}\cdot 5)$$

where $A^{-1}_{\lambda\mu}(\omega_\alpha)$ is the inverse matrix of

$$A_{\lambda\mu}(\omega)=\delta_{\lambda\mu}+\sum_{\beta}\sum_{k''}\frac{g_{\beta\lambda}g_{\beta\mu}F_{\beta}^{2}(\omega_{\beta}'')/\omega_{\beta}''}{\omega_{\beta}''-\omega_{\alpha}-i\varepsilon}. \tag{AI·6}$$

The scattering matrix is obtained as

$$S_{\beta\alpha}(k',k)=\delta_{k'k}\delta_{\beta\alpha}-2\pi i\delta(\omega_{\beta}'-\omega_{\alpha})\,T_{\beta\alpha}(\omega), \tag{AI·7}$$

$$T_{\beta\alpha}(\omega)=(1/\omega)\sum_{\lambda,\mu}g_{\beta\lambda}F_{\beta}(\omega)\,A_{\lambda\mu}^{-1}(\omega)\,g_{\alpha\mu}F_{\alpha}(\omega). \tag{AI·8}$$

Further from (2·11) and (2·12) one obtains

$$\begin{aligned}\overline{\mathcal{H}}_{k\alpha,k'\beta}&\equiv(\Psi_{k\alpha}\mathcal{H}\Psi_{k'\beta})\quad(\mathcal{H}=dH/dg)\\&=(1/g)(\omega_{\alpha}+\omega_{\beta}')\sum_{\lambda}\varphi_{\lambda\alpha}^{*}(k)\,\varphi_{\lambda\beta}(k').\end{aligned} \tag{AI·9}$$

$\overline{\mathcal{H}}$ is related to the phase shift by

$$\sum_{\alpha}d/dg\;\eta_{\alpha}(\omega)=-2\sum_{\alpha}\sum_{k''}\overline{\mathcal{H}}_{\alpha k'',\alpha k''}\delta(\omega_{\alpha}''-\omega), \tag{AI·10}$$

so that substituting it in (AI·9) one obtains

$$\sum_{\alpha}d/dg\;\eta_{\alpha}(\omega)=-(4\omega/g)\sum_{\lambda}\sum_{\alpha}\sum_{k''}|\varphi_{\lambda\alpha}(k'')|^{2}\delta(\omega_{\alpha}''-\omega). \tag{AI·11}$$

Further the determinant for this model is expressed as

$$\begin{aligned}\Delta(E)&=\det\left|P\left\{1+\frac{1}{H_{0}-E}H_{1}\right\}P\right|\\&=\det|A_{\mu\nu}(E)|=\det\left|\delta_{\mu\nu}+\sum_{\beta}\sum_{k''}\frac{g_{\beta\mu}g_{\beta\nu}F_{\beta}^{2}(\omega_{\beta}'')/\omega_{\beta}''}{\omega_{\beta}''-E}\right|,\end{aligned} \tag{AI·12}$$

where P is the projection operator of the eigenspace to which V and (N,θ_{α}) belong.

Appendix II

The completeness in Lee model and the self-energy

The notations are to be referred to I, § 3. The condition that the space spanned by $\{\Psi_{V},\Psi_{N\theta}\}$ coincides with that by $\{\Phi_{V},\Phi_{N\theta}\}$ is written as

$$\sum_{i}\varphi_{il}\varphi_{ij}^{\dagger}=\delta_{ij}\begin{cases}N^{2}+\sum_{k}|\varphi(k)|^{2}=1, & \text{(AII·1)}\\ gN^{2}F(\omega)/\omega+\sum_{k''}\varphi(k'')\,\varphi^{+}(\boldsymbol{k}'',\boldsymbol{k})=0, & \text{(AII·2)}\\ g^{2}N^{2}F(\omega)/\omega\cdot F(\omega')/\omega'\\ \quad+\sum_{k''}\varphi(\boldsymbol{k},\boldsymbol{k}'')\,\varphi^{\dagger}(\boldsymbol{k}'',\boldsymbol{k}')=\delta(\boldsymbol{k}-\boldsymbol{k}). & \text{(AII·3)}\end{cases}$$

It is seen that (AII·1)—(A II·3) are replaced by (AII·1) and

$$gF(\omega)/\omega+\varphi(k)-gF(\omega)/\omega\cdot\sum_{k''}|\varphi(k'')|^{2}\omega''/(\omega''-\omega-i\varepsilon)=0. \tag{AII·4}$$

$\varphi(k)$ has been given by

$$\varphi(k)=-g(F(\omega)/\omega)\Big/\Big\{1+g^2\sum_{k''}\frac{F^2(\omega'')}{\omega''}\frac{1}{\omega''-\omega-i\varepsilon}\Big\}. \qquad \text{(AII·5)}$$

We can directly ascertain (AII·1) and (AII·4) as follows. Put

$$\Delta(E)=1+g^2\sum_{k''}\frac{F^2(\omega'')}{\omega''}\frac{1}{\omega''-E}, \qquad \text{(AII·6)}$$

and consider the contour integral in the complex E-plane

$$\frac{1}{2\pi i}\int_C\frac{dE'}{E'}\frac{1}{\Delta(E')}$$

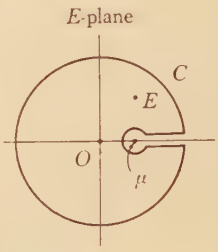

Fig. 2.

then

i) the contribution from the infinite circle equals 1, since $\Delta(E)\to 1$ for $E\to\infty$,

ii) the contribution from the pole at $E=0$ equals

$$1/(1+g^2\sum_{k''}F^2(\omega'')/\omega'')=N^2$$

and

iii) the contribution from the cut on the real axis equals

$$-\int_\mu^\infty d\omega''\sum_{k''}|\varphi(k'')|^2\delta(\omega''-\omega)=-\sum_{k''}|\varphi(k'')|^2,$$

on account of

$$1/\Delta(\omega)-1/\Delta^*(\omega)=-(1/|\Delta(\omega)|^2)\cdot(\Delta(\omega)-\Delta^*(\omega)),$$

and

$$\Delta(\omega)-\Delta^*(\omega)=\sum_{k''}g^2F^2(\omega'')/\omega''\cdot 2\pi i\delta(\omega''-\omega).$$

Hence one gets

$$\sum_{k''}|\varphi(k'')|^2+N^2=1.$$

Similarly by considering the contour integral

$$\frac{1}{2\pi i}\int_C\frac{dE'}{E'-E}\frac{1}{\Delta(E')},$$

one obtains

$$-\sum_{k''}|\varphi(k'')|^2\omega''/(\omega''-\omega-i\varepsilon)+1=1/\Delta(\omega), \qquad \text{(AII·7)}$$

which is identical with (AII·4).

Further the expansion of the both sides of (AII·7) at $\omega\sim\infty$ yields

$$\text{the left-hand side} \sim 1+(1/\omega)\sum_{k''}|\varphi(k'')|^2\omega'',$$

and (AII·8)

$$\text{the right-hand side} \sim 1+(1/\omega)g^2\sum_{k''}F^2(\omega'')/\omega''.$$

Therefore by comparing the coefficients of $1/\omega$ one obtains

$$\sum_{k''}\omega''|\varphi(k'')|^2=g^2\sum_{k''}F^2(\omega'')/\omega''=-\delta m=-(g/2)\,d/dg\,\delta m. \quad \text{(AII·9)}$$

Appendix III

The form of $\Delta^j(E)$ in a general case

Proof of the form (1·8) for $\Delta^j(E)$ in a general case is given.

Δ be the determinant consisting of elements $\Delta_{\mu\nu}$ which contain a parameter g:

$$\Delta=\det|\Delta_{\mu\nu}|, \quad \text{(AIII·1)}$$

then the following identities hold:

$$\Delta\delta_{\mu\nu}=\sum_{\lambda}\Delta_{\mu\lambda}A_{\lambda\nu}, \quad \text{(AIII·2)}$$

$$d\Delta/dg=\sum_{\mu,\nu}A_{\mu\nu}\,d/dg\,\Delta_{\nu\mu}, \quad \text{(AIII·3)}$$

where $A_{\lambda\nu}$ is the cofactor of $\Delta_{\nu\lambda}$. If $\Delta^{-1}_{\nu\lambda}$ is the inverse of $\Delta_{\nu\lambda}$,

$$A_{\lambda\nu}=\Delta^{-1}_{\lambda\nu}/\Delta. \quad \text{(AIII·4)}$$

As a formal extension, for an arbitrary operator U which contains a parameter g, the determinant

$$\Delta=\det|U| \quad \text{(AIII·5)}$$

may be defined through the differential eqation

$$(d\Delta/dg)/\Delta=\mathrm{Sp}\,[(dU/dg)\,U^{-1}]. \quad \text{(AIII·6)}$$

On account of

$$\left\{1+\frac{1}{H_0-E}H_1\right\}^{-1}\cdot\frac{d}{dg}\left\{1+\frac{1}{H_0-E}H_1\right\}=\frac{1}{H-E}\mathcal{H}, \quad \text{(AIII·7)}$$

$$\Delta^j(E)=\det\left|P_j\left\{1+\frac{1}{H_0-E}H_1\right\}P_j\right|$$

satisfies or may be defined through

$$\frac{1}{\Delta^j(E)}\frac{d\Delta^j(E)}{dg}=\mathrm{Sp}\left[P_j\frac{1}{H-E}\mathcal{H}P_j\right]. \quad \text{(AIII·8)}$$

Now the spur can be calculated by introducing the set of energy eigenstates, yielding

$$\frac{1}{\Delta^J(E)}\frac{d\Delta^J(E)}{dg}=\sum_\alpha\frac{1}{\omega_\alpha-E}\frac{d\omega_\alpha}{dg}-\frac{1}{\pi}\int_{\omega_0}^{\infty}d\omega\frac{1}{\omega-E}\frac{d}{dg}\eta_J(\omega), \quad \text{(AIII·9)}$$

on account of the relation between the interaction Hamiltonian and the phase shift. By integrating on g under the conditions that

$$\Delta^J(E)\,]_{g=0}=1,$$

and that $\Delta^J(E)$ should be continuous with respect to g, one obtains

$$\log\Delta^J(E)=\sum_\alpha\log\frac{E-\omega_\alpha}{E-\omega_0}-\frac{1}{\pi}\int_{\omega_0}^{\infty}d\omega\frac{\eta_J(\omega)}{\omega-E}. \quad \text{(AIII·10)}$$

Appendix IV

$\eta_J(\omega)$ *in terms of the S-matrix*

$\eta_J(\omega)$ has been defined compactly through

$$e^{2i\eta_J(\omega)}=\det|P_JS_{III}(\omega)P_J|. \quad \text{(AIV·1)}$$

In cases where inelastic processes compete, the variable I takes continuously infinite different values, so that the determinant is only a formal expression. In such a case we can make use of the relation

$$d\eta_J/d\omega=(1/2i)\,\mathrm{Sp}[P_J(dS/d\omega)\,S^\dagger P_J]=-(1/2i)\,\mathrm{Sp}[P_JS(dS^\dagger/d\omega)P_J],$$

hence, if ω_1 is the energy of the production threshold,

$$\eta_J(\omega)-\eta_J(\omega_1)=(1/4i)\int_{\omega_1}^{\omega}d\omega'\,\mathrm{Sp}[P_J\{(dS/d\omega')\,S^\dagger(\omega')-S(\omega')\,(dS^\dagger/d\omega')\}P_J], \quad \text{(AIV·2)}$$

from which one can evaluate $\eta_J(\omega)$ in terms of the observed scattering matrix.

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On the Quaternary Intermediate Boson in the Weak Interaction

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A hypothesis on the quaternary intermediate boson in the weak interaction is formulated along a device of localization of transformations. Two transformations which belong to the generalized full symmetric group between proton, neutron and Λ-particle are utilized: The first causes changes between left-handed proton and neutron, while the second does between left-handed neutron and Λ-particle. Three constituents of the quaternary boson are introduced so as to harmonize with the localization of the first transformation, and the remaining one is deduced from the localization of the second transformation.

The strongly interacting system is arranged to be invariant under the generalized full symmetric transformation so as to be unaffected by both transformations, while the weakly interacting system which is charge independent under the covariance of the first transformation becomes charge dependent after the second transformation.

The first approximation of the resultant weak interaction is in harmony with $|\Delta I|=1/2$ law, i.e. just agrees with Lee-Yang's theory. The bare mass of the quarternary boson is small but finite in virtue of the coexistence of eleven mesons besides pions and kaons.

The quaternary boson uniquely determines the conserved vector and axial-vector currents of the strongly interacting system, the violations of which are discussed by changing meson masses.

§ 1. Introduction

A hypothesis on the intermediate boson in the weak interacttion was originated from an analogy with the electromagnetic field interacting universally with the charged particles.[1] One of the aims of this hypothesis, then, was to give a logical foundation of the empirical universality of the coupling constants in the weak phenomena.

However, any consistent theory of the intermediate boson has not been formulated, because of conflicts between requirements imposed on the intermediate boson. They are summarized as follows:

[A] The intermediate boson (here-after called W-field) should mediate all existing weak phenomena.

[B] The usual Fermi interactions can be realized by the secondary effect of W-field. Then their V-A coupling nature should result from the interaction of W-field with baryons and also leptons.

[C] Coupling constants between W-field and nucleon pair should be insensitive against the reaction of its strong interaction so as to be nearly equal to coupling constants between W-field and lepton pairs.

[D] W-field should make the decay interactions between baryons and mesons hold the so-called $|\Delta I|=1/2$ law at least approximately.

[E] W-field should not realize any decays to like lepton pairs with comparative order of one to unlike lepton pairs.

[F] The intervention of W-field should not raise the rate of muon electron radiative decay from the usual Fermi coupling theory.

[G] The mass of W-field should be large enough to be consistent with any observable evidence.

We may take the following properties of W-field from these requirements: i) W-field may have at least positive and negative charged partners from [A] (Ogawa's assumption[2]); ii) W-field may have vector and axial-vector characters from [B] (Gell-Mann's assumption[3]); iii) Since the requirement [C] can be understood from the conservation laws of vector and axial vector nucleon currents,

$$\partial_\mu \boldsymbol{V}_\mu^{N}=0, \quad \partial_\mu \boldsymbol{P}_\mu^{N}\simeq 0, \tag{1·1}$$

part of W-field interacting with these currents must be satisfied by

$$\partial_\mu \omega_\mu^{+}\simeq 0, \quad \partial_\mu \omega_\mu^{-}\simeq 0, \tag{1·2}$$

which means that this part may belong to the pure state of spin 1. (Feynman Gell-Mann[4] and Goldberger-Treimann[5]'s assumption); iv) An ingenious device which satisfies the requirement [D] was proposed by Lee and Yang.[6] They assumed two neutral partners of W-field besides two charged ones. The quaternary W-field, then, reveals dual nature in iso-spin transformation so as to give the $|\Delta I|=1/2$ law automatically. The existence of neutral partners of W-field may be essential about this law (Lee-Yang's assumption); v) The neutral W-field, however, can not play any important role in the interaction with leptons from the requirement [E] (Lee-Yang's assumption); vi) A restriction is also imposed on the interaction between charged W-field and leptons from the requirement [F], because a larger value of the muon radiative decay must be expected in an easy-going scheme.[7] One less dangerous device is to distinguish neutrino interacting with muon from the beta neutrino (Oneda-Pati's assumption[8]); vii) The mass of W-field may be determined from the observed empirical facts, for instance, from the Michel parameter of muon decay[9]

$$\rho-0.75\simeq\frac{1}{3}\left(\frac{m_\mu}{M}\right)^2, \tag{1·3}$$

if the coupling constant G is adjusted by the Fermi coupling constant

$$F_V\simeq\frac{G^2}{M^2}. \tag{1·4}$$

An ambitious proposal was made by Schwinger and Lopes,[10] that the coupling constant G is equated with a root of the fine structure constant α. Thus the

mass should be so large ($M^2 \sim 10^3 m_N^2$) that we can not detect any observable effect of W-field (Schwinger and Lopes' assumption).

The conflicts in these requirements reveal themselves in the dualities in the W-field. The first is that the W-field has vector and axial-vector nature in usual space, because there may be strong interaction which defines its parity. The second is the duality of iso-spin raised by Lee-Yang's theory. And the third duality, which may be the most important for the original intuition of the W-field hypothesis, is about its spin state, that is, W-field gives spin 1 state together with implicit spin 0 state when it interacts with other particles.* If W-field would be pure state of spin 1, any decay of spin 0 particle could not be realized through this field.[11] The purity of state is closely related with the conservation laws which must be taken from the universality of coupling constants; that is the contribution of spin 0 state is ineffective in virture of the conservation laws. However, the conservation laws do not hold exactly, besides the vector nucleon-current. For instance, a break-down of the axial-vector nucleon-current is interpreted from the ratio of beta decay coupling constant $F_A/F_V = -1.20$. This break-down implies W-field to be mixed state of spin 1 and 0, the latter of which gives pion decay process. On the other hand, the so-called Golberger-Treiman relation

$$\partial_\mu \boldsymbol{P}_\mu^N = i \,\text{const} \cdot \mu_\pi^2 \boldsymbol{\pi} \,, \tag{1·5}$$

which minimizes a break-down of the axial-vector nucleon current is in harmony with the pion decay rate.[12] This enforces us to construct the theory with the approximate conservation laws, deviation of which is minimized as small as possibles.

A consideration presented here starts with the strongly interacting system with the conservation laws of both vector and axial-vector currents. For the conservation laws of nucleon part, the system must be invariant under the transformation

$$N \to \exp[i(\boldsymbol{\alpha} + \gamma_5 \boldsymbol{\beta})\boldsymbol{\tau}]N; \qquad \boldsymbol{\alpha}, \boldsymbol{\beta} = \text{const. isovector.} \tag{1·6}$$

We can construct the system by introducing some suitable meson fields if the couplings between baryons and mesons are non-derivative as illustrated by Gell-Mann and Lévy,[13] and if the couplings are both non-derivative and derivative as has been stated by the author.[14]

The three constituents of W-field are introduced in this system so as to retain the invariance under the localized transformation

$$N \to \exp[i\boldsymbol{\alpha}(x)\boldsymbol{\tau}(1+\gamma_5)]N \tag{1·7}$$

just along the line of Yang and Mills.[15),16] Then the form of conserved vector

* This is easily seen from the occurrence of the contact interaction among baryons in the interaction representation.

and axial-vector currents can be determined uniquely. Since a conservation law follows from weaker invariance (under transformation with constant phase) than a local transformation which is reinforced by a certain field,[17] it is not destroyed by changing the mass of field. The conservation of currents of strongly interacting system is not violated by changing the mass of W-field, but their axial-vector part breaks its conserved character by changing the mass of some particle, for instance, by changing the mass of pion from 0 to μ_π, its violation takes the form of (1·5).

The system thus introduced is still far from the reality, because W-field can not interact with other baryons besides nucleon and the system is charge independent. In order to remove the former defect, we introduce Λ-particle as a reprepresentative of strange particles and also take a wider transformation group between proton, neutron and Λ-particle which is a generalization of full symmmetric group.[18] The charge independence of the system can be destroyed by changing it to other system through the local transformation

$$\Psi \to \exp\left[if(x)R(1+\gamma_5)\right]\Psi \qquad (1\cdot8)$$

where

$$\Psi=\begin{bmatrix} p \\ n \\ \Lambda \end{bmatrix}, \qquad R=\begin{bmatrix} & & \\ & & -i \\ & i & \end{bmatrix}.$$

Since the system with which we start is not invariant under a generalized full symmetric group, the transformation (1·8) gives observable effects not only from its local character but also from its rotaton. The localization effect gives the system a new variable which is interpreted as one part of W-field and the rotation effect creates the interaction between W-field and Λ-particle.

One important modification of the starting system is, however, necessary. Since the strongly interacting part of the system must be charge independent after the transformation (1·8) as before the transformation, its part must be arranged to be unaffected by this procedure. The invariance under the transformation (1·6) is insufficient from this requirement. The desired system must have more symmetrical nature, that is, the starting system must be constructed to be invariant under the generalized full symmetric transformation. The necessity of the invariance under the generalized full symmetric transformation in the system is, then, closely connected with the transformation (1·8) which is characterized by the coupling ratio of Λ-particle-nucleon pair ($\bar{G}$) to nucleon pair with W-field (G).

$$f \sim \text{const}\,\frac{\bar{G}}{G}. \qquad (1\cdot9)$$

The ratio is usually to be smaller than 1.

A new variable which is introduced by the localization effect of (1·8) can

produce Lee-Yang's interaction in the first approximation of (1·9), which makes the interpretation of the fourth constituent of W-field possible.

The way of discussion taken in this article starts with the construction of the ideal strongly interacting system with the generalized full symmetry (§ 2). Then three constituents of W-field are introduced in relation with the invariance of transformotion (1·7) (§ 3) and finally the fourth of W-field is produced when the system is transformed by transformation (1·8) (§ 4). A similar procedure is applied also to the leptonic interaction of W-field (§ 5). The conservation laws thus obtained and their violations are discussed by changing masses of particles so as to have reality (§ 6). The last section (§ 7) is devoted to discussions on the meanings of the procedure taken in connection with the concept of degenerate vacuum.

§ 2. The generalized full symmetric system

A system which consists of proton, neutron and Λ-particle interacting with pion and kaon is described by

$$L^0=-\overline{\Psi}\left[\gamma\partial+\frac{2m_N+m_\Lambda}{3}-\frac{m_\Lambda-m_N}{\sqrt{3}}\lambda_8\right]\Psi \tag{2·1}$$

$$-\frac{1}{2}\sum_{j=1}^{7}[(\partial_\mu\pi_j)^2+\mu_j^2\pi_j^2]-i\overline{\Psi}\gamma_5[g_\pi\sum_{j=1}^{3}\lambda_j\pi_j+g_K\sum_{j=4}^{7}\lambda_j\pi_j]\Psi\,,$$

$$\mu_1=\mu_2=\mu_3\equiv\mu_\pi,\ \mu_4=\mu_5=\mu_6=\mu_7\equiv\mu_k\,, \tag{2·2}$$

where λ_j's ($j=1, \cdots 8$) are 3×3 matrices which satisfy the relations,

$$\mathrm{Tr}\,(\lambda_i\lambda_j)=2\delta_{ij},\ [\lambda_i,\lambda_j]_-=2if_{ijk}\lambda_k\,,$$
$$[\lambda_i,\lambda_j]_+=\frac{4}{3}\delta_{ij}+2d_{ijk}\lambda_k\,. \tag{2·3}$$

Their concrete representation and the coefficients f_{ijk} and d_{ijk} are to be referred to Gell-Mann[19] (Also see appendix I). The system is invariant under the transformation

$$\Psi\to\exp\left[i\sum_{j=1}^{3}\alpha_j\lambda_j\right]\Psi\,,\qquad \alpha_j=\text{const.}, \tag{2·4}$$

because λ_1, λ_2 and λ_3 are understood to be the iso-spin matrix in the usual sense.

The full symmetry proposed by Ikeda, Ogawa and Ohnuki is interpreted that a system is invariant under the transformation

$$\Psi\to\exp\left[i\sum_{j=1}^{8}\alpha_j\lambda_j\right]\Psi,\qquad \alpha_j=\text{const.} \tag{2·5}$$

A suitable system must have unique mass m for three baryons and unique mass μ for eight mesons including a new meson π_8 besides pion and kaon and also unique coupling constant g;

$$L^I=-\bar{\Psi}(\gamma\partial+m)\Psi-\frac{1}{2}\sum_{j=1}^{8}[(\partial_\mu\pi_j)^2+\mu^2\pi_j^2] \tag{2·6}$$

$$-ig\,\bar{\Psi}\gamma_5\sum_{j=1}^{8}\lambda_j\pi_j\Psi.$$

A generalization proposed here is to make the system invariant under the transformation which we call the generalized full symmetric transformation

$$\Psi\to\exp\Big[i\sum_{j=1}^{8}(\alpha_j+\gamma_5\beta_j)\lambda_j\Big]\Psi,\qquad \alpha_j,\beta_j=\text{const.} \tag{2·7}$$

The desired system is given by

$$L=-\bar{\Psi}\Big[\gamma\partial+\frac{2m_N+m_A}{3}-\frac{m_A-m_N}{\sqrt{3}}\lambda_8\Big]\Psi \tag{2·8}$$

$$-\frac{1}{2}\sum_{j=0}^{8}[(\partial_\mu\pi_j)^2+\mu_j^2\pi_j^2]-\frac{1}{2}\sum_{j=0}^{8}[(\partial_\mu\sigma_j)^2+\bar{\mu}_j^2\sigma_j^2]$$

$$-g\bar{\Psi}\Big[\Big\{\Big(\frac{2}{3}+\frac{\lambda_8}{\sqrt{3}}\Big)\sigma_0+\sqrt{2}\Big(\frac{1}{3}-\frac{\lambda_8}{\sqrt{3}}\Big)\sigma_8+\sum_{j=1}^{7}\lambda_j\sigma_j\Big\}$$

$$+i\gamma_5\Big\{\Big(\frac{2}{3}+\frac{\lambda_8}{\sqrt{3}}\Big)\pi_0+\sqrt{2}\Big(\frac{1}{3}-\frac{\lambda_8}{\sqrt{3}}\Big)\pi_8+\sum_{j=1}^{7}\lambda_j\pi_j\Big\}\Big]\Psi,$$

where $\sigma_j(j=0,1\cdots8)$ together with π_0 and π_8 are newly introduced mesons. In order to get high symmetry of (2·7) the masses of particles must satisfy strong restrictions. There exist two possible cases of restrictions: One case is to make baryon masses zero

$$\text{(I)}\qquad m_N=m_A=0,\quad \mu_j=\bar{\mu}_j\equiv\mu\quad(j=0,\cdots8), \tag{2·9}$$

which has been considered in usual literatures. Another possible case is to make meson masses zero

$$\text{(II)}\qquad m_N\neq m_A,\quad \mu_j=\bar{\mu}_j=0\quad(j=0,\cdots8) \tag{2·10}$$

which has been proposed in connection with the axial-vector current conservation. For the latter case, since the invariant quantity against the transformation (2·7) is only

$$\Big(\sigma_0+\frac{m_N}{g}\Big)^2+\Big(\sigma_8+\frac{m_A}{\sqrt{2}\,g}\Big)^2+\pi_0^2+\pi_8^2+\sum_{j=1}^{7}(\sigma_j^2+\pi_j^2), \tag{2·11}*$$

we can add some portions to masses of σ_0 and σ_8 without a violation of invariance, for instance,

$$\bar{\mu}_0'=km_N,\quad \bar{\mu}_8'=\frac{k}{\sqrt{2}}m_A,\qquad k=\text{const.} \tag{2·12}$$

* The transformation character of each field variable is to be referred to the work of Gell-Mann and Lévy,[12] and of the author.[14]

The deviation of mass of any other particle from (2·9) or (2·10) may generally destroy the invariance which gives rise to the conservation laws. The relation between mass alteration and violation of conservation laws will be discussed in the succeeding section (§ 6).

§ 3. First localized transformation: triplet *W*-field

The first transformation which is now localized relates with the generalized iso-spin rotation, the transformation between left-handed neutron and proton. The triplet *W*-field can be introduced in order to retain the system (2·8) invariant under the transformation

$$\text{(F)} \qquad \Psi \to \exp\left[i\sum_{j=1}^{3}\alpha_j(x)\lambda_j(1+\gamma_5) \right]\Psi, \tag{3·1}$$

just along the line of Yang-Mills' procedure.

The procedure is accomplished by replacing every derivatives with absolute derivatives tabled in Appendix II. For instance, the absolute derivatives of nucleon and Λ-particle are given by

$$D_\mu N=\partial_\mu N+iG(1+\gamma_5)\boldsymbol{\tau}\cdot\boldsymbol{\omega}_\mu N, \quad D_\mu \Lambda=\partial_\mu \Lambda, \tag{3·2}$$

where $\boldsymbol{\omega}_\mu(\omega_{1\mu}, \omega_{2\mu}, \omega_{3\mu})$ is the triplet *W*-field and $\boldsymbol{\tau}$ is usual iso-spin.

The system is given by

$$\begin{aligned}
L^F=&-N(\gamma_\mu D_\mu+m_N)N-\bar{\Lambda}(\gamma_\mu\partial_\mu+m_\Lambda)\Lambda-\frac{1}{4}\boldsymbol{\omega}^2{}_{\mu\nu}\\
&-\frac{1}{2}[(D_\mu\sigma_0)^2+\bar{\mu}_0{}^2\sigma_0{}^2]-\frac{1}{2}[(\partial_\mu\sigma_8)^2+\bar{\mu}_8{}^2\sigma_8{}^2]-\frac{1}{2}[(D_\mu\boldsymbol{\sigma})^2+\mu_\sigma{}^2\boldsymbol{\sigma}^2]\\
&-\frac{1}{2}[(D_\mu\pi_0)^2+\mu_0{}^2\pi_0{}^2]-\frac{1}{2}[(\partial_\mu\pi_8)^2+\mu_8{}^2\pi_8{}^2]-\frac{1}{2}[(D_\mu\boldsymbol{\pi})^2+\mu_\pi{}^2\boldsymbol{\pi}^2]\\
&-[D_\mu\bar{K}\cdot D_\mu K+\mu_K{}^2\bar{K}K]-[D_\mu\bar{L}\cdot D_\mu L+\mu_L{}^2\bar{L}L]\\
&-g[\bar{N}\{(\sigma_0+i\gamma_5\pi_0)+\boldsymbol{\tau}\cdot(\boldsymbol{\sigma}+i\gamma_5\boldsymbol{\pi})\}N+\sqrt{2}\,\bar{\Lambda}(\sigma_8+i\gamma_5\pi_8)\Lambda\\
&+\sqrt{2}\,\bar{N}(L+i\gamma_5 K)\Lambda+\sqrt{2}\,\bar{\Lambda}(\bar{L}+i\gamma_5\bar{K})N],
\end{aligned} \tag{3·3}$$

where

$$\begin{aligned}
&\boldsymbol{\sigma}=(\sigma_1, \sigma_2, \sigma_3), \quad \boldsymbol{\pi}=(\pi_1, \pi_2, \pi_3),\\
&K=\begin{bmatrix} (\pi_4-i\pi_5)/\sqrt{2} \\ (\pi_6-i\pi_7)/\sqrt{2} \end{bmatrix}, \quad L=\begin{bmatrix} (\sigma_4-i\sigma_5)/\sqrt{2} \\ (\sigma_6-i\sigma_7)/\sqrt{2} \end{bmatrix},\\
&\bar{K}=[(\pi_4+i\pi_5)/\sqrt{2} \quad (\pi_6+i\pi_7)/\sqrt{2}], \quad \bar{L}=[(\sigma_4+i\sigma_5)/\sqrt{2} \quad (\sigma_6+\sigma_7)/\sqrt{2}],\\
&\boldsymbol{\omega}_{\mu\nu}=D_\mu\boldsymbol{\omega}_\nu-D_\nu\boldsymbol{\omega}_\mu
\end{aligned} \tag{3·4}$$

and the relations between masses are again (2·9) or (2·10).

The equation of triplet W-field becomes

$$\partial_\nu \boldsymbol{\omega}_{\mu\nu} - 4G\boldsymbol{\omega}_\nu \times \boldsymbol{\omega}_{\mu\nu} + \left(\frac{2G}{g} m_N\right)^2 \omega_\mu + iG(\boldsymbol{V}_\mu + \boldsymbol{P}_\mu) = 0, \tag{3·5}$$

where

$$\begin{aligned} \boldsymbol{V}_\mu = {} & \bar{N}\gamma_\mu \boldsymbol{\tau} N + 2i(\boldsymbol{\sigma} \times D_\mu \boldsymbol{\sigma} + \boldsymbol{\pi} \times D_\mu \boldsymbol{\pi}) \\ & - (\bar{K}\boldsymbol{\tau} D_\mu K - D_\mu \bar{K}\boldsymbol{\tau} K) - (\bar{L}\boldsymbol{\tau} D_\mu L - D_\mu \bar{L}\boldsymbol{\tau} L), \end{aligned} \tag{3·6}$$

$$\begin{aligned} \boldsymbol{P}_\mu = {} & \bar{N}\gamma_\mu \gamma_5 \boldsymbol{\tau} N + \frac{2im_N}{g} D_\mu \boldsymbol{\pi} + \frac{4iG}{g^2} m_N{}^2 \boldsymbol{\omega}_\mu \\ & + 2i(\boldsymbol{\sigma} D_\mu \pi_0 - D_\mu \boldsymbol{\sigma} \cdot \pi_0 - \boldsymbol{\pi} D_\mu \sigma_0 + D_\mu \boldsymbol{\pi} \cdot \sigma_0) \\ & - i(\bar{K}\boldsymbol{\tau} D_\mu L - D_\mu \bar{K}\boldsymbol{\tau} L) + i(\bar{L}\boldsymbol{\tau} D_\mu K - D_\mu \bar{L}\boldsymbol{\tau} K). \end{aligned} \tag{3·7}$$

It is remakable that the mass of W-field is given by

$$M_0{}^2 = \left(\frac{2G}{g} m_N\right)^2 \tag{3·8}$$

which is small but finite in the case of (2·10) ($m_N \neq 0$). The divergences of $\boldsymbol{V}_\mu$ and $\boldsymbol{P}_\mu$ are

$$\partial_\mu \boldsymbol{V}_\mu = 2G\boldsymbol{\omega}_\mu \times (\boldsymbol{V}_\mu + \boldsymbol{P}_\mu), \tag{3·9}$$

$$\begin{aligned} \partial_\mu \boldsymbol{P}_\mu = {} & \frac{2im_N}{g} \mu_\pi{}^2 \boldsymbol{\pi} + 2i(\mu_0{}^2 - \mu_\sigma{}^2)\pi_0 \boldsymbol{\sigma} - 2i(\bar{\mu}_0{}^2 - \mu_\pi{}^2)\sigma_0 \boldsymbol{\pi} \\ & + i(\mu_K{}^2 - \mu_L{}^2)(\bar{K}\boldsymbol{\tau} L + \bar{L}\boldsymbol{\tau} K) + \frac{4iG}{g^2} m_N{}^2 \partial_\mu \boldsymbol{\omega}_\mu \\ & + 2G\boldsymbol{\omega}_\mu \times (\boldsymbol{V}_\mu + \boldsymbol{P}_\mu), \end{aligned} \tag{3·10}$$

which reduces to zero in the limit of $G=0$ in virtue of mass relations (2·9) or (2·10). These quantities can be interpreted as the conserved vector and axial-vector current in the strongly interacting system. The conservation laws including the weak interaction are recovered by adding a term as

$$\partial_\mu (\boldsymbol{V}_\mu + 2i\boldsymbol{\omega}_\nu \times \boldsymbol{\omega}_{\mu\nu}) = 0, \tag{3·11}$$

$$\partial_\mu (\boldsymbol{P}_\mu + 2i\boldsymbol{\omega}_\nu \times \boldsymbol{\omega}_{\mu\nu}) = -\frac{4iG}{g^2} m_N{}^2 \partial_\mu \boldsymbol{\omega}_\mu. \tag{3·12}$$

It is not unsatisfactory that the right-hand side of (3·12) remains, because, if we took

$$\partial_\mu \boldsymbol{\omega}_\mu = 0, \quad \partial_\nu (\partial_\mu \boldsymbol{\omega}_\mu) = 0 \tag{3·13}$$

as the initial conditions, the former would be the constant of motion in virtue of

$$\Box (\partial_\mu \boldsymbol{\omega}_\mu) = 0, \tag{3·14}$$

which follows from the field equation (3·5). The nature of the conservation law remains unchanged if we increase the mass of W-field, but they may break down if we change the masses of other particles from the relations (2·9) or (2·10) or more loosely from

$$m_N=0, \quad \mu_\pi=\bar{\mu}_0, \quad \mu_\sigma=\mu_0, \quad \mu_K=\mu_L \tag{3·15}$$

or

$$m_N\neq 0, \ \mu_\pi=\bar{\mu}_0=0, \ \mu_\sigma=\mu_0, \ \mu_K=\mu_L, \tag{3·16}$$

which are weaker conditions than (2·9) or (2·10).

§ 4. Second localized transformation: singlet W-field

The interacting system with triplet W-field presented in § 3 is insufficient, because of the lack of interaction between Λ-particle and W-field and of the conservation of iso-spin rotation. The real system must be produced by introducing the Λ-particle interaction and by violating the invariance of iso-spin rotation.

This can be done by transforming the system through the following localized transformation:

$$(S) \qquad \Psi\to\exp\left[if(x)\lambda_7\frac{1+\gamma_5}{2}\right]\Psi. \tag{4·1}$$

Since the part of strong interaction in the system is arranged to be invariant under the transformation $f(x)=\text{const}$, the deviation of the transformed system from the original system can be confined to the modification of W-field interactions.

If we take

$$f(x)=\frac{\bar{G}}{G}\exp\left[2G\omega^*(x)\right], \tag{4·2}$$

a part of baryon field is changed as

$$\begin{aligned}
&-\bar{\Psi}\gamma_\mu D_\mu\Psi\equiv-\bar{\Psi}\gamma_\mu\partial_\mu\Psi-iG\bar{\Psi}\gamma_\mu(1+\gamma_5)\left(\lambda_1\omega_{1\mu}+\lambda_2\omega_{2\mu}+\lambda_3\omega_{3\mu}\right)\Psi\\
&\to-\bar{\Psi}\gamma_\mu\Delta_\mu\Psi\equiv-\bar{\Psi}\gamma_\mu\partial_\mu\Psi\\
&\qquad-iG\bar{\Psi}\gamma_\mu(1+\gamma_5)\left[\left(\lambda_1\omega_{1\mu}+\lambda_2\omega_{2\mu}\right)\cos f+\left(\lambda_4\omega_{1\mu}+\lambda_5\omega_{2\mu}\right)\sin f\right.\\
&\qquad\left.+\frac{3+\cos 2f}{4}\lambda_3\omega_{3\mu}+\frac{\sqrt{3}\,(1-\cos 2f)}{4}\lambda_8\omega_{3\mu}-\frac{\sin 2f}{2}\lambda_6\omega_{3\mu}+\frac{\bar{G}}{G}\lambda_7\omega_\mu^*\right]\Psi.
\end{aligned} \tag{4·3}$$

Assuming that the new constant $\bar{G}$ is smaller than G

$$\bar{G}/G<1, \quad G<1, \tag{4·4}$$

we get in the first approximation

$$-\bar{\Psi}\gamma_\mu\partial_\mu\Psi-iG\bar{\Psi}\gamma_\mu(1+\gamma_5)[\lambda_1\omega_{1\mu}+\lambda_2\omega_{2\mu}+\lambda_3\omega_{3\mu}]\Psi$$
$$-i\bar{G}\bar{\Psi}\gamma_\mu(1+\gamma_5)[\lambda_4\omega_{1\mu}+\lambda_5\omega_{2\mu}-\lambda_6\omega_{3\mu}+\lambda_7\omega_\mu^*]\Psi. \quad (4\cdot5)$$

Then $\bar{G}$ is interpreted as the coupling constant between Λ-particle-nucleon pair and W-field. The resultant interaction does destroy the charge independence and the triplet W-field and singlet W-field from a quartet which assures the $|\Delta I|=1/2$ law as stated by Lee-Yang.

The charge operator which takes a form

$$Q=\frac{1}{3}+\frac{\lambda_3}{2}+\frac{\lambda_8}{2\sqrt{3}} \quad (4\cdot6)$$

is invariant under the second transformation. The electromagnetic properties of the transformed system may retain to be same as the previous system.

The absolute derivatives of fields are generally transformed so as to harmonize the localization effect of the second transformation and their concrete forms are listed in Appendix III. The transformed system has the same form with (3·3) where all absolute derivatives are replaced by transformed ones and a new free term of singlet W-field (if it is necessary) must be added.

The quaternery W-field has the following equations:

$$\partial_\nu\omega_{i\mu\nu}-4G(\omega_\nu\times\omega_{\mu\nu})_i+\frac{(4G^2+2\bar{G}^2)m_N^2+2\bar{G}^2m_\Lambda^2}{g^2}\omega_{i\mu} \quad (4\cdot7)$$
$$+iG(V_{i\mu}+P_{i\mu})+i\bar{G}(V_{i+3\mu}+P_{i+3\mu})=0,$$
$$i=1,\ 2,\ 3$$

$$\partial_\mu\omega_\mu^*=-\frac{i\bar{G}}{1-2\bar{G}^2(m_\Lambda^2+m_N^2)/g^2}\partial_\mu(V_{7\mu}+P_{7\mu}), \quad (4\cdot8)$$

where V_i and $P_i(i=1,\ 2,\ 3,)$ are the nucleon part of vector and axial-vector currents, while V_i and $P_i(i=4,\ 5,\ 6,\ 7)$ are the Λ-particle-nucleon part of vector and axial-vector currents. The mass of triplet W-field is transformed to

$$M^2=\frac{4G^2+2\bar{G}^2}{g^2}m_N^2+\frac{2\bar{G}^2}{g^2}m_\Lambda^2, \quad (4\cdot9)$$

which is still small but finite for the case of (2·10) $(m_Nm_\Lambda\neq0)$. Any field of Yang-Mills' type is generally considered that its mass becomes zero regardless of the reaction. It is true for the case of the original Yang-Mills formulation where the invariance of localized iso-spin rotation holds, because the mass term breaks down its invariance. This is not the case with parity changing transformation proposed here. The circumstance does correspond just with the one for nucleon mass.[20] We have two alternative restrictions, one of which is characterized by the zero mass of baryons, while the other by the zero mass of bosons. If we could realize the baryon masses in the invariant system under parity changing transformation as indicated by Nambu in an analogy with the

superconductivity, this system might allow W-field to have a small but finite mass. The mass of W-field appearing in the system, however, does not increase as far as the invariance of the first localized transformation survives. The second transformation introduced in this section does change this restriction, because it breaks the invariance. It may thus be probable to get the reactive mass from the interaction. Unfortunately, the interaction which causes the reactive mass is usually considered to be weak. The answer may depend on the high-energy behavior of this weak interaction.[21]

§ 5. Interaction between W-field and lepton

A symmetrical treatment between baryons and leptons may be impossible from the standpoint of $|\Delta I|=1/2$ law, though the beautiful symmetry between three baryons, proton, neutron and Λ-particle and three leptons, neutrino, electron and muon was emphasized by Gamba, Marshak and Okubo.[22]

We take two pairs of leptons in which neutrino interacting with muon is distinguished from beta-neutrino so as to make muon relatively stable for radiative decay $\mu\to e+\gamma$. They are expressed by

$$\chi_1=\begin{bmatrix}\nu\\ e\end{bmatrix}, \qquad \chi_2=\begin{bmatrix}\nu'\\ \mu\end{bmatrix}. \tag{5·1}$$

When the interaction between W-field and lepton-pairs is introduced, the transformation of W-field induced by the first localized transformation of baryon-meson system

$$\boldsymbol{\omega}_\mu\to\boldsymbol{\omega}_\mu-4\boldsymbol{\alpha}\times\boldsymbol{\omega}_\mu-\frac{1}{G}\partial_\mu\boldsymbol{\alpha}, \quad \boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3) \tag{5·2}$$

enforces the lepton system to be also invariant under the simultaneous transformation

$$\chi_i\to\exp\left[i\boldsymbol{\alpha}(x)\boldsymbol{\tau}(1+\gamma_5)\right]\chi_i, \quad i=1,2, \tag{5·3}$$

that is, we must have

$$L_l=-\sum_{i=1}^{2}\bar{\chi}_i\gamma\partial\chi_i-iG\sum_{i=1}^{2}\bar{\chi}_i\gamma_\mu(1+\gamma_5)\boldsymbol{\tau}\boldsymbol{\omega}_\mu\chi_i. \tag{5·4}$$

The system then has the following unwanted interactions:

$$iG[\bar{e}\gamma_\mu(1+\gamma_5)e-\bar{\nu}\gamma_\mu(1+\gamma_5)\nu + \bar{\mu}\gamma_\mu(1+\gamma_5)\mu-\bar{\nu}'\gamma_\mu(1+\gamma_5)\nu']\omega_{3\mu}. \tag{5·5}$$

However, since the baryon-meson system has lost the invariance of (5·2) after the second localized transformation, by which the transformed system is only invariant under the transformation

$$\mathcal{F}\to\exp\left[i\alpha Q(1+\gamma_5)\right]\mathcal{F},\quad \alpha=\text{const.} \tag{5·6}$$

This suggests us that the full interaction (5·4) may be unnecessary. In order to omit (5·5) more similarly with the device in the baryon-meson system, we assume that the lepton system is simultaneously transformed by

$$\chi_i\to\exp\left[i\frac{1+\gamma_5}{2}g(x)\tau_3\right]\chi_i,\qquad g(x)=2G\int_0^\infty dx_\mu\,\omega_{3\mu}, \tag{5·7}$$

when the second localized transformation for baryon-meson system is performed. Then the system is changed to

$$\begin{aligned}L_l'=&-\sum_{i=1}^{2}\bar{\chi}_i\gamma_\mu\partial_\mu\chi_i-iG\sum_{i=1}^{2}\bar{\chi}_i[(\omega_{1\mu}\cos 2g+\omega_{2\mu}\sin 2g)\tau_1 \\ &+(\omega_{2\mu}\cos 2g-\omega_{1\mu}\sin 2g)\tau_2]\chi_i \\ \simeq&-\sum_{i=1}^{2}\bar{\chi}_i\gamma\partial\chi_i-iG\sum_{i=1}^{2}\bar{\chi}_i(\tau_1\omega_{1\mu}+\tau_2\omega_{2\mu})\chi_i\,.\end{aligned} \tag{5·8}$$

The third constituent of the triplet W-field plays quite a similar role to one of the singlet W-fields in the baryon-meson system.

§ 6. Violation of the conservation laws

The quantities of $V_{i\mu}$ and $P_{i\mu}(i=1,\cdots 7)$ satisfy the so-called conservation laws

$$\partial_\mu V_{i\mu}=0,\qquad \partial_\mu P_{i\mu}=0 \tag{6·1}$$

as far as they concern with the strong interactions. The foundation of the conservation laws dues to the idealized mass relations

$$m_N=m_A=0,\quad \mu_i=\bar{\mu}_i=\mu\quad i=0,\,1\cdots 8 \tag{6·2}$$

or

$$m_N\neq m_A,\quad \mu_i=\bar{\mu}_i=0\,. \tag{6·3}$$

The existence of such quantities is easily understood from the invariance of (2·7) which assures the 16 conserved (2 quantities do not appear explicitly). The nature of conserved quantities is insensitive to a change of W-field mass. However, it is very sensitive to any deviation of the mass of other particles from (6·2) or (6·3), and exceptions are three vector currents $V_{i\mu}(i=1,\,2,\,3)$. The general deviations for each quantity are listed in Appendix IV.

It is interesting to see the conservations when every meson takes the same mass value. Then we have

$$\begin{aligned}&\partial_\mu V_{1\mu}=0 && \text{(similar for } i=2,\,3),\\ &\partial_\mu V_{4\mu}=i(m_A-m_N)/g\cdot\mu_L{}^2\sigma_5 && \text{(similar for } i=5,\,6,\,7),\\ &\partial_\mu P_{1\mu}=2im_N/g\cdot\mu_\pi{}^2\pi_1 && \text{(similar for } i=2,\,3),\end{aligned} \tag{6·4}$$

$$\partial_\mu P_{4\mu} = i(m_\Lambda + m_N)/g \cdot \mu_K^2 \pi_4 \qquad \text{(similar for } i=5, 6, 7),$$

which are Goldberger-Treiman's relation. They show that each meson is mainly responsible for the deviation of each conservation law. If each meson has mass difference from others as in the real world, the deviation increases. For instance, for the case of nucleon axial vector current, its deviation is mainly determined by pion mass and successively modified by mass differences between other particles, in which the difference μ_K^2-μ_L^2 may be essential because μ_L^2 is probably too large to be detectable.

The violation of the conservation laws may give rise to the mixture of spin states for W-field. The same equation as (3·14) is now given by

$$\Box(\partial_\mu \boldsymbol{\omega}_\mu) = -\frac{2G}{g} m_N \mu_\pi^2 \boldsymbol{\pi}, \tag{6·5}$$

which means that the initial conditions

$$\partial_\mu \boldsymbol{\omega}_\mu = 0, \qquad \partial_\nu(\partial_\mu \boldsymbol{\omega}_\mu) = 0 \tag{6·6}$$

are destroyed in the course of interaction. Then the spin state of W-field is obliged to be mixture of spin 1 and spin 0. The same effect is also expected from the interaction among W-field and leptons, because in the real world the lepton masses deviate from

$$m_\mu = m_e = m_\nu = 0 \tag{6·7}$$

to give rise to a violation of invariance under the transformation proposed.

The appearence of spin 0 state in W-field gives rise to the leptonic decays of spin 0 particles (pion and kaon).*) It is very reasonable that Goldberger-Treiman's relation can also explain the lifetime of pion decay quantitatively.*)

§ 7. Summary and discussion

We can summarize the remarkable points which may present one possibility for understanding the nature of weak interaction.

I. The conserved quantities in the idealized system are uniquely determined by the introduction of quaternary W-field. And their violation is expressed by changing the mass of bosons.

II. The $|\Delta I| = 1/2$ law in the baryon boson weak interaction approximately holds as the introduction of quaternary W-field should be.

III. The roles of constituents of W-field are clarified by two types of localized transformations. The triplet is interpreted from a generalization of

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* An effect of spin 0 part would be expected from the contact term in the Hamiltonian of interaction representation. It gives, however, no effect on decay in the case of strict conservation law.

Yang-Mills' idea, while the other, the singlet, is related with the special gauge which transforms the system to other in the full symmetric group space.

IV. The triplet W-field has a small but finite mass in the case of finite baryon masses and a possibility of increasing its amount by field reaction depends on the high-energy behavior weak interaction.

V. The violation of conservation laws gives rise to mixture of spin 0 and spin 1 state, which is responsible for the decay of spin 0 particles.

Besides these points, there are still many defects in the assumptions taken. The first obstacle may be the introduction of a generalized full symmetry, by which the strongly interacting system is unchanged under the introduction of the singlet W-field. Since the requirement is closely connected with the deviation form the original system which is characterized by the coupling ratio of Λ-particle-nucleon pair to nucleon-nucleon pair with W-field $(\bar{G}\,G)$, the necessity of generalized full symmetry may have equal-footing. This symmetry requires the co-existence of eleven new mesons besides usual seven mesons, all of which are still unobserved. Besides the question of their existence, there may be another possible model which holds the generalized full symmetry without introducing new mesons if we assume the gradient coupling in the strong intearction as stated by Taylor[23] and Gell-Mann-Lévy, which is too complicated from the viewpoint of transformation.

The second point which is one of the essential points in this paper is the question what the second transformation means. The procedure is to take a kind of special gauge for the system. The more fundamental understanding of the existence of many (infinite) different systems for each gauge may enforce us to consider the infinite degeneracy of vacuum as emphasized by Dürr and Heisenberg.[24] In fact the original state in which $\Delta I=0$ holds is transformed to one whith $|\Delta I|=1/2$. This may be understood if the vacuum carries a portion of iso-spin.

The third is concerned with the mass of W-field. The question is how we can get its mass large enough to be consistent with the observed phenomena. An idea of the drastic change of weak interaction at high energy may be welcome for getting a large reactive mass. This is not improbable, because the interaction of W-field is generally unrenormalizable because of the existence of bare mass.[25]

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Appendix I

Gell-Mann matrices

A set of matrices λ_i is represented by

$$\lambda_1=\begin{bmatrix} & 1 & \\ 1 & & \\ & & \end{bmatrix} \quad \lambda_2=\begin{bmatrix} & -i & \\ i & & \\ & & \end{bmatrix} \quad \lambda_3=\begin{bmatrix} 1 & & \\ & -1 & \\ & & \end{bmatrix}$$

$$\lambda_4=\begin{bmatrix} & & 1 \\ & & \\ 1 & & \end{bmatrix} \quad \lambda_5=\begin{bmatrix} & & -i \\ & & \\ i & & \end{bmatrix} \quad \lambda_6=\begin{bmatrix} & & \\ & & 1 \\ & 1 & \end{bmatrix}$$

$$\lambda_7=\begin{bmatrix} & & \\ & & -i \\ & i & \end{bmatrix} \quad \lambda_8=\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix}$$

and the following relations are obtained:

$$\lambda_1^2=\lambda_2^2=\lambda_3^2=\frac{2}{3}+\frac{\lambda_8}{\sqrt{3}}, \quad \lambda_8^2=\frac{2}{3}-\frac{\lambda_8}{\sqrt{3}},$$

$$\lambda_4^2=\lambda_5^2=\frac{2}{3}+\frac{\lambda_3}{2}-\frac{\lambda_8}{2\sqrt{3}}, \quad \lambda_6^2=\lambda_7^2=\frac{2}{3}-\frac{\lambda_3}{2}-\frac{\lambda_8}{2\sqrt{3}},$$

$$\lambda_1\lambda_2=i\lambda_3, \ \lambda_2\lambda_3=i\lambda_1, \ \lambda_3\lambda_1=i\lambda_2,$$

$$\lambda_1\lambda_8=\lambda_1/\sqrt{3}, \ \lambda_2\lambda_8=\lambda_2/\sqrt{3}, \ \lambda_3\lambda_8=\lambda_3/\sqrt{3},$$

$$\lambda_4\lambda_8=-(\lambda_4+3i\lambda_5)/2\sqrt{3}, \ \lambda_5\lambda_8=-(\lambda_5-3i\lambda_4)/2\sqrt{3},$$

$$\lambda_6\lambda_8=-(\lambda_6+3i\lambda_7)/2\sqrt{3}, \ \lambda_7\lambda_8=-(\lambda_7-3i\lambda_6)/2\sqrt{3},$$

$$\lambda_4\lambda_5=i(\lambda_3+\sqrt{3}\lambda_8)/2, \ \lambda_6\lambda_7=-i(\lambda_3-\sqrt{3}\lambda_8)/2,$$

$$\lambda_4\lambda_6=-i\lambda_4\lambda_7=i\lambda_5\lambda_6=\lambda_5\lambda_7,$$
$$=-(\lambda_1+i\lambda_2)/2,$$

$$\lambda_1\lambda_4=i\lambda_1\lambda_5=-i\lambda_2\lambda_4=\lambda_2\lambda_5=-\lambda_3\lambda_6=-i\lambda_3\lambda_7$$
$$=(\lambda_6+i\lambda_7)/2,$$

$$\lambda_1\lambda_6=i\lambda_1\lambda_7=i\lambda_2\lambda_6=-\lambda_2\lambda_7=\lambda_3\lambda_4=i\lambda_3\lambda_5$$
$$=(\lambda_4+i\lambda_5)/2.$$

Appendix II

Absolute derivatives for fields under the first transformation (F)

$$D_\mu\boldsymbol{\omega}_\nu=\partial_\mu\boldsymbol{\omega}_\nu-2G\boldsymbol{\omega}_\mu\times\boldsymbol{\omega}_\nu,$$

$$D_\mu N=\partial_\mu N+iG(1+\gamma_5)\boldsymbol{\tau}\cdot\boldsymbol{\omega}_\mu N,$$

$$D_\mu \Lambda = \partial_\mu \Lambda,$$

$$D_\mu \pi_0 = \partial_\mu \pi_0 - 2G\boldsymbol{\omega}_\mu \boldsymbol{\sigma},$$

$$D_\mu \pi_8 = \partial_\mu \pi_8,$$

$$D_\mu \sigma_0 = \partial_\mu \sigma_0 + 2G\boldsymbol{\omega}_\mu \boldsymbol{\pi},$$

$$D_\mu \sigma_8 = \partial_\mu \sigma_8,$$

$$D_\mu \boldsymbol{\pi} = \partial_\mu \boldsymbol{\pi} - 2G\left(\boldsymbol{\omega}_\mu \times \boldsymbol{\pi} + \boldsymbol{\omega}_\mu\left(\sigma_0 + \frac{m_N}{g}\right)\right),$$

$$D_\mu \boldsymbol{\sigma} = \partial_\mu \boldsymbol{\sigma} - 2G(\boldsymbol{\omega}_\mu \times \boldsymbol{\sigma} - \boldsymbol{\omega}_\mu \pi_0),$$

$$D_\mu \frac{K+iL}{\sqrt{2}} = \partial_\mu \frac{K+iL}{\sqrt{2}} + 2iG\boldsymbol{\tau}\boldsymbol{\omega}_\mu \frac{K+iL}{\sqrt{2}}$$

$$D_\mu \frac{K-iL}{\sqrt{2}} = \partial_\mu \frac{K-iL}{\sqrt{2}},$$

$$D_\mu \frac{\bar{K}-i\bar{L}}{\sqrt{2}} = \partial_\mu \frac{\bar{K}-i\bar{L}}{\sqrt{2}} - 2iG\frac{\bar{K}-i\bar{L}}{\sqrt{2}}\ \boldsymbol{\tau}\boldsymbol{\omega}_\mu,$$

$$D_\mu \frac{\bar{K}+i\bar{L}}{\sqrt{2}} = \partial_\mu \frac{\bar{K}+i\bar{L}}{\sqrt{2}}.$$

Appendix III

Transformed absolute derivatives for fields by the second transformation (S)

$$\Delta_\mu \boldsymbol{\omega}_\nu = \partial_\mu \boldsymbol{\omega}_\nu - 2G\boldsymbol{\omega}_\mu \times \boldsymbol{\omega}_\nu, \qquad \text{(unchanged)}$$

$$\Delta_\mu \pi_0 = \partial_\mu \pi_0 - 2G[\omega_{1\mu}\sigma_1 + \omega_{2\mu}\sigma_2 + \omega_{3\mu}\sigma_3]$$
$$+ \bar{G}[-\omega_{1\mu}(\pi_5+\sigma_4) + \omega_{2\mu}(\pi_4-\sigma_5) + \omega_{3\mu}(\pi_7+\sigma_6) + \omega_\mu{}^*(\pi_6-\sigma_7)],$$

$$\Delta_\mu \pi_8 = \partial_\mu \pi_8 + \sqrt{2}\bar{G}[\omega_{1\mu}(\pi_5-\sigma_4) - \omega_{2\mu}(\pi_4+\sigma_5) - \omega_{3\mu}(\pi_7-\sigma_6) - \omega_\mu{}^*(\pi_6+\sigma_7)],$$

$$\Delta_\mu \sigma_0 = \partial_\mu \sigma_0 + 2G[\omega_{1\mu}\pi_1 + \omega_{2\mu}\pi_2 + \omega_{3\mu}\pi_3]$$
$$+ \bar{G}[\omega_{1\mu}(\pi_4-\sigma_5) + \omega_{2\mu}(\pi_5+\sigma_4) - \omega_{3\mu}(\pi_6-\sigma_7) + \omega_\mu{}^*(\pi_7+\sigma_6)],$$

$$\Delta_\mu \sigma_8 = \partial_\mu \sigma_8 + \sqrt{2}\bar{G}[\omega_{1\mu}(\pi_4+\sigma_5) + \omega_{2\mu}(\pi_5-\sigma_4) - \omega_{3\mu}(\pi_6+\sigma_7) + \omega_\mu{}^*(\pi_7-\sigma_6)],$$

$$\Delta_\mu \pi_1 = \partial_\mu \pi_1 + 2G\left[-\omega_{1\mu}\left(\sigma_0 + \frac{m_N}{g}\right) - \omega_{2\mu}\pi_3 + \omega_{3\mu}\pi_2\right]$$
$$+ \bar{G}[-\omega_{1\mu}(\pi_7+\sigma_6) + \omega_{2\mu}(\pi_6-\sigma_7) + \omega_{3\mu}(\pi_5+\sigma_4) + \omega_\mu{}^*(\pi_4-\sigma_5)]$$

$$\Delta_\mu \pi_2 = \partial_\mu \pi_2 + 2G\left[\omega_{1\mu}\pi_3 - \omega_{2\mu}\left(\sigma_0 + \frac{m_N}{g}\right) - \omega_{3\mu}\pi_1\right]$$
$$+ \bar{G}[-\omega_{1\mu}(\pi_6-\sigma_7) - \omega_{2\mu}(\pi_7+\sigma_6) - \omega_{3\mu}(\pi_4-\sigma_5) + \omega_\mu{}^*(\pi_5+\sigma_4)],$$

$$\Delta_\mu\pi_3=\partial_\mu\pi_3+2G\Big[-\omega_{1\mu}\pi_2+\omega_{2\mu}\pi_1-\omega_{3\mu}\Big(\sigma_0+\frac{m_N}{g}\Big)\Big]$$
$$+\bar{G}[-\omega_{1\mu}(\pi_5+\sigma_4)+\omega_{2\mu}(\pi_4-\sigma_5)-\omega_{3\mu}(\pi_7+\sigma_6)-\omega_\mu{}^*(\pi_6-\sigma_7)],$$
$$\Delta_\mu\sigma_1=\partial_\mu\sigma_1+2G[\omega_{1\mu}\pi_0-\omega_{2\mu}\sigma_3+\omega_{3\mu}\sigma_2]$$
$$+\bar{G}[\omega_{1\mu}(\pi_6-\sigma_7)+\omega_{2\mu}(\pi_7+\sigma_6)-\omega_{3\mu}(\pi_4-\sigma_5)+\omega_\mu{}^*(\pi_5+\sigma_4)],$$
$$\Delta_\mu\sigma_2=\partial_\mu\sigma_2+2G[\omega_{1\mu}\sigma_3+\omega_{2\mu}\pi_0-\omega_{3\mu}\sigma_1]$$
$$+\bar{G}[-\omega_{1\mu}(\pi_7+\sigma_6)+\omega_{2\mu}(\pi_6-\sigma_7)-\omega_{3\mu}(\pi_5+\sigma_4)-\omega_\mu{}^*(\pi_4-\sigma_5)],$$
$$\Delta_\mu\sigma_3=\partial_\mu\sigma_3+2G[-\omega_{1\mu}\sigma_2+\omega_{2\mu}\sigma_1+\omega_{3\mu}\pi_0]$$
$$+\bar{G}[\omega_{1\mu}(\pi_4-\sigma_5)+\omega_{2\mu}(\pi_5+\sigma_4)+\omega_{3\mu}(\pi_6-\sigma_7)-\omega_\mu{}^*(\pi_7+\sigma_6)],$$
$$\Delta_\mu\pi_4=\partial_\mu\pi_4+G[\omega_{1\mu}(\pi_7-\sigma_6)+\omega_{2\mu}(\pi_6+\sigma_7)+\omega_{3\mu}(\pi_5-\sigma_4)]$$
$$+\bar{G}[-\omega_{1\mu}(\sigma_3+\sigma_0+\sqrt{2}\,\sigma_8+\frac{m_N+m_\Lambda}{g})-\omega_{2\mu}(\pi_3+\pi_0-\sqrt{2}\pi_8)$$
$$+\omega_{3\mu}(\pi_0+\sigma_1)-\omega_\mu{}^*(\pi_1-\sigma_2)],$$
$$\Delta_\mu\pi_5=\partial_\mu\pi_5+G[-\omega_{1\mu}(\pi_6+\sigma_7)+\omega_{2\mu}(\pi_7-\sigma_6)-\omega_{3\mu}(\pi_4+\sigma_5)]$$
$$+\bar{G}[\omega_{1\mu}(\pi_3+\pi_0-\sqrt{2}\,\pi_8)-\omega_{2\mu}(\sigma_3+\sigma_0+\sqrt{2}\,\sigma_8+\frac{m_N+m_\Lambda}{g})$$
$$-\omega_{3\mu}(\pi_1-\sigma_2)-\omega_\mu{}^*(\pi_2+\sigma_1)]$$
$$\Delta_\mu\pi_6=\partial_\mu\pi_6+G[\omega_{1\mu}(\pi_5-\sigma_4)-\omega_{2\mu}(\pi_4+\sigma_5)-\omega_{3\mu}(\pi_7-\sigma_6)]$$
$$+\bar{G}[\omega_{1\mu}(\pi_2-\sigma_1)-\omega_{2\mu}(\pi_1+\sigma_2)-\omega_{3\mu}(\sigma_3-\sigma_0-\sqrt{2}\,\sigma_8-\frac{m_N+m_\Lambda}{g})$$
$$+\omega_\mu{}^*(\pi_3-\pi_0+\sqrt{2}\,\pi_8)],$$
$$\Delta_\mu\pi_7=\partial_\mu\pi_7+G[-\omega_{1\mu}(\pi_4+\sigma_5)-\omega_{2\mu}(\pi_5-\sigma_4)+\omega_{3\mu}(\pi_6+\sigma_7)]$$
$$+\bar{G}[\omega_{1\mu}(\pi_1+\sigma_2)+\omega_{2\mu}(\pi_2-\sigma_1)+\omega_{3\mu}(\pi_3-\pi_0+\sqrt{2}\,\pi_8)$$
$$+\omega_\mu{}^*(\sigma_3-\sigma_0-\sqrt{2}\,\sigma_8-\frac{m_N+m_\Lambda}{g})],$$
$$\Delta_\mu\sigma_4=\partial_\mu\sigma_4+G[\omega_{1\mu}(\pi_6+\sigma_7)-\omega_{2\mu}(\pi_7-\sigma_6)+\omega_{3\mu}(\pi_4+\sigma_5)]$$
$$+\bar{G}[\omega_{1\mu}(\pi_3+\pi_0+\sqrt{2}\,\pi_8)-\omega_{2\mu}(\sigma_3+\sigma_0-\sqrt{2}\,\sigma_8-\frac{m_\Lambda-m_N}{g})$$
$$-\omega_{3\mu}(\pi_1-\sigma_2)-\omega_\mu{}^*(\pi_2+\sigma_1)],$$
$$\Delta_\mu\sigma_5=\partial_\mu\sigma_5+G[\omega_{1\mu}(\pi_7-\sigma_6)+\omega_{2\mu}(\pi_6+\sigma_7)+\omega_{3\mu}(\pi_5-\sigma_4)]$$
$$+\bar{G}[\omega_{1\mu}(\sigma_3+\sigma_0-\sqrt{2}\sigma_8-\frac{m_\Lambda-m_N}{g})+\omega_{2\mu}(\pi_3+\pi_0+\sqrt{2}\,\pi_8)$$
$$-\omega_{3\mu}(\pi_2+\sigma_1)+\omega_\mu{}^*(\pi_1-\sigma_2)],$$

$$\Delta_\mu\sigma_6=\partial_\mu\sigma_6+G[\omega_{1\mu}(\pi_4+\sigma_5)+\omega_{2\mu}(\pi_5-\sigma_4)-\omega_{3\mu}(\pi_6+\sigma_7)]$$
$$+\bar{G}[\omega_{1\mu}(\pi_1+\sigma_2)+\omega_{2\mu}(\pi_2-\sigma_1)+\omega_{3\mu}(\pi_3-\pi_0-\sqrt{2}\,\pi_8)$$
$$+\omega_\mu^*(\sigma_3-\sigma_0+\sqrt{2}\,\sigma_8+\frac{m_A-m_N}{g})],$$

$$\Delta_\mu\sigma_7=\partial_\mu\sigma_7+G[\omega_{1\mu}(\pi_5-\sigma_4)-\omega_{2\mu}(\pi_4+\sigma_5)-\omega_{3\mu}(\pi_7-\sigma_6)]$$
$$+\bar{G}[-\omega_{1\mu}(\pi_2-\sigma_1)+\omega_{2\mu}(\pi_1+\sigma_2)+\omega_{3\mu}(\sigma_3-\sigma_0+\sqrt{2}\sigma_8+\frac{m_A-m_N}{g})$$
$$-\omega_\mu^*(\pi_3-\pi_0-\sqrt{2}\pi_8)].$$

Appendix IV.

The vector and axial-vector currents and their divergence

The following abbreviation is used:

$$\Delta_\mu[A,B]=\Delta_\mu A\cdot B-A\cdot\Delta_\mu B.$$

Vector currents:

$$V_{1\mu}=\bar{\Psi}\gamma_\mu\lambda_1\Psi-2i[\Delta_\mu[\pi_2,\pi_3]+\Delta_\mu[\sigma_2,\sigma_3]]$$
$$-i[\Delta_\mu[\pi_4,\pi_7]-\Delta_\mu[\pi_5,\pi_6]+\Delta_\mu[\sigma_4,\sigma_7]-\Delta_\mu[\sigma_5,\sigma_6]],$$
$$V_{2\mu}=\bar{\Psi}\gamma_\mu\lambda_2\Psi-2i[\Delta_\mu[\pi_3,\pi_1]+\Delta_\mu[\sigma_3,\sigma_1]]$$
$$-i[\Delta_\mu[\pi_4,\pi_6]+\Delta_\mu[\pi_5,\pi_7]+\Delta_\mu[\sigma_4,\sigma_6]+\Delta_\mu[\sigma_5,\sigma_7]],$$
$$V_{3\mu}=\bar{\Psi}\gamma_\mu\lambda_3\Psi-2i[\Delta_\mu[\pi_1,\pi_2]+\Delta_\mu[\sigma_1,\sigma_2]]$$
$$-i[\Delta_\mu[\pi_4,\pi_5]-\Delta_\mu[\pi_6,\pi_7]+\Delta_\mu[\sigma_4,\sigma_5]-\Delta_\mu[\sigma_6,\sigma_7]],$$
$$V_{4\mu}=\bar{\Psi}\gamma_\mu\lambda_4\Psi+i[\Delta_\mu[\pi_1,\pi_7]+\Delta_\mu[\pi_2,\pi_6]+\Delta_\mu[\pi_3+\pi_0-\sqrt{2}\pi_8,\pi_4]$$
$$+\Delta_\mu[\sigma_1,\sigma_7]+\Delta_\mu[\sigma_2,\sigma_6]+\Delta_\mu[\sigma_3+\sigma_0-\sqrt{2}\sigma_8-\frac{m_A-m_N}{g},\sigma_5]]$$
$$+\frac{i\bar{G}}{g^2}(m_A-m_N)^2\omega_{1\mu},$$
$$V_{5\mu}=\bar{\Psi}\gamma_\mu\lambda_5\Psi+i[-\Delta_\mu[\pi_1,\pi_6]+\Delta_\mu[\pi_2,\pi_7]-\Delta_\mu[\pi_3+\pi_0-\sqrt{2}\pi_8,\pi_4]$$
$$-\Delta_\mu[\sigma_1,\sigma_6]+\Delta_\mu[\sigma_2,\sigma_7]-\Delta_\mu[\sigma_3+\sigma_0-\sqrt{2}\sigma_8-\frac{m_A-m_N}{g},\sigma_4]]$$
$$+\frac{i\bar{G}}{g^2}(m_A-m_N)^2\omega_{2\mu},$$
$$V_{6\mu}=-\bar{\Psi}\gamma_\mu\lambda_6\Psi+i[-\Delta_\mu[\pi_1,\pi_5]+\Delta_\mu[\pi_2,\pi_4]+\Delta_\mu[\pi_3-\pi_0+\sqrt{2}\pi_8,\pi_7]$$
$$-\Delta_\mu[\sigma_1,\sigma_5]+\Delta_\mu[\sigma_2,\sigma_4]+\Delta_\mu[\sigma_3-\sigma_0+\sqrt{2}\sigma_8+\frac{m_A-m_N}{g},\sigma_7]]$$
$$+\frac{i\bar{G}}{g^2}(m_A-m_N)^2\omega_{3\mu},$$

$$V_{7\mu}=\bar{\Psi}\gamma_\mu\lambda_7\Psi+i[-\Delta_\mu[\pi_1,\,\pi_4]-\Delta_\mu[\pi_2,\,\pi_5]+\Delta_\mu[\pi_3-\pi_0+\sqrt{2}\,\pi_8,\,\pi_6]$$
$$-\Delta_\mu[\sigma_1,\,\sigma_4]-\Delta_\mu[\sigma_2,\,\sigma_5]+\Delta_\mu[\sigma_3-\sigma_0+\sqrt{2}\,\sigma_8+\frac{m_A-m_N}{g},\,\sigma_6]]$$
$$+\frac{i\bar{G}}{g^2}(m_A-m_N)^2\omega_\mu^*.$$

Axial-vector currents:

$$P_{1\mu}=\bar{\Psi}\gamma_\mu\gamma_5\lambda_1\Psi-2i[\Delta_\mu[\sigma_1,\,\pi_0]+\Delta_\mu[\sigma_0+\frac{m_N}{g},\,\pi_1]]$$
$$-i[\Delta_\mu[\sigma_6,\,\pi_4]+\Delta_\mu[\sigma_7,\,\pi_5]+\Delta_\mu[\sigma_4,\,\pi_6]+\Delta_\mu[\sigma_5,\,\pi_7]]+\frac{4iG}{g^2}\,m_N{}^2\omega_{1\mu},$$

$$P_{2\mu}=\bar{\Psi}\gamma_\mu\gamma_5\lambda_2\Psi-2i[\Delta_\mu[\sigma_2,\,\pi_0]+\Delta_\mu[\sigma_0+\frac{m_N}{g},\,\pi_2]]$$
$$+i[\Delta_\mu[\sigma_7,\,\pi_4]-\Delta_\mu[\sigma_6,\,\pi_5]-\Delta_\mu[\sigma_5,\,\pi_6]+\Delta_\mu[\sigma_4,\,\pi_7]]+\frac{4iG}{g^2}\,m_N{}^2\omega_{2\mu},$$

$$P_{3\mu}=\bar{\Psi}\gamma_\mu\gamma_5\lambda_3\Psi-2i[\Delta_\mu[\sigma_3,\,\pi_0]+\Delta_\mu[\sigma_0+\frac{m_N}{g},\,\pi_3]]$$
$$-i[\Delta_\mu[\sigma_4,\,\pi_4]+\Delta_\mu[\sigma_5,\,\pi_5]-\Delta_4[\sigma_6,\,\pi_6]-\Delta_\mu[\sigma_7,\,\pi_7]+\frac{4iG}{g^2}\,m_N{}^2\omega_{3\mu},$$

$$P_{4\mu}=\bar{\Psi}\gamma_\mu\gamma_5\lambda_4\Psi-i[\Delta_\mu[\sigma_6,\,\pi_1]-\Delta_\mu[\sigma_7,\,\pi_2]+\Delta_\mu[\sigma_4,\,\pi_3+\pi_0+\sqrt{2}\,\pi_8]$$
$$+\Delta_\mu[\sigma_3+\sigma_0+\sqrt{2}\,\sigma_8+\frac{m_A+m_N}{g},\,\pi_4]+\Delta_\mu[\sigma_1,\,\pi_6]-\Delta_\mu[\sigma_2,\,\pi_7]]$$
$$+\frac{i\bar{G}}{g^2}(m_A+m_N)^2\omega_{1\mu},$$

$$P_{5\mu}=\bar{\Psi}\gamma_\mu\gamma_5\lambda_5\Psi-i[\Delta_\mu[\sigma_7,\,\pi_1]+\Delta_\mu[\sigma_6,\,\pi_2]+\Delta_\mu[\sigma_5,\,\pi_3+\pi_0+\sqrt{2}\,\pi_8]$$
$$+\Delta_\mu[\sigma_3+\sigma_0+\sqrt{2}\,\sigma_8+\frac{m_A+m_N}{g},\,\pi_5]+\Delta_\mu[\sigma_2,\,\pi_6]+\Delta_\mu[\sigma_1,\,\pi_7]]$$
$$+\frac{i\bar{G}}{g^2}(m_A+m_N)^2\omega_{2\mu},$$

$$P_{6\mu}=-\bar{\Psi}\gamma_\mu\gamma_5\lambda_6\Psi+i[\Delta_\mu[\sigma_4,\,\pi_1]+\Delta_\mu[\sigma_5,\,\pi_2]-\Delta_\mu[\sigma_6,\,\pi_3-\pi_0-\sqrt{2}\,\pi_8]$$
$$-\Delta_\mu[\sigma_3-\sigma_0-\sqrt{2}\,\sigma_8-\frac{m_A+m_N}{g},\,\pi_6]+\Delta_\mu[\sigma_1,\,\pi_4]+\Delta_\mu[\sigma_2,\,\pi_5]],$$
$$+\frac{i\bar{G}}{g^2}(m_A+m_N)^2\omega_{3\mu}$$

$$P_{7\mu}=\bar{\Psi}\gamma_\mu\gamma_5\lambda_7\Psi-i[\Delta_\mu[\sigma_5,\pi_1]-\Delta_\mu[\sigma_4,\pi_2]-\Delta_\mu[\sigma_7,\pi_3-\pi_0-\sqrt{2}\,\pi_8]$$
$$-\Delta_\mu[\sigma_3-\sigma_0-\sqrt{2}\,\sigma_8-\frac{m_\Lambda+m_N}{g},\pi_7]-\Delta_\mu[\sigma_2,\pi_4]+\Delta_\mu[\sigma_1,\pi_5]]$$
$$+\frac{i\bar{G}}{g^2}(m_\Lambda+m_N)^2\omega_\mu^*.$$

The divergence of the vector and axial-vector currents in limit of $G=0$ takes the following:

$$\partial_\mu V_{1\mu}=\partial_\mu V_{2\mu}=\partial_\mu V_{3\mu}=0,$$

$$\partial_\mu V_{4\mu}=\frac{i(m_\Lambda-m_N)}{g}\mu_L^2\sigma_5+i(\mu_\sigma^2-\mu_L^2)(\sigma_1\sigma_7+\sigma_2\sigma_6+\sigma_3\sigma_5)$$
$$+i(\mu_\pi^2-\mu_K^2)(\pi_1\pi_7+\pi_2\pi_6+\pi_3\pi_5)+i(\bar{\mu}_0^2-\mu_L^2)\sigma_0\sigma_5$$
$$+i(\mu_0^2-\mu_K^2)\pi_0\pi_5-i\sqrt{2}\,(\bar{\mu}_8^2-\mu_L^2)\sigma_8\sigma_5-i\sqrt{2}\,(\mu_8^2-\mu_K^2)\pi_8\pi_5,$$

$\partial_\mu V_{5\mu}$

$\partial_\mu V_{6\mu}$ similar,

$\partial_\mu V_{7\mu}$

$$\partial_\mu P_{1\mu}=\frac{2im_N}{g}\mu_\pi^2\pi_1-2i(\mu_\sigma^2-\mu_0^2)\sigma_1\pi_0+2i(\mu_\pi^2-\bar{\mu}_0^2)\sigma_0\pi_1$$
$$+i(\mu_K^2-\mu_L^2)(\sigma_6\pi_4+\sigma_7\pi_5+\sigma_8\pi_6+\sigma_5\pi_7),$$

$\partial_\mu P_{2\mu}$
similar,
$\partial_\mu P_{3\mu}$

$$\partial_\mu P_{4\mu}=\frac{i(m_\Lambda+m_N)}{g}\mu_K^2\pi_4-i(\mu_\sigma^2-\mu_K^2)(\sigma_3\pi_4+\sigma_1\pi_6-\sigma_2\pi_7)$$
$$+i(\mu_\pi^2-\mu_L^2)(\sigma_6\pi_1-\sigma_7\pi_2+\sigma_4\pi_3)+i(\bar{\mu}_0^2-\mu_K^2)\sigma_0\pi_4$$
$$-i(\mu_0^2-\mu_L^2)\sigma_4\pi_0+i\sqrt{2}\,(\bar{\mu}_8^2-\mu_K^2)\sigma_8\pi_4-i\sqrt{2}\,(\mu_8^2-\mu_L^2)\sigma_4\pi_8,$$

$\partial_\mu P_{5\mu}$

$\partial_\mu P_{6\mu}$ similar.

$\partial_\mu P_{7\mu}$

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Nuclear Radiation from Solids

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An operator identity concerning the Hamiltonian of a harmonic oscillator enables one to calculate the distribution of the energy of nuclear radiation from solids, which includes the effect of multiple phonon processes exactly. The moments of the recoil energy or of the energy of radiation are obtained to any order in terms of the moments of the frequency of lattice vibration. These relations suggest a method of experimental determination of the frequency distribution of lattice vibration.

§ 1. Introduction

When a particle or a gamma ray is emitted from a nucleus, the residual nucleus recoils to balance the momentum carried away by the radiation. If, for example, a nucleus is initially at rest, the energy of radiation emitted from it is reduced, in comparison to that when the nucleus is fixed, by an amount of $k_0^2/2M$, where k_0 is the magnitude of momentum of the radiation from fixed nucleus and M is the mass of the nucleus.*

On the other hand, in order that a radiation is absorbed by a free nucleus initially at rest, the energy of the radiation must be larger by $k_0^2/2M$ than that necessary for the radiation to be absorbed by fixed nucleus.

In the case when nuclei are bound in a crystal lattice, the energy of radiation they emit depends on the state of motion in which the emitter finds itself at the instant of emission, and the observed distribution of the energy is the average of individual ones with respect to the thermal motion of nuclei inside the crystal. For example, the position of the peak and the width of the energy distribution of nuclear gamma ray from solid are lower and wider, respectively, than those of the gamma rays from fixed nuclei of the same kind.[1]

At low temperatures, however, there exists under certain conditions a considerable amount of probability for the crystal as a whole to accept the recoil of nuclear radiation without inducing any internal excitation of the lattice. In this case, because of the large mass of the crystal, the effect of recoil is negligible and the distribution of the radiation energy is practically the same as that from fixed nuclei.

The existence of such recoilless gamma rays, which was discovered by Mössbauer,[2] seems to provide us with a variety of its applications both in

* More exactly M is the reduced mass of the particle emitted and the residual nucleus.

nuclear and solid state physics. In particular, a method has been developed by Mössbauer[2] and others[3] to determine the mean lives of long-lived isomers, and the internal electromagnetic fields of solids can be measured from fine structures of recoilless gamma rays.[4] Also a possibility has been proposed to examine the validity of the general theory of relativity.[5]

A theory of resonant absorption or emission of neutrons by nuclei bound in a crystal lattice was given by Lamb,[6] and his results were modified by Mössbauer[2] to the case of gamma rays for the analysis of his experiments. The effect of the lattice vibrations on the distribution of gamma-ray energy was also considered by Visscher[7] and Lipkin.[8]

Visscher used the Debye model for the lattice vibration, and gave the results of numerical calculation for the energy distribution of the 129 kev gamma rays from Ir^{191} in a crystal of natural iridium; while Lipkin made a general consideration which is independent of the specific model of lattice vibration, and obtained the average recoil energy of the lattice and the probability for the recoilless emission of gamma rays.

Although the experiments so far performed deal with the recoilless gamma rays, it might be of interest to investigate those with recoil, because in this case the distribution of the gamma-ray energy is closely related to the thermal motion of the emitting nucleus.

In the present paper an attempt is made to find explicit relations between the distribution of the radiation energy and the density of the frequency of lattice vibration. In § 2 a generating function is obtained, which yields the moments of recoil energy to any order in terms of the moments of frequency of lattice vibration. The inclusion of the multiple phonon processes is made automatically by virtue of an operator identity, the proof of which is given in the Appendix. In § 3 the distribution of the energy of radiation, which is exact in the above sense, is obtained and is shown to be identical with the approximate one obtained by Lamb.[6] The moments of the energy of radiation with respect to its distribution are then obtained. § 4 is devoted to the evaluation of the nuclear level width for nuclei bound in a crystal lattice, and the conditions are obtained under which the width can be approximated by the level width when the nucleus is fixed. Some concluding remarks are given in § 5.

Throughout our calculation a crystal lattice is regarded as an assembly of harmonic oscillators, and the effects caused by the anharmonicity of lattice vibration are not considered. By nuclear radiation we mean gamma rays or particles like neutrons, as they are treated in equal terms, unless otherwise stated.

§ 2. Moments of recoil energy

The matrix element for the transition in which a nucleus bound in a crystal lattice emits a particle or a gamma ray is[8]

$$\mathcal{H}_s^*(\boldsymbol{k})\cdot\langle\{m_\alpha\}|\exp(-i\boldsymbol{k}\cdot\boldsymbol{Q})|\{n_\alpha\}\rangle, \tag{1}$$

where the first factor $\mathcal{H}_s^*(\boldsymbol{k})$ is the nuclear matrix element for the emission of the radiation with momentum $\boldsymbol{k}$ and spin projection s, and the second is the matrix element for the lattice which changes its state from $|\{n_\alpha\}\rangle$ to $|\{m_\alpha\}\rangle$ due to the recoil by the radiation. $\boldsymbol{Q}$ is the displacement of the nucleus from its equilibrium position $\boldsymbol{Q}_0$, and can be decomposed into normal coordinates as[9]

$$\boldsymbol{Q}=\sum_\alpha \frac{\boldsymbol{\epsilon}_\alpha}{(NM\omega_\alpha)^{1/2}}\{q_\alpha\cos(\boldsymbol{\kappa}_\alpha\cdot\boldsymbol{Q}_0)+p_\alpha\sin(\boldsymbol{\kappa}_\alpha\cdot\boldsymbol{Q}_0)\}, \tag{2}$$

where p's and q's satisfy the usual commutation relations,*

$$[p_\alpha, q_\beta]=p_\alpha q_\beta-q_\beta p_\alpha=-i\delta_{\alpha\beta},\quad [p_\alpha, p_\beta]=[q_\alpha, q_\beta]=0.$$

In Eq. (2) N is the number of atoms in the crystal, M the mass of the nucleus under consideration, and $\boldsymbol{\epsilon}_\alpha$, $\boldsymbol{\kappa}_\alpha$, and ω_α are, respectively, the polarization, momentum, and frequency of the α-th normal vibration (phonon).

The wave function $|\{n_\alpha\}\rangle$ of the lattice vibration is an eigenfunction of the lattice Hamiltonian H_L with the eigenvalue $E(\{n_\alpha\})$, where

$$H_L=\sum_\alpha H_\alpha \qquad \text{with } H_\alpha=\frac{\omega_\alpha}{2}(p_\alpha^2+q_\alpha^2), \tag{3}$$

and

$$E(\{n_\alpha\})=\sum_\alpha E_{n_\alpha} \quad \text{with} \quad E_{n_\alpha}=\left(n_\alpha+\frac{1}{2}\right)\omega_\alpha. \tag{4}$$

Thus

$$|\{n_\alpha\}\rangle=\prod_\alpha |n_\alpha\rangle \qquad \text{and} \qquad H_\alpha|n_\alpha\rangle=E_{n_\alpha}|n\rangle. \tag{5}$$

The normalized probability of the transition of the lattice is given by[8]

$$P(\{m_\alpha\},\ \{n_\alpha\})=|\langle\{m_\alpha\}|\exp(-i\boldsymbol{k}\cdot\boldsymbol{Q})|\{n_\alpha\}\rangle|^2. \tag{6}$$

Actually, the nuclear matrix element also depends upon the lattice states through $\boldsymbol{k}$ because of the conservation of energy. However, we shall tentatively disregard this dependence in this section, and concern ourselves only with the lattice matrix element, since in the following section the distribution and moments of the energy of emitted radiation will be discussed.

From Eqs. (2), (5) and (6), with the origin of the coordinate system taken at the equilibrium position of the nucleus, $P(\{m_\alpha\},\ \{n_\alpha\})$ can be rewritten as

* In the following calculations the natural unit system in which $c=\hbar=1$ is used, and the Boltzmann constant is also set equal to unity so that temperatures are measured in the unit of energy.

$$P(\{m_\alpha\}, \{n_\alpha\}) = \prod_\alpha P_{m_\alpha n_\alpha}, \tag{7}$$

where $P_{m_\alpha n_\alpha}$ is the normalized transition probability for the α-th normal mode, and is given by

$$P_{m_\alpha n_\alpha} = |\langle m_\alpha | \exp(-i x_\alpha q_\alpha) | n_\alpha \rangle|^2, \tag{8}$$

with

$$x_\alpha = \frac{(\boldsymbol{\epsilon}_\alpha \cdot \boldsymbol{k}_0)}{(NM\omega_\alpha)^{1/2}}. \tag{9}$$

The average value of the recoil energy of the lattice is

$$\sum_{\{m_\alpha\}} \sum_{\{n_\alpha\}} \{E(\{m_\alpha\}) - E(\{n_\alpha\})\} P(\{m_\alpha\}, \{n_\alpha\}) W(\{n_\alpha\}), \tag{10}$$

where $W(\{n_\alpha\})$ is the Boltzmann factor for the initial state at temperature T, namely,

$$W(\{n_\alpha\}) = \prod_\alpha W_{n_\alpha}(T), \tag{11}$$

with

$$W_{n_\alpha}(T) = \exp(-E_{n_\alpha}/T) / \sum_{n_\alpha} \exp(-E_{n_\alpha}/T). \tag{12}$$

We generalize Eq. (10) to define the n-th order moment of recoil energy as the average value of its n-th power,

$$J_n(T) = \sum_{\{m_\alpha\}} \sum_{\{n_\alpha\}} \{E(\{m_\alpha\}) - E(\{n_\alpha\})\}^n P(\{m_\alpha\}, \{n_\alpha\}) W(\{n_\alpha\}). \tag{13}$$

Instead of calculating Eq. (13) directly, we introduce a generating function for the moments as

$$J(\lambda, T) = \sum_{\{m_\alpha\}} \sum_{\{n_\alpha\}} \exp[\lambda\{E(\{m_\alpha\}) - E(\{n_\alpha\})\}] \times P(\{m_\alpha\}, \{n_\alpha\}) W(\{n_\alpha\}), \tag{14}$$

so that the n-th order moment $J_n(T)$ is given by

$$J_n(T) = \frac{\partial^n J(\lambda, T)}{\partial \lambda^n}\bigg|_{\lambda=0}. \tag{15}$$

From Eqs. (4), (7) and (11), Eq. (14) can be expressed as

$$J(\lambda, T) = \prod_\alpha J_\alpha(\lambda, T), \tag{16}$$

where

$$J_\alpha(\lambda, T) = \sum_{m_\alpha} \sum_{n_\alpha} \exp\{\lambda(E_{m_\alpha} - E_{n_\alpha})\} P_{m_\alpha n_\alpha} W_{n_\alpha}(T). \tag{17}$$

The generating function $J_\alpha(\lambda, T)$ for a single harmonic oscillator can be obtained in the following way.

By the use of Eqs. (5), (8) and (12), Eq. (17) becomes

$$J_\alpha(\lambda, T)=tr[\exp(ix_\alpha q_\alpha)\exp(\lambda H_\alpha)\exp(-ix_\alpha q_\alpha)\exp(-(\lambda+1/T)H_\alpha)] /tr[\exp(-(1/T)H_\alpha)], \tag{18}$$

where tr means the sum of diagonal elements of the matrix. Since $\exp(ix_\alpha q_\alpha)\times\exp(\lambda H_\alpha)\exp(-ix_\alpha q_\alpha)$ can be regarded as a unitary transform of $\exp(\lambda H_\alpha)$, the expression (18) can be written as

$$J_\alpha(\lambda, T)=\frac{tr[\exp(\lambda\omega_\alpha H(-x_\alpha, 0))\cdot\exp(-(\lambda\omega_\alpha+\omega_\alpha/T)H(0, 0))]}{tr[\exp(-(\omega_\alpha/T)H(0, 0))]}, \tag{19}$$

where $H(\xi, \eta)$ is, as defined in the Appendix (A·1),

$$H(\xi, \eta)=(1/2)\{(p+\xi)^2+(q+\eta)^2\}.$$

For the reduction of Eq. (19) we make use of the identity (A·2) through (A·8), which is proved in the Appendix, and note that the eigenvalue sequence of $H(\xi, \eta)$ is the same as that of $H(0, 0)$. We thus obtain

$$J_\alpha(\lambda, T)=\exp\left[-\frac{x_\alpha^2}{2}\left\{\coth\left(\frac{\omega_\alpha}{2T}\right)-\frac{\cosh(\omega_\alpha/2T+\lambda\omega_\alpha)}{\sinh(\omega_\alpha/2T)}\right\}\right]. \tag{20}$$

From Eqs. (9), (16), and (20), it is seen that

$$J(\lambda, T)=\exp\left[-\sum_\alpha\frac{(\boldsymbol{\epsilon}_\alpha\cdot\boldsymbol{k}_0)^2}{2NM\omega_\alpha}\coth\left(\frac{\omega_\alpha}{2T}\right) +\sum_\alpha\frac{(\boldsymbol{\epsilon}_\alpha\cdot\boldsymbol{k}_0)^2}{2NM\omega_\alpha\sinh(\omega_\alpha/2T)}\cosh\left(\frac{\omega_\alpha}{2T}+\lambda\omega_\alpha\right)\right]. \tag{21}$$

In performing the summation with respect to α over the $3N$ different modes of lattice vibration, we define the density of frequency of the lattice vibration, $\rho_i(\omega)$ $(i=1, 2, 3)$, where $N\rho_i(\omega)d\omega$ is the number of normal modes with polarization ϵ_i, and frequency between ω and $\omega+d\omega$, so that $\rho_i(\omega)$ is normalized in the sense that

$$\int_0^\infty\rho_i(\omega)d\omega=1.$$

Eq. (21) can then be written as

$$J(\lambda, T)=\exp\left[-\sum_{i=1}^3\frac{(\boldsymbol{\epsilon}_i\boldsymbol{k}_0)^2}{2M}\int_0^\infty\frac{\rho_i(\omega)d\omega}{\omega\tanh(\omega/2T)} +\sum_{i=1}^3\frac{(\boldsymbol{\epsilon}_i\cdot\boldsymbol{k}_0)^2}{2M}\int_0^\infty\frac{\rho_i(\omega)d\omega}{\omega\sinh(\omega/2T)}\cosh\left(\frac{\omega}{2T}+\lambda\omega\right)\right], \tag{22}$$

where the summation is now taken over the three different directions of polarization.

Having obtained the generating function $J(\lambda, T)$, we can now calculate by Eq. (15) the moments of the recoil energy to any order. For example, the first three moments are

$$J_0(T)=1, \quad J_1(T)=k_0^2/2M \tag{23}$$

and

$$J_2(T)=\sum_{i=1}^{3}\frac{(\boldsymbol{\epsilon}_i\cdot\boldsymbol{k}_0)^2}{2M}\int_0^\infty d\omega\,\rho_i(\omega)\,\omega\coth\left(\frac{\omega}{2T}\right)+\left(\frac{k_0^2}{2M}\right)^2. \tag{24}$$

The first of Eq. (23) shows that the probability is properly normalized, while the second one, which was obtained also by Lipkin,[8] expresses that the average value of the recoil energy is the same as that of the free nucleus initially at rest. Although these two moments are independent of the temperature and the frequency distribution of lattice vibration, the second and the higher order moments, which determine the shape of the distribution of the recoil energy, depend upon the temperature, the frequency density, and the direction of the radiation.

In order to see the relations between the moments of the recoil energy and the density of the frequency of lattice vibration, we define $F_m(T)$ by

$$F_m(T)=\sum_{i=1}^{3}\frac{(\boldsymbol{\epsilon}_i\cdot\boldsymbol{k}_0)^2}{2M}\int_0^\infty d\omega\,\rho_i(\omega)\,\omega^m\begin{cases}1 & \text{for even } m\\ \coth\left(\dfrac{\omega}{2T}\right) & \text{for odd } m.\end{cases} \tag{25}$$

Then the $(n+1)$-th order moment of the recoil energy is, from Eqs. (15), (22), and (25),

$$J_{n+1}(T)=\sum_{m=0}^{n}\frac{n!}{m!\,(n-m)!}F_m(T)\,J_{n-m}(T). \tag{26}$$

If $F_m(T)$ is known, for example, by assuming the Debye model for the lattice vibration, then $J_n(T)$ is obtained by Eq. (26). Conversely, if it is possible to determine $J_n(T)$ experimentally, then $F_m(T)$ is obtained by solving Eq. (26), or by

$$F_m(T)=\frac{\partial^{m+1}}{\partial\lambda^{m+1}}\log J(\lambda, T)\Big|_{\lambda=0}.$$

$F_m(T)$ with even m is of special interest, since it does not depend on temperature, and is proportional to the m-th order moment of the lattice frequency with respect to its density. It has been shown by Montroll[10] that such moments of even power of the frequency of lattice vibration can be obtained theoretically from the dynamics of lattice vibration, without an explicit knowledge of the density of frequency.

For practical purposes it is more convenient to obtain the moments of the recoil energy around its average value. If we define a_n by

$$a_n=\sum_{\{m_\alpha\}}\sum_{\{n_\alpha\}}\left[[E(\{m_\alpha\})-E(\{n_\alpha\})-\frac{k_0^2}{2M}\right]^n P(\{m_\alpha\},\ \{(n_\alpha\})\,W(\{n_\alpha\}),$$

then the first several moments are, from Eq. (26),

$$a_0=1,\ a_1=0,\ a_2=F_1(T),\ a_3=F_2(T),\ a_4=3[F_1(T)]^2+F_3(T),$$
$$a_5=10F_1(T)F_2(T)+F_4(T),\ a_6=15[F_1(T)]^3+15F_1(T)F_3(T)+10[F_2(T)]^2+F_5(T).$$

Under the assumption of the Debye model of lattice vibration, namely,

$$\rho_i(\omega)=\frac{3\omega^2}{\theta^3} \qquad \omega\leq\theta$$
$$=0 \qquad \omega>\theta,$$

where θ is the Debye temperature of the crystal, the above moments are, for $T\ll\theta$,

$$a_2=\frac{3}{4}\theta\left(\frac{k_0^2}{2M}\right), \qquad a_3=\frac{3}{5}\theta^2\left(\frac{k_0^2}{2M}\right),$$
$$a_4=\frac{27}{16}\theta^2\left(\frac{k_0^2}{2M}\right)+\frac{1}{2}\theta^3\left(\frac{k_0^2}{2M}\right), \qquad a_5=\frac{9}{2}\theta^3\left(\frac{k_0^2}{2M}\right)^2+\frac{3}{7}\theta^4\left(\frac{k_0^2}{2M}\right),$$
$$a_6=\frac{305}{64}\theta^3\left(\frac{k_0^2}{2M}\right)^3+\frac{69}{40}\theta^4\left(\frac{k_0^2}{2M}\right)^2+\frac{3}{8}\theta^5\left(\frac{k_0^2}{2M}\right),$$

or for $T\gg\theta$,

$$a_2=2T\left(\frac{k_0^2}{2M}\right), \qquad a_3=\frac{3}{5}\theta^2\left(\frac{k_0^2}{2M}\right)$$
$$a_4=12T^2\left(\frac{k_0^2}{2M}\right)^2+\frac{6}{5}T\theta^2\left(\frac{k_0^2}{2M}\right), \qquad a_5=12T\theta^2\left(\frac{k_0^2}{2M}\right)+\frac{3}{7}\theta^4\left(\frac{k_0^2}{2M}\right),$$
$$a_6=120T^3\left(\frac{k_0^2}{2M}\right)^3+36T^2\theta^2\left(\frac{k_0^2}{2M}\right)^2+\frac{6}{7}T\theta^4\left(\frac{k_0^2}{2M}\right).$$

The first terms of the moments of even order correspond to those due to the Gaussian distribution, whereas the remaining terms to the deviation from the Gaussian. We see that the latter is large unless $k_0^2/2M$ is much larger than θ, or θ^2/T, respectively.

In the case of $T\gg\theta$ the results of Montroll's methods can be used instead of the Debye model, and the result is, for a simple cubic lattice,

$$a_2=2T\left(\frac{k_0^2}{2M}\right), \qquad a_3=2(\alpha+4\beta)\left(\frac{k_0^2}{2M}\right)$$
$$a_4=12T^2\left(\frac{k_0^2}{2M}\right)^2+4T(\alpha+4\beta)\frac{k_0^2}{2M},$$

$$a_5 = 40T(\alpha+4\beta)\left(\frac{k_0^2}{2M}\right)^2 + 2T(3\alpha^2+16\alpha\beta+40\beta^2)\left(\frac{k_0^2}{2M}\right)$$

$$a_6 = 120T^3\left(\frac{k_0^2}{2M}\right)^3 + 120T^2(\alpha+4\gamma)\left(\frac{k_0^2}{2M}\right)^2 + 4T(3\alpha^2+16\alpha\beta+40\beta^2)\left(\frac{k_0^2}{2M}\right),$$

where α and β are, respectively, force constants of the interaction between the nearest and the second nearest neighbors.

§ 3. Distribution of energy of radiation

The distribution of energy E of the emitted particle or gamma ray is given by the Breit-Wigner or Weisskopf-Wigner formula summed and averaged over the final and initial states of the lattice:

$$w_s(E\,;\,\boldsymbol{n}) = \sum_{\{m_\alpha\}}\sum_{\{n_\alpha\}} \frac{|\mathcal{H}_s(\boldsymbol{k})|^2 P(\{m_\alpha\},\{n_\alpha\})\,W(\{n_\alpha\})}{\{E-E_0+E(\{m_\alpha\})-E(\{n_\alpha\})\}^2+(1/4)\gamma(\{n_\alpha\})^2}, \quad (27)$$

where E_0 is the energy of radiation when the nucleus is fixed, $\boldsymbol{n}$ is a unit vector in the direction of $\boldsymbol{k}$, and $\gamma(\{n_\alpha\})$ is the width of the initial state which is given by

$$\gamma(\{n_\alpha\}) = 2\pi\sum_{s\cdot\boldsymbol{k}}\sum_{\{m_\alpha\}}|\mathcal{H}_s(\boldsymbol{k})|^2 P(\{m_\alpha\},\{n_\alpha\})\,\delta(E-E_0+E(\{m_\alpha\})-E(\{n_\alpha\})). \quad (28)$$

Following Lamb[6] we use the following formula,

$$\mathcal{R}e\int_0^\infty \exp[i(a+ib)t] = \frac{b}{a^2+b^2} \quad (b>0),$$

and rewrite Eq. (27) as

$$w_s(E\,;\,\boldsymbol{n}) = \frac{2}{\bar{\gamma}}|\mathcal{H}_s(\boldsymbol{k})|^2\int_0^\infty dt\,\exp\left[i\left(E-E_0+\frac{i}{2}\bar{\gamma}\right)t\right]$$

$$\times\sum_{\{m_\alpha\}}\sum_{\{n_\alpha\}}\exp[i\{E(\{m_\alpha\})-E(\{n_\alpha\})\}t]P(\{m_\alpha\},\{n_\alpha\})\,W(\{n_\alpha\}), \quad (29)$$

where $\gamma(\{n_\alpha\})$ has been replaced by its average value $\bar{\gamma}$. With the help of the definition of $J(\lambda, T)$ by Eq. (14), the above expression becomes

$$w_s(E\,;\,\boldsymbol{n}) = \frac{2}{\bar{\gamma}}|\mathcal{H}_s(\boldsymbol{k})|^2\,\mathcal{R}e\int_0^\infty dt\,\exp\left[i\left((E-E_0+\frac{i}{2}\bar{\gamma}\right)t\right]J(it,T), \quad (30)$$

or using the explicit expression of $J(\lambda, T)$ given by Eq. (22), we find

$$w_s(E\,;\,\boldsymbol{n}) = \frac{2}{\bar{\gamma}}|\mathcal{H}_s(\boldsymbol{k})|^2\exp\left[-\frac{k^2}{2M}\sum_{i=1}^{3}(\epsilon_i\boldsymbol{n})^2\int_0^\infty\frac{\rho_i(\omega)\,d\omega}{\omega\tanh(\omega/2T)}\right]\times$$

$$\times \mathcal{R}e\int_0^\infty dt\,\exp\left[i\left(E-E_0+\frac{i}{2}\bar{\gamma}\right)t+\frac{k^2}{2M}\sum_{i=1}^{3}{}'(\epsilon_i n)^2\int_0^\infty \frac{\rho_i(\omega)\,d\omega}{\omega}\frac{\cosh\left(\frac{\omega}{2T}+i\omega t\right)}{\sinh(\omega/2T)}\right]. \tag{31}$$

The above result is exact except that $\gamma(\{n_\alpha\})$ has been approximated by $\bar{\gamma}$ in order to carry out the summation in Eq. (27). Especially, the effect of the multiple phonon processes is included without any approximation. Eq. (31) is seen to be identical with the result obtained by Lamb,[6] where the multiple phonon processes are taken into account only approximately. Singwi and Sjölander[11] also obtained an expression similar to Eq. (31), by using the method of a space-time self-correlation function developed by Van Hove,[12] which is often used in the theory of neutron diffraction. In fact our formula coincides with their result in the case of a cubic Bravais lattice, to which their consideration was restricted.

Although they had to use a round-about method of introducing the physically complicated notion of a space-time self-correlation function, in our formulation Eq. (31) is obtained directly from Eq. (27) once the intuitive identity (A·1) through (A·8) is established. Furthermore, in Eq. (30) the distribution of the energy of radiation is expressed in a simple way in terms of the generating function of the moments of recoil energy.

Unfortunately Eq. (31) is so complicated that the integrals involved can be performed only numerically under the assumption of a simple model for the lattice vibration, or semi-quantitative features of the distribution can be obtained for a few extreme cases of the temperature. In this situation it seems worthwhile to investigate not the shape of the distribution itself but its moments.

For this purpose let us now expand $J(it, T)$ in Eq. (30) with respect to t, noting the relation (15),

$$J(it, T)=\sum_{l=0}^{\infty}\frac{(it)^l}{l!}J_l(T).$$

Then the integration over t can be performed easily and the result is

$$w_s(E\,;\boldsymbol{n})=W_s(E\,;\boldsymbol{n})+W_s^*(E\,;\boldsymbol{n}), \tag{32}$$

where

$$W_s(E\,;\boldsymbol{n})=\frac{-i}{\bar{\gamma}}\,|\mathcal{K}_s(\boldsymbol{k})|^2\sum_{l=0}^{\infty}(-1)^l J_l(T)\,(E-E_0-\frac{i}{2}\bar{\gamma})^{-(l+1)}. \tag{33}$$

The n-th order moment of the energy of radiation is defined by

$$(E-E_0)^n=\sum_s{}'\int_0^\infty (E-E_0)^n w_s(E\,;\boldsymbol{n})\,f(E)\,dE\Big/\sum_s{}'\int_0^\infty w_s(E\,;\boldsymbol{n})\,f(E)\,dE, \tag{34}$$

where $\sim$ means the directional average over a small solid angle around the vector

$\boldsymbol{n}$, and $f(E)$ is the density of states of the radiation, namely $f(E)=const\times E^{1/2}$ in the case of particle emission, or $f(E)=const\times E^2$ for gamma-ray emisson.

The integrands in Eq. (34) are originally defined for the real positive value of E. We shall, however, extend their definition over the whole complex E-plane by analytic continuation. The integrands have then singularities at $E_0\pm i/2\cdot\bar{\gamma}$ and possibly at infinity. In the case of particle emission $E=0$ is a branch point of $f(E)$, whereas there is no such branch point in the case of gamma-ray emission.

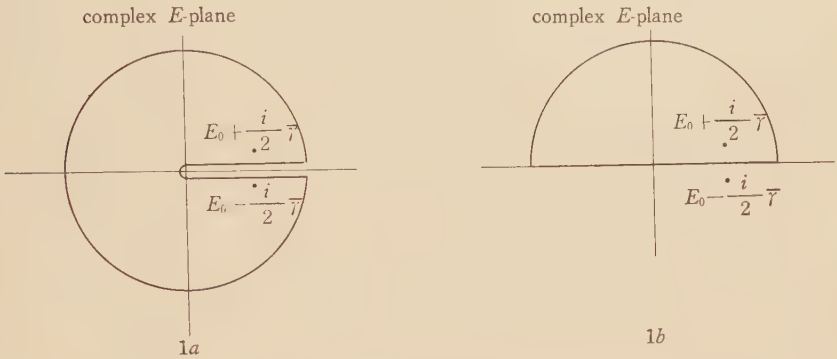

Fig. 1. The contour C of the integral in Eq. (35) in the case of: 1*a*. particle emission, and 1*b*. gamma-ray emission.

By residue formation we find

$$\int\left(E-E_0-\frac{i}{2}\bar{\gamma}\right)^n \tilde{W}_s(E;\boldsymbol{n})\,f(E)\,dE=\frac{2\pi}{\bar{\gamma}}\,|\tilde{\mathcal{K}}_s(\boldsymbol{k})|^2 f(E)\,(-1)^n J_n(T),\quad {\scriptstyle |E=E_0+(i/2)\bar{\gamma}} \tag{35}$$

where the closed contour C is chosen as shown in Fig. 1*a* in the case of particle emission, or as shown in Fig. 1*b* in the case of gamma-ray emission. If $|\tilde{\mathcal{K}}_s(\boldsymbol{k})|^2 f(E)$ as a function of E approaches zero faster than any negative power of E as E increases, then the contribution to the left-hand side of Eq. (35) from the integral along the large circle becomes negligible as its radius is increased, and the remaining integral can be taken along the real axis. After this procedure $\bar{\gamma}$ can be neglected if it is sufficiently smaller than E_0.

In the case of particle emission we thus obtain

$$\int_0^\infty (E-E_0)^n \tilde{W}_s(E;\boldsymbol{n})\,f(E)\,dE=\frac{\pi}{\bar{\gamma}}\,|\tilde{\mathcal{K}}_s(\boldsymbol{k}_0)|^2 f(E_0)\,(-1)^n J_n(T),$$

or by Eq. (32),

$$\int_0^\infty (E-E_0)^n \tilde{w}_s(E;\boldsymbol{n})\,f(E)\,dE=\frac{2\pi}{\bar{\gamma}}\,|\tilde{\mathcal{K}}_s(\boldsymbol{k}_0)|^2 f(E_0)\,(-1)^n J_n(T). \tag{36}$$

In this case the n-th order moment of the energy of the particle as defined by Eq. (34) is given by

$$\overline{(E-E_0)^n}=(-1)^n J_n(T). \tag{37}$$

In the case of gamma-ray emission we obtain, instead of Eq. (36),

$$\int_{-\infty}^{+\infty}(E-E_0)^n \bar{w}_s(E\,;\boldsymbol{n})\, f(E)\, dE=\frac{2\pi}{\bar{\gamma}}|\mathcal{H}_s(\boldsymbol{k}_0)|^2 f(E_0)\ (-1)^n J_n(T),$$

and we have to assume further that the contribution of the integral from the negative value of E is small enough to be neglected, in order that Eq. (37) is valid also in this case.

We have shown that, under certain assumptions as to the behavior of the nuclear matrix element as a function of energy, the moments of the energy of radiation is given by Eq. (37). Since these moments are measurable, and $J_n(T)$ is related to the moments of the frequency of lattice vibration, it turns out that the latter can be obtained experimentally, once the distribution of the energy of nuclear radiation from the crystal is measured.

§ 4. Evaluation of width

In this section we consider the width $\bar{\gamma}$ which appeared in the distribution function of the radiation energy Eq. (31). Since the width $\gamma(\{n_\alpha\})$ is given by Eq. (28), its statistical average can be easily obtained by the method similar to that used in the preceding section. Thus

$$\begin{aligned}\bar{\gamma}&=\sum_{\{n_\alpha\}}\gamma(\{n_\alpha\})\, W(\{n_\alpha\})\\ &=\sum_{s,\boldsymbol{k}}|\mathcal{H}_s(\boldsymbol{k})|^2\int_{-\infty}^{+\infty}dt\,\exp[i(E-E_0)t]J(it,T).\end{aligned} \tag{38}$$

In order to know the exact dependence of $\bar{\gamma}$ on temperature, the integral must be carried out numerically. The rough estimate of $\bar{\gamma}$, however, is obtained by expanding $J(it,T)$ into the power series of t. Thus we obtain

$$\bar{\gamma}=\sum_{n=0}^{\infty}\frac{1}{n!}\gamma_n(E_0)J_n(T), \tag{39}$$

where

$$\gamma_0(E_0)=2\pi\sum_{s,\boldsymbol{k}}|\mathcal{H}_s(\boldsymbol{k})|^2\delta(E-E_0) \tag{40}$$

is the level width when the nucleus is fixed, and

$$\gamma_n(E_0)=\frac{d^n\gamma_0(E)}{dE^n}\Big|_{E=E_0}. \tag{41}$$

If $|\mathcal{H}_s(\boldsymbol{k})|^2$ varies very slowly with energy E in the neighborhood of E_0, we can retain only the first few terms of the expansion (39), and we obtain

$$\bar{\gamma}=\gamma_0+\frac{k_0^2}{2M}\gamma_1. \tag{42}$$

In this approximation $\bar{\gamma}$ is independent of temperature, and roughly equals the width of the isolated nucleus.

Another way of seeing the above result is to expand the function in the exponent of the expression (22) for $J(\lambda, T)$ with respect to $\lambda=it$, keeping terms up to the quadratic in t. If this approximation is allowed, $\bar{\gamma}$ becomes, after the integration with respect to t,

$$\bar{\gamma}=\sqrt{2\pi}\sum_{s,k}|\mathcal{H}_s(\boldsymbol{k})|^2\left[\frac{k^2}{2M}\sum_{i=1}^{3}(\boldsymbol{\epsilon}_i\cdot\boldsymbol{n})^2\int_0^\infty\frac{\rho_i(\omega)\omega d\omega}{\tanh(\omega/2T)}\right]^{-1/2}$$

$$\times\exp\left[-\left(E-E_0+\frac{k^2}{2M}\right)^2/2\cdot\frac{k^2}{2M}\sum_{i=1}^{3}(\boldsymbol{\epsilon}_i\cdot\boldsymbol{n})^2\int_0^\infty\frac{\rho_i(\omega)\,\omega\,d\omega}{\tanh(\omega/2T)}\right], \tag{43}$$

or

$$\bar{\gamma}\approx 2\pi\sum_{s,k}|\mathcal{H}_s(\boldsymbol{k})|^2\delta\left(E-E_0+\frac{k^2}{2M}\right).$$

When $k_0^2/2M$ is much smaller than E_0, we again obtain Eq. (42).

In order that the approximation considered above is valid, it can be shown that the following condition has to be fulfilled,

$$\frac{k_0^2}{2M}\gg\frac{1}{T}\left(\int_0^\infty\rho_i(\omega)\,\omega\,d\omega\right)^2 \qquad \text{for } T\gg\int_0^\infty\rho_i(\omega)\,\omega\,d\omega,$$

$$\frac{k_0^2}{2M}\gg\int_0^\infty\rho_i(\omega)\,\omega\,d\omega \qquad \text{for } T\ll\int_0^\infty\rho_i(\omega)\,\omega\,d\omega,$$

which is the so-called weak binding case often considered in the theory of neutron diffraction.[13]

§ 5. Conclusion

It has become clear from the foregoing arguments that the moments of the energy of nuclear radiation from solid are related in a rather simple way to the moments of the frequency of lattice vibration. The calculations have been based on the Breit-Wigner of Weisskopf-Wigner formula, and the transition matrix has been assumed to be separable into two parts, one is the nuclear and the other the lattice transition matrix.

The nuclear matrix element has been assumed to vary slowly with energy in the neighborhood of the resonance energy, but to decrease very fast as the energy turns away from the resonance value. Also the nuclear level width has to be small compared with the resonance energy.

Since the moments of the frequency of lattice vibration are important parameters in the theory of crystal dynamics, it is hoped that future experiments would enable one to find the values of those moments from the observed distribution of the radiation energy.

In the present calculation the effect of multiple phonon processes has been taken into account correctly, and our method seems to be applicable to the problem of neutron diffraction,[13] since the matrix element of the latter is essentially the same as ours.

Appendix

In this Appendix a proof is given of the following operator identity:
Let $H(\xi, \eta)$ be defined by

$$H(\xi, \eta) = (1/2)\{(p+\xi)^2 + (q+\eta)^2\}, \tag{A·1}$$

with p and q satisfying the commutation relation $[p, q] = -i$, and ξ, and η being arbitrary c-numbers, then

$$\exp(i\alpha_1 H(\xi_1, \eta_1)) \cdot \exp(i\alpha_2 H(\xi_2, \eta_2)) = \exp(i\alpha H(\xi, \eta)) + iS), \tag{A·2}$$

where the parameters in the right-hand side are given as functions of those in the left-hand side as follows,

$$\alpha = \alpha_1 + \alpha_2, \tag{A·3}$$

$$\xi = \frac{1}{2}(\xi_1 + \xi_2) - u(\alpha_1, \alpha_2)(\xi_1 - \xi_2) + v(\alpha_1, \alpha_2)(\eta_1 - \eta_2), \tag{A·4}$$

$$\eta = \frac{1}{2}(\eta_1 + \eta_2) - u(\alpha_1, \alpha_2)(\eta_1 - \eta_2) - v(\alpha_1, \alpha_2)(\xi_1 - \xi_2), \tag{A·5}$$

with

$$u(\alpha_1, \alpha_2) = \frac{\cos\alpha_1 - \cos\alpha_2}{2(1-\cos\alpha)}, \tag{A·6}$$

$$v(\alpha_1, \alpha_2) = \frac{\sin\alpha - \sin\alpha_1 - \sin\alpha_2}{2(1-\cos\alpha)}, \tag{A·7}$$

and

$$S = -v(\alpha_1, \alpha_2)\{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2\}. \tag{A·8}$$

In order to prove the above identity we first note that for arbitrary

operators A and B the following identity holds,

$$\exp(A)\exp(B)=\exp(A+B+C), \tag{A·9}$$

where

$$C=\frac{1}{2}[A, B]+\frac{1}{12}\{[A, [A, B]]+[[A, B], B]\}+\cdots. \tag{A·10}$$

In applying Eqs. (A·9) and (A·10) to the left-hand side of Eq. (A·2), we note that the commutator between $H(\xi_1, \eta_1)$ and $H(\xi_2, \eta_2)$ and their multiple commutators are linear in p and q, and quadratic in ξ's and η's. We therefore see that the form of the right-hand side of Eq. (A·2) follows from its left-hand side, and that S is a quadratic expression of ξ's and η's.

To determine α, ξ, and η we start from the identities,

$$\begin{aligned}&\exp(i\alpha H(0, 0))p\exp(-i\alpha H(0, 0))\\&\quad=p\cos\alpha-q\sin\alpha,\end{aligned} \tag{A·11}$$

and

$$\begin{aligned}&\exp(i\alpha H(0, 0))q\exp(-i\alpha H(0, 0))\\&\quad=p\sin\alpha+q\cos\alpha,\end{aligned} \tag{A·12}$$

which can be verified by differentiation of both sides with respect to α. By applying a unitary transformation, $p\to p+\xi$, and $q\to q+\eta$, to Eqs. (A·11) and (A·12), we obtain

$$\begin{aligned}&\exp(i\alpha H(\xi, \eta))p\exp(-i\alpha H(\xi, \eta))\\&\quad=p\cos\alpha-q\sin\alpha-\xi(1-\cos\alpha)-\eta\sin\alpha,\end{aligned} \tag{A·13}$$

and

$$\begin{aligned}&\exp(i\alpha H(\xi, \eta))q\exp(-i\alpha H(\xi, \eta))\\&\quad=p\sin\alpha+q\cos\alpha+\xi\sin\alpha-\eta(1-\cos\alpha).\end{aligned} \tag{A·14}$$

By successive application of these transformations, we obtain

$$\begin{aligned}&\exp(i\alpha_1 H(\xi_1, \eta_1))\exp(i\alpha_2 H(\xi_2, \eta_2))p\exp(-i\alpha_2 H(\xi_2, \eta_2))\exp(-i\alpha_1 H(\xi_1, \eta_1))\\&\quad=p\cos(\alpha_1+\alpha_2)-q\sin(\alpha_1+\alpha_2)+\xi_1\{\cos(\alpha_1+\alpha_2)-\cos\alpha_2\}\\&\quad-\eta_1\{\sin(\alpha_1+\alpha_2)-\sin\alpha_2\}-\xi_2(1-\cos\alpha_2)-\eta_2\sin\alpha_2,\end{aligned} \tag{A·15}$$

and

$$\begin{aligned}&\exp(i\alpha_1 H(\xi_1, \eta_1))\exp(i\alpha_2 H(\xi_2, \eta_2))q\exp(-i\alpha_2 H(\xi_2, \eta_2))\exp(-i\alpha_1 H(\xi_1, \eta_1))\\&\quad=p\sin(\alpha_1+\alpha_2)+q\cos(\alpha_1+\alpha_2)+\xi_1\{\sin(\alpha_1+\alpha_2)-\sin\alpha_2\}\\&\quad+\eta_1\{\cos(\alpha_1+\alpha_2)-\cos\alpha_2\}+\xi_2\sin\alpha_2-\eta_2(1-\cos\alpha_2).\end{aligned} \tag{A·16}$$

In order that Eq. (A·2) holds these two expressions (A·15) and (A·16) must be identical with (A·13) and (A·14), respectively. Thus we find Eqs. (A·3) through (A·7).

By applying a unitary transformation $p \to p+a$ and $q \to q+b$ to both sides of Eq. (A·2), we see that S is invariant with respect to any parallel translation in the ξ-η plane, namely $\xi_i \to \xi_i + a$, $\eta_i \to \eta_i + b$. Likewise, by a unitary transformation $p \to p\cos\varphi - q\sin\varphi$ and $q \to p\sin\varphi + q\cos\varphi$, we find that S is invariant with respect to any proper rotation in the ξ-η plane, namely $\xi_i \to \xi_i \cos\varphi + \eta_i \sin\varphi$ and $\eta_i \to -\xi_i \sin\varphi + \eta_i \cos\varphi$. Recalling that S is quadratic in ξ's and η's, we see that S must be of the form

$$S = f(\alpha_1, \alpha_2)\{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2\}, \tag{A·17}$$

where the function $f(\alpha_1, \alpha_2)$ is yet to be determined.

We next note that

$$\frac{\partial}{\partial \alpha} \exp(i\alpha H(\xi, \eta)) = iH(\xi, \eta)\exp(i\alpha H(\xi, \eta)), \tag{A·18}$$

$$\frac{\partial}{\partial \xi} \exp(i\alpha H(\xi, \eta)) = i[\exp(i\alpha H(\xi, \eta)), q], \tag{A·19}$$

and

$$\frac{\partial}{\partial \eta} \exp(i\alpha H(\xi, \eta)) = -i[\exp(i\alpha H(\xi, \eta)), p]. \tag{A·20}$$

Differentiation of both sides of Eq. (A·2) by α_1 and α_2, with the use of the above identities (A·18) through (A·20), yields, respectively,

$$\begin{aligned}\frac{\partial S}{\partial \alpha_1} &= H(\xi_1, \eta_1) - H(\xi, \eta) \\ &+ \frac{\partial \eta}{\partial \alpha_1}\{\exp(i\alpha H(\xi, \eta))p\exp(-i\alpha H(\xi, \eta)) - p\} \\ &- \frac{\partial \xi}{\partial \alpha_1}\{\exp(i\alpha H(\xi, \eta))q\exp(-i\alpha H(\xi, \eta)) - q\},\end{aligned} \tag{A·21}$$

and

$$\begin{aligned}\frac{\partial S}{\partial \alpha_2} &= H(\xi_2, \eta_2) - H(\xi, \eta) \\ &- \frac{\partial \eta}{\partial \alpha_2}\{\exp(-i\alpha H(\xi, \eta))p\exp(i\alpha H(\xi, \eta)) - p\} \\ &+ \frac{\partial \xi}{\partial \alpha_2}\{\exp(-i\alpha H(\xi, \eta))q\exp(i\alpha H(\xi, \eta)) - q\}.\end{aligned} \tag{A·22}$$

From Eqs. (A·1), (A·13) and (A·14) the above expressions can be shown to become

$$\frac{\partial S}{\partial \alpha_1} = -\{(\xi-\xi_1)+\frac{\partial \bar{\xi}}{\partial \alpha_1}\sin\alpha+\frac{\partial \eta}{\partial \alpha_1}(1-\cos\alpha)\}(p+\bar{\xi})$$
$$-\{(\eta-\eta_1)-\frac{\partial \bar{\xi}}{\partial \alpha_1}(1-\cos\alpha)+\frac{\partial \eta}{\partial \alpha_1}\sin\alpha\}(q+\eta)$$
$$+\frac{1}{2}\{(\bar{\xi}-\bar{\xi}_1)^2+(\eta-\eta_1)^2\}, \qquad (A\cdot 23)$$

and

$$\frac{\partial S}{\partial \alpha_2} = -\{(\bar{\xi}-\bar{\xi}_2)+\frac{\partial \bar{\xi}}{\partial \alpha_2}\sin\alpha-\frac{\partial \eta}{\partial \alpha_2}(1-\cos\alpha)\}(p+\bar{\xi})$$
$$-\{(\eta-\eta_2)+\frac{\partial \bar{\xi}}{\partial \alpha_2}(1-\cos\alpha)+\frac{\partial \eta}{\partial \alpha_2}\sin\alpha\}(q+\eta)$$
$$+\frac{1}{2}\{(\bar{\xi}-\bar{\xi}_2)^2+(\eta-\eta_2)^2\}. \qquad (A\cdot 24)$$

Because S does not contain p and q, the first two terms of both Eqs. (A·23) and (A·24) must vanish, as can be verified by the use of Eqs. (A·3) through (A·7). We therefore find using Eqs. (A·4) and (A·5) that

$$\frac{\partial S}{\partial \alpha_1} = \{(1/2+u(\alpha_1,\alpha_2))^2+v(\alpha_1,\alpha_2)^2\}\ \{(\bar{\xi}_1-\bar{\xi}_2)^2+(\eta_1-\eta_2)^2\}, \qquad (A\cdot 25)$$

and

$$\frac{\partial S}{\partial \alpha_2} = \{(1/2-u(\alpha_1,\alpha_2))^2+v(\alpha_1,\alpha_2)^2\}\{(\bar{\xi}_1-\bar{\xi}_2)^2+(\eta_1-\eta_2)^2\}, \qquad (A\cdot 26)$$

or, recalling Eqs. (A·6), (A·7) and (A·17), we see that

$$\frac{\partial f}{\partial \alpha_1} = \frac{1-\cos\alpha_2}{2(1-\cos\alpha)}, \qquad (A\cdot 27)$$

and

$$\frac{\partial f}{\partial \alpha_2} = \frac{1-\cos\alpha_1}{2(1-\cos\alpha)}. \qquad (A\cdot 28)$$

Integration of these equations under the initial conditions $f(0,\alpha_2)=f(\alpha_1,0)=0$ finally gives $f(\alpha_1,\alpha_2)$ as

$$f(\alpha_1,\alpha_2) = -v(\alpha_1,\alpha_2), \qquad (A\cdot 29)$$

and this concludes the proof of Eq. (A·8).

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Spin and Statistics

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A general discussion of the vacuum expectation value for the product of two fields shows how the structure depends upon the integral or half-integral nature of the spin. When this is compared with the statement of statistics as the requirement that fields commute or anti-commute for points in space-like relation, at distances greater than any specified amount, the connection between spin and statistics follows.

This note is a continuation of an earlier discussion[1] on the general basis in relativistic quantum field theory for the connection between spin and statistics, and the TCP theorem. We have nothing to add to the second topic, but do wish to reconsider the question of spin and statistics since the previous treatment was unnecessarily complicated by the intrusion of two other aspects of relativistic field theory which are grave problems in their own right. These are the limiting high energy behavior of fields and the occurrence of superfluous field components. It suffices to require of the fields that, characteristic of the statistics, they commute (Bose-Einstein) or anticommute (Fermi-Dirac) for any two points in space-like relation at spatial separations in excess of some finite distance which need not be zero. Covariance under the proper, orthochronous Lorentz group is demanded, of course, together with the hypothesis that the ground state of the system is the unique invariant vacuum state.

We denote by $\chi(x)$ a Hermitian field of finite multiplicity (individual components: $\chi_a(x)$, $a=1\cdots m$). Nothing needs to be specified about the part of this multiplicity that refers to internal degrees of freedom such as electrical charge, With regard to the spin aspect of the multiplicity we require just spin homogeneity, in the sense that only integral values, or half-integral values, of the spin S will be represented. There is no restriction concerning irreducibility of the field. It is also permissible that $\chi(x)$ stands for the product of several other fields at the common point x. The coordinate dependence of the field is represented by the unitary transformation

$$\chi(x)=e^{-iPx}\chi e^{iPx},$$

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where

$$Px = P^\mu x_\mu = \boldsymbol{P}\cdot\boldsymbol{x} - P^0 x^0.$$

With the exception of the vacuum state $\langle$, for which

$$\langle P^\mu = 0,$$

the spectrum of the energy-momentum vector is time-like, or null, with positive energy component,

$$-P^{2\prime} \geq 0, \quad P^{0\prime} > 0.$$

To every homogeneous, proper, orthochronous Lorentz transformation,

$$\bar{x}^\mu = l_\nu{}^\mu x^\nu,$$

$$\det l = +1, \quad l_0{}^0 \geq 1,$$

there is associated a unitary operator $U(l)$, and a finite real matrix $L(l)$, such that

$$\bar{\chi}(\bar{x}) = L(l)\chi(x)$$

and

$$\bar{\chi}(x) = U(l)^{-1}\chi(x)\, U(l),$$

or, equivalently,

$$U(l)^{-1}\chi U(l) = L(l)\,\chi$$

and

$$U(l)^{-1} P^\mu U(l) = l_\nu{}^\mu P^\nu.$$

In particular, for the infinitesimal transformation

$$l_\nu{}^\mu = \delta_\nu{}^\mu + \delta\omega_\nu{}^\mu,$$

$$\delta\omega_{\mu\nu} = -\delta\omega_{\nu\mu},$$

we have

$$U(\delta\omega) = 1 + i\,\frac{1}{2}\,\delta\omega^{\mu\nu} J_{\mu\nu},$$

$$L(\delta\omega) = 1 + i\,\frac{1}{2}\,\delta\omega^{\mu\nu} S_{\mu\nu},$$

where the Hermitian operators $J_{\mu\nu}$ and the finite imaginary matrices $S_{\mu\nu}$ obey the same commutation relations. The latter are identical with those of the attached four-dimensional Euclidean rotation group, which is obtained by the transformation

$$x_4 = i x^0$$

and, for example,

$$S_{k4} = i S_{0k}.$$

The invariant vacuum state also obeys

$$\langle J_{\mu\nu}=0.$$

This is not an independent property but is implied by the null eigenvalues of P^μ, according to

$$\langle U(l)^{-1}P^\mu=0,$$

and the assumed uniqueness of the vacuum state, for zero is the only possible value in a one-dimensional representation of the $J_{\mu\nu}$ commutation relations. It follows that the vacuum expectation value of χ obeys

$$L(l)\langle\chi\rangle=\langle\chi\rangle$$

for arbitrary l, and therefore

$$\langle\chi\rangle=0,$$

with the possible exception of invariant components of χ. For these a null value can be attained by the addition of suitable constants, which we assume has been done.

The basic tool in our discussion is the vacuum expectation value of the product of two fields, the matrix

$$\langle\chi(x)\chi(x')\rangle=\langle\chi e^{iP(x-x')}\chi\rangle.$$

The decomposition of all states into the various subspaces labelled by the energy-momentum values

$$p^\mu=P^{\mu\prime}$$

is expressed by

$$e^{iP(x-x')}=\int(dp)\,e^{ip(x-x')}M(p),$$

where $M(p)$ is a projection operator and thus is a Hermitian, non-negative operator. The vacuum state will be represented here by a term in $M(p)$ containing the four-dimensional delta function $\delta(p)$ and $\rangle\langle$. The latter will not contribute to the two field expectation value, however, since $\langle\chi\rangle=0$. We indicate this effective limitation of the decomposition to states with real mass $\kappa\geq 0$ and positive energy by writing

$$e^{iP(x-x')}=\int(dp)\int_0^\infty d\kappa^2\,e^{ip(x-x')}\eta_+(p)\,\delta(p^2+\kappa^2)\,M(p),$$

where

$$\eta_+(p)=\begin{cases}1, & p^0>0\\ 0, & p^0<0\end{cases}.$$

Accordingly,

$$\langle \chi(x)\,\chi(x')\rangle = \int \frac{(dp)}{(2\pi)^3}\, d\kappa^2\, e^{ip(x-x')}\,\eta_+(p)\,\delta(p^2+\kappa^2)\,m(p),$$

in which

$$m(p) = (2\pi)^3 \langle \chi M(p) \chi \rangle$$

is a finite-dimensional Hermitian, non-negative matrix,

$$m(p)^{T*} = m(p) \geq 0,$$

which represents the properties of the projection operator $M(p)$. If the field $\chi(x)$ is to have any dynamical significance at all, $m(p)$ must be a positive matrix over some region of the p-space. This general remark, excluding the possibility that

$$\langle \chi = \chi \rangle = 0,$$

can be made much more specific when $\chi(x)$ is a fundamental field, which contains among its components the generators of a complete operator basis, for the property of completeness implies the ability of the field operators to produce all other states from the vacuum state.

The unitary transformation expressing the vector nature of the operators P^μ implies that

$$M(lp) = U(l)\, M(p)\, U(l)^{-1}$$

and, correspondingly,

$$m(lp) = L(l)\, m(p)\, L(l)^T.$$

The infinitesimal form of the last statement,

$$\frac{1}{i}\left(p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu}\right) m(p) + S_{\mu\nu} m(p) + m(p)\, S_{\mu\nu}^T = 0,$$

is an assertion of rotational invariance for a system with orbital and spin angular momenta. If one introduces $p_4 = ip^0$ and $S_{k4} = iS_{0k}$, this becomes four-dimensional Euclidean rotational invariance. The most general matrix function $m(p)$ is obtained by combining arbitrary functions of the invariant $-p^2 = \kappa^2$, finite numerical matrices, and four-dimensional spherical harmonics. The latter are homogeneous polynomials in p^μ that are irreducible with respect to the contraction $-p^2 = \kappa^2$. If S is the highest spin value contained in χ, the maximum degree of any spherical harmonic is $2S$. All that we shall require here is the knowledge that the matrix $m(p)$ can be constructed from such finite irreducible polynomials in p^μ, and arbitrary functions of κ^2. This we express by writing

$$m(p) = m(p, \kappa^2).$$

A statement of infinitesimal Euclidean rotational invariance implies invariance under every proper orthogonal transformation, for, with finite bounded matrices,

there can be no barrier to constructing any transformation of the proper Euclidean rotation group from the infinitesimal ones. In particular, we can produce the proper orthogonal transformation which is the four-dimensional reflection $p^\mu \to -p^\mu$. A corresponding L matrix is

$$R_{st}=e^{\pi i S_{12}}\, e^{\pi i S_{34}}$$

since the transformation is a rotation through the angle π in the 12 and the 34 planes, for example. All the spin matrices $S_{\mu\nu}$, $\mu, \nu=1\cdots 4$ have the same spectra and therefore

$$e^{2\pi i S_{12}}=e^{2\pi i S_{34}}=(-1)^{2S}=\begin{cases}+1: S & \text{integer},\\ -1: S & \text{half odd integer},\end{cases}$$

which implies that

$$R_{st}^2=1.$$

We also note that iS_{12} is a real matrix, while $iS_{34}=-S_{03}$ is imaginary. Accordingly

$$R_{st}^*=e^{\pi i S_{12}}\, e^{-\pi i S_{34}}=(-1)^{2S} R_{st}$$

and the matrix R_{st} is real and imaginary, respectively, for integral and half-integral S. We do not need to specify the symmetry properties of this matrix.

What we learn in this way is the continuation of the algebraic structure of $m(p, \kappa^2)$ from positive to negative values of p^0,

$$m(-p, \kappa^2)=R_{st}\, m(p, \kappa^2)\, R_{st}^T=(-1)^{2S} R_{st}\, m(p, \kappa^2)\, R_{st}^{T*}.$$

Since the non-negativeness of the matrix $m(p)$ is maintained in the product $R_{st}\, m(p)\, R_{st}^{T*}$, the continuation is such that

$$p^0>0: \quad (-1)^{2S} m(-p, \kappa^2)\geq 0.$$

If p^0 is allowed to assume both signs, these attributes are summarized by

$$S, \text{ integer}: \qquad m(p, \kappa^2)\geq 0,$$

$$S, \text{ half odd integer}: \quad \epsilon(p)\, m(p, \kappa^2)\geq 0,$$

where

$$\epsilon(p)=\begin{cases}+1, & p^0>0,\\ -1, & p^0<0.\end{cases}$$

The order of factors in the vacuum expectation value is reversed by complex conjugation,

$$\langle \chi(x)\, \chi(x')\rangle^*=\int \frac{(dp)}{(2\pi)^3}\, d\kappa^2\, e^{ip(x-x')}\, \eta_-(p)\, \delta(p^2+\kappa^2)\, m(-p, \kappa^2)^*,$$

in which we have introduced

$$\eta_-(p)=\eta_+(-p).$$

From this follows the commutator, $[\ ]_-$, and anticommutator, $[\ ]_+$, expectation values

$$\langle|\chi(x),\chi(x')]_\pm\rangle=\int\frac{(dp)}{(2\pi)^3}d\kappa^2\,e^{ip(x-x')}\delta(p^2+\kappa^2)$$
$$\times[\eta_+(p)\,m(p,\kappa^2)\pm\eta_-(p)\,m(-p,\kappa^2)^*].$$

We can also use the relation

$$\eta_\pm(p)=\frac{1}{2}\,[1\pm\epsilon(p)\,]$$

to write

$$\langle[\chi(x),\chi(x')]_\pm\rangle=\int\frac{(dp)}{(2\pi)^3}d\kappa^2\,e^{ip(x-x')}\delta(p^2+\kappa^2)$$
$$\times\frac{1}{2}[m(p,\kappa^2)\pm m(-p,\kappa^2)^*+\epsilon(p)\,(m(p,\kappa^2)\mp m(-p,\kappa^2)^*)].$$

Since complex conjugation of a matrix does not alter the quality of non-negativeness, these combinations have the basic property

$$S,\text{ integer}:\qquad m(p,\kappa^2)+m(-p,\kappa^2)^*\geq 0$$
$$S,\text{ half odd integer}:\quad \epsilon(p)\,(m(p,\kappa^2)-m(-p,\kappa^2)^*)\geq 0,$$

and there are sets of p values for which the matrices are positive.

The polynomial nature of the p-dependence in $m(p,\kappa^2)$ is utilized in writing

$$\langle[\chi(x),\chi(x')]_\pm\rangle=\int_0^\infty d\kappa^2\,\frac{1}{2}\left[m\left(\frac{1}{i}\partial,\kappa^2\right)\pm m\left(\frac{1}{i}\partial,\kappa^2\right)^*\right]\Delta^{(1)}(x-x',\kappa^2)$$
$$+\int_0^\infty d\kappa^2\frac{1}{2i}\left[m\left(\frac{1}{i}\partial,\kappa^2\right)\mp m\left(\frac{1}{i}\partial,\kappa^2\right)^*\right]\Delta(x-x',\kappa^2),$$

where Δ and $\Delta^{(1)}$ are the well-known invariant functions

$$\Delta(x,\kappa^2)=i\int\frac{(dp)}{(2\pi)^3}e^{ipx}\epsilon(p)\,\delta(p^2+\kappa^2),$$
$$\Delta^{(1)}(x,\kappa^2)=\int\frac{(dp)}{(2\pi)^3}e^{ipx}\delta(p^2+\kappa^2).$$

For these vacuum expectation values to exist when the points are in time-like relation, the growth of $m(p,\kappa^2)$ with increasing κ^2 can be no more than algebraic.

The factor $\epsilon(p)$ ensures that $\Delta(x-x',\kappa^2)$ vanishes for equal times or, more generally, when the two points are in space-like relation,

$$(x-x')^2>0:\ \Delta(x-x',\kappa^2)=0,$$

whereas for the $\Delta^{(1)}$ function we have the integral representation

$$x^2>0 : \Delta^{(1)}(x, \kappa^2) = \frac{1}{2\pi^2}\int_0^\infty d\lambda\, e^{-\lambda x^2-\kappa^2/4\lambda} > 0.$$

If the commutator or anticommutator is to vanish for spatial separations greater than some specified distance, $(x-x')^2>a^2$, so should the corresponding term involving $\Delta^{(1)}$. Since an irreducible polynomial in $1/i\cdot\partial_\mu$ cannot annihilate a scalar function of $(x-x')^2$, the real function $m(\kappa^2)$ that can be associated with each polynomial must be such that

$$x^2>a^2 : \int_0^\infty d\kappa^2\, m(\kappa^2)\, \Delta^{(1)}(x, \kappa^2) = 0.$$

More precisely, it is a derivative with respect to the variable x^2, of an order corresponding to the degree of the associated polynomial, that vanishes. However, this weighted integral of $\Delta^{(1)}(x, \kappa^2)$ cannot be a polynomial in x^2 or a non-zero constant, for that would contradict the asymptotic behavior of $\Delta^{(1)}$,

$$\Delta^{(1)}(x, \kappa^2) = \frac{1}{2}(\pi x)^{-2},\ \kappa^2=0$$

$$x^2\to\infty : \quad \sim\kappa^{1/2}(2\pi x)^{-3/2}e^{-\kappa x},\ \kappa^2>0.$$

We now prove that $m(\kappa^2)$ must vanish identically. The restriction to the class of functions $m(\kappa^2)$ that, at most, grow algebraically with κ^2 enables the null condition to be presented in the form

$$x^2>a^2 : \int_0^\infty d\lambda\, e^{-\lambda x^2}\int_0^\infty d\kappa^2\, e^{-\kappa^2/4\lambda}\, m(\kappa^2) = 0.$$

We appeal to the following lemma about real Laplace transforms. If

$$\int_0^\infty dx e^{-tx} f(x) = 0,\ t\geq 1,$$

where $f(x)$ belongs to the class of real functions characterized by

$$\int_0^\infty dx e^{-x} f(x)^2 < \infty,$$

then $f(x)=0$. On writing $t=1+s$, the first condition appears as

$$\int_0^\infty dx e^{-sx} e^{-x} f(x) = 0,\ s\geq 0,$$

from which we infer, for the class of functions under consideration, that

$$\int_0^\infty dx\, x^n e^{-x} f(x) = 0, \quad n = 0, 1, \cdots.$$

An equivalent statement is

$$\int_0^\infty dx\, L_n(x)\, e^{-x} f(x) = 0, \quad n = 0, 1, \cdots,$$

where $L_n(x)$ is the n^{th} Laguerre polynomial, the set of which is complete and orthonormal with respect to the weight factor e^{-x}. As a member of the indicated class, $f(x)$ can be expanded in a series of Laguerre polynomials with coefficients that have a finite norm, and every coefficient is zero, which proves the lemma. We can now conclude, successively, that

$$\int_0^\infty d\kappa^2 e^{-\kappa^2/4\lambda} m(\kappa^2) = 0, \quad \lambda > 0,$$

and then,

$$m(\kappa^2) = 0.$$

The vanishing of commutator $(-)$ or anticommutator $(+)$ for sufficient space-like separations thus requires that

$$m(p, \kappa^2) \mp m(-p, \kappa^2)^* = 0.$$

But, on referring to the positiveness properties that are associated with the spin type, it is seen that the commutator condition, characteristic of Bose-Einstein statistics, is impossible for half-integral S while the anticommutator condition, or Fermi-Dirac statistics, is impossible for integral S. This proves, in a unified way, the connection between spin and statistics.†

In view of the equivalence between the spin and the statistics classification of fields, we can write the restriction on $m(p, \kappa^2)$ in the form

$$m(p, \kappa^2) = (-1)^{2S} m(-p, \kappa^2)^*.$$

Alternative presentations are

$$m(p, \kappa^2)^* = R_{st}\, m(p, \kappa^2)\, R_{st}^T{}^*$$

and

$$m(p, \kappa^2) = (-1)^{2S} R_{st}\, m(p, \kappa^2)^*\, R_{st}^T,$$

the first of which shows the compatibility of the reality and positiveness prop-

† We are not unaware of other recent discussions[2),3)] that have exploited similar physical requirements with rather different mathematical methods. Although our treatment at least has the merit of being more elementary, the principal difference is one of attitude towards the foundations of quantum field theory. The quantum mechanical basis assumed in this paper is described in a continuing series of short notes in the Proc. Nat. Acad. Sci. U. S.[4)]

erties of $m(p, \kappa^2)$, while the second, transcribed into

$$(-1)^{2S} R_{st} \langle \chi(-x)\, \chi(-x') \rangle^* R^T_{st} = \langle \chi(x)\, \chi(x') \rangle,$$

is just the statement of TCP invariance for the two field vacuum expectation value.

Let us also note that the vacuum expectation value of the commutator, for integral spin, and the anticommutator, for half-integral spin, can both be presented as

$$\langle [\chi(x), \chi(x')]_{\pm} \rangle = -i \int_0^\infty d\kappa^2 m\left(\frac{1}{i}\partial, \kappa^2\right) \Delta(x-x', \kappa^2)$$

and therefore vanish for all space-like intervals. In order to specialize this result to equal times, we use the relation $(p^0)^2 = \boldsymbol{p}^2 + \kappa^2$ to write $m(p, \kappa^2)$ as

$$m(p, \kappa^2) = m_a(\boldsymbol{p}, \kappa^2) + p^0 m_b(\boldsymbol{p}, \kappa^2),$$

where, among other properties, we have

$$S, \text{ integer}: \quad m_a(\boldsymbol{p}, \kappa^2) \geq 0$$

$$S, \text{ half-integer}: \quad m_b(\boldsymbol{p}, \kappa^2) \geq 0.$$

This gives

$$\langle [\chi(x), \chi(x')]_{\pm} \rangle = \int_0^\infty d\kappa^2 \left[-i m_a\left(\frac{1}{i}\nabla, \kappa^2\right) + m_b\left(\frac{1}{i}\nabla, \kappa^2\right) \partial_0 \right] \Delta(x-x', \kappa^2),$$

and, in the limit of equal times,

$$x^0 = x^{0\prime} : \langle [\chi(x), \chi(x')]_{\pm} \rangle = \int_0^\infty d\kappa^2 m_b\left(\frac{1}{i}\nabla, \kappa^2\right) \delta(\boldsymbol{x}-\boldsymbol{x}'),$$

provided, of course, that the matrix polynomial in $\boldsymbol{p}$ exists,

$$\int_0^\infty d\kappa^2 m_b(\boldsymbol{p}, \kappa^2) < \infty.$$

It is natural to suppose that this restriction distinguishes fundamental fields from arbitrary local field products. A related statement is

$$x^0 = x^{0\prime} : \langle [i\partial_0 \chi(x), \chi(x')]_{\pm} \rangle = \int_0^\infty d\kappa^2 m_a\left(\frac{1}{i}\nabla, \kappa^2\right) \delta(\boldsymbol{x}-\boldsymbol{x}'),$$

provided that

$$\int_0^\infty d\kappa^2 m_a(\boldsymbol{p}, \kappa^2) < \infty,$$

although the latter is less compelling for a fundamental field that obeys first order differential equations.

Despite the apparent generality of this discussion there is one significant element of incompleteness. It does not apply directly to fields bearing electric charge if the dynamical effect of the electromagnetic field is included. That is a consequence of the operator gauge transformation which must accompany a Lorentz transformation, if the electromagnetic requirement of three-dimensional transversality is to be maintained in each coordinate system. We have chosen not to insist on this point in order to simplify the discussion. The necessary elaboration will be considered elsewhere.

But one can give an elementary argument that the spin-statistics connection also governs charged fields if it is assumed that, while the permissible spin-statistics categories for charged fields may involve the dynamical presence of the electromagnetic field, they do not depend upon the specific magnitude of the charge. Then we can multiply various charged fields and produce electrically neutral combinations to which the preceding discussion does apply. For example, let $\chi^{(1)}$ be a B. E. field carrying unit charge (± 1) and integral spin, while $\chi^{(2)}$ is a B. E. field of unit charge and half-integral spin. From these we form an electrically neutral combination $\chi^{(1)}(x)\chi^{(2)}(x)$ that constitutes a B. E. field of half-integral spin, which is impossible. In this way we conclude that charged fields can either be B. E. with integral spin and F. D. with half-integral spin, or, B. E. with half-integral spin and F. D. with integral spin, since these mutually exclusive possibilities are the only ones that lead to acceptable bilinear neutral combinations. Now it suffices to consider an electrically neutral combination $\chi^{(1)}(x)\chi^{(2)}(x)\chi^{(3)}(x)$, formed from unit and doubly charged fields, in which all fields belong to the same spin-statistics category. If this is B. E. statistics with half-integral spin, the result is a neutral field with those characteristics, which is excluded, as is the consequence of F. D. statistics, integer spin.

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Integral Representations for Scattering Amplitudes in Perturbation Theory. II

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The precise supports of the weight functions in the integral representation proposed previously are found for the equal-mass-particle, pion-pion and nucleon-nucleon scatterings. From these results the maximal momentum transfers below which dispersion relations are derived are much improved.

§ 1. Statement of results

In a previous paper,[1] which will be referred to as I, we have proved an integral representation (similar to the Mandelstam representation[2]),

$$\int_0^\infty d\alpha \int_0^1 dx \frac{\rho_{12}(\alpha, x)}{\alpha - xs - (1-x)t - i\varepsilon} + \int_0^\infty d\beta \int_0^1 dy \frac{\rho_{23}(\beta, y)}{\beta - yt - (1-y)u - i\varepsilon}$$

$$+ \int_0^\infty d\gamma \int_0^1 dz \frac{\rho_{31}(\gamma, z)}{\gamma - zu - (1-z)s - i\varepsilon} \qquad (1\cdot1)$$

for some practical cases of the two-particle scattering in every order of perturbation theory. Here s, t and u are the three invariant squares of energy and momentum transfers in this process. The support properties of weight functions were investigated also in I, but their precise boundaries were not given in it. The aim of this paper is to present the precise supports* of the weight functions in cases of the equal-mass-particle scattering, the pion-pion scattering and the nucleon-nucleon scattering.

The results obtained in this paper are as follows. We can show the following support properties under a certain general conjecture whose validity is very affirmable.

i) Equal-mass-particle scattering (mass μ):

$$\alpha \geqq (2\mu)^2 + (2\mu)^2 \sqrt{x(1-x)} \qquad (1\cdot2)$$

and likewise for β and γ.

ii) Pion-pion scattering:

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* Precise support means such a region that the weight function always vanishes outside it for every graph and is non-vanishing inside it for at least one graph.

$$\alpha \geqq \mathrm{Min}[(2m_\pi)^2 x+(4m_\pi)^2(1-x)+(4m_\pi)^2\sqrt{x(1-x)},$$
$$(4m_\pi)^2 x+(2m_\pi)^2(1-x)+(4m_\pi)^2\sqrt{x(1-x)}] \tag{1·3}$$

and likewise for β and γ.

iii) Nucleon-nucleon scattering:

$$\alpha \geqq (2m_N)^2 x+(2m_\pi)^2(1-x)+(2m_\pi)^2\sqrt{x(1-x)}, \tag{1·4a}$$
$$\beta \geqq \mathrm{Min}[(2m_\pi)^2 y+(2m_N)^2(1-y)+(2m_\pi)^2\sqrt{y(1-y)},$$
$$(2m_N)^2 y+(2m_\pi)^2(1-y)+(2m_\pi)^2\sqrt{y(1-y)}], \tag{1·4b}$$
$$\gamma \geqq (2m_\pi)^2 z+(2m_N)^2(1-z)+(2m_\pi)^2\sqrt{z(1-z)}. \tag{1·4c}$$

The regions $(1\cdot2)\sim(1\cdot4)$ are precise supports, and from them the Mandelstam bounding curves[3] are derived as will be seen in § 3.

iv) Pion-nucleon scattering:

$$\alpha \geqq (m_N+m_\pi)^2 x+(2m_\pi)^2(1-x)+(2m_\pi)^2\sqrt{x(1-x)}, \tag{1·5a}$$
$$\beta \geqq (2m_\pi)^2 y+(m_N+m_\pi)^2(1-y)+(2m_\pi)^2\sqrt{y(1-y)}, \tag{1·5b}$$
$$\gamma \geqq \mathrm{Max}[(m_N+m_\pi)^2 z+\{(m_N-m_\pi)^2+(2m_\pi)^2\}(1-z)$$
$$+(2m_\pi)^2\sqrt{z(1-z)},\quad \{(m_N-m_\pi)^2+(2m_\pi)^2\}z$$
$$+(m_N+m_\pi)^2(1-z)+(2m_\pi)^2\sqrt{z(1-z)}]. \tag{1·5c}$$

Unlike the former three cases, they are not precise supports. They should be improved by closer investigation.

As was pointed out in I, when t is fixed to $t=-4\mathcal{A}^2$, the integral representation (1·1) is reduced to a single dispersion relation by putting

First term:

$$s'=\frac{\alpha+4\mathcal{A}^2(1-x)}{x}, \tag{1·6a}$$

Second term:

$$u'=\frac{\beta+4\mathcal{A}^2 y}{1-y}, \tag{1·6b}$$

Third term:

$$s'=\frac{\gamma-zM_\mathcal{A}^2}{1-2z} \quad \text{for } z<1/2, \tag{1·6c}$$
$$u'=\frac{\gamma-(1-z)M_\mathcal{A}^2}{2z-1} \quad \text{for } z>1/2$$

with

$$M_\mathcal{A}^2 \equiv M_a^2+M_b^2+M_c^2+M_d^2+4\mathcal{A}^2=s+u, \tag{1·7}$$

where M_a, M_b, M_c, M_d are the external-particle masses. So, in order that the

normal dispersion relation is derived from the integral representation (1·1), it is necessary and sufficient that

$$\alpha \geqq M_{AB}^2 x - 4\Delta^2(1-x), \tag{1·8a}$$

$$\beta \geqq -4\Delta^2 y + M_{AD}^2(1-y), \tag{1·8b}$$

$$\gamma \geqq \begin{cases} (M_A^2 - M_{AB}^2)z + M_{AB}^2(1-z) & \text{for } z < 1/2, \\ M_{AD}^2 z + (M_A^2 - M_{AD}^2)(1-z) & \text{for } z > 1/2. \end{cases} \tag{1·8c}$$

Therefore if

$$\alpha \geqq M_{AB}^2 x + M_{AC}^2(1-x), \tag{1·9a}$$

$$\beta \geqq M_{AC}^2 y + M_{AD}^2(1-y), \tag{1·9b}$$

$$\gamma \geqq M^2 \cdot \mathrm{Min}(z, 1-z), \tag{1·9c}$$

then the normal dispersion relation is obtained for

$$-M_{AC}^2 \leqq 4\Delta^2 \leqq 4\Delta_{max}^2 \tag{1·10}$$

with

$$\Delta_{max}^2 \equiv \frac{1}{4}(M^2 - M_a^2 - M_b^2 - M_c^2 - M_d^2). \tag{1·11}$$

In the above, M_{AB}, M_{AC} and M_{AD} are the lowest intermediate-state masses* associated with s, t and u, respectively. Using (1·11) we get the following results from (1·2) ~ (1·5).

i) Equal-mass-particle scattering:

$$\Delta_{max}^2 = 2\mu^2. \tag{1·12}$$

ii) Pion-pion scattering:

$$\Delta_{max}^2 = 8m_\pi^2. \tag{1·13}$$

iii) Nucleon-nucleon scattering:

$$\Delta_{max}^2 = 2m_\pi^2. \tag{1·14}$$

iv) Pion-nucleon scattering:

$$\Delta_{max}^2 = 2m_\pi^2. \tag{1·15}$$

§ 2. Derivation

As was shown in I, the denominator function V of the integrand in the Feynman parametric integral formula is written as

$$V = \sum_{i=1}^{N} x_i m_i^2 - \zeta_A M_a^2 - \zeta_B M_b^2 - \zeta_C M_c^2 - \zeta_D M_d^2 - \zeta_{AB} s - \zeta_{AC} t - \zeta_{AD} u \tag{2·1}$$

* Except for one-particle intermediate states.

with

$$s+t+u=M_a^2+M_b^2+M_c^2+M_d^2,$$

where

N: number of internal lines,

x_i: Feynman parameter,

m_i: internal mass,

$\zeta_h \equiv W_h/U \geqq 0$ $(h=A, B, C, D, AB, AC, AD)$.

The definitions of the parametric functions $U(\geqq 0)$ and $W_h(\geqq 0)$ are given in I. We shall present various detailed properties of W_h in Appendix A.

The integration variables in (1·1) are defined by

$$\alpha=\frac{\sum_{i=1}^{N} x_i m_i^2-(\zeta_A+\zeta_{AD})M_a^2-(\zeta_B+\zeta_{AD})M_b^2-(\zeta_C+\zeta_{AD})M_c^2-(\zeta_D+\zeta_{AD})M_d^2}{(\zeta_{AB}-\zeta_{AD})+(\zeta_{AC}-\zeta_{AD})},$$

and

$$x=\frac{\zeta_{AB}-\zeta_{AD}}{(\zeta_{AB}-\zeta_{AD})+(\zeta_{AC}-\zeta_{AD})} \tag{2·2}$$

for $\zeta_{AD}<\mathrm{Min}(\zeta_{AB}, \zeta_{AC})$, and so on.

Now, the central importance of the present paper is an inequality

$$W_A W_D \geqq W_{AB} W_{AC}. \tag{2·3}$$

Unfortunately, the general proof of (2·3) is not yet accomplished, but the validity of this conjecture is very affirmable as will be seen in Appendix A. Its validity was checked for all the Feynman graphs in which the number of internal lines is less than ten (i. e. $N<10$) (of course no special restrictions are made for interaction types). Five other inequalities similar to (2·3) are also obtained by interchanging A, B, C, D appropriately.

If the inequality (2·3) is accepted, we get

$$\begin{aligned}(W_A+W_D+2W_{AD})^2 &\geqq (W_A+W_D)^2+4W_{AD}^2 \geqq 4(W_A W_D+W_{AD}^2)\\ &\geqq 4(W_{AB}W_{AC}+W_{AD}^2) \geqq 4(W_{AB}-W_{AD})(W_{AC}-W_{AD}),\end{aligned} \tag{2·4}$$

and therefore

$$\zeta_A+\zeta_D+2\zeta_{AD} \geqq 2\sqrt{(\zeta_{AB}-\zeta_{AD})(\zeta_{AC}-\zeta_{AD})}, \tag{2·5a}$$

provided that $\zeta_{AD}<\mathrm{Min}(\zeta_{AB}, \zeta_{AC})$. Likewise we get

$$\zeta_B+\zeta_C+2\zeta_{AD} \geqq 2\sqrt{(\zeta_{AB}-\zeta_{AD})(\zeta_{AC}-\zeta_{AD})}. \tag{2·5b}$$

Inequalities of the following type are often proved for the purpose of proving dispersion relations:

$$\sum_{i=1}^{N} x_i m_i^2 - (\zeta_A+\zeta_{AD})(M_a^2+K_1^2) - (\zeta_B+\zeta_{AD})(M_b^2+K_2^2)$$
$$- (\zeta_C+\zeta_{AD})(M_c^2+K_2^2) - (\zeta_D+\zeta_{AD})(M_d^2+K_1^2)$$
$$- (\zeta_{AB}-\zeta_{AD})M_1^2 - (\zeta_{AC}-\zeta_{AD})M_2^2 \geqq 0, \tag{2·6}$$

since in order to prove such inequalies it is essential that external momenta are Euclidean. Then from (2·2) and (2·5) we get

$$\alpha \geqq M_1^2 x + M_2^2(1-x) + 2(K_1^2+K_2^2)\sqrt{x(1-x)}. \tag{2·7}$$

We have similar results for β and γ.

i) Equal-mass-particle scattering:

For any Feynman graph we can put

$$M_1^2 = M_2^2 = (2\mu)^2,$$
$$K_1^2 = K_2^2 = \mu^2 \tag{2·8}$$

as was shown in (3·4) of I. So we get (1·2) on account of (2·7).

ii) Pion-pion scattering:

For any Feynman graph we can put *either*

$$M_1^2 = (2m_\pi)^2, \quad M_2^2 = (4m_\pi)^2,$$
$$K_1^2 = K_2^2 = (2m_\pi)^2 \tag{2·9a}$$

or

$$M_1^2 = (4m_\pi)^2, \quad M_2^2 = (2m_\pi)^2,$$
$$K_1^2 = K_2^2 = (2m_\pi)^2 \tag{2·9b}$$

as will be shown in Appendix B.

iii) Nucleon-nucleon scattering:

Similarly to the equal-mass case, the result obtained in I is improved to (1·4) according to (2·7) with

$$K_1^2 = K_2^2 = m_\pi^2. \tag{2·10}$$

iv) Pion-nucleon scattering:

The inequalities (1·5) are obtained in exactly the same way as the above.

As is easily verified, the equality cases in (1·2), (1·3) and (1·4) are realized in the respective non-trivial lowest order graphs. Therefore they are the precise supports for the equal-mass, pion-pion and nucleon-nucleon scatterings, respectively.

§ 3. Relation with the Mandelstam bounding curves

We have seen in (2·7) that the weight function usually vanishes unless

$$\alpha \geqq ax + b(1-x) + c\sqrt{x(1-x)}, \tag{3·1}$$

where a, b and c are some positive constants. Since our variables α and x are related to the Mandelstam ones s' and t' through

$$\alpha = xs' + (1-x)t' \tag{3·2}$$

as is seen from (1·1), the bounding curve of the weight function in (s', t') plane is obtained as the envelope of the family of straight lines

$$x(s'-a) + (1-x)(t'-b) = c\sqrt{x(1-x)}. \tag{3·3}$$

Some elementary calculation leads to

$$(s'-a)(t'-b) = c^2/4 \tag{3·4}$$

with

$$s' > a.$$

This is the positive branch of a rectangular hyperbola having asymptotes $s'=a$ and $t'=b$. For the equal-mass-particle, pion-pion and nucleon-nucleon scatterings, the bounding curves which are represented by (3·4) coincide with those of the Mandelstam weight functions derived by Mandelstam,[3] who used essentially the lowest order graphs only.

Conversely, if the bounding curve of the Mandelstam weight function is

$$t' = \varphi(s') \quad \text{with } s' > a, \tag{3·5}$$

then the corresponding support of our integral representation will become

$$\alpha \geqq \operatorname*{Min}_{s'>a}[xs' + (1-x)\varphi(s')]. \tag{3·6}$$

As for the pion-nucleon scattering, Mandelstam[3] derived the bounding curves by making use of some lower order graphs which contain intermediate states having the lowest masses. Unlike the above three cases, since there are many graphs whose structures are essentially different from the graphs considered by him, it is not known whether or not his bounding curves are correct in every order of perturbation theory.

Appendix A. On the validity of (2·3)

A1. Fundamental Properties of W_h

We first summarize the fundamental properties of W_h. We consider a connected Feynman graph G in which N and n are the numbers of internal lines and of independent circuits, respectively. The set of external lines in G is denoted by g, and h stands for its subset. The complement of h is denoted by h^*. We denote by $\mathbf{S}(h)$ the set of intermediate states dividing g into h and h^*. Then the expression for W_h is given by

$$W_h \equiv \sum_{S \in \mathbf{S}(h)} (U^{(H_1)} U^{(H_2)} \prod_{i \in S} x_i). \tag{A·1}$$

Here H_1 and H_2 are the two subgraphs which are obtained by opening the lines of the intermediate state S, and $U^{(H)}$ is the U-function* for the graph H.

(a) W_h is an $n+1$-th order homogeneous polynomial consisting of positive terms only.

(b) W_h is linear with respect to every x_i, namely,

$$\forall i \quad \partial^2 W_h/\partial x_i^2=0. \tag{A·2}$$

The above two properties follow from the definition of W_h immediately (see I).

(c) If a line i is a one-particle line[4] in G, we have

$$\partial W_h/\partial x_i=0 \tag{A·3}$$

as far as $\{i\}\equiv S\notin \mathbf{S}(h)$. If not so, $\partial W_h/\partial x_i$ is equal to the W_h-function of the graph G' which is obtained from G by *eliminating* the line i.

Proof. Since the U-function cannot contain a parameter which corresponds to a one-particle line, (A·3) follows from (A·1) immediately. So we consider the case in which i is not a one-particle line.

The correspondence between intermediate states in G and in G' are as follows.

i) If $S\ni i$, then $S'\equiv S-\{i\}$ is an intermediate state in G'.

ii) If $S\not\ni i$ and if any line of S is connected with any external lines without passing through the line i, then $S'\equiv S$ is an intermediate state in G', because any external lines then are mutually connected in G' when any proper subset of S' is opened.

iii) If $S\not\ni i$ and if a line of S cannot be connected with an external line without passing through the line i, then S is not an intermediate state in G', because a proper subset of S then becomes an intermediate state in G'.

Conversely, for an intermediate state S' in G', $S\equiv S'+\{i\}$ becomes an intermediate state in G if G does not separate when S' is opened in G, otherwise S' is an intermediate state in G. Thus the correspondence is one-to-one except for the case iii).

We will show that the expression

$$\partial/\partial x_i[U^{(H_1)}U^{(H_2)}\prod_{j\in S}x_j] \tag{A·4}$$

is equal to the term corresponding to S' in G' for the cases i) and ii). This is evident for the former case. For the latter case it is sufficient in general (because of $H_1\cap H_2=\phi$) to show that $\partial U/\partial x_i$ is equal to the U-function in the graph which is obtained by eliminating the line i. From Theorem 5–1 of N,** putting $H=G-\{i\}$ (where G should be identified with H_1 or H_2 in the present application) we get

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* $U\equiv\sum x_{\nu_1}x_{\nu_2}\cdots x_{\nu_n}$, where the summation goes over all basic sets $(\nu_1,\nu_2,\cdots,\nu_n)$. Here n stands for the number of independent circuits, and a basic set means a set of n internal lines such that any circuit contains at least one of its elements.

** N represents reference 4).

$$U=U^{(H)}x_i+x_i\text{-independent terms,} \tag{A·5}$$

namely, $\partial U/\partial x_i=U^{(H)}$.

Finally, we must show that (A·4) vanishes for the case iii). We observe that the line i does not belong to any circuit in the subgraphs H_1 and H_2 when S is opened, because if $\exists C\ i\in C\subset H_1+H_2$, then any line of S would be connected with any external line *via* $C-\{i\}$ instead of *via* i if connected *via* i. So $U^{(H_1)}U^{(H_2)}$ does not contain x_i. This means (A·4) is zero. Q. E. D.

(d) If a line i alone composes a circuit, we have

$$W_h=0 \quad \text{when } x_i=0. \tag{A·6}$$

If not so, $W_h(x_i=0)$ is equal to the W_h-function of the graph $\hat{G}$ which is obtained from G by *shrinking* the line i.

Proof. For the former case (A·6) is evident because every term in (A·1) must contain the factor x_i. For the latter case the correspondence between intermediate states in G and in $\hat{G}$ is as follows.

i) If $S\ni i$, S is not an intermediate state in $\hat{G}$.

ii) If $S\not\ni i$, the intermediate states in G are the same as those in $\hat{G}$, because to shrink the line i is a continuous operation irrelevant to S, which cannot change connection properties.

For the case i) $\prod_{j\in S}x_j$ obviously vanishes when $x_i=0$.

For the case ii), from Theorem 5–1 of N, putting $H=\{i\}$ we see that the part irrelevant to x_i in U is $U^{(R)}$ in general where R is the reduced graph which is obtained by shrinking the line i. Therefore the expression $U^{(H_1)}U^{(H_2)}\prod_{j\in S}x_j$ is reduced to the corresponding one of $\hat{G}$ when $x_i=0$. Q. E. D.

(e) If $h\neq k$, k^*, then W_h and W_k have no common terms.

Proof. This is self-evident for $N=0$. So we use mathematical induction with respect to N. If W_h and W_k had a common term $x_{\nu_1}x_{\nu_2}\cdots x_{\nu_{n+1}}$, $\partial W_h/\partial x_{\nu_1}$ and $\partial W_k/\partial x_{\nu_1}$ would have a common term. If ν_1 is a one-particle line, at least one of them must vanish on account of (A·3). If not so, they are the W_h and W_k in a graph in which the number of internal lines is $N-1$. In both cases the existence of a common term leads to inconsistency. Q. E. D.

A2. Use of mathematical induction

Our aim is to show the inequality (2·3). The general proof for it is not yet accomplished, but the problem is much simplified by the use of mathematical induction with respect to N.

We write

$$R_1\supset R_2 \tag{A·7}$$

for two polynomials R_1 and R_2 if R_1-R_2 contains no negative term. $R_1\supset R_2$ follows from (A·7) when $x_i\geq 0$, though the inverse is not true in general. What we want to show is

$$W_A W_D \supset W_{AB} W_{AC}. \tag{A·8}$$

Because of the property (b), $W_h W_k$ is at most quadratic with respect to every x_i. We write the linear part and the remaining one of a polynomial R by $L(R)$ and $Q(R)$ respectively, i. e.

$$R = L(R) + Q(R)$$

with

$$\forall i \quad \partial^2 L(R)/\partial x_i^2 = 0. \tag{A·9}$$

Then we have

$$Q(W_A W_D) \supset Q(W_{AB} W_{AC}). \tag{A·10}$$

This is shown in the following way.

If (A·10) did not hold, there would be a term $x_{\lambda_1} x_{\lambda_2} \cdots x_{\lambda_{2n+2}}$ for which the numerical coefficient in the right-hand side is larger than that in the left-hand side. By the definition of Q at least two among $\lambda_1, \lambda_2, \cdots, \lambda_{2n+2}$ must be identical. We denote such a line by i. Then we get

$$(\partial^2/\partial x_i^2)\, Q(W_A W_D) \not\supset (\partial^2/\partial x_i^2)\, Q(W_{AB} W_{AC}). \tag{A·11}$$

On the other hand, the property (b) leads to

$$\frac{\partial^2}{\partial x_i^2} Q(W_h W_k) = \frac{\partial^2}{\partial x_i^2} (W_h W_k) = 2 \frac{\partial W_h}{\partial x_i} \frac{\partial W_k}{\partial x_i}. \tag{A·12}$$

From the property (c) and the assumption of induction we obtain

$$\frac{\partial W_A}{\partial x_i} \frac{\partial W_D}{\partial x_i} \supset \frac{\partial W_{AB}}{\partial x_i} \frac{\partial W_{AC}}{\partial x_i}. \tag{A·13}$$

(Note that if i is a one-particle line then we always have

$$\frac{\partial W_h}{\partial x_i} \frac{\partial W_k}{\partial x_i} = 0 \tag{A·14}$$

as far as $h \neq k, k^*$ because $S \equiv \{i\}$ cannot be contained in both $\boldsymbol{S}(h)$ and $\boldsymbol{S}(k)$.) (A·11) with (A·12) is inconsistent with (A·13). Thus (A·10) must be true.

Therefore we have only to prove

$$L(W_A W_D) \supset L(W_{AB} W_{AC}). \tag{A·15}$$

If $N < 2n+2$, it is obvious that

$$L(W_A W_D) = L(W_{AB} W_{AC}) = 0. \tag{A·16}$$

Next, we consider the case $N > 2n+2$. If the right-hand side of (A·15) were larger than the left-hand side in the coefficient of $x_{\lambda_1} x_{\lambda_2} \cdots x_{\lambda_{2n+2}}$, then this property would survive also when $x_i = 0$, where i is one of the lines different from any one of $\lambda_1, \lambda_2, \cdots, \lambda_{2n+2}$. But according to the property (d) both sides of (A·15) *either* vanish *or* are reduced to those for a graph in which the num-

ber of internal lines is $N-1$. This is inconsistent with the assumption of induction.

Thus *our task is reduced to comparing the coefficients of* $\prod_{i=1}^{N} x_i$ *in* $W_A W_D$ *and in* $W_{AB} W_{AC}$ *for the case* $N=2n+2$ *only.*

A3. Further reduction of the problem

Let $G^{(AD)}$ be the graph in which the two external vertices A and D are identified, and $W^{(AD)}$ be the U-function for the graph $G^{(AD)}$.

Lemma 1

$$W_A + W_D + W_{AB} + W_{AC} \subset W^{(AD)}. \tag{A·17}$$

Proof. Non-overlapping in the left-hand side is clear from the property (e).

Now, consider a term $\prod_{i\in H} x_i \in W_A$, for instance. According to the definition of W_A we have

$$\exists S \in \boldsymbol{S}(A) \quad H = F_1 + F_2 + S, \tag{A·18}$$

where F_1 and F_2 are some basic sets in H_1 and in H_2, respectively. We will show that H is a basic set in $G^{(AD)}$ (then $W_A \subset W^{(AD)}$ follows).

A circuit in $G^{(AD)}$ corresponds to a path[4] $P(AD)$ or a circuit C in G. It is evident that $P(AD) \cap S \neq \phi$. For a circuit C such that $C \cap S = \phi$, because of either $C \subset H_1$ or $C \subset H_2$ we get $C \cap (F_1 + F_2) \neq \phi$. Therefore any circuit in $G^{(AD)}$ always intersects H. Thus H is a basic set in $G^{(AD)}$ (of course $\nu(H) = n+1$). Q. E. D.

We consider a graph G in which $N=2n+2$. We call a pair (H, H') satisfying the following properties a "division of G".

$$\begin{aligned} &\text{i)} \quad H \cap H' = \phi, \\ &\text{ii)} \quad H + H' = G, \\ &\text{iii)} \quad \nu(H) = \nu(H') = n+1. \end{aligned} \tag{A·19}$$

In particular, if

$$\prod_{i\in H} x_i \in W^{(AD)} \quad \text{and} \quad \prod_{i\in H'} x_i \in W^{(AD)}, \tag{A·20}$$

then the division (H, H') is called "allowable". Moreover, if

$$\prod_{i\subset H} x_i \in W_h \quad \text{and} \quad \prod_{i\in H'} x_i \in W_k, \tag{A·21}$$

then it is called "(h, k)-allowable". Then what we should show is that the number of (AB, AC)-allowable divisions is not more than that of (A, D)-allowable divisions.

Lemma 2. In order that a division (H, H') is allowable, it is necessary and sufficient that both H and H' are connected graphs in $G^{(AD)}$ and touch all the vertices of $G^{(AD)}$. Here the statement that H *touches* a vertex E means that H contains an internal line whose end point is E.

Proof. If (A·20) holds, both H and H' are basic sets in $G^{(AD)}$. If H did not touch a vertex E in $G^{(AD)}$, the star[4] corresponding to E would be included in H'. This is inconsistent with the fact that H' is a basic set. Hence H must touch every vertex in $G^{(AD)}$.

H cannot include any circuit, because $\exists C\ C\subset H$ implies $H'\cap C=\phi$ which means H' is not a basic set. So the number of the vertices in $G^{(AD)}$ which H touches is

$$\nu(H)+l=n+1+l, \tag{A·22}$$

where l is the number of disconnected parts of H in $G^{(AD)}$. On the other hand the number of the vertices in $G^{(AD)}$ is $N-(n+1)+1=n+2$. We therefore get $l=1$, namely, H is a connected graph in $G^{(AD)}$. H' is similar.

Next, we prove the inverse proposition. If $\exists C\subset G^{(AD)}\ C\cap H=\phi$, we get $C\subset H'$. However, since H' is a connected graph in $G^{(AD)}$ which consists of $n+1$ lines and tonches $n+2$ vertices, it cannot include any circuit C. Therefore we must have $\forall C\subset G^{(AD)}\ C\cap H\neq\phi$. This means H is a basic set in $G^{(AD)}$. H' is similar. Q. E. D.

Lemma 3. If (H, H') is allowable, the intermediate state of G included in H(or H') is unique and contained in $\boldsymbol{S}(A)+\boldsymbol{S}(D)+\boldsymbol{S}(AB)+\boldsymbol{S}(AC)$.

Proof. If H is a basic set in $G^{(AD)}$, all paths $P(AD)$ must intersect H. Hence H is a generalized intermediate state[1] in G. So H includes an intermediate state S in G.

If $S\in\boldsymbol{S}(B)+\boldsymbol{S}(C)+\boldsymbol{S}(AD)$, H' could not connect the external vertex $A(\equiv D)$ with B or C in $G^{(AD)}$. This is inconsistent with Lemma 2. So $S\in\boldsymbol{S}(A)+\boldsymbol{S}(D)+\boldsymbol{S}(AB)+\boldsymbol{S}(AC)$.

If $\exists S, S'\subset H(S\neq S')S,\ S'\in\boldsymbol{S}(A)+\boldsymbol{S}(D)+\boldsymbol{S}(AB)+\boldsymbol{S}(AC)$, then H' could not connect A with a vertex lying between S and S'. This is inconsistent with Lemma 2. Q. E. D.

We call S the "characteristic intermediate state of H".

Lemma 4. If the characteristic intermediate state S belongs to $\boldsymbol{S}(h)$ $(h=A, D, AB, AC)$, then we have

$$\prod_{i\in H} x_i\in W_h. \tag{A·23}$$

Proof. Since (H, H') is allowable, any circuit C in G intersects H. So any circuit C such that $C\cap S=\phi$ intersects $H-S$. Let H_1 and H_2 be the two subgraphs in $G-S$. Then we have

$$\prod_{i\in H} x_i\in U^{(H_1)}U^{(H_2)}\prod_{i\in S} x_i\subset W_h. \tag{A·24}$$

Q. E. D.

We can thus find the numerical coeffecients of $\prod_{i=1}^{N} x_i$ in W_AW_D and in $W_{AB}W_{AC}$ purely graphically without remembering the complicated expression (A·1).

Namely, *in order to find the coefficient of* $\prod_{i=1}^{N} x_i$ *in* $W_h W_k$ *we have only to count the number of divisions* (H, H') *such that*

i) when A *and* D *are identified, both* H *and* H' *become connected graphs and touch all vertices, and*

ii) H *includes an intermediate state* $S \in \mathbf{S}(h)$, *while* H' *does* $S' \in \mathbf{S}(k)$.

A4. Lower order graphs

Since the general proof is not yet accomplished, we will check the validity of (2·3) for lower order graphs. In doing this, we may not pay regard to the following graphs:

i) The graph in which A and D or B and C are connected by only one line.

ii) The graph which involves a 1-vertex or an external 2-vertex or a circuit consisting of only one line.

iii) The graph which involves an internal 2-vertex or a circuit consisting of two lines.

For the graphs stated in i) and ii) no (AB, AC)-allowable division is possible.*

For the case iii), denoting by i and j the lines which touch the 2-vertex or are included in the circuit mentioned above, we see that *either* $i \in H$ and $j \in H'$ *or* $j \in H$ and $i \in H'$ for any allowable division (H, H') (see Lemma 2 and its proof). So the problem is reduced to that for the graph which is obtained from G by eliminating i and shrinking j.

In Fig. 1 we list all non-trivial graphs for $n=1, 2, 3$. Since (2·3) is symmetric under the interchanges of A and D and/or B and C, we have no need to specify A and D or B and C. Table I displays the numbers of (A, D)-allowable and (AB, AC)-allowable divisions for the graphs shown in Fig. 1. In the first row we write the numbers

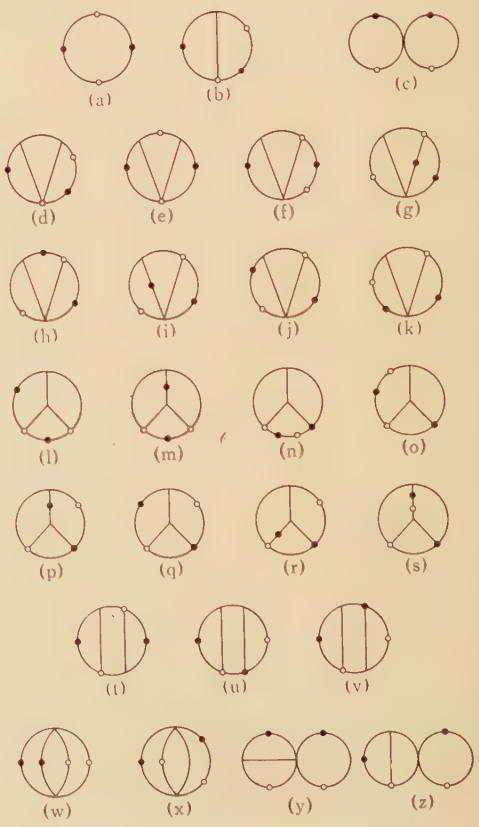

Fig. 1.

* As for the graph having an external 2-vertex, the existence of a one-particle line contradicts the uniqueness of the characteristic intermediate state for an (AB, AC)-allowable division.

of (A, D)-allowable divisions in the case in which white small circles are identified with A and D and black ones are B and C. In the second row we write those in the case in which the identifications are reversed. As for (AB, AC)-allowable divisions no such specification is necessary, because $W_{AB}W_{AC}$ is symmetric under the interchange of pairs (A, D) and (B, C). Their numbers are written in the third row.

Table I. Coefficients of the term $\prod_{i=1}^{N} x_i$

graph	a	b c	d e f g h i j k	l m n o p q r s	t u v	w x	y z
$W_A W_D$	1	1 1	1 1 3 2 2 2 2 1	3 2 6 3 3 3 3 2	1 2 1	4 3	3 1
$W_B W_C$	1	2 1	4 5 6 4 4 3 3 2	6 6 6 4 2 3 3 2	5 2 1	4 3	3 1
$W_{AB} W_{AC}$	1	1 0	1 1 3 2 1 0 1 1	3 2 6 3 1 2 1 1	1 1 0	0 1	0 0

From Table I and the above consideration we see that our conjecture (2·3) is always valid at least as far as $N<10$. We checked its validity also for some higher order graphs.

Appendix B. Proof of (2·6) with (2·9) for the pion-pion scattering

We make use of the method developed in Part III of N. Our task is to show that for *either*

$$k_A=(2, 1),\quad k_B=(-2, 1),\quad k_C=(2, -1),\quad k_D=(-2, -1) \tag{B·1a}$$

or

$$k_A=(2, 1),\quad k_B=(2, -1),\quad k_C=(-2, 1),\quad k_D=(-2, -1), \tag{B·1b}$$

where

$$(a, b)\equiv a k_1+b k_2,\quad -k_1^2=-k_2^2=m_\pi^2,\quad k_1 k_2=0, \tag{B·2}$$

we can always choose internal momenta (a_i, b_i) satisfying

$$\forall i\quad a_i^2+b_i^2\leqq (m_i/m_\pi)^2 \tag{B·3}$$

under the momentum conservation at each vertex.

In doing this, it is essential to make use of the pseudoscalarity of the pion. We assume that all other particles interacting with the pion have masses larger than $2m_\pi$. If the kaon is scalar, it should be replaced by two pions. If there are baryon closed loops, each baryon line should be replaced by one or two pion lines alternately. Then the number of lines which start from each vertex becomes always even, and the condition (B·3) is satisfied if

$$\forall i\quad a_i^2+b_i^2\leqq 1 \tag{B·4}$$

in this new graph. From the above property we immediately see that

$$\forall S \in \boldsymbol{S}(A)+\boldsymbol{S}(B)+\boldsymbol{S}(C)+\boldsymbol{S}(D), \quad \nu(S) \text{ is odd,} \tag{B·5a}$$

$$\forall S \in \boldsymbol{S}(AB)+\boldsymbol{S}(AC)+\boldsymbol{S}(AD), \quad \nu(S) \text{ is even.} \tag{B·5b}$$

(A) The case $\exists S \in \boldsymbol{S}(AB)\ \nu(S)=2$.

Opening S, we get two subgraphs G_I and G_F. For definiteness we consider G_I. The external lines of G_I are A, B, E, F where E and F are the two lines of S. From the consideration in § 15 of N, G_I must have one of the graphs shown in Fig. 16 of N as its skeleton. But in the present case, since the number of internal lines which start from each external vertex is odd we can start from A along another line, and always proceed along the lines which are not drawn in Fig. 16 of N until we reach another external vertex, because the number of lines which start from each vertex is even. So our skeletons are modified as shown in Fig. 2

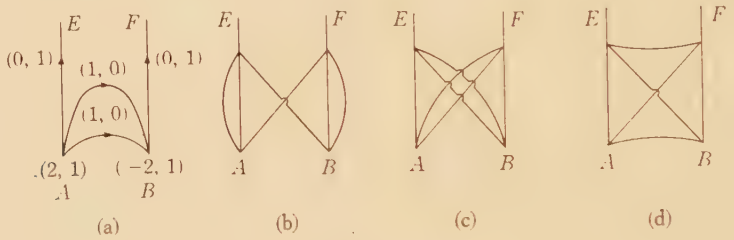

Fig. 2.

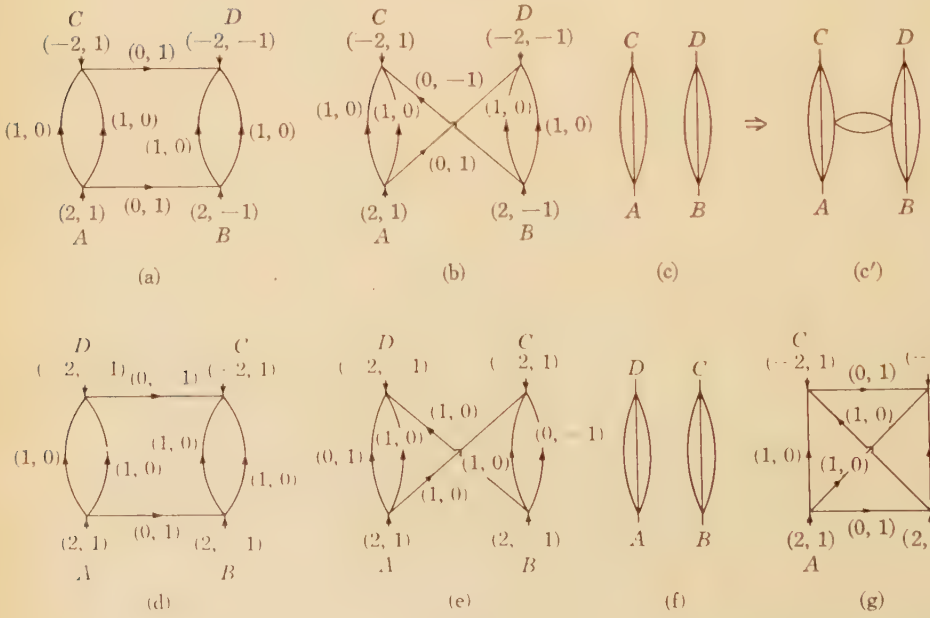

Fig. 3.

Fig. 2 (b), (c) and (d) are special cases of Fig. 2 (a). Thus, after all, as a skeleton of G_I we have only to consider Fig. 2 (a) as expected intuitively. For this skeleton we choose (a_i, b_i) as indicated in Fig. 2 (a). We can deal with G_F in the same way.

(B) The case $\forall S \in \boldsymbol{S}(AB)$ $\nu(S) \geqq 4$ (including the case $\boldsymbol{S}(AB) = \phi$)

According to Theorem 13–4 of N, there are four non-overlapping paths:

$$\text{i)} \quad P_1(AC),\ P_2(AC),\ P_1(BD),\ P_2(BD)$$

$$\text{or} \quad \text{ii)} \quad P_1(AD),\ P_2(AD),\ P_1(BC),\ P_2(BC)$$

$$\text{or} \quad \text{iii)} \quad P(AC),\ P(AD),\ P(BC),\ P(BD).$$

Using the above argument, we get seven skeletons shown in Fig. 3. Since our graph is a connected one, Fig. 3 (c) should be replaced by Fig. 3(c′), which is a special case of Fig. 3 (a). Fig. 3 (f) is similar. For the other graphs we choose (a_i, b_i) as indicated in respective graphs.

Our proof is thus completed.

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Internal Structure of Very Massive Stars

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Some models are computed for very massive stars ($M=46.8M_\odot$) consisting of four regions: (I) a homogeneous radiative envelope, (II) a semiconvective intermediate zone, (III) a radiative intermediate zone and (IV) a homogeneous convective core. The semiconvective intermediate zone is formed in the consequence of growing convective instability which is characteristic in very massive stars and the distribution of composition in this region is determined so as to maintain the convective neutrality taking account of the effect of the gradient of mean molecular weight. It is shown that the very massive stars evolve essentially in the same way as less massive stars in their early phase and the growing convective instability has little influence on the effective temperature and luminosity of stars.

§ 1. Introduction

In the previous paper,[1]† we proposed some models for stars of $46.8\,M_\odot$ in their early evolutional phase of hydrogen burning. One of the difficulties in constructing the models of very massive stars is to find the change in the distribution of the hydrogen content which is consistent with the growing convective instability. That is, as a star heavier than 10 solar masses consumes hydrogen in its central region, the boundary of convective core moves outwards.[1),2)]

Whether the newly arised convective instability induces the mixing of a large fraction of stellar mass or not remains open. Schwarzschild and Härm[2)] constructed the stellar models of this type as follows. The growing convective instability induces a large scale of mixing between the radiative envelope and the convective core and thus there sets up a new semiconvective intermediate zone where the chemical composition is adjusted by convective mixing so as to maintain the convective neutrality. Their treatment, however, does not seem to be well justified since the condition of the convective neutrality used by them did not contain a term arising from the existence of a gradient of mean molecular weight so that the large scale of mixing assumed by them must be due to a long distant overshooting of the convective motion into the intermediate zone which is stable against convection.

In "I", the effect of the gradient of mean molecular weight was taken into account. The construction of the model, however, is not satisfactory because

† This paper will be referred to as "I" in the following.

the radiative energy flux on the boundary between the convective core and the intermediate zone is not continuous, and so the convective motion, which has a small but finite velocity on this boundary, will overshoot into the semiconvective zone. The convective neutrality in the latter zone will break down as a consequence of this mixing and the inner convective core grows until the chemical composition takes a distribution such that the inner boundary of the intermediate zone will be in radiative equilibrium instead of being convective neutral. This means that the fitting condition in "I" at the inner boundary of intermediate zone is not correct. In this paper a consistent model is proposed to remove the above difficulty.

§ 2. The construction of model and the results

The model consists of four regions as schematically shown in Fig. 1: (I) a homogeneous radiative envelope, (II) a semiconvective outer intermediate zone (satisfying Eq. (1) in "I"), (III) a radiative inner intermediate zone, and (IV) a homogeneous convective core. The following subscripts are used to denote the quantities in each part: 0 for the point where the mass fraction is equal to that of the convective core of the initial model, 1 for the interface between radiative envelope and outer semiconvective intermediate zone, 2 for the inner boundary of semiconvective intermediate zone, 3 for the outer boundary of radiative intermediate zone and 4 for the interface between the radiative intermediate zone and convective core.

The construction of an inhomogeneous model is performed in an approximate way as follows: First, the assumed intermediate zone in radiative equilibrium, which would be formed between radiative envelope and convective core if there were no growing convective instability as in the case of less massive star, is determined as usual by the inward integration from the point q_0 with the assumption that the composition parameter l is expressed as $l=(q/q_0)^{-\nu}$. The assumed intermediate zone joins the region from q_0 to q_4 continuously and the variation of the composition which is expressed with the parameter ν is determined by the fitting condition at the point q_4, that is, the continuity of U, V and $(n+1)=d\log P/d\log T$.

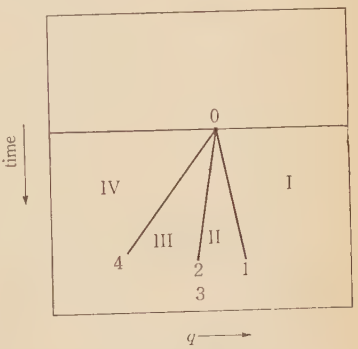

Fig. 1. The developement of the model is shown schematically. (I) a homogeneous radiative envelope, (II) a semiconvective zone formed in the consequence of growing convective instability, (III) a radiative intermediate zone and (IV) a homogeneous convective core.

Secondly, the intermediate zone is modified by taking into account the mixing between envelope and intermediate zone caused by growing convective instability.

The modification is made by the procedure that the integration over the semiconvective zone is carried out from the point q_1 inwards and then the solution obtained is fitted to the above-mentioned radiative intermediate zone. The distribution of composition is determined by the following equation,[1]

$$\frac{1}{(n+1)_{rad}} = \frac{1}{(n+1)_{ad}} + \frac{\beta}{4-3\beta}\frac{d\log\mu}{d\log P}$$

or more explicitly,

$$\frac{d\log\mu}{d\log x} = \left\{\frac{d\log t}{d\log x} - \left(\frac{8-6\beta}{32-24\beta-3\beta^2}\right)\cdot\frac{d\log p}{d\log x}\right\}\frac{(4-3\beta)}{\beta}.$$

This means that the semiconvective zone is adjusted so as to maintain convective neutrality taking account of the gradient of mean molecular weight.

During the mixing through the semiconvective zone, the total amount of hydrogen or helium must be kept constant because of the lacking of nuclear reactions in this region. The mixing leads to the depletion of helium or addition of hydrogen at the innermost region of the semiconvective zone as it grows outwards. The gradient of mean molecular weight in the neighbourhood of the interface between the semiconvective and the radiative intermediate zone becomes steeper by this mixing and thus we can approximate the variation of mean molecular weight in this region by the discontinuous one as shown in Fig. 2. The mass fraction $q_2(=q_3)$ is determined by the condition

$$\int_{q_1}^{q_2} X_{s.c.}(q)\,dq = \int_{q_0}^{q_3} X_{a.r.}(q)\,dq,$$

where $X_{s.c.}(q)$ is the distribution of hydrogen in semiconvective zone and $X_{a.r.}(q)$ is that in the assumed radiative zone. The fitting of the two solutions (radiative intermediate zone and semiconvective zone) at q_2 is made on the U-V plane according to the condition $U_3/U_2 = V_3/V_2 = \mu_3/\mu_2$. The other procedures for constructing the model are the same as employed in "I".

It must be noted here that in convective motion the driv-

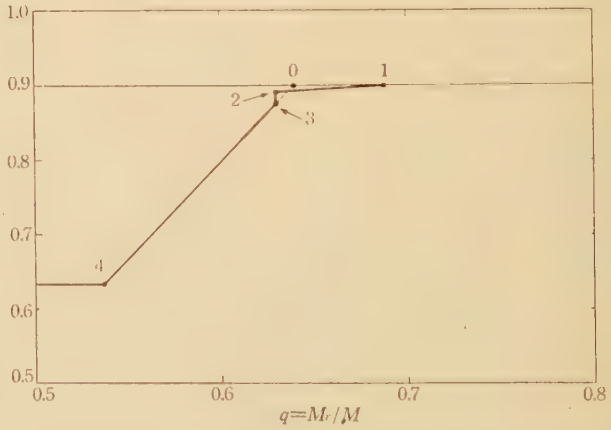

Fig. 2. The distribution of composition in star. Subscript 0=the point where the mass fraction is equal to that of the convective core, 1=interface between radiative envelope and outer semiconvective intermediate zone, 2=inner boundary of radiative intermediate zone and 4=the interface between the radiative intermediate zone and convective core.

ing force acting at the point p_b on the mass element which started from the point p_a is expressed as

$$F=\frac{GM_r}{r^2}\left\{\frac{d\log T}{d\log p}-\frac{8-6\beta}{32-24\beta-3\beta^2}-\frac{\beta}{4-3\beta}\frac{d\log\mu}{d\log P}\right\}\frac{\Delta p}{p},$$

where $\Delta p=p_b-p_a$. Thus in our model $F=0$ for $q_1>q>q_2$, and $F<0$ (the force is against the convection) for $q_4<q<q_3$. Suppose that there were some overshooting of mass motion into the radiative intermediate zone, for example, by the motion from outer or inner convective zone or from the perturbations arising from stellar rotation, and let the mean length of this overshooting be l. If $l>r_{24}$, where r_{24} is the distance from q_4 to q_2, the overshooting mass motion will induce the mixing of large scale and this will correspond to the case investigated by Schwarzschild and Härm. On the other hand, the case of $l<r_{24}$ corresponds to ours. Thus, even if the semiconvective zone investigated by Schwarzschild and Härm sets up in very early phase of little depletion of hydrogen in the core, it will develope to our scheme sooner or later as the hydrogen in the core is exhausted.

Table I. Mathematical and physical characteristics of the models.

	Homogeneous	Inhomogeneous			
	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$
$\log C$	−3.6185		−3.500		
U_i	1.49	1.07	1.16	1.18	1.59
V_i	4.63	4.59	4.07	4.13	4.03
$(n+1)_i$	3.35	3.47			3.54
$\log x_i$	−0.374	−0.416			−0.500
$\log q_i$	−0.193	−0.162	−0.200	−0.200	−0.269
$\log p_i$	0.616	0.706			1.059
$\log t_i$	−0.571	−0.523			−0.421
β_i	0.819	0.770			0.739
X_i	0.900	0.900	0.894	0.870	0.634
β_c	0.735		0.671		
$\log T_c$	7.645		7.666		
$\log L/L_\odot$	5.368		5.487		
$\log R/R_\odot$	0.850		0.930		
$\log T_e$	4.679		4.667		
ν_{34}			1.12		

The results obtained, i.e. the mathematical and physical characteristics of the models are shown in Table I and the distribution of the hydrogen in the star is plotted in Fig. 2. We see that the very massive stars evolve essentially

in the same way as less massive stars at least in their early phase and the growing convective instability has little influence on the observed physical quantities such as T_c and L of stars and modifies the shape of the distribution of the composition in them only slightly.

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Nature of the First Excited States of the Vibrational Even-Even Nuclei

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The energy of the first excited states of even-even nuclei, and the probability of the electromagnetic transition between this and the ground states are calculated. The calculation covers almost all the known vibrational even-even nuclei, and is performed by assuming that the effective interaction is a sum of the pair and Q-Q interactions, and by employing the BCS and Sawada approximations. The agreement of the obtained results with experiments are in general good, although it gets poorer for cases which have a smaller number of nucleons outside closed shells. For such cases rigorous shell model calculation is also performed and comparing the results with the above approximate ones, discussions on the validity of those approximations are made.

§ 1. Introduction

Recent development of the theory of many-body problems has made it possible to perform fairly detailed numerical analyses on the properties of lower states of nuclei in various mass number regions. In particular the method of Bardeen, Cooper and Schrieffer,[1] and Bogoliubov,[2] developed for the description of the superconductor and found to be very useful in treating the effect of the pair interaction, turned out to be the most powerful and has been applied extensively to calculate various quantities related to the rotational[3] and S. C. S. nuclei.[4] The purpose of the present paper is to extend particularly the work of KS, in which the Q-Q interaction is considered in addition to the pair interaction, to the description of the properties of the first excited 2^+ states of even-even nuclei in the vibrational regions.

When there exist both the pair and the Q-Q interactions, it is known[5] that the most convenient way is to apply first the Bogoliubov transformation[2),6] to the system of free particles, and then apply Sawada's method[7] to the system of thus obtained quasi-particles. The argument so far developed along this line[5] are, however, mostly limited to the cases in which protons and neutrons are treated separately. This assumption would be allowable when there exists only the pair interaction, but might not necessarily be appropriate when also the Q-Q interaction is to be included. It is because the Q-Q interaction is a long range force and under the above assumption works so as to polarize the protons and neutrons independently, resulting in states of rather high energy due to the increased symmetry energy. To get a correlated polarization and thus lower energy states, it would be reasonable to assume that similar Q-Q interaction works between

protons and neutrons as well.*

The derivation of the dispersion equation to determine the energy ω of the first excited 2^+ state of even-even nuclei above the ground state is explained fairly in detail in § 2, for the case in which the Q-Q interaction works also between protons and neutrons. It is seen that the obtained dispersion equation has a very reasonable form in that (i) when the Q-Q interaction between the protons and neutrons is turned off, our dispersion equation is factored into two independent dispersion equations which describe separate collective excitations of protons and neutrons, and (ii) when the above interaction is turned on, the resulting lowest 2^+ (collective) state is lower than any of the 2^+ (collective) states obtained separately for protons and neutrons.** In § 2 it is also discussed how to calculate the transition probability $B(\mathrm{E2}; 0^+ \to 2^+)$ of the γ-ray between the ground and the 2^+ collective states.

The numerical calculations of ω and $B(\mathrm{E2})$ are performed for almost all the known vibrational nuclei, by using the IBM 704 computer available at the Japan Meteorological Agency in Tokyo. The results of the calculation are presented in § 3, together with their comparison with experiment. As is seen there, fairly good agreement with experiment is obtained in general, although our present calculation does not necessarily reproduce much finer details of the experiments.

It is also found there that the agreement of our calculation with experiment gets poorer for cases in which the numbers of both the protons and neutrons are close to their respective magic numbers. For those cases some investigations on the validity of the BCS and Sawada approximations are made in § 4, by comparing the calculation of § 3 with that of the ordinary shell model one. Finally concluding remarks are given in § 5.

§ 2. Formulation of the calculation

As was mentioned in the Introduction we assume that the effective interaction that works between nucleons is given as a sum of the pair and Q-Q interactions. Therefore our Hamiltonian H is written as

$$H = H_0 + H_{pair} + H_{Q\text{-}Q}. \tag{1}$$

In (1) H_0 is the part of the Hamiltonian which describes the single-particle motion in a self-consistent potential, which we take essentially in the same way as was done by KS, i.e. the non-interacting j-j coupling shell model in a harmonic oscillator potential. H_{pair} is also taken as in KS except that in our case

* Consideration of the Q-Q interaction between protons and neutrons is made by Baranger,[5] but no detailed discussion on the dispersion equation with inclusion of this interaction has been given.

** This point has been discussed in our previous note.[8]

we treat the protons and neutrons at the same time. As we assume, however, that there is no pair interaction between protons and neutrons, they can be treated separately and thus the mathematics involved is exactly the same as in KS. Thus we shall summarize here only the main results of KS which are relevant to our present calculation.

First we have to solve the following BCS equations:

$$(G/2)\sum_j (j+1/2)/\sqrt{(\varepsilon_j-\lambda)^2+\Delta^2}=1, \tag{2-1}$$

$$\sum_j (j+1/2)\,(1-(\varepsilon_j-\lambda)/\sqrt{(\varepsilon_j-\lambda)^2+\Delta^2})=N, \tag{2-2}$$

the meaning of the notations is the same as in KS, except that G stands for G_P when we consider protons, while it stands for G_N when we consider neutrons.

The Hamiltonian $H_{Q\cdot Q}$ which induces the collective vibration is assumed to have the form

$$H_{Q\cdot Q}=-\sum_{i,j}(V_0+V_1(\tau_i\cdot\tau_j))\,r_i^2\cdot r_j^2\sum_\mu Y^*_{2\mu}(i)\,Y_{2\mu}(j). \tag{3}$$

If this Hamiltonian is expressed in the second quantized form, with respect to the non-interacting j-j coupling shell model particles, it becomes*

$$H_{Q\cdot Q}=-\frac{1}{2}\sum\langle\nu_1\tau_1,\,\nu_2\tau_2|\,(V_0+V_1(\tau_1\cdot\tau_2))\,r_1^2\cdot r_2^2\sum_\mu Y^*_{2\mu}(1)\,Y_{2\mu}(2)\,|\nu_3\tau_3,\,\nu_4\tau_4\rangle \tag{4}$$

$$\times a^+_{\nu_1\tau_1}\,a^+_{\nu_2\tau_2}\,a_{\nu_3\tau_3}\,a_{\nu_4\tau_4}.$$

Here $\nu_i=(n_i l_i j_i m_i)$ describes a set of quantum numbers of a single-particle orbit, except τ_i which specifies whether the i-th particle is a proton or a neutron.

By performing the Bogoliubov transformation[2),6)] on $a_{\nu_i\tau_i}$ and $a^+_{\nu_i\tau_i}$, and then collecting the terms which are to be retained under the random-phase approximation,[7)] $H_{Q\cdot Q}$ of (2) is reduced to

$$H_c=H^c_{PP}+H^c_{NN}+H^c_{NP}+H^{c,\,o\cdot o}_{NP\cdot NP}, \tag{5}$$

where

$$H^c_{PP}=-\frac{G_0}{2}\sum_{\beta\gamma}g_P(\beta)\,g_P(\gamma)\sum_q\{2P^{\beta+}_{2q}P^{\gamma}_{2q}+(-)^q(P^{\beta+}_{2q}P^{\gamma+}_{2-q}+P^{\beta}_{2-q}P^{\gamma}_{2q})\}, \tag{5-1}$$

$$H^c_{NN}=-\frac{G_0}{2}\sum_{\beta\gamma}g_N(\beta)\,g_N(\gamma)\sum_q\{2N^{\beta+}_{2q}N^{\gamma}_{2q}+(-)^q(N^{\beta+}_{2q}N^{\gamma+}_{2-q}+N^{\beta}_{2-q}N^{\gamma}_{2q})\}, \tag{5-2}$$

$$H^c_{NP}=-\frac{G_1}{2}\sum_{\beta\gamma}g_P(\beta)\,g_N(\gamma)\sum_q\{2P^{\beta+}_{2q}N^{\gamma}_{2q}+(-)^q(P^{\beta+}_{2q}N^{\gamma+}_{2-q}+P^{\beta}_{2-q}N^{\gamma}_{2q})\}$$

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* The matrix element $\langle 1,2|H_{Q\cdot Q}(1,2)|3,4\rangle$ is defined as

$$\langle 1,2|H_{Q\cdot Q}(1,2)|3,4\rangle=\int\Psi^*_{\nu_1\tau_1}(1)\,\Psi^*_{\nu_2\tau_2}(2)\,H_{Q\cdot Q}(1,2)\,\Psi_{\nu_3\tau_3}(2)\,\Psi_{\nu_4\tau_4}(1)\,dv_1dv_2.$$

$$-\frac{G_1}{2}\sum_{\beta\gamma} g_N(\beta) g_P(\gamma) \sum_q \{2N^{\beta+}_{2q} P^{\gamma}_{2q} + (-)^q (N^{\beta+}_{2q} P^{\gamma+}_{2-q} + N^{\beta}_{2-q} P^{\gamma}_{2q})\}, \quad (5\text{–}3)$$

and

$$\begin{aligned} G_0 &= V_0 + (1/4)\, V_1, \\ G_1 &= V_0 - (1/4)\, V_1. \end{aligned} \quad (5\text{–}4)$$

In (5) $g_{P(N)}(\beta) = \langle \nu_1 \| r^2 Y_2 \| \nu_2 \rangle (u_{\nu_1} v_{\nu_2} + v_{\nu_1} u_{\nu_2}) / \sqrt{5(1+\delta_{\nu_1\nu_2})}$, while u_j and v_j are the well-known fractional occupation parameters and $\beta = (\nu_1 \nu_2)$ means a pair of proton (or neutron) orbits. $P^{\beta+}_{2q} = (1+\delta_{j_1 j_2})^{-1/2} \sum \langle j_1 j_2 m_1 m_2 | 2q \rangle \alpha^+_{j_2 m_2} \alpha^+_{j_1 m_1}$ is an operator which creates a proton pair β coupled to an angular momentum 2, while $N^{\beta+}_{2l}$ is the corresponding operator for a neutron pair. Finally, $H^{c.o-o}_{NP-NP}$ is effective, within the approximation employed in the present paper, only in odd-odd nuclei and thus can be discarded.

To set up the dispersion equation, we first calculate the commutation relations of $N^{\alpha+}_{2l}$, etc., with the total Hamiltonian. Under the Boson approximation,[7] the results are

$$\begin{aligned} [N^{\alpha+}_{2q}, H] &= -E^N_\alpha N^{\alpha+}_{2q} + G_0 g_N(\alpha) \sum_\beta g_N(\beta) (N^{\beta+}_{2q} + (-)^q N^{\beta}_{2-q}) + G_1 g_N(\alpha) \sum_\beta g_P(\beta) \\ &\quad \times (P^{\beta+}_{2q} + (-)^q P^{\beta}_{2-q}), \\ [P^{\alpha+}_{2l}, H] &= -E^P_\alpha P^{\alpha+}_{2q} + G_0 g_P(\alpha) \sum_\beta g_P(\beta) (P^{\beta+}_{2q} + (-)^q P^{\beta}_{2-q}) + G_1 g_P(\alpha) \sum_\beta g_N(\beta) \\ &\quad \times (N^{\beta+}_{2q} + (-)^q N^{\beta}_{2-q}), \\ [(-)^q N^{\alpha}_{2-q}, H] &= E^N_\alpha (-)^q N^{\alpha}_{2-q} - [N^{\alpha+}_{2q}, H] - E^N_\alpha N^{\alpha+}_{2q}, \\ [(-)^q P^{\alpha}_{2-q}, H] &= E^P_\alpha (-)^q P^{\alpha}_{2-q} - [P^{\alpha+}_{2l}, H] - E^P_\alpha P^{\alpha+}_{2l}, \end{aligned} \quad (6)$$

where $E_\alpha = E_{j_1} + E_{j_2} (\alpha = (j_1 j_2))$, while E_{j_1} is the energy of a quasi-particle in an orbit j_1.

Define an operator B^{n+}_{2q} by

$$B^{n+}_{2q} = \sum_\alpha \{\Psi^n_\alpha N^{\alpha+}_{2q} + \Phi^n_\alpha (-)^q N^{\alpha}_{2-q} + H^n_\alpha P^{\alpha+}_{2q} + \Lambda^n_\alpha (-)^q P^{\alpha}_{2-q}\},$$

the superscript n specifying each normal mode, and compute the commutator of B^{n+}_{2q} with H:

$$\begin{aligned} [B^{n+}_{2q}, H] = & \sum_\alpha \Psi^n_\alpha \{-E^N_\alpha N^{\alpha+}_{2q} + G_0 g_N(\alpha) \sum_\beta g_N(\beta) (N^{\beta+}_{2q} + (-)^q N^{\beta}_{2-q}) \\ & + G_1 g_N(\alpha) \sum_\beta g_P(\beta) (P^{\beta+}_{2q} + (-)^q P^{\beta}_{2-q})\} \\ & + \sum_\alpha \Phi^n_\alpha \{E^N_\alpha (-)^q N^{\alpha}_{2-q} - G_0 g_N(\alpha) \sum_\beta g_N(\beta) (N^{\beta+}_{2q} + (-)^q N^{\beta}_{2-q}) \\ & - G_1 g_N(\alpha) \sum_\beta g_P(\beta) (P^{\beta}_{2q} + (-)^q P^{\beta}_{2-q})\} \end{aligned} \quad (7)$$

$$+\sum_{\alpha} \Pi_{\alpha}^{n}\{-E_{\alpha}^{P} P_{2q}^{\alpha+}+G_0 g_P(\alpha)\sum_{\beta} g_P(\beta)\,(P_{2q}^{\beta+}+(-)^q P_{2-q}^{\beta})$$

$$+G_1 g_P(\alpha)\sum_{\beta} g_N(\beta)\,(N_{2q}^{\beta+}+(-)^q N_{2-q}^{\beta})\}$$

$$+\sum_{\alpha} \Lambda_{\alpha}^{n}\{E_{\alpha}^{P}(-)^q P_{2-q}^{\alpha}-G_0 g_P(\alpha)\sum_{\beta} g_P(\beta)\,(P_{2q}^{\beta+}+(-)^q P_{2-q}^{\beta})$$

$$-G_1 g_P(\alpha)\sum_{\beta} g_N(\beta)\,(N_{2q}^{\beta+}+(-)^q N_{2-q}^{\beta})\}.$$

Putting (7) equal to $-\omega_n B_{2q}^{n+}$ and equating the coefficients of $N_{2q}^{\alpha+}$, etc., on both sides of this equation, the following coupled equation is obtained:

$$-\omega_n \Psi_{\alpha}^{n}=-E_{\alpha}^{N}\Psi_{\alpha}^{n}+G_0 g_N(\alpha)\sum_{\beta} g_N(\beta)\,(\Psi_{\beta}^{n}-\Phi_{\beta}^{n})+G_1 g_N(\alpha)\sum_{\beta} g_P(\beta)\,(\Pi_{\beta}^{n}-\Lambda_{\beta}^{n}), \tag{8-1}$$

$$-\omega_n \Phi_{\alpha}^{n}=E_{\alpha}^{N}\Phi_{\alpha}^{n}+G_0 g_N(\alpha)\sum_{\beta} g_N(\beta)\,(\Psi_{\beta}^{n}-\Phi_{\beta}^{n})+G_1 g_N(\alpha)\sum_{\beta} g_P(\beta)\,(\Pi_{\beta}^{n}-\Lambda_{\beta}^{n}), \tag{8-2}$$

$$-\omega_n \Pi_{\alpha}^{n}=-E_{\alpha}^{P}\Pi_{\alpha}^{n}+G_0 g_P(\alpha)\sum_{\beta} g_P(\beta)\,(\Pi_{\beta}^{n}-\Lambda_{\beta}^{n})+G_1 g_P(\alpha)\sum_{\beta} g_N(\beta)\,(\Psi_{\beta}^{n}-\Phi_{\beta}^{n}), \tag{8-3}$$

$$-\omega_n \Lambda_{\alpha}^{n}=E_{\alpha}^{P}\Lambda_{\alpha}^{n}+G_0 g_P(\alpha)\sum_{\beta} g_P(\beta)\,(\Pi_{\beta}^{n}-\Lambda_{\beta}^{n})+G_1 g_P(\alpha)\sum_{\beta} g_N(\beta)\,(\Psi_{\beta}^{n}-\Phi_{\beta}^{n}). \tag{8-4}$$

Subtract (8–2) from (8–1), then one gets

$$-\omega_n(\Psi_{\alpha}^{n}-\Phi_{\alpha}^{n})=-E_{\alpha}^{n}(\Psi_{\alpha}^{n}+\Phi_{\alpha}^{n}),$$

or

$$(E_{\alpha}^{N}-\omega_n)\Psi_{\alpha}^{n}=-(E_{\alpha}^{N}+\omega_n)\Phi_{\alpha}^{n}. \tag{9}$$

Thus putting (9) equal to ψ_{α}^{n}, say, we get

$$\Psi_{\alpha}^{n}=\psi_{\alpha}^{n}/(E_{\alpha}^{N}-\omega_n) \quad \text{and} \quad \Phi_{\alpha}^{n}=-\psi_{\alpha}^{n}/(E_{\alpha}^{N}+\omega_n). \tag{10-1}$$

Similarly we get

$$\Pi_{\alpha}^{n}=\varphi_{\alpha}^{n}/(E_{\alpha}^{P}-\omega_n) \quad \text{and} \quad \Lambda_{\alpha}^{n}=-\varphi_{\alpha}^{n}/(E_{\alpha}^{P}+\omega_n). \tag{10-2}$$

Substituting (10–1) and (10–2) in (8–1) and (8–3), a coupled equation for ψ_{α}^{n} and φ_{α}^{n} is obtained as

$$\psi_{\alpha}^{n}-G_0 g_N(\alpha)\sum_{\beta}\frac{2E_{\beta}^{N} g_N(\beta)}{(E_{\beta}^{N})^2-\omega_n^2}\psi_{\beta}^{n}-G_1 g_N(\alpha)\sum_{\beta}\frac{2E_{\beta}^{P} g_P(\beta)}{(E_{\beta}^{P})^2-\omega_n^2}\varphi_{\beta}^{n}=0,$$

$$\varphi_{\alpha}^{n}-G_0 g_P(\alpha)\sum_{\beta}\frac{2E_{\beta}^{P} g_P(\beta)}{(E_{\beta}^{P})^2-\omega_n^2}\varphi_{\beta}^{n}-G_1 g_P(\alpha)\sum_{\beta}\frac{2E_{\beta}^{N} g_N(\beta)}{(E_{\beta}^{N})^2-\omega_n^2}\psi_{\beta}^{n}=0. \tag{11}$$

Further put

$$\psi_{\alpha}^{n}=C_N^{n} g_N(\alpha) \quad \text{and} \quad \varphi_{\alpha}^{n}=C_P^{n} g_P(\alpha), \tag{12}$$

and insert into (11). Then if X_N^{n} and X_P^{n} are defined by

$$X_N^n=\sum_\beta \frac{2E_\beta^N \bar{g}_N^2(\beta)}{(E_\beta^N)^2-\omega_n^2} \text{ and } X_P^n=\sum_\beta \frac{2E_\beta^P \bar{g}_P^2(\beta)}{(E_\beta^P)^2-\omega_n^2}, \tag{13}$$

and $\bar{g}_N(\beta)$ by

$$\bar{g}_N(\beta)=g_N(\beta)/(\sqrt{5/4\pi}\langle j|r^2|j\rangle), \tag{14}$$

(11) is reduced to*

$$\begin{aligned} &C_N(1-\bar{G}_0X_N)-C_P\bar{G}_1X_P=0,\\ &C_P(1-\bar{G}_0X_P)-C_N\bar{G}_1X_N=0. \end{aligned} \tag{15}$$

The condition that C_N and C_P are non-vanishing gives the following secular equation :[8]

$$(1-\bar{G}_0X_N)(1-\bar{G}_0X_P)-\bar{G}_1^2X_NX_P=0,$$

or

$$1-\bar{G}_0(X_N+X_P)+(\bar{G}_0^2-\bar{G}_1^2)X_NX_P=0. \tag{16}$$

Here $\bar{G}_0$ and $\bar{G}_1$ are defined by

$$\bar{G}_0=G_0(5/4\pi)\overline{\langle j|r^2|j\rangle}^2, \text{ and } \bar{G}_1=G_1(5/4\pi)\overline{\langle j|r^2|j\rangle}^2, \tag{17}$$

$\overline{\langle j|r^2|j\rangle}$ being the expectation value of r^2 averaged over the single-particle states in the relevant unfilled shell.

The relation between C_N and C_P can be obtained from (15) as

$$\frac{C_N}{C_P}=\frac{\bar{G}_1X_P}{1-\bar{G}_0X_N}=\frac{1-\bar{G}_0X_P}{\bar{G}_1X_N},$$

or

$$\left(\frac{C_N}{C_P}\right)^2=\bar{G}_1X_P(1-\bar{G}_0X_P)/\bar{G}_1X_N(1-\bar{G}_0X_N). \tag{18}$$

It is thus seen that

$$C_N=\sqrt{\bar{G}_1X_P(1-\bar{G}_0X_P)} \text{ and } C_P=\sqrt{\bar{G}_1X_N(1-\bar{G}_0X_N)}. \tag{19}$$

This section will be closed by giving the expression for the probability of the E2 transition between the ground and the first excited 2^+ states.

The operator of the E2 transition is given, after Bogoliubov transformation and retaining the terms which are consistent with our present approximation, by

$$\begin{aligned} \mathfrak{M}(\mathrm{E2},\mu)=&-\{e_P\sum_\alpha \bar{g}_P(\alpha)(P_{2\mu}^{\alpha+}+(-)^\mu P_{2-\mu}^\alpha)\\ &+e_N\sum_\alpha \bar{g}_N(\alpha)(N_{2\mu}^{\alpha+}+(-)^\mu N_{2-\mu}^\alpha)\}\sqrt{5/4\pi}\ \overline{\langle j|r^2|j\rangle}. \end{aligned} \tag{20}$$

* Hereafter we drop the superscript n for simplicity.

To calculate $B(\mathrm{E2})$, it is necessary first to normalize the collective state. In other words we have to calculate N, which appear in

$$B^{+}_{2q}=\frac{1}{N}\sum_{\alpha}\{\Psi_{\alpha}N^{\alpha+}_{2q}+\Phi_{\alpha}(-)^{q}N^{\alpha}_{2-q}+\Pi_{\alpha}P^{\alpha+}_{2q}+\Lambda_{\alpha}(-)^{q}P^{\alpha}_{2-q}\}, \tag{21}$$

so that $[B_{2q}, B^{+}_{2q}]=1$. This requires that

$$\begin{aligned}
N^{2}&=\sum_{\alpha}\{(\Psi^{2}_{\alpha}-\Phi^{2}_{\alpha})+(\Pi^{2}_{\alpha}-\Lambda^{2}_{\alpha})\}\\
&=C^{2}_{N}\sum_{\alpha}\bar{g}^{2}_{N}(\alpha)\left(\frac{1}{(E^{N}_{\alpha}-\omega)^{2}}-\frac{1}{(E^{N}_{\alpha}+\omega)^{2}}\right)+C^{2}_{P}\sum_{\alpha}\bar{g}^{2}_{P}(\alpha)\left(\frac{1}{(E^{P}_{\alpha}-\omega)^{2}}-\frac{1}{(E^{P}_{\alpha}+\omega)^{2}}\right)\\
&=C^{2}_{N}\sum_{\alpha}4\omega E^{N}_{\alpha}\bar{g}^{2}_{N}(\alpha)/((E^{N}_{\alpha})^{2}-\omega^{2})^{2}+C^{2}_{P}\sum_{\alpha}4\omega E^{P}_{\alpha}\bar{g}^{2}_{P}(\alpha)/((E^{P}_{\alpha})^{2}-\omega^{2})^{2}\\
&\equiv C^{2}_{N}X'_{N}+C^{2}_{P}X'_{P}.
\end{aligned} \tag{22}$$

Thus we get

$$N=\sqrt{C^{2}_{N}X'_{N}+C^{2}_{P}X'_{P}}. \tag{23}$$

X'_{N} and X'_{P} are defined by the final version of (22) and they are related to the X_{N} and X_{P} of (13) by

$$X'_{N}=\left(\frac{dX_{N}(\omega)}{d\omega}\right) \text{ and } X'_{P}=\left(\frac{dX_{P}(\omega)}{d\omega}\right). \tag{24}$$

The $B(\mathrm{E2})$ is defined as

$$B(\mathrm{E2}\,;\,0^{+}\to 2^{+})=\left|\frac{\langle I_{i}|\mathfrak{M}(\mathrm{E2},0)|I_{f}\rangle}{\langle 22\,;\,M_{f}0|00\rangle}\right|^{2}$$

with

$$\langle 22\,;\,M_{f}0|00\rangle=\langle 22\,;\,00|00\rangle=1/\sqrt{5},$$

and thus

$$B(\mathrm{E2}\,;\,0^{+}\to 2^{+})=5|\langle I_{i}|\mathfrak{M}(\mathrm{E2},0)|I_{f}\rangle|^{2}.$$

Now

$$\langle I_{i}|\mathfrak{M}(\mathrm{E2},0)|I_{f}\rangle=\langle \Psi_{0}|\mathfrak{M}(\mathrm{E2},0)|B^{+}_{20}\Psi_{0}\rangle=\langle \Psi_{0}|[\mathfrak{M}(\mathrm{E2},0), B^{+}_{20}]|\Psi_{0}\rangle,$$

and

$$\begin{aligned}
&[\mathfrak{M}(\mathrm{E2},0), B^{+}_{20}]\\
&=-\frac{1}{N}\{e_{P}\sum_{\alpha}\bar{g}_{P}(\alpha)(\Pi_{\alpha}-\Lambda_{\alpha})+e_{N}\sum_{\alpha}\bar{g}_{N}(\alpha)(\Psi_{\alpha}-\Phi_{\alpha})\}\sqrt{5/4\pi}\,\langle j|r^{2}|j\rangle\\
&=-\frac{1}{N}\left\{e_{P}C_{P}\sum_{\alpha}\frac{2E^{P}_{\alpha}\bar{g}^{2}_{P}(\alpha)}{(E^{P}_{\alpha})^{2}-\omega^{2}}+e_{N}C_{N}\sum_{\alpha}\frac{2E^{N}_{\alpha}\bar{g}^{2}_{N}(\alpha)}{(E^{N}_{\alpha})^{2}-\omega^{2}}\right\}\sqrt{5/4\pi}\,\langle j|r^{2}|j\rangle\\
&=-\frac{1}{N}\sqrt{5/4\pi}\,\langle j|r^{2}|j\rangle\,(e_{P}C_{P}X_{P}+e_{N}C_{N}X_{N}),
\end{aligned} \tag{25}$$

with $e_P=e+e_{eff}$ and $e_N=e_{eff}$, e_{eff} being the effective charge introduced by Mottelson.[9]

Thus summarizing, we get

$$B(\mathrm{E2}\,;0^+\to 2^+)=5\cdot\frac{5}{4\pi}\langle j|r^2|j\rangle^2\,\frac{\{(e+e_{eff})C_PX_P+e_{eff}C_NX_N\}^2}{C_N^2X_N'+C_P^2X_P'}. \quad (26)$$

§ 3. Numerical calculations and comparison with experiments

The numerical calculations to be reported in this section are for ω, the energy of the first excited 2^+ state and for $B(\mathrm{E2})$, the reduced transition probability of the E2 γ-ray from the ground state to the above 2^+ state. The calculation covers almost all the known vibrational nuclei and the results will be compared with experiment in subsections B and C. In subsection A discussions are given on the values of the parameters used in the calculation.

3A. Parametric values

The calculation proceeds as follows. We first solve the BCS equation (2) to get the energy and the fractional occupation parameters u and v of the quasi-particle states. To do this we have to know the energies ε_ν of the free shell-model states, and also the magnitudes of the strength parameters G_P and G_N of the pair interaction. The energy ω is then obtained as the lowest possible solution of the dispersion equation (16), and to do this two strength parameters $\bar{G}_0$ and $\bar{G}_1$ of the Q-Q interaction have to be known. Finally $B(\mathrm{E2})$ is calculated by using (26) if the magnitude of the effective charge, e_{eff}, is given.

Appropriate values of G_P and G_N were found for S.C.S. nuclei by KS mostly from the analyses of the even-odd mass differences, and then affirmed by comparing various quantities with experiment. They find that $G_P\fallingdotseq G_N\fallingdotseq 19\,A^{-1}$ Mev is the best for Sn isotopes and nuclei with $N=28$ and 50, while $G_P\fallingdotseq G_N\fallingdotseq 23\,A^{-1}$ Mev is the best for Pb isotopes and nuclei with $N=82$. For the values to be employed in the present calculation, it seems reasonable to use the same order of magnitudes as in KS because the known even-odd mass difference and the magnitude of the energy gap are not very much different in the vibrational and the S.C.S. nuclei. Thus we search for the best values in the range $(19\text{–}23)\,A^{-1}$ Mev both for G_P and G_N in our calculation.

Once the magnitude of G_P and G_N and the energies ε_ν are fixed, then ω becomes a function of two parameters $\bar{G}_0$ and $\bar{G}_1$. One of them, $\bar{G}_0$, is to have just the same meaning as the strength parameter X of KS, for which they find that $110\,A^{-1}$ Mev is the most appropriate value in the S.C.S. nuclei. Our dispersion equation (16) can certainly be applied to S.C.S. nuclei too, and it is easy to see that in this case all the terms in (16) multiplied with $\bar{G}_1$ vanish, thus making ω a function only of $\bar{G}_0$. In other words it is possible to find an appropriate value of $\bar{G}_0$ by comparing our theoretical ω with experiment for

S.C.S. nuclei. It will be seen below that we get $\bar{G}_0=80\,A^{-1}$ Mev as the best value for Sn isotopes which is certainly much weaker than the KS value. In general it is expected, as can also be proved formally, that the Sawada approximation requires weaker $\bar{G}_0$ than does the adiabatic approximation used by KS, in accordance with the above Sn case, and thus we consider it appropriate to seek for values of $\bar{G}_0$ in the range, say, $(70\sim110)\,A^{-1}$ Mev. Similar range of value is considered also for $\bar{G}_1$, because it is expected that the long range force acts as strongly between different kinds of nucleons as it does between the same kind of nucleons.

Corresponding to the situation mentioned above that the Sawada approximation requires weaker $\bar{G}_0$ than the adiabatic approximation does, it is expected that the value of e_{eff} to be used in our calculation could be smaller than that used by KS which was $e_{eff}=1.0$. As the machine time to be required for the calculation of $B(\mathrm{E2})$ is very short, however, our code was set up so as to give $B(\mathrm{E2})$ for e_{eff} from 0.0 to 2.0 with the step of 0.2.

As for the energies ε_ν of the single-particle states, the values used by KS are also employed here, so long as they are available. The use of KS's ε_ν, however, sometimes results in ω which do not necessarily agree well with experiment, and in such cases some arbitrary modifications are made on ε_ν so as to give better results;* or Schröders results[10] are used who calculated the energies of the single-particles contained in a potential which has diffuse surface. Use is also made of Schröder's results when no information on ε_ν is available from KS. The values of ε_ν which give the best values of ω for different mass number regions are shown in the tables given in the Appendix, together with the sources from where they are taken. As is seen from these tables the single-particle orbits considered in the calculation are only those which lie in the outermost unfilled major shells.

3B. Results on ω

The results of our calculation on ω are compared with experiment[11] in Figs. 1∼6, separately for nuclei which lie in different mass number regions. Those which are considered in Fig. 1 lie in the middle of the proton shell $Z=40\sim50$ and of the neutron shell $N=50\sim80$, while those given in Fig. 2, i. e. Sn isotopes has a magic proton number $Z=50$. Those which are given in Figs. 3∼6 all have proton numbers within the shell $Z=50\sim82$. Among them those shown in Figs. 3∼4 contain isotopes which have a magic neutron number $N=82$, while those shown in Fig. 5 contain isotopes with $N=88$ and $N=90$ where it is known that a sudden transition from the vibrational to rotational nuclei occurs.

Looking at these figures it is seen that the general agreement with experi-

* In particle in the Sn region KS's ε_ν are modified so as to give better fit to the spacings of the one quasi-particle states in odd Sn isotopes.

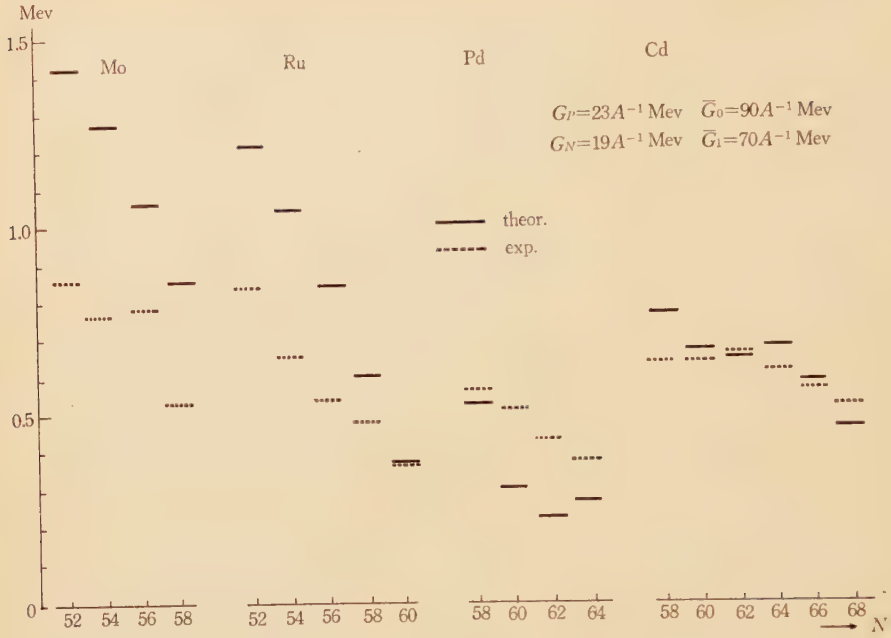

Fig. 1. ω as a function of the neutron number N for Mo, Ru, Pd and Cd isotopes. The theoretical values are shown by the full lines and the experimental values by the dotted lines. The used values of the strength parameters of the pair and the Q-Q interactions are also shown in the figure. These explanations apply also to Figs. 2~6.

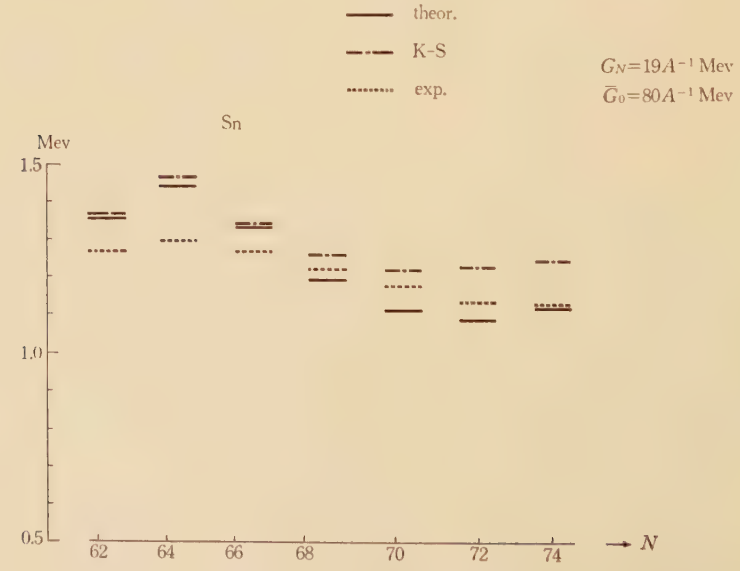

Fig. 2. ω for Sn isotopes. The results of the calculations of Kisslinger and Sorensen are also given for comparison.

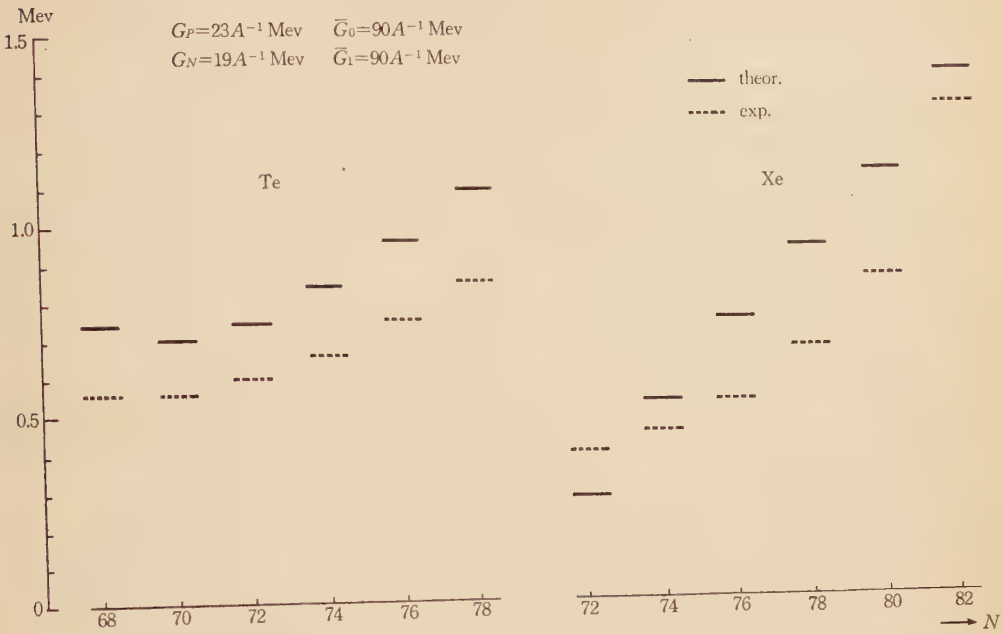

Fig. 3. ω for Te and Xe isotopes.

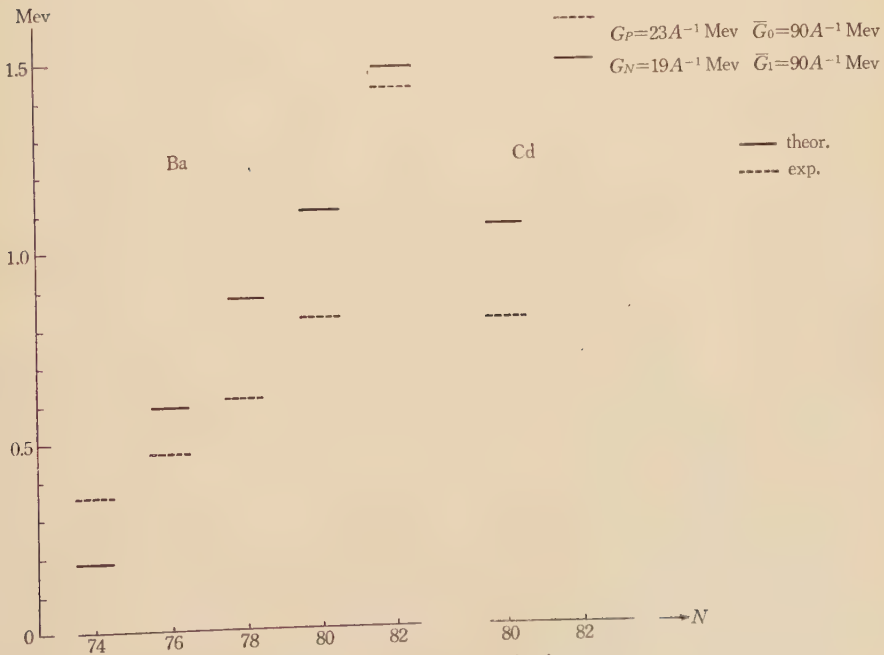

Fig. 4. ω for Ba and Ce isotopes.

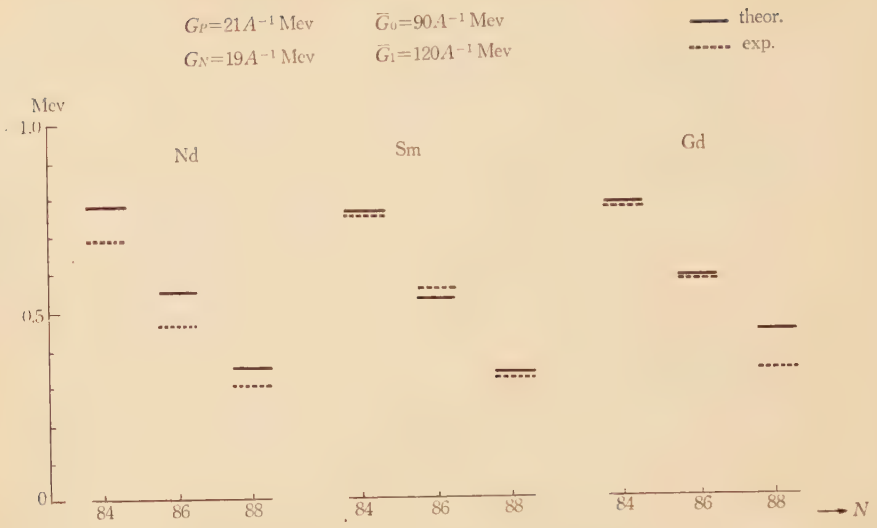

Fig. 5. ω for Nd, Sm and Cd isotopes.

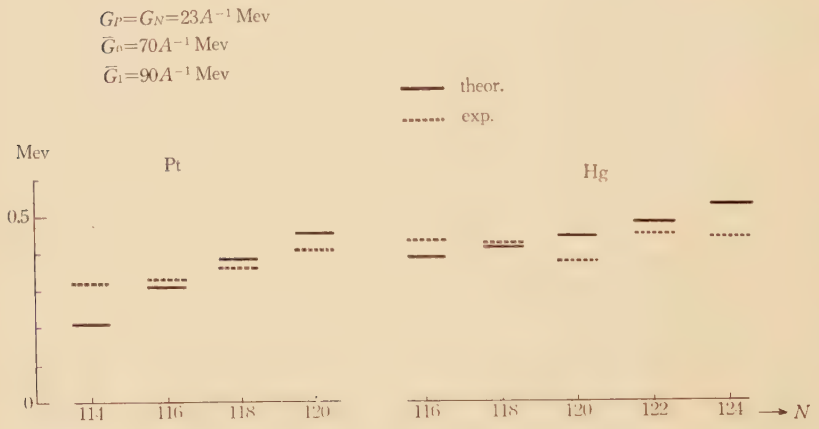

Fig. 6. ω for Pt and Hg isotopes.

ment is fairly good, and especially that the following important systematic features of the experimental ω are well reproduced: (i) Experimentally ω increases as the proton and/or neutron numbers approach the magic number and our theoretical ω also has this trend. (ii) The sudden transition from the vibrational to rotational nuclei, which occurs for the change of N from 88 to 90, is well explained in the present calculation* in the sense that for the chosen parametric values real ω are obtained as solutions of the dispersion equation (16) for $N\leqq 88$, while imaginary solutions are obtained for $N=90$. (iii) For those nuclei which

* It is to be noted that $h_{11/2}$ orbit is not considered here, and thus our explanation is different from that considered by Mottelson and Nilsson.[12]

have neutron numbers within the shell $N=50\sim82$, it is seen that ω takes a maximum value for $N\fallingdotseq64$ and a minimum value at $N\fallingdotseq70$. Such a trend is also seen in our theoretical results.

In spite of these general agreement with experiment, there remain several defects in the results so far obtained which may be enumerated as follows: (i) The agreement with experiment of the value of ω gets poorer as the number of particles (or holes) outside the closed shell becomes small. (ii) Experimentally it is seen that in several neighbouring isotopes the dependence of ω on the neutron number N are not the same, as can be seen by comparing, e.g. Hg with Pt, Sn with Cd, Ru with Mo and Sn with Te. The present calculation failed to reproduce these detailed structure of ω except for the last case. Thus in Te ω is increased when N changes from 70 to 72, while in Sn it is decreased, and this feature is well reproduced in our calculation.

3C. Results for $B(\mathrm{E2}\,;0^+\to2^+)$

As our expression of $B(\mathrm{E2})$ given by (26) contains e_{eff} as an adjustable parameter, it is possible to get exact agreement of $B(\mathrm{E2})$ with experiment,[11] by giving an appropriate value to e_{eff}. However, considering that e_{eff} reflects the core polarization, it is thought that it should have an essentially constant value for nuclei which have the same core, i. e. for nuclei which lie in the same mass number region. It thus seems meaningful to see whether this constancy of e_{eff},

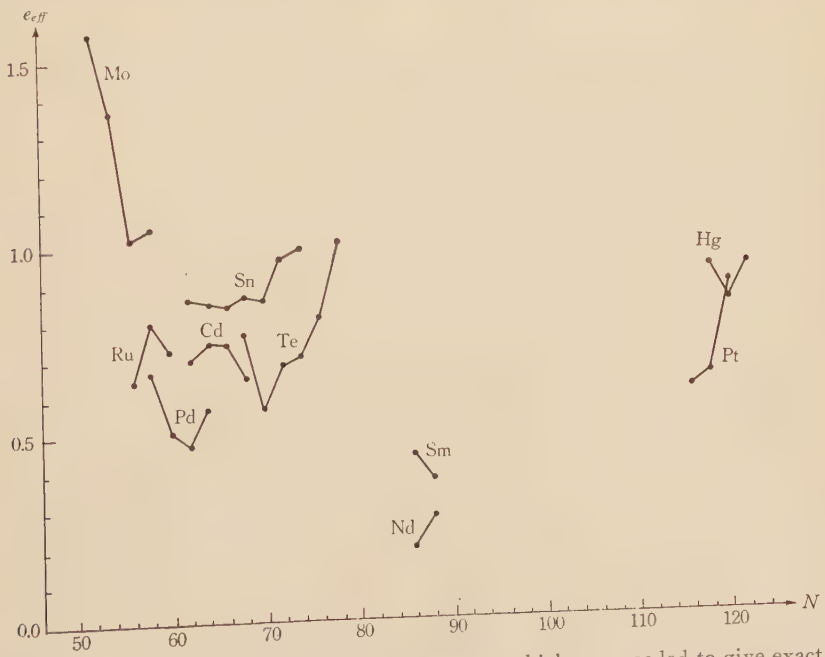

Fig. 7. The values of the effective charge e_{eff}, which are needed to give exact agreement of the theoretical $B(\mathrm{E2})$ with experiment. The abscissa is the neutron number N.

which gives exact agreement of $B(E2)$ with experiment in each nucleus, prevails or not. Such values of e_{eff}, obtained by interpolating the results calculated in the manner explained in § 2, are plotted in Fig. 7 against the neutron number N, and the points which correspond to each isotope are connected with straight lines.

It is seen that the constancy of e_{eff} is not necessarily very well established, but it is nevertheless seen that, except for Mo, Sm and Nd, e_{eff} lies in the range $0.5 \sim 1.0$. It is to be noted that the experimental uncertainty in $B(E2)$ is usually of the order of $\pm 10\%$, and in some cases as large as $\pm 20\%$.

§ 4. Discussions on the calculated results

As we have seen in § 3, the results of our calculation explain the general features of experimental data fairly well, but there still remain several defects, and in this section their possible origins will be discussed.

The origin of the defects may be enumerated as follows: (i) Oversimplification in the assumption that the effective interaction between nucleons is represented simply as a sum of the pair and the Q-Q interactions. (ii) The poorness of the BCS and Sawada approximations. (iii) Lack in the search for the various parametric values so far performed in the present work.

It is very hard to estimate the error caused by the simplifying assumption mentioned in the above item (i). From the standpoint that the simpler the model, the more useful it is, however, it seems better to try to retain this simplified model as much as possible. The discussion on this point would thus have to be postponed until more detailed analyses are performed, based on this assumption; so it will not be done here.

Concerning the above second point, i.e. the validity of the BCS and the Sawada approximations, it should be mentioned first that both of them are good approximations when the density of the single-particle states is high and the number of particles participating in the correlation is large. As has been mentioned in the end of §§ 3B, the particles which are considered to interact with each other are only those which lie in the outermost unclosed shell. This number is comparatively large when this unclosed shell is about half filled, but becomes rather small when the nucleon numbers approach the magic numbers. The situation is certainly the worst when both of the proton and neutron numbers are close to their respective magic numbers, and this fact might be the reason why the agreement of the calculated ω with experiment is so poor, e.g. for some of the lighter isotopes of Mo and Ru. (The situation is also reflected in the fact that e_{eff} for these cases are too large compared with that in other nuclei.)

It is well known that any state obtained as a solution of the BCS equation is a mixture of single-particle states, which belong to different nucleon numbers, and it is expected that the error caused by this fact becomes rather serious

when the number of nucleons gets very few. To see how strong this mixture is in such cases, we take up the neutron part of the ground state wave function* of Mo^{94} which has only two neutrons outside the closed shell. For this case the probability $P(n)$, that the state with a definite neutron number n is found in the ground state, is $P(0)=0.343$, $P(2)=0.394$ and $P(4)=0.198$; i.e. the probability of the admixed state with $n=0$ is comparable to that of the state with $n=2$, and that of the state with $n=4$ is not negligibly small.

On the other hand the main simplifications which are made in the Sawada (or the random phase) approximation are (a) to replace the total Q-Q interaction given in (2) by the effective one given in (3), which means to neglect the scattering term and; (b) to employ the Boson approximation in the evaluation of the commutation relation (4), which means to neglect the exchange term. Although the neglect of the scattering term might not be so serious in many of the cases considered in § 3, where the particle numbers outside the closed shells are comparatively large,** it is expected that it will become rather serious when the number of particles gets closer to the magic numbers. In particular when there exist only two nucleons outside the closed shells, clearly no particle-hole state can be constructed and thus the effective Q-Q interaction (3) gives vanishing contribution. In other words, if Q-Q interaction were to give finite effect in this particular case in one way or another, it means that the contribution from the scattering term must be 100%.

The reason why we obtained finite contribution of the Q-Q interaction from the neutron in the calculation of § 3, e. g. for Mo^{94}, in spite of the neglect of the scattering terms, is that, as mentioned above, there are non-vanishing admixture of the four particle states in the quasi-particle description of this nucleus, and this part gives room for a non-vanishing contribution of the particle-hole interaction. Thus what we have calculated for Mo^{94}, in § 3 is mostly the contributions from the spurious parts of the BCS wave function, although because of the existence of protons outside the closed shell, the P-N (and P-P) part of the effective Q-Q interaction gives rise to non-spurious particle-hole interactions.

It would thus be of value to treat the problem in a non-spurious way and compare the result with that of the above calculation. For this purpose the usual shell model calculation is made for Mo^{94} by using the same parametric values as used in § 3, which gave there $\omega=1.42$ Mev for this element.

First we completely neglect the existence of protons. Then if further the Q-Q interaction is neglected for the moment, the pair interaction is treated easily by using the method of Bayman,[14] and the lowest 2^+ state thus obtained

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* Here the ground state means that of the BCS system; i. e. the Q-Q interaction has not yet been considered.

** In fact Yoshida[12] estimated the effect of the scattering term on ω for the case of Sn^{118}. In this nucleus the protons are considered as inert, and so only the neutrons are considered whose number lies just in the middle of the neutron shell $N=50\sim82$. The estimated effect is found to be about 5% of that of the particle-hole interaction term.

turns out to appear at 1.61 Mev above the ground state. This is to be compared with 1.74 Mev, the energy of the lowest two quasi-particle state obtained as the solution of the BCS equation (2).

We next take into account the contribution of the Q-Q interaction, and this is done by performing a matrix diagonalization.* The lowest 2^+ state thus obtained is found at 2.25 Mev,** which is much larger than the previous value 1.42 Mev.

To make the matter more realistic, we finally take into account the existence of two protons, by assuming that they have a definite configuration $(g_{9/2}^2)$. Assuming further that the energy $E((g_{9/2}^2)_{2^+}) - E((g_{9/2}^2)_{0^+})$ equals 1.61 Mev, i. e. the energy of the above mentioned lowest 2^+ neutron state without the Q-Q interaction, and taking into account the N-P part of the Q-Q interaction, again a matrix diagonalization is performed. The energy of thus obtained lowest 2^+ state is 1.50 Mev,*** which is still larger than the previous value 1.42 Mev of § 3.

This result may mean that it is hard to get agreement with experiment at the same time for off closed shell nuclei and nuclei very close to doubly closed shells, if the same parametric values are to be used in both cases.

We might thus be forced to assume weaker values of G_P and G_N for the latter kind of nuclei.

This section will be closed by giving some consideration concerning the third point of the possible source of the defect, as mentioned in the beginning of this section, i.e. whether we have already searched for all the conceivable ranges of the parametric values.

It is not very easy to answer this question for certainty, but so long as the standpoint is taken that we should use parametric values which are fixed in each mass-number region (except the A^{-1} dependence), it is our impression that the ranges of the values for which we have so far searched G_P, G_N, $\bar{G}_0$ and $\bar{G}_1$ have been wide enough. If it were the case, the only parameters for which values we might have not yet made a thorough search are ε_ν. To show how sensitively the results depend on ε_ν, we show in Fig. 8 the values of ω for Cd isotopes computed with three different choices of ε_ν. As the comparison particularly of the cases (b) and (c) shows, between which only one ε_ν, i.e. $\varepsilon_{g_{7/2}}$ for the neutron is changed very slightly, results are rather sensitive to ε_ν. It might thus be possible to explain within the framework of the present analysis the very detailed N-dependence of ω, of which we mentioned in the end of §§ 3B, if a more detailed search of the values of ε_ν were made. Unfortunately we could

* For this calculation we take $\bar{G}_0 = 1.82$ Mev, which is large compared with $\bar{G}_0 = 0.96$ Mev used in § 3 for Mo^{94}. The New value gives $\omega = 1.42$ Mev for the solution of (16) computed by taking $X_P = 0$.

** In our original manuscript we gave much smaller value for this quantity. It was caused as we did not take into account the depression of the ground state due to the Q-Q interaction. We are indebted to Dr. A. Arima for drewing our attention to this point.

*** For this calculation all the parametric values are the same as used in § 3.

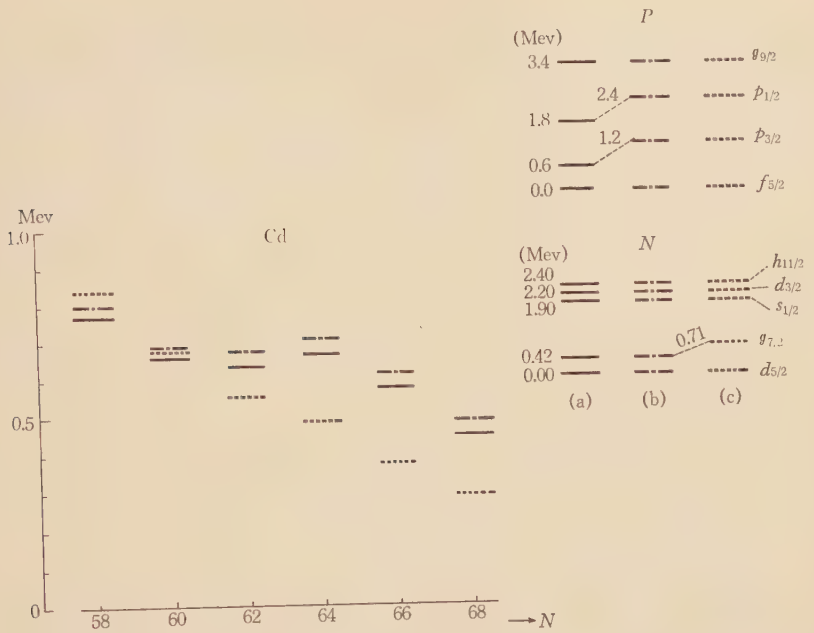

Fig. 8. ω for Cd isotopes for various choices of the set of ε_ν. The values of the strength parameters are the same as used in obtaining the results of Fig. 1. Three different sets, (a), (b) and (c), of ε_ν are shown in the upper right corner of the figure, separately for protons and neutrons. ω corresponding to these choices are plotted as functions of N, respectively, by the full, dott-dash and dotted lines, among which the first one is the same as that given in Fig. 1.

not so far make such an extensive calculation, as the machine time of the computer available to us has been rather limited. It is also expected that the neglected terms in the Sawada approximation might cause further coupling of the first excited 2^+ state with quasi-particle states and states of higher phonon numbers, which might not change the values of ω obtained in the present paper drastically, but is strong enough so as to affect the finer N dependence of ω.

§ 5. Concluding remarks

It is found that the calculation, based on the assumption that effective interaction is a sum of the pair and the Q-Q interactions and made by employing the BCS and the Sawada approximations, explains fairly well the general properties of the first excited 2^+ states of various even-even nuclei in the vibrational regions.

To be able to make a comparison with experiment in finer detail, however, it might be necessary to perform further calculations with different choices of the values ε_ν, the energy of the single-particle orbitals of the shell model.

It is also hoped that such a more detailed calculation is performed in com-

bination with the analysis of the neighbouring odd A nuclei, and further by using approximations which are improved than those used in the present paper. In particular the improvement on the Sawada approximation by taking into account also the effect of the scattering and exchange terms, will make it possible to treat the nature of the two (or higher) phonon states as well based on the first principles and in that way make the analysis on the first 2^+ state more reliable. Such a calculation is under way by the present authors.

Appendix

In this appendix the parametric values ε_{lj}, the energy of the shell model orbitals are given, together with the values of λ, the Fermi energy and of Δ, half the energy gap.

Table A. I

Configuration and ε (Mev)				Fermi energy λ and half-energy gap Δ (in Mev)							
Proton (ref. 4)		Neutron (ref. 4)			Mo		Ru		Pd		Cd	
				N	λ	Δ	λ	Δ	λ	Δ	λ	Δ
$f_{5/2}$	0.00	$d_{5/2}$	0.00	52	−0.560	0.667	−0.539	0.650				
$p_{3/2}$	0.60	$g_{7/2}$	0.42	54	−0.292	0.874	−0.277	0.851				
$p_{1/2}$	1.80	$s_{1/2}$	1.90	56	−0.025	0.986	−0.015	0.960				
$g_{9/2}$	3.40	$d_{3/2}$	2.20	58	0.244	1.039	0.250	1.011	0.256	0.983	0.261	0.957
		$h_{11/2}$	2.40	60			0.525	1.018	0.527	0.988	0.528	0.960
				62					0.825	0.957	0.823	0.925
				64					1.176	0.923	1.178	0.887
				66							1.535	0.915
				68							1.818	0.951
				p	2.931	0.754	3.285	0.804	3.567	0.691	3.820	0.535

Table A. II.

Configuration and ε (Mev)				Fermi energy λ and half-energy gap Δ (in Mev)		
		Neutron (ref. 4)			Sn	
				N	λ	Δ
		$d_{5/2}$	0.00	62	0.822	0.894
		$g_{7/2}$	0.42	64	1.181	0.852
		$s_{1/2}$	1.90	66	1.542	0.886
		$d_{3/2}$	2.20	68	1.823	0.926
		$h_{11/2}$	2.40	70	2.057	0.936
				72	2.265	0.916
				74	2.456	0.867

Table A. III

Configuration and ε (Mev)				Fermi energy λ and half-energy gap Δ (in Mev)								
Proton (ref. 10)		Neutron (ref. 4)			Te		Xe		Ba		Ce	
				N	λ	Δ	λ	Δ	λ	Δ	λ	Δ
$g_{7/2}$	0.00	$d_{5/2}$	0.00	68	1.828	0.902						
$d_{5/2}$	0.78	$g_{7/2}$	0.42	70	2.058	0.915						
$h_{11/2}$	2.18	$s_{1/2}$	1.90	72	2.264	0.897	2.262	0.879				
$d_{3/2}$	2.88	$d_{3/2}$	2.20	74	2.452	0.850	2.448	0.834	2.444	0.818		
$s_{1/2}$	3.20	$h_{11/2}$	2.40	76	2.627	0.772	2.621	0.758	2.614	0.744		
				78	2.792	0.656	2.784	0.645	2.775	0.634		
				80			2.938	0.472	2.928	0.464	2.918	0.457
				p	−0.476	0.600	−0.190	0.749	0.085	0.820	0.376	0.802

Table A. IV

Configuration and ε (Mev)				Fermi energy λ and half-energy gap Δ (in Mev)						
Proton (ref. 10)		Neutron (ref. 10)			Nd		Sm		Gd	
				N	λ	Δ	λ	Δ	λ	Δ
$g_{7/2}$	0.00	$f_{7/2}$	0.00	84	−0.324	0.440	−0.316	0.431	−0.307	0.422
$d_{5/2}$	0.78	$h_{9/2}$	1.11	86	−0.113	0.573	−0.107	0.560	−0.101	0.548
$h_{11/2}$	2.18	$p_{3/2}$	1.84	88	0.120	0.643	0.125	0.627	0.129	0.611
$d_{3/2}$	2.88	$i_{13/2}$	2.56	p	0.674	0.652	0.964	0.624	1.342	0.577
$s_{1/2}$	3.20	$p_{1/2}$	2.73							
		$f_{5/2}$	3.07							

Table A. V

Configuration and ε (Mev)				Fermi energy λ and half-energy gap Δ (in Mev)				
Proton (ref. 10)		Neutron (ref. 4)			Pt		Hg	
				N	λ	Δ	λ	Δ
$g_{7/2}$	0.000	$h_{9/2}$	0.000	114	1.806	0.937		
$d_{5/2}$	1.600	$f_{7/2}$	0.400	116	1.978	0.859	1.973	0.845
$h_{11/2}$	2.000	$i_{13/2}$	1.116	118	2.152	0.769	2.147	0.758
$d_{3/2}$	2.954	$p_{3/2}$	1.850	120	2.327	0.665	2.321	0.654
$s_{1/2}$	3.304	$f_{5/2}$	2.180	122			2.498	0.524
		$p_{1/2}$	2.750	124			2.704	0.341
				p	2.983	0.364	3.226	0.252

In the first and second columns of each table presented in this appendix are given the quantum numbers (lj) which specify the proton orbitals in an unfilled major shell and the corresponding energies ε, while in the third and fourth columns similar quantities are given for the neutron orbitals. These quantities are all fixed in one mass number region.

The fifth column gives the neutron number N, and then the following pairs of columns give λ and Δ for the neutron part of the wave function of various isotopes, in the order of increasing atomic numbers. In the lowest row of these columns are given λ and Δ for the proton part of the wave function as calculated for the lightest element of each set of isotopes, considering that their dependence on N will be negligible. As the same ε are used for all the elements in one mass number region, the variation of λ and Δ (for protons) from one set of isotopes to another is simply due to the A^{-1} dependence of G_P, the strength parameter of the pair interaction between protons.

Tables A. I and A. II respectively refer to nuclei for which ω are given in Figs. 1 and 2 of the text. As in the latter, i.e. in the Sn isotopes, protons are considered to be inert, no quantities referring to protons are given in Table A. II. As the same values of ε_{lj} are used, for nuclei for which ω are given in Figs. 3 and 4 the quantities referring to them are given together in Table A. III. Finally Tables A. IV and A. V respectively refer to nuclei for which ω are given in Figs. 5 and 6 in the text.

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Viscosity of Fermi Particle Systems

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The quantum-statistical expression for the viscosity coefficient in terms of time correlation function is calculated for Fermi gases, and an interpolation formula valid in the entire region of temperature is derived. The formula for the viscosity gives the temperature dependence of T^{-2} in the extremely degenerate case, and leads to Uehling's expression for the quantum correction in the slightly degenerate case, and to the Enskog-Chapman theory in the classical limit. The behaviour in the intermediate temperature region between the extremely and slightly degenerate cases is determined with numerical calculation by using the S-wave approximation. It is shown that the minimum of the viscosity occurs at about $0.6\,T_c$, T_c being the degeneracy temperature.

These results are used to discuss Zinov'eva's experiments on the viscosity of liquid He^3 by employing Landau's theory of Fermi liquid. The effective mass and the degeneracy temperature are determined from recent experiments on the heat capacity, as $m=2.34m_0$ and $T_C=2.13$ °K. A good agreement is obtained below about $T=0.6$ °K by assuming $a=1.158\times10^{-8}$ cm for the scattering amplitude describing the collision between quasi-particles.

§ 1. Introduction

The recent theories[1),2)] of linear dissipative systems provide us with exact expressions for transport coefficients, which are valid for condensed as well as dilute systems. These expressions are formulated in terms of the time correlation functions of the corresponding dynamical fluxes, and serve as a systematic approach to transport properties. The coefficient of shear viscosity of isotropic fluids can be expressed as[2)]

$$\eta=\frac{1}{VkT}\int_0^\infty \Psi(t)\,dt, \tag{1·1}$$

in terms of the time correlation function

$$\Psi(t)=\langle\{FF(t)\}\rangle\equiv\frac{1}{2}\langle FF(t)+F(t)F\rangle, \tag{1·2}$$

where the angular brackets mean the average over the canonical ensemble of

the system with the temperature T and the volume V, and k is the Boltzmann constant. F is the viscosity flux defined by

$$F=\sum_{i=1}^{N}\frac{p_{ix}p_{iy}}{m_i}+\sum_{i<j}^{N}\sum^{N}(F_{ij})_x\,(r_{ij})_y, \tag{1·3}$$

p_{ix} being the x-component of the momentum of particle i, $(r_{ij})_y$ the y-component of the relative coordinate between i and j, $(F_{ij})_x$ the x-component of the intermolecular force between i and j, and N the number of particles. $F(t)$ is the value of F after time t and should, in the quantum-mechanical case, be read as the Heisenberg operator.

By investigating the time dependence of the correlation function, one of the authors[3] calculated the coefficients of viscosity and thermal conductivity for quantum dilute gases. The procedure consists of the determination of the spectrum of the master collision operator for spherically symmetric intermolecular forces, and it was shown that in the classical limit it yields the same results as Enskog-Chapman's theory. The calculation was, however, limited to the non-degenerate case, so that the density-dependent symmetry effects were neglected.

The present paper is, therefore, concerned with the formulation of the density-dependent symmetry effects, which play a decisive role below the degeneracy temperature. We consider the viscosity of the Fermi particles and investigate its temperature dependence over the entire region of temperature. When the associated de-Broglie wavelength is of the order of magnitude of the average distance between particles, the symmetry effects become important, so that we have, in general, to treat the many-body collisions which come from the overlapping of the wave packets of more than three particles. We employ, however, the binary collision approximation, taking the many-body effects into account by introducing the density-dependent symmetry effects in the collision term.

In the high temperature region where Enskog-Chapman's theory is valid, the viscosity is proportional to $1/T$ in the case of hard sphere molecules. As the temperature decreases, quantum effects appear. The first quantum effects are the diffraction effect for the interaction mechanism and the consideration of the statistics in the scattering cross section. Calculation of the viscosity in this temperature region has been performed by several authors.[4] As the temperature still goes down, the system starts to be degenerate and the second "statistics" effect, that is, the density-dependent symmetry effect becomes important. Uehling[5] has given an expression for the viscosity coefficient in the slightly degenerate case. In the very low temperatures, the density-dependent symmetry effects become essential, and the viscosity of the Fermi gases increases as the temperature decreases. Thus in the low temperature limit, the limiting law that the viscosity is proportional to T^{-2} becomes valid. This temperature dependence in the extremely degenerate case has been shown by Tomonaga[6] on the basis of the Uhlenbeck-Uehling equation. We can see, therefore, that the temperature

dependence of the viscosity should have a minimum below the degeneracy temperature. One of the aims of the present paper is the derivation of an interpolation formula which is valid between the extremely degenerate and slightly degenerate cases, and the determination of the minimum.

In 1949, Weinstock et al.[7] observed the viscosity of liquid He^3 and obtained 22.0 micropoise at 2.79°K and 30.4 micropoise at 1.04°K. Recently, Taylor and Dash[8] obtained an empirical formula, $\eta=18.7(3.35/T)^{0.38}$ micropoise, over 1.2 —3.35°K, and Zinov'eva[9] measured in the range of 0.35—3.2°K and obtained more definite results. In the observed range of temperature, the experiments on liquid He^3 do not show the T^{-2} law. However, the observed behaviour that viscosity rises as temperature decreases is possibly considered to be due to the effects of the Fermi-Dirac statistics. This was suggested by Buckingham and Temperley[10] by referring to the T^{-2} law. We shall discuss the viscosity of liquid He^3, employing Landau's theory of Fermi liquid, and compare the numerical calculation of the interpolation formula with Zinov'eva's results.

In § 2, we shall investigate the diagonal elements of the Heisenberg operator for the flux in the number representation, and establish a kinetic equation for the diagonal elements in terms of a master collision operator. In § 3, an approximate eigenvalue problem for the master collision operator is considered, and we obtain an expression for the viscosity in terms of the master collision operator. An evaluation of this expression is given in § 4 and we derive an interpolation formula for the viscosity valid over the entire temperature region. §§ 5 and 6 are devoted to the reduction of the general interpolation formula in the extremely and slightly degenerate cases. In § 7, we express the interpolation formula in terms of reduced quantities and evaluate this formula by numerical calculation, using the S-wave approximation. Comparison with experiments on liquid He^3 is made in § 8, by employing Landau's theory of Fermi liquid. The last section is devoted to some remarks.

§ 2. Correlation function in terms of collision operator

We consider a Fermi gas of one component with spherically symmetric interactions between particles. The contribution due to the intermolecular force to the viscosity flux, (1·3), can be neglected in comparison with that due to the momentum transfer:

$$F=\frac{1}{m}\sum_{i=1}^{N} p_{ix}\, p_{iy}. \tag{2·1}$$

In order to introduce the symmetry effects explicitly, we employ the number representation so that Eq. (2·1) takes the form[11]

$$F=\frac{1}{m}\sum_{p} p_x\, p_y\, n_p\,, \tag{2·2}$$

where n_p is the number operator of particles belonging to the momentum eigenfunction $(1/\sqrt{V})\exp(i\boldsymbol{p}\boldsymbol{x}/\hbar)$. The eigenfunction of the number operators, $|n_1, n_2, \cdots\rangle$, is denoted by $|n\rangle$ for simplicity. Neglecting the interaction energy in the Hamiltonian in the Boltzmann factor, the correlation function can be written as

$$\Psi(t)=\langle\{F_n\langle n|F(t)|n\rangle\}\rangle, \tag{2·3}$$

where F_n are the eigenvalues of F corresponding to the eigenstates $|n\rangle$ and $\langle n|F(t)|n\rangle$ the diagonal elements of the Heisenberg operator $F(t)$ in the number representation, and

$$\langle B\rangle\equiv\sum_{n_s}\cdots\sum B\exp\left(\frac{\mu\sum n_s-\sum\varepsilon_s n_s}{kT}\right)\times\left\{\sum_{n_s}\cdots\sum\exp\left(\frac{\mu\sum n_s-\sum\varepsilon_s n_s}{kT}\right)\right\}^{-1} \tag{2·4}$$

expresses the average of the quantity B over the grand canonical ensemble,[12] where $\varepsilon_s=p_s^2/2m$ is the kinetic energy per particle and μ the chemical potential.

The equation of motion for the Heisenberg operator $F(t)$ is

$$F(t)=U^{\dagger}(t)FU(t),\quad U(t)=\exp(-iHt/\hbar), \tag{2·5}$$

where H is the Hamiltonian of the system and $U^{\dagger}(t)$ the Hermitian conjugate of $U(t)$. The diagonal elements of (2·5) can be written as[3]

$$\langle n|F(t)|n\rangle=F_n+\sum_{n'(\neq n)}W(n',\,n\,;t)\,\{F_{n'}-F_n\}, \tag{2·6}$$

where

$$W(n',\,n\,;t)=|\langle n'|U(t)|n\rangle|^2,\quad n'\neq n, \tag{2·7}$$

expresses the transition probability of the system from state $|n\rangle$ to state $|n'\rangle$ after time t. For a short time interval t, the transition probability is evaluated to yield

$$\langle n|F(t)|n\rangle=(1-t\Lambda)F_n, \tag{2·8}$$

where we have defined the master collision operator Λ:

$$\Lambda F_n=-\sum_{p<q}(1/V)\int d\Omega\,\frac{|\boldsymbol{p}-\boldsymbol{q}|}{m}\sigma(|\boldsymbol{p}-\boldsymbol{q}|,\theta)\times n_p n_q(1-n_r)(1-n_s)(1/m)(r_x r_y+s_x s_y-p_x p_y-q_x q_y), \tag{2·9}$$

$\boldsymbol{p}$, $\boldsymbol{q}$ being the momenta of the colliding particles and $\boldsymbol{r}$, $\boldsymbol{s}$ the corresponding momenta after the collision, and $d\Omega$ the element of solid angle in the direction of scattering (θ, φ) and $\sigma(|\boldsymbol{p}-\boldsymbol{q}|,\theta)$ the scattering cross section for the binary collision. Eq. (2·8) is valid for the short time intervals t such that $\tau_c<t<\tau_f$, where τ_c and τ_f are the mean collision time and the mean free time of particles, respectively. The above procedure can be applied to a short time interval

$(t\sim t+\Delta t)$ by neglecting the contribution due to the off-diagonal elements of $F(t)$ to yield

$$\langle n|F(t+\Delta t)|n\rangle=(1-\Delta t\Lambda)\langle n|F(t)|n\rangle.$$

Thus we obtain

$$\langle n|F(t)|n\rangle=\exp(-t\Lambda)F_n, \tag{2·10}$$

which is combined with Eq. (2·3) to yield

$$\Psi(t)=\langle F_n\exp(-t\Lambda)F_n\rangle. \tag{2·11}$$

Eq. (2·11) describes the temporal development of the correlation function in terms of the collision operator Λ. Introducing the relative and centre-of-mass momenta

$$\left.\begin{aligned}&\boldsymbol{p}^r=(1/2)(\boldsymbol{q}-\boldsymbol{p}),\quad \boldsymbol{p}^{r'}=(1/2)(\boldsymbol{s}-\boldsymbol{r}),\\&\boldsymbol{p}^c=\boldsymbol{p}+\boldsymbol{q},\end{aligned}\right\} \tag{2·12}$$

we can write as

$$\begin{aligned}\Lambda F_n=&\sum_{p<q}(1/V)\int d\Omega\frac{2p^r}{m}\sigma(p^r,\theta)\\&\times n_pn_q(1-n_r)(1-n_s)(2/m)(p_x^rp_y^r-p_x^{r'}p_y^{r'}).\end{aligned} \tag{2·13}$$

§ 3. Approximate eigenvalue of collision operator

If F_n is an eigenfunction of the master collision operator with an eigenvalue λ,

$$\Lambda F_n=\lambda F_n, \tag{3·1}$$

and if λ is independent of n, then the correlation function (2·11) can be written as

$$\Psi(t)=\langle F_n^2\rangle\exp(-t\lambda), \tag{3·2}$$

and λ can be determined by

$$\lambda=\frac{\langle F_n\Lambda F_n\rangle}{\langle F_n^2\rangle}. \tag{3·3}$$

Then, the viscosity coefficient is given, from Eq. (1·1), as

$$\eta=\frac{\langle F_n^2\rangle}{VkT\lambda}. \tag{3·4}$$

In the classical case, it has been shown[3] that Eq. (3·1) holds exactly for the Maxwellian molecules, and, although Eq. (3·1) is only approximately true in the general case, Eqs. (3·3) and (3·4) lead to Enskog-Chapman's expression for the viscosity in the first approximation. In the quantum-mechanical case, especially when the density-dependent symmetry effects become important, the situation becomes complicated. It will be shown, however, in Appendix A, that

Eq. (3·1) is still approximately true in the extremely degenerate case. Thus, in the high temperature limit and in the extremely degenerate case, the eigenvalue equation (3·1) holds approximately. Therefore, it is natural to assume that Eqs. (3·3) and (3·4) are approximately true over the entire temperature region.

§ 4. Interpolation formula

We calculate the viscosity from Eqs. (3·3) and (3·4), and investigate its temperature dependence.

With the aid of the relations

$$\langle n_p n_{p'}\rangle = f_p f_{p'} + \delta(\boldsymbol{p}-\boldsymbol{p}')\{f_p - f_p^2\}, \tag{4·1}$$

and

$$\sum_p f_p \frac{p_x p_y}{m} = 0, \tag{4·2}$$

where f_p is the Fermi distribution function, we obtain

$$\begin{aligned}\langle F_n^2\rangle &= \left\langle \sum_p n_p \frac{p_x p_y}{m} \sum_{p'} n_{p'} \frac{p_x' p_y'}{m} \right\rangle \\ &= \frac{1}{m^2}\sum_p p_x^2 p_y^2 f_p(1-f_p) \\ &= 2V\left(\frac{2\pi mkT}{h^2}\right)^{3/2}(kT)^2 G_{3/2}(\zeta),\end{aligned} \tag{4·3}$$

where $\zeta = \mu/kT$ and $G_s(\zeta)$ is the Sommerfeld integral

$$G_s(\zeta) = \frac{1}{\Gamma(s+1)}\int_0^\infty \frac{x^s}{e^{x-\zeta}+1}dx, \tag{4·4}$$

$\Gamma(s)$ being the gamma function of s-th order.

Combination of Eqs. (2·2) and (2·13) leads to

$$\begin{aligned}\langle F_n \Lambda F_n\rangle = \sum_{p<q}(1/V)\int d\Omega (2p^r/m)\sigma(p^r,\theta)\times f_p f_q(1-f_r)(1-f_s) \\ \times (1/m)\{p_x p_y(1-f_p) + q_x q_y(1-f_q) - r_x r_y f_r - s_x s_y f_s\} \\ \times (2/m)(p_x^r p_y^r - p_x^{r'} p_y^{r'}),\end{aligned} \tag{4·5}$$

where the relation

$$\begin{aligned}\langle n_{p'} n_p n_q (1-n_r)(1-n_s)\rangle = f_{p'} f_p f_q (1-f_r)(1-f_s) \\ + \delta(\boldsymbol{p}'-\boldsymbol{p}) f_p (1-f_p) f_q (1-f_r)(1-f_s) \\ + \delta(\boldsymbol{p}'-\boldsymbol{q}) f_p f_q (1-f_q)(1-f_r)(1-f_s)\end{aligned}$$

$$-\delta(\boldsymbol{p}'-\boldsymbol{r})f_pf_qf_r(1-f_r)(1-f_s)$$
$$-\delta(\boldsymbol{p}'-\boldsymbol{s})f_pf_q(1-f_r)\,f_s(1-f_s), \tag{4·6}$$

for $\boldsymbol{p}\neq\boldsymbol{q}\neq\boldsymbol{r}\neq\boldsymbol{s}$, and Eq. (4·2) have been used. In Eq. (4·5), we replace the sums over $\boldsymbol{p}$ and $\boldsymbol{q}$ by the integrals with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$. Then, we transform the integration variables from $(\boldsymbol{p}, \boldsymbol{q})$ to $(\boldsymbol{r}, \boldsymbol{s})$, considering the inverse collision, and we make use of the principle of detailed balance and the reciprocal theorem. The transformed equation thus obtained is added to Eq. (4·5), dividing by two. Thus we arrive at

$$\langle F_n\Lambda F_n\rangle=(2V/h^6)\int d\boldsymbol{p}^c\,d\boldsymbol{p}^r\,d\Omega\,(2p^r/m)\sigma(p^r,\theta)$$
$$\times f_pf_q(1-f_r)(1-f_s)(2/m^2)(p_x^rp_y^r-p_x^{r'}p_y^{r'})^2, \tag{4·7'}$$

where the relation $d\boldsymbol{p}d\boldsymbol{q}=d\boldsymbol{p}^cd\boldsymbol{p}^r$ has been used. In the same way, we can rewrite as

$$\langle F_n\Lambda F_n\rangle=(2V/h^6)\int d\boldsymbol{p}^c\,d\boldsymbol{p}^r\,d\Omega\,(2p^r/m)\sigma(p^r,\theta)$$
$$\times f_pf_q(1-f_r)(1-f_s)(4/m^2)p_x^rp_y^r(p_x^rp_y^r-p_x^{r'}p_y^{r'}), \tag{4·7}$$

which is more convenient in the following calculation.

We now introduce the dimensionless quantities:

$$\boldsymbol{\gamma}=\sqrt{2c}\,\boldsymbol{p}^r,\quad \boldsymbol{\gamma}'=\sqrt{2c}\,\boldsymbol{p}^{r'},\quad \boldsymbol{\lambda}=\sqrt{c/2}\,\boldsymbol{p}^c, \tag{4·8}$$

with $c=1/2mkT$. Then, Eq. (4·7) can be transformed to be

$$\langle F_n\Lambda F_n\rangle=\frac{2^{3/2}V}{m^3h^6c^{11/2}}\int_0^\infty d\gamma\gamma^3\int_0^\infty d\lambda\,\lambda^2\int_0^\pi d\alpha\,\sin\alpha f_\tau f_{\tau_1}$$
$$\times\int_0^{2\pi}d\varepsilon\int_0^\pi d\theta\,\sin\theta\int_0^{2\pi}d\delta\gamma_x\gamma_y'\int_0^\pi d\theta\,\sin\theta\,\sigma(\gamma\sqrt{mkT},\theta)$$
$$\times\int_0^{2\pi}d\varphi(1-f_{\tau'})(1-f_{\tau_1'})(\gamma_x\gamma_y-\gamma_x'\gamma_y'), \tag{4·9}$$

where, as is shown in Fig. 1,

θ, δ: Polar and azimuthal angles of vector $\boldsymbol{\lambda}$ in a fixed coordinate system;

α, ε: Polar and azimuthal angles of vector $\boldsymbol{\gamma}$ in such a new coordinate system that the new x-axis agrees with the $\boldsymbol{\lambda}$ vector, and the new xy-plane is determined by the old x-axis and the $\boldsymbol{\lambda}$ vector;

α', ε': Polar and azimuthal angles of vector $\boldsymbol{\gamma}'$ in the above new coordinate system;

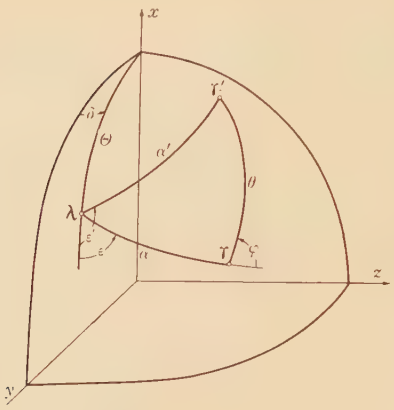

Fig. 1. Three vectors $\boldsymbol{\lambda}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$.

and

$$\left.\begin{aligned}
&\lambda_x=\lambda\cos\theta, \qquad \lambda_y=\lambda\sin\theta\cos\delta,\\
&\gamma_x=\gamma(\cos\theta\cos\alpha-\sin\theta\sin\alpha\cos\varepsilon),\\
&\gamma_y=\gamma(\sin\theta\cos\delta\cos\alpha+\cos\theta\cos\delta\sin\alpha\cos\varepsilon\\
&\qquad\qquad -\sin\delta\sin\alpha\sin\varepsilon),\\
&\gamma_x'=\gamma\cos\theta(\cos\theta\cos\alpha-\sin\theta\sin\alpha\cos\varepsilon)\\
&\quad -\gamma\sin\theta\cos\varphi(\cos\theta\sin\alpha+\sin\theta\cos\alpha\cos\varepsilon)\\
&\qquad\qquad +\gamma\sin\theta\sin\varphi\sin\theta\sin\varepsilon,\\
&\gamma_y'=\gamma\cos\theta(\sin\theta\cos\delta\cos\alpha+\cos\theta\cos\delta\sin\alpha\cos\varepsilon\\
&\quad -\sin\delta\sin\alpha\sin\varepsilon)-\gamma\sin\theta\cos\varphi(\sin\theta\cos\delta\sin\alpha\\
&\quad -\cos\theta\cos\delta\cos\alpha\cos\varepsilon+\sin\delta\cos\alpha\sin\varepsilon)\\
&\quad -\gamma\sin\theta\sin\varphi(\cos\theta\cos\delta\sin\varepsilon+\sin\delta\cos\varepsilon),
\end{aligned}\right\} \tag{4·10}$$

and

$$\left.\begin{aligned}
f_\tau&=\{e^{(\gamma^2+\lambda^2)/2-\zeta-\gamma\lambda\cos\alpha}+1\}^{-1},\\
f_{\tau_1}&=\{e^{(\gamma^2+\lambda^2)/2-\zeta+\gamma\lambda\cos\alpha}+1\}^{-1}.
\end{aligned}\right\} \tag{4·11}$$

The quantities with primes denote the correspondents after collisions. In order to perform the integrations in Eq. (4·9), we expand the Fermi functions in terms of the Legendre polynomials:

$$\left.\begin{aligned}
&f_\tau f_{\tau_1}=\frac{e^{-\rho}}{2}\sum_{l=0}^{\infty}a_l(\gamma,\lambda;\zeta)P_l(\cos\alpha),\\
&(1-f_{\tau'})(1-f_{\tau_1'})=\frac{e^{\rho}}{2}\sum_{l=0}^{\infty}a_l(\gamma,\lambda;\zeta)P_l(\cos\alpha'),
\end{aligned}\right\} \tag{4·12}$$

where

$$\rho=\frac{1}{2}(\gamma^2+\lambda^2)-\zeta, \tag{4·13}$$

and the terms of odd l should be dropped because the left-hand sides of Eqs. (4·12) are even with respect to $\cos\alpha$ or $\cos\alpha'$. By making use of the addition theorem:

$$P_l(\cos\alpha')=P_l(\cos\alpha)\,P_l(\cos\theta)$$
$$+2\sum_{m=1}^{l}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\alpha)\,P_l^m(\cos\theta)\cos m(\pi-\varphi), \tag{4·14}$$

where

$$\cos\alpha'=\cos\alpha\cos\theta+\sin\alpha\sin\theta\cos(\pi-\varphi), \tag{4·15}$$

we carry out the integrations with respect to φ, ε, δ and θ in turn. (Details are given in Appendix B.) Thus we have

$$\langle F_n\Lambda F_n\rangle=\frac{2^{7/2}\pi^2 V}{15m^3h^6c^{11/2}}\int_0^\infty d\gamma\,\gamma^7\int_0^\infty d\lambda\,\lambda^2$$
$$\times(2\pi)\int_0^\pi d\theta\,\sin\theta\,\sigma(\gamma\sqrt{mkT},\theta)\{1-P_2(\cos\theta)\}\sum_{l=0}^{\infty}\frac{a_l^2}{2l+1}P_l(\cos\theta). \tag{4·16}$$

Eq. (4·16) is rigorous apart from the two-body collision approximation, which has been introduced in the derivation of Eq. (2·9). The last factor involving the Legendre polynomials represents the contribution from the scattering-angle dependence of the Fermi distribution function in the transition probability. We neglect, however, the terms with the Legendre polynomials of $l\geqq 2$. It should be remarked here that this does not mean the neglect of the contributions from the higher order partial waves to the scattering cross section. This approximation corresponds to the procedure that the factor $(1-f_r)(1-f_s)$ in Eq. (4·7) is replaced by its average with respect to the scattering solid angle (θ, φ). This approximation is valid in the extremely degenerate case, because the above factor becomes insensitive for the change of the angles, as will be pointed out in the next section, and is also reasonable in the slightly degenerate case in the sense of Uehling's approximation, as will be shown in Appendix C. In the classical limit, the above approximation is exact. Thus we obtain

$$\langle F_n\Lambda F_n\rangle=\frac{2^8\pi^2 m^{5/2}(kT)^{11/2}V}{5h^6}\int_0^\infty d\gamma\,\gamma^7\,\Omega^{(2)}(\gamma\sqrt{mkT})\times\int_0^\infty d\lambda\,\lambda^2[a_0(\gamma,\lambda;\zeta)]^2, \tag{4·17}$$

where

$$\Omega^{(2)}(\gamma\sqrt{mkT}) = 2\pi\int_0^\pi \sigma(\gamma\sqrt{mkT}, \theta)(1-\cos^2\theta)\sin\theta\, d\theta, \tag{4·18}$$

and

$$a_0(\gamma, \lambda; \zeta) = \frac{1}{\gamma\lambda \sinh\rho} \log\frac{\cosh(1/2)(\gamma\lambda+\rho)}{\cosh(1/2)(\gamma\lambda-\rho)}. \tag{4·19}$$

With the aid of Eqs. (4·3) and (4·17), we arrive at an interpolation formula for the viscosity coefficient η, which is approximately valid over the entire temperature region:

$$\eta = \frac{1}{VkT} \frac{(\langle F_n^2\rangle)^2}{\langle F_n \Lambda F_n\rangle}. \tag{4·20}$$

On the basis of this interpolation formula, we shall first discuss the viscosity in the extremely and slightly degenerate cases. It will be shown that, in the slightly degenerate case, this formula leads to Uehling's results, and, in the extremely degenerate case, to the T^{-2} law. In the intermediate temperature region, we shall perform numerical calculation of this formula.

§ 5. Viscosity at very low temperatures

The system becomes extremely degenerate at very low temperature. In this case, we have

$$\zeta \gg \delta, \tag{5·1}$$

δ being the half-value width of the region where the degenerate Fermi distribution function deviates from the step function. The factor $f_\tau f_{\tau_1}(1-f_{\tau'})(1-f_{\tau_1'})$ in Eq. (4·9) does not vanish if and only if the following relations are satisfied simultaneously:

$$\left.\begin{aligned} &\rho-\gamma\lambda\cos\alpha<\delta, \qquad \rho+\gamma\lambda\cos\alpha<\delta,\\ &\rho-\gamma\lambda\cos\alpha'>-\delta, \qquad \rho+\gamma\lambda\cos\alpha'>-\delta, \end{aligned}\right\} \tag{5·2}$$

which lead to

$$2(\zeta-\delta)<\gamma^2+\lambda^2<2(\zeta+\delta), \tag{5·3}$$

and

$$\left.\begin{aligned} &-2\delta<\gamma\lambda\cos\alpha<2\delta,\\ &-2\delta<\gamma\lambda\cos\alpha'<2\delta. \end{aligned}\right\} \tag{5·4}$$

Therefore, the contribution to the integrals with respect to γ and λ in Eq. (4·17) comes from the region where $\gamma^2+\lambda^2\sim O(\zeta)$. It is plausible, however, to assume that the major contribution comes especially from such a region that

$$\gamma^2 \sim O(\zeta) \text{ and } \lambda^2 \sim O(\zeta), \tag{5·5}$$

provided that the scattering cross section does not take extraordinarily large values out of this region. Then, from (5·4), $\cos\alpha$ and $\cos\alpha'$ should be very small as compared with unity. This fact means physically that the majority of the effective collisions occur in the neighborhood of the Fermi surface, so that the values of velocity of two particles before and after collisions are almost equal to each other, and therefore $\boldsymbol{p}^r$ and $\boldsymbol{p}^{r'}$ are nearly parpendicular to $\boldsymbol{p}^c$. This physical situation is true in the extremely degenerate case. Therefore, the assumption (5·5) seems to be very natural in our case.

From (5·5), $\gamma\lambda$ is very large as compared with unity, so that we can expand in Eq. (4·19) as

$$\log\frac{\cosh(1/2)(\gamma\lambda+\rho)}{\cosh(1/2)(\gamma\lambda-\rho)} = \rho - e^{-\gamma\lambda}(e^{\rho}-e^{-\rho}) + \cdots. \tag{5·6}$$

Taking the first term, we obtain

$$a_0(\gamma, \lambda; \zeta) = \frac{2\rho}{\gamma\lambda(e^{\rho}-e^{-\rho})}. \tag{5·7}$$

Now we employ γ and ρ as new integration variables instead of γ and λ. Then, with the use of Eq. (5·7), Eq. (4·17) can be written as

$$\langle F_n \Lambda F_n \rangle \cong \frac{2^{10}\pi^2 m^{5/2}(kT)^{11/2}V}{5h^6}\int_0^{\infty} d\gamma\, \gamma^5\, \Omega^{(2)}(\gamma\sqrt{mkT}) \times \int_{-(1/2)(2\zeta-\gamma^2)}^{\infty} \frac{d\rho}{\sqrt{2\rho+2\zeta-\gamma^2}}\,\frac{\rho^2}{(e^{\rho}-e^{-\rho})^2}. \tag{5·8}$$

Since $2\zeta-\gamma^2$ is of the order of magnitude δ, we can neglect 2ρ in the factor $\sqrt{2\rho+2\zeta-\gamma^2}$. Therefore, the integration range in the $\gamma\rho$-plane can be approximated as

$$\left.\begin{aligned} &-\infty<\rho<\infty, \\ &0<\gamma<\sqrt{2\zeta}, \end{aligned}\right\} \tag{5·9}$$

because the integrand of (5·8) does not vanish only in the zone of the width δ along the γ-axis. Since

$$\int_{-\infty}^{\infty}\frac{\rho^2}{(e^{\rho}-e^{-\rho})^2}d\rho = \frac{\pi^2}{12}, \tag{5·10}$$

Eq. (5·8) becomes

$$\langle F_n \Lambda F_n \rangle \cong \frac{2^8\pi^4 V(2m\mu_0)^{5/2}Q}{15h^6}(kT)^3, \tag{5·11}$$

where

$$Q=\int_0^1 \frac{x^5 Q^{(2)}(x\sqrt{2m\mu_0})}{\sqrt{1-x^2}}\,dx$$

$$=2\pi\int_0^1 dx\,\frac{x^5}{\sqrt{1-x^2}}\int_0^\pi d\theta\,\sin^3\theta\,\sigma(x\sqrt{2m\mu_0},\,\theta), \tag{5·12}$$

and μ has been replaced by the Fermi energy, μ_0, at 0°K.

On the other hand, in the extremely degenerate case, $G_s(\zeta)$ in Eq. (4·4) is expanded as

$$G_s(\zeta)=\frac{\zeta^{s+1}}{\Gamma(s+2)}+\frac{\pi^2}{6}\frac{\zeta^{s-1}}{\Gamma(s)}+\cdots. \tag{5·13}$$

Taking the first term of Eq. (5·13), Eq. (4·3) becomes

$$\langle F_n^2\rangle=\frac{2^4\pi V(2m)^{3/2}\mu_0^{5/2}}{15h^3}kT. \tag{5·14}$$

Thus, from Eqs. (4·20), (5·11) and (5·14), we obtain the following approximate expression for the viscosity coefficient:

$$\eta\cong\frac{\sqrt{2m}\,\mu_0^{5/2}}{15\pi^2(kT)^2Q}. \tag{5·15}$$

This expression gives the T^{-2} temperature dependence, which is the same as obtained by Tomonaga.[6] They differ a little, however, in the definition of the cross section Q, which is due to the difference of the approximations employed.

§ 6. Viscosity at high temperatures

We next consider the slightly degenerate case, where we have

$$e^\xi<1. \tag{6·1}$$

In this case, the expansion of Eq. (4·19) in terms of the quantity e^ξ leads to

$$a_0^2=4e^{2\xi-(\gamma^2+\lambda^2)}\left\{1-\frac{4}{\gamma\lambda}e^{\xi-(1/2)(\gamma^2+\lambda^2)}\sinh(\gamma\lambda)\right\}$$
$$+O(e^{4\xi}), \tag{6·2}$$

retaining only the first two terms. The first term corresponds to the Boltzmann distribution and the second expresses the first order correction due to the Fermi statistics. The equivalent to this situation has been used also in Uehling's treatment.[5] On the basis of Eq. (6·2), we derive an expression for viscosity from the interpolation formula. A calculation directly from Eq. (4·9) based on Uehling's approximation will be shown in Appendix C in order to justify the approximation which has been employed to obtain the interpolation formula.

Substitution of (6·2) in Eq. (4·17) and integration with respect to λ lead to

$$\langle F_n \Lambda F_n \rangle = \frac{2^8 \pi^{5/2} m^{5/2} (kT)^{11/2} V e^{2\zeta}}{5h^6} \left\{ \int_0^\infty \gamma^7 e^{-\gamma^2} \Omega^{(2)} \, d\gamma \right.$$

$$\left. - \frac{8\sqrt{2}\, e^{\zeta}}{3\sqrt{3}} \int_0^\infty \gamma^7 e^{-(4/3)\gamma^2} \Omega^{(2)} \, d\gamma \right\}. \tag{6·3}$$

If we define the quantities as

$$\Omega^{(l,s)}(T) = \sqrt{\frac{kT}{\pi m}} \int_0^\infty e^{-\gamma^2} \gamma^{2s+3} \Omega^{(l)}(\gamma \sqrt{mkT}) \, d\gamma, \tag{6·4}$$

$$\Upsilon^{(l,s)}(T) = \sqrt{\frac{kT}{\pi m}} \int_0^\infty e^{-(4/3)\gamma^2} \gamma^{2s+3} \Omega^{(l)}(\gamma \sqrt{mkT}) \, d\gamma, \tag{6·5}$$

Eq. (6·3) can be written as

$$\langle F_n \Lambda F_n \rangle = \frac{32}{5} \frac{(kT)^2 V}{} \left(\frac{2\pi mkT}{h^2} \right)^3 e^{2\zeta} \Omega^{(2,2)} \times \left\{ 1 - \frac{8\sqrt{2}}{3\sqrt{3}} e^{\zeta} \frac{\Upsilon^{(2,2)}}{\Omega^{(2,2)}} \right\}. \tag{6·6}$$

Next, we expand Eq. (4·4) in terms of the quantity e^{ζ}:

$$G_s(\zeta) = e^{\zeta} \{ 1 - 2^{-(s+1)} e^{\zeta} \} + O(e^{3\zeta}), \tag{6·7}$$

which is inserted into Eq. (4·3) to yield

$$\langle F_n^2 \rangle = 2V \left(\frac{2\pi mkT}{h^2} \right)^{3/2} (kT)^2 e^{\zeta} (1 - 2^{-5/2} e^{\zeta}). \tag{6·8}$$

Insertion of Eqs. (6·6) and (6·8) into Eq. (4·20) leads to

$$\eta = \frac{5kT}{8\Omega^{(2,2)}} \left\{ 1 - e^{\zeta} \left(2^{-3/2} - \frac{8\sqrt{2}}{3\sqrt{3}} \frac{\Upsilon^{(2,2)}}{\Omega^{(2,2)}} \right) \right\}. \tag{6·9}$$

We have, for the Fermi gas,

$$\frac{N}{V} = 4\pi \left(\frac{2m}{h^2} \right)^{3/2} \int_0^\infty \frac{\varepsilon^{1/2} d\varepsilon}{e^{\beta(\varepsilon - \mu)} + 1} = 2 \left(\frac{2\pi mkT}{h^2} \right)^{3/2} G_{1/2}(\zeta), \tag{6·10}$$

which can be written, with the aid of Eq. (6·7), as

$$\frac{N}{V} = 2 \left(\frac{2\pi mkT}{h^2} \right)^{3/2} e^{\zeta} \left(1 - \frac{1}{2\sqrt{2}} e^{\zeta} + \cdots \right). \tag{6·11}$$

Therefore, we obtain

$$e^{\xi}=\frac{N}{2V}\left(\frac{h^2}{2\pi mkT}\right)^{3/2}\left\{1+\frac{1}{4\sqrt{2}}\frac{N}{V}\left(\frac{h^2}{2\pi mkT}\right)^{3/2}+\cdots\right\}, \tag{6·12}$$

where the first term suffices for our approximation. Thus we arrive at the following expression for the viscosity coefficient:

$$\eta=\frac{5kT}{8\Omega^{(2,2)}}\left\{1-\frac{N}{2V}\left(\frac{h^2}{2\pi mkT}\right)^{3/2}\left(2^{-3/2}-\frac{8\sqrt{2}}{3\sqrt{3}}\frac{\Gamma^{*(2,2)}}{\Omega^{(2,2)}}\right)\right\}. \tag{6·13}$$

This equation is the same as Uehling's result.[5), 13)] The first term of Eq. (6·13),

$$\eta=\frac{5kT}{8\Omega^{(2,2)}}, \tag{6·14}$$

corresponds to the nondegenerate case.[4)] In the classical limit, where the quantum effects in the scattering cross section vanish, Eq. (6·14) agrees with Enskog-Chapman's result.

§ 7. Numerical calculations

It has been shown in the preceding sections that the interpolation formula (4·20) with Eqs. (4·3) and (4·17) satisfactorily describes the temperature dependence of viscosity in the extremely and slightly degenerate cases. In this section, we deal with the behaviour of viscosity in the intermediate temperature range about the degeneracy temperature.

To perform numerical calculation, we rewrite the interpolation formula in a reduced form using the following units: T_c (degeneracy temperature) for temperature, m for mass, $h/\sqrt{m\mu_0}$ for length, and h/μ_0 for time. Eq. (4·20) thus takes the form

$$\eta=\frac{5}{2}(m\mu_0T^*)^{1/2}\left(\frac{2}{3}F_{3/2}(\zeta)\right)^2\Big/\int_0^\infty d\gamma\,\gamma^7\,\Omega^{(2)}\int_0^\infty d\lambda\,\lambda^2\,a_0^2, \tag{7·1}$$

where the reduced temperature, $T^*=T/T_c$, and the Fermi function, $F_s(\zeta)=\Gamma(s+1)G_s(\zeta)$, have been defined. The transport cross section $\Omega^{(2)}(p^r)$ can be written in terms of the phase shifts η_l of partial waves as follows:

$$\Omega^{(2)}(p^r)=\frac{3}{4}\Omega^{(2)}_{F.D.}+\frac{1}{4}\Omega^{(2)}_{B.E.}, \tag{7·2}$$

where

$$\Omega^{(2)}_{B.E.}=2\left(\frac{2\pi}{\kappa^2}\right)\sum_{even}\frac{(l+1)(l+2)}{(l+3/2)}\sin^2(\eta_{l+2}-\eta_l), \tag{7·3}$$

$$\Omega^{(2)}_{F.D.}=2\left(\frac{2\pi}{\kappa^2}\right)\sum_{odd}\frac{(l+1)(l+2)}{(l+3/2)}\sin^2(\eta_{l+2}-\eta_l), \tag{7·4}$$

κ being the magnitude of the corresponding wave vector to the relative momentum,

$\hbar\boldsymbol{\kappa}=(\boldsymbol{q}-\boldsymbol{p})/2$. In the case of hard sphere interaction, η_l is expressed as

$$\tan\eta_l(\kappa)=\frac{j_l(\kappa a)}{n_l(\kappa a)}, \tag{7·5}$$

$j_l(\kappa a)$ and $n_l(\kappa a)$ being the spherical Bessel functions, and a the diameter of hard sphere. Let us take the S-wave phase shift alone. Then we have

$$\eta_0=-\kappa a, \tag{7·6}$$

which is combined with Eqs. (7·2), (7·3) and (7·4) to lead to

$$Q^{(2)}(\kappa)=\frac{4}{3}\pi a^2. \tag{7·7}$$

We define the reduced form of the coefficient of viscosity η^* as follows:

$$\eta^*=(a^2/\sqrt{m\mu_0})\eta. \tag{7·8}$$

From Eqs. (7·1) and (7·7), we obtain

$$\eta^*=\frac{15\sqrt{T^*}}{64\sqrt{2\pi}}\frac{((2/3)F_{3/2}(\zeta))^2}{H(\zeta)}, \tag{7·9}$$

where

$$H(\zeta)=\int_0^\infty dx\int_0^{\pi/2}d\omega\,\frac{x^6\cos^5\omega}{[\sinh(x^2-\zeta)]^2}\left[\log\frac{\cosh(1/2)\{x^2(1+\sin2\omega)-\zeta\}}{\cosh(1/2)\{x^2(1-\sin2\omega)-\zeta\}}\right]^2. \tag{7·10}$$

The integral $H(\zeta)$ has been derived from the denominator of Eq. (7·1) by making the transformation of integration variables from (γ, λ) to (x, ω) by the following relations:

$$\left.\begin{aligned}\gamma&=\sqrt{2}\,x\cos\omega,\\ \lambda&=\sqrt{2}\,x\sin\omega.\end{aligned}\right\} \tag{7·11}$$

We evaluated $H(\zeta)$ numerically. The Fermi functions $F_{1/2}(\zeta)$ and $F_{3/2}(\zeta)$ have been tabulated by McDougall and Stoner,[14] and T^* is obtained from

$$T^*=\frac{kT}{\mu_0}=\left\{\frac{3}{2}F_{1/2}(\zeta)\right\}^{-2/3}. \tag{7·12}$$

The values of η^* thus obtained are shown in Table I. From Table I, we can see that as the temperature increases η^* takes a minimum at about $T^*=0.61$, where $\zeta\simeq1.0$, i. e. $\mu\simeq kT$. This behaviour of viscosity is due to the density-dependent symmetry effects of the Fermi particle system and has been anticipated from the intuitive arguments as stated in § 1. The curve η^* *versus* T^* is illustrated in Fig. 2.

Table I. Reduced viscosity η^* calculated with the use of the S-wave approximation

$\zeta(=\mu/kT)$	T^*	η^*
8.1	0.121909	0.14176
6.9	0.142421	0.11972
4.8	0.200883	0.088726
3.0	0.304021	0.069048
2.1	0.400117	0.062946
1.4	0.517253	0.060629
1.0	0.610855	0.060495
0.9	0.638288	0.060615
0.7	0.698895	0.061058
0.4	0.806469	0.062251
0.0	0.988733	0.064927
−0.8	1.54647	0.074502

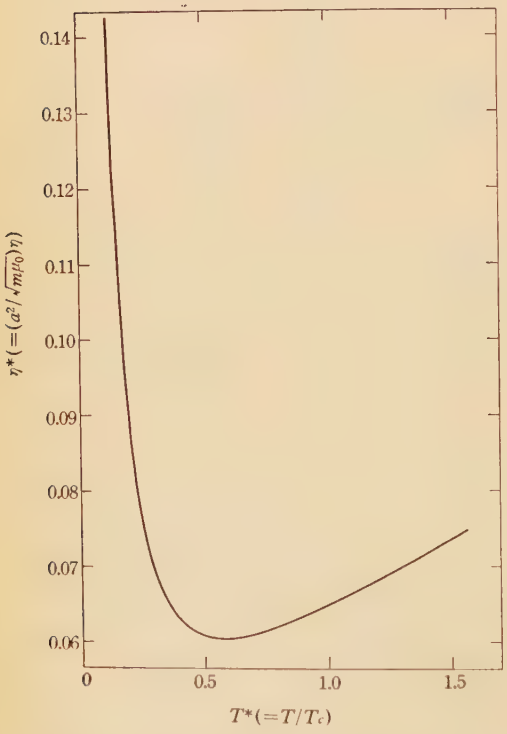

Fig. 2. Viscosity η^* *versus* temperature T^* curve based on the S-wave approximation.

§ 8. Comparison with experiments on liquid He^3

There have recently been performed a number of experiments on the viscosity of liquid He^3.[7),8),9)] Abrikosov and Khalatnikov[15)] calculated the viscosity of liquid He^3, employing Landau's theory of Fermi liquid,[16),17)] and obtained the T^{-2} law in the extremely degenerate case. The experiments, however, do not reach the limiting region where the T^{-2} law is valid. Therefore, we shall discuss Zinov'eva's experiments by applying the results obtained in the previous section.

According to Landau,[17)] Fermi liquid can be described by Fermi-type excitations. At low temperatures, the spectrum of the excitation has, in the simple case, the form of

$$\varepsilon(p) \simeq \frac{p^2}{2m}, \tag{8·1}$$

where m is the effective mass. We assume this spectrum to be valid for liquid He^3 at low temperatures and apply our calculations in the previous sections to

liquid He^3. Then the quantity a should be read as the scattering amplitude describing the collision between the excitations.

Recent experiments suggest that the heat capacity of liquid He^3 has the linear temperatrue dependence below about $T=0.1°K$, showing the validity of Landau's theory. From this, we can determine the effective mass m with the aid of the fact that the number of excitations is equal to the number of He^3 atoms. The heat capacity per particle takes the form

$$c=\frac{C}{N}=\gamma T, \tag{8·2}$$

where γ is related to μ_0 and m by the relations

$$\gamma=\frac{\pi^2}{2}\frac{k^2}{\mu_0}=\left(\frac{\pi V}{3N}\right)^{2/3}\frac{k^2}{\hbar^2}\,m. \tag{8·3}$$

From the experimental data[18] we find that $\gamma=3.2035\times10^{-16}$ erg deg^{-2} per particle. This leads to the degeneracy temperature

$$T_c=\frac{\mu_0}{k}=\frac{\pi^2}{2}\frac{k}{\gamma}\simeq 2.13°K. \tag{8·4}$$

The factor $m\mu_0$ involved in the viscosity (7·8) becomes

$$m\mu_0=\frac{\hbar^2}{2}\left(\frac{3N}{8\pi V}\right)^{2/3}=34.404\times10^{-40}, \tag{8·5}$$

where the number density N/V has been determined to be $N/V=1.6443\times10^{22}$, by making use of the extrapolated value 0.08235 gr/cm^3 for the mass density of liquid He^3 at 0°K.[19] Thus, from Eqs. (7·8) and (8·5), we obtain

$$\eta=5.8655\times10^{-20}\times a^{-2}\times\eta^*. \tag{8·6}$$

Let us take $a=1.158\times10^{-8}$ cm for the scattering amplitude so that η may fit well with the experimental curve. The relations of the viscosity η *versus* the temperature $T(=T_cT^*)$ thus obtained are shown in Table II and illustrated in Fig. 3.

Fig. 3 shows a good agreement of the theoretical curve with the experiments below about $T=0.6°K$. The deviation from the experiments becomes larger as the temperature increases. Especially, the experiments do not show any minimum. This may be understood as follows. At very low temperatures, contribution from the "symmetry effects" is superior to that from the interaction mechanism so that the Fermi gas spectrum (8·1) yields a good agreement with the experiments. As the temperature increases, however, the symmetry effect goes down and the deviation from (8·1) due to interaction becomes important. Thus, the experimental curve continues to decrease, without any minimum, in accordance with the general behavior of the viscosity of liquid.

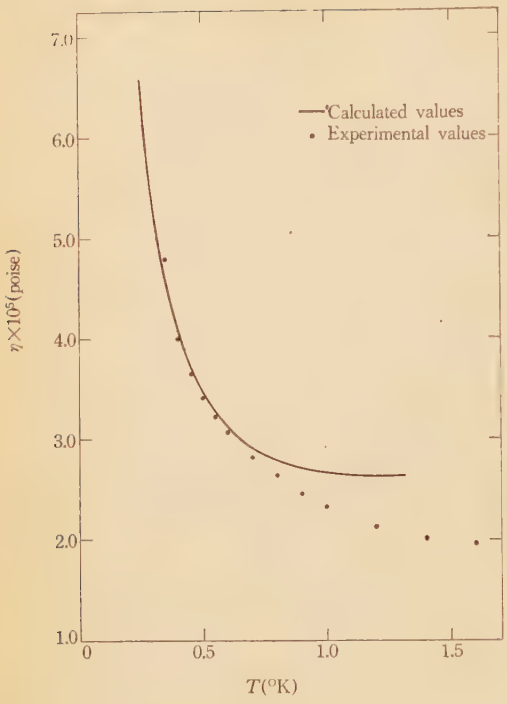

Table II. Viscosity η *versus* temperature T

T(°K)	$\eta\cdot10^5$(poise)
0.2597	6.201
0.3034	5.237
0.4279	3.881
0.6476	3.020
0.8522	2.753
1.1017	2.652
1.3011	2.646

Fig. 3. Temperature dependence of the viscosity of liquid He^3. The curve shows calculated values from Table II and • is Zinov'eva's observed values.

§ 9. Some remarks

The expression for the viscosity coefficient in terms of time-correlation function was calculated for the Fermi gases, and we obtained an approximate expression for the viscosity which is valid over the entire region of temperature. In the derivation, we made the following two approximations: (*a*) averaging out the factor $(1-f_r)(1-f_s)$ with respect to the scattering solid angle, and (*b*) taking only the *S*-wave phase shift in order to obtain a simplified interpolation formula which describes satisfactorily the behavior of the viscosity in the intermediate temperature range between the extremely and slightly degenerate cases. These approximations are, of course, unnecessary so far as we are concerned with Eq. (4·16) to obtain more rigorous results. Pais and Uhlenbeck[20] investigated the second virial coefficient of quantum gases at low temperatures and noted that at 1°K the *D*-state contributions are already appreciable for the Lennard-Jones type potential. However, the *S*-wave approximation may suffice to determine the approximate behavior in the intermediate temperature region.

We performed a numerical calculation of the interpolation formula and determined the minimum of the viscosity to be at about $T=0.6T_c$. With the use of the numerical calculation, we discussed Zinov'eva's experiments on liquid He^3, determining the effective mass of the quasi-particles from the recent experiments on the heat capacity. The scattering amplitude a was chosen so that

the theoretical curve of the viscosity fits the experiments well. According to the theory of weakly interacting Fermi particles, we obtain the following expression for the effective mass m in terms of the scattering length a_0 of the original particles:[17]

$$\frac{m_0}{m}=1-\frac{8}{15}\left(\frac{3}{\pi}\right)^{2/3}(7\log 2-1)a_0^2\left(\frac{N}{V}\right)^{2/3},$$

where m_0 is the mass of the original particles. Using the value of the effective mass determined from Eqs. (8·3) and (8·4), we thus obtain $a_0=2.1\times10^{-8}$ cm. It turns out therefore that the value of a we employed is of the same order of magnitude as the scattering length a_0.

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Appendix A

Approximate eigenvalue problem at the extremely degenerate case

Eq. (2·13) may be written approximately as

$$\Lambda F_n=\sum_{p<q}n_pn_q(1/V)\int d\Omega\,(2p^r/m)\sigma(p^r,\theta)$$
$$\times f_pf_q(1-f_r)(1-f_s)(2/m)(p_x^{\,r}p_y^{\,r}-p_x^{\,r\prime}p_y^{\,r\prime}). \quad (A\cdot1)$$

In the extremely degenerate case, the effective collisions between particles occur in the neighborhood of the Fermi surface. Therefore, the factor $f_pf_q(1-f_r)\times(1-f_s)$ in Eq. (A·1) may be replaced by its average value with respect to scattering angles (θ,φ), $\overline{f_pf_q(1-f_r)(1-f_s)}$, because the factor can be considered to be insensitive for the change of angles as pointed out in § 5.

Thus we obtain

$$\Lambda F_n=\sum_{p<q}n_pn_q(2/m)p_x^{\,r}p_y^{\,r}\,\overline{f_pf_q(1-f_r)(1-f_s)}\lambda_{pq}^{(2)}, \quad (A\cdot2)$$

where we have defined

$$\lambda_{pq}^{(2)}=(2\pi/V)\int_0^\pi d\theta\,\sin\theta\,(2p^r/m)\sigma(p^r,\theta)\{1-P_2(\cos\theta)\}. \quad (A\cdot3)$$

We assume now that on the average the deviation of $\overline{f_pf_q(1-f_r)(1-f_s)}\lambda_{pq}^{(2)}$, which is dependent on the relative momenta before and after collisions, is small from some average value of it with respect to the momenta, $\langle f_pf_q(1-f_r)(1-f_s)\lambda_{pq}^{(2)}\rangle$.

Then, taking account of

$$\sum_{p<q} n_p n_q \frac{2p_x^r p_y^r}{m} = \frac{N}{2} F_n, \tag{A·4}$$

where we have put the total momentum $\sum_p n_p \boldsymbol{p}$ to be zero, we obtain approximately

$$\Lambda F_n = \lambda F_n, \tag{A·5}$$

with

$$\lambda = \frac{N}{2} \overline{\langle f_p f_q (1-f_r)(1-f_s) \lambda_{pq}^{(2)} \rangle}, \tag{A·6}$$

the average $\langle \cdots \rangle$ being to be determined from Eq. (3·3).

Appendix B

Integrations in Eq. (4·9)

Making use of Eqs. (4·10), the second equation of (4·12), and (4·14), the integration with respect to φ leads to

$$\begin{aligned}
&\int_0^{2\pi} (1-f_{r'})(1-f_{r_1'}) (\gamma_x \gamma_y - \gamma_{x_1}' \gamma_y') \, d\varphi \\
&= (e^2/2) 2\pi [\gamma_x \gamma_y \{1 - P_2(\cos\theta)\} \sum_{l=0}^{\infty} a_l P_l(\cos\alpha) P_l(\cos\theta) \\
&\quad + B \sin\theta \cos\theta \sum_{l=2}^{\infty} \frac{a_l}{l(l+1)} P_l^1(\cos\alpha) P_l^1(\cos\theta) \\
&\quad - \frac{1}{2}(D-E) \sin^2\theta \sum_{l=2}^{\infty} \frac{a_l}{(l-1)l(l+1)(l+2)} P_l^2(\cos\alpha) P_l^2(\cos\theta)],
\end{aligned} \tag{B·1}$$

where

$$\begin{aligned}
\gamma_x \gamma_y = \gamma^2 \Big\{ &\sin\theta \cos\theta \cos\delta (\cos^2\alpha - \sin^2\alpha \cos^2\varepsilon) \\
&+ \cos 2\theta \cos\delta \sin\alpha \cos\alpha \cos\varepsilon \\
&- \cos\theta \sin\delta \sin\alpha \cos\alpha \sin\varepsilon \\
&+ \sin\theta \sin\delta \sin^2\alpha \sin\varepsilon \cos\varepsilon \Big\},
\end{aligned} \tag{B·2}$$

$$\begin{aligned}
B = \gamma^2 \Big\{ &-\sin\theta \cos\theta \cos\delta \sin 2\alpha (1+\cos^2\varepsilon) \\
&+ \cos 2\theta \cos\delta \cos 2\alpha \cos\varepsilon \\
&- \cos\theta \sin\delta \cos 2\alpha \sin\varepsilon
\end{aligned}$$

$$+\sin\theta\sin\delta\sin 2\alpha\sin\varepsilon\cos\varepsilon\Big\}, \tag{B·3}$$

$$\begin{aligned} D-E=\gamma^2\Big\{&\sin\theta\cos\theta\cos\delta(\sin^2\alpha-\cos^2\alpha\cos^2\varepsilon+\sin^2\varepsilon)\\ &-\cos 2\theta\cos\delta\sin\alpha\cos\alpha\cos\varepsilon\\ &+\cos\theta\sin\delta\sin\alpha\cos\alpha\sin\varepsilon\\ &+\sin\theta\sin\delta(1+\cos^2\alpha)\sin\varepsilon\cos\varepsilon\Big\}. \end{aligned} \tag{B·4}$$

Next, let us carry out the integrations with respect to ε, δ and θ in turn. Then, contributions from each term of Eq. (B·1) to Eq. (4·9) become as follows:

$$\left.\begin{aligned} &\int_0^{\pi}d\theta\sin\theta\int_0^{2\pi}d\delta\int_0^{2\pi}d\varepsilon(\gamma_x\gamma_y)^2=\frac{8\pi^2}{15}\gamma^4,\\ &\int_0^{\pi}d\theta\sin\theta\int_0^{2\pi}d\delta\int_0^{2\pi}d\varepsilon\,\gamma_x\gamma_yB=0,\\ &\int_0^{\pi}d\theta\sin\theta\int_0^{2\pi}d\delta\int_0^{2\pi}d\varepsilon\,\gamma_x\gamma_y(D-E)=0. \end{aligned}\right\} \tag{B·5}$$

Thus, using the first equation of (4·12) and the orthogonality of the Legendre polynomials, we obtain Eq. (4·16).

In Eq. (4·16), $a_l(\gamma,\lambda;\zeta)$ is given by

$$a_l=\frac{2l+1}{2}\int_0^{\pi}\frac{P_l(\cos\alpha)\sin\alpha\,d\alpha}{\cosh\rho+\cosh(\gamma\lambda\cos\alpha)}, \tag{B·6}$$

which leads to Eq. (4·19) for $l=0$.

Appendix C

Calculation directly from Eq. (4·9) using Uehling's approximation

According to (6·1), we obtain the following expansion:

$$\begin{aligned} &f_\tau f_{\tau_1}(1-f_{\tau'})(1-f_{\tau_1'})=e^{2\zeta}e^{-(\gamma^2+\lambda^2)}\\ &\times[1-2e^{\zeta-(1/2)(\gamma^2+\lambda^2)}\{\cosh(\gamma\lambda\cos\alpha)+\cosh(\gamma\lambda\cos\alpha')\}]\\ &+O(e^{4\zeta}). \end{aligned} \tag{C·1}$$

Let us call (C·1) Uehling's approximation,[5] neglecting terms of higher order represented by $O(e^{4\zeta})$.

In order to evaluate the contribution from the third term of (C·1) to Eq. (4·9), let us expand it as follows:

$$\cosh(\gamma\lambda\cos\alpha')=\sum_{l=0}^{\infty}b_l(\gamma,\lambda)P_l(\cos\alpha'). \tag{C·2}$$

We take account of the addition theorem (4·14) and carry out the integrations with respect to φ, ϑ, θ and ξ, which is similar to the calculations carried out in Appendix B. Then we obtain

$$-\frac{2^{11/2}\pi^2 Ve^{3\zeta}}{15m^3h^6c^{11/2}}\int_0^{\infty}d\gamma\,\gamma^7 e^{-(3/2)\gamma^2}\int_0^{\infty}d\lambda\,\lambda^2 e^{-(3/2)\lambda^2}\int_0^{\pi}d\alpha\sin\alpha$$

$$\times\sum_{l=0}^{\infty}b_lP_l(\cos\alpha)(2\pi)\int_0^{\pi}d\theta\sin\theta\,\sigma(p^r,\theta)\{1-P_2(\cos\theta)\}P_l(\cos\theta), \tag{C·3}$$

and find that this is the same as the contribution from the second term of (C·1), since all but the term of $l=0$ vanish if the integration with respect to α is carried out.

Thus we obtain

$$\langle F_n\Lambda F_n\rangle=\frac{2^{5/2}\pi^{5/2}Ve^{2\zeta}}{5m^3h^6c^{11/2}}\left\{\int_0^{\infty}d\gamma\,\gamma^7e^{-\gamma^2}\Omega^{(2)}\right.$$

$$\left.-\frac{8}{\sqrt{\pi}}e^{\zeta}\int_0^{\infty}d\gamma\,\gamma^7e^{-(3/2)\gamma^2}\Omega^{(2)}\int_0^{\infty}d\lambda\,\lambda^2e^{-(3/2)\lambda^2}\int_0^{\pi}d\alpha\sin\alpha\cosh(\gamma\lambda\cos\alpha)\right\}, \tag{C·4}$$

which leads to Eq. (6·3). We see, therefore, that Eq. (6·3) can be obtained without any other approximations than (C·1). In § 6, we have derived it from the interpolation formula with the use of the approximation (6·2) which is equivalent to (C·1). Thus we can say that the neglect of all terms other than that of $l=0$, which was made in § 4 to obtain the interpolation formula, leads to rigorous results in this slightly degenerate case.

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Stellar Synthesis of the Proton-Rich Heavy Elements

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The proton-rich heavy nuclei are produced by proton-capture and/or photo-nuclear reactions starting with the products of neutron capture.

The stellar conditions in which the proton-rich nuclei are produced must be such that the proton density $\rho_p=10\sim10^2\mathrm{gcm}^{-3}$, the neutron density $\rho_n=10^{-5}\sim10^{-4}\mathrm{gcm}^{-3}$ and the other matter density $\rho\lesssim10^{-5}\mathrm{gcm}^{-3}$ at the temperature $T=2.5\sim3\times10^9{}^\circ\mathrm{K}$. It is, however, difficult to explain the abundance of all proton-rich nuclei for the equal temperature. It is possible if there exist the regions of different temperature. To form these nuclei the nuclear formation by the effective (p, γ) reaction occurs in the higher temperature region and the nuclear formation by the effective (γ, n) reaction in the lower temperature region.

§ 1. Introduction

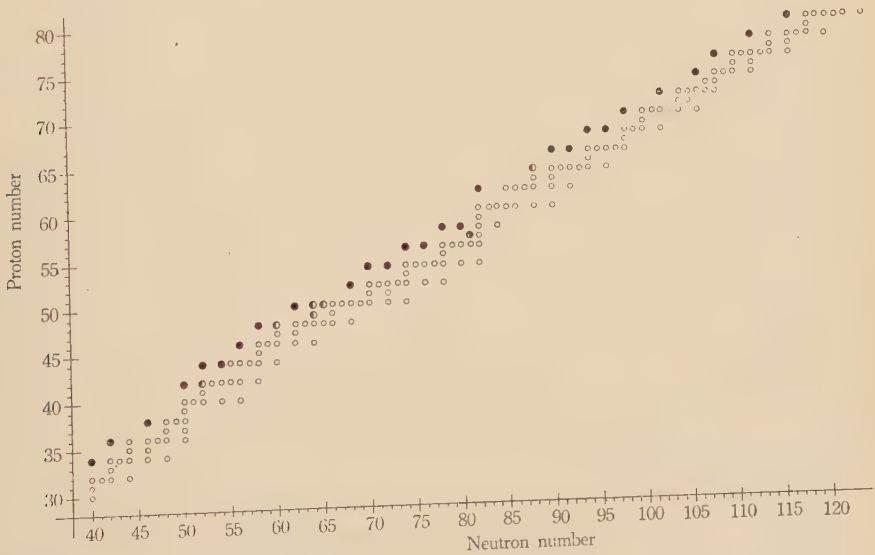
Burbidge et al.[1] and Cameron[2] have explained the observed data by assuming that the heavy elements existing in the universe are formed through the neutron capture process. They divided its process into two types: The one is the slow neutron capture process (s-process) which occurs at a comparatively low temperature in the stellar interior, the other the rapid neutron capture process (r-process) which takes place during the initial stage of supernova explosion. The heavy nuclei being produced through these processes are subjected to beta-decays, so that they are located on the stability curve and on its right-hand side in the N-Z plane (see Fig. 1).

However, the proton-rich heavy nuclei which are located on the left-hand side of the stability curve cannot at all be produced by the neutron capture; hence it has been suggested that these nuclei are produced by proton capture and/or photo-nuclear reactions starting with the products of neutron capture processes. Burbidge et al.[1] explained qualitatively the data of Suess and Urey[3] assuming that these nuclei were initially produced by (p, γ) reactions operating on about 1% of the material which has already been synthesized through neutron capture processes. They showed that these reactions occur when the temperature $T=2\sim3\times10^9{}^\circ\mathrm{K}$ and the proton density $\rho_p\gtrsim10^2\,\mathrm{gcm}^{-3}$. It is impossible, however, to explain the relative abundances of all the proton-rich nuclei under the same stellar conditions and the nucleus ${}^{138}_{57}\mathrm{La}$ cannot be produced for a low temperature.

Moreover, for the temperatures above 2.5×10^9 degree the (γ, n) reaction time, $\tau_{\gamma n}$, is smaller than the (p, γ) reaction time, $\tau_{p\gamma}$, in the very heavy element

region; hence the formation of the proton-rich nuclei by the (p, γ) reaction would be almost impossible there.

In this paper we shall quantitatively deal with the process of forming the proton-rich heavy nuclei in the supernova envelope, and investigate whether the abundances of all proton-rich nuclei are explained under the same stellar conditions. As was suggested by Burbidge et al.[1] we assume the (p, γ) and (γ, n) reactions as the processes producing these nuclei. The comparison between the reaction times of two processes, (p, γ) and (γ, n), forming the proton-rich heavy nuclei from the seed nuclei which have already been synthesized tells us that $\tau_{\gamma n} \sim \tau_{p\gamma}$ at the temperatures of $2.5 \sim 3 \times 10^9$ °K in the region of elements having large mass number A and vice versa (see Fig. 5). However, at the temperature of 3×10^9 °K only the (p, γ) reaction of the reactions on the nuclei having the small mass number is responsible for the formation of the proton-rich nuclei, while at the temperature of 2.5×10^9 °K only the (γ, n) reaction of the reactions on the nuclei having the large mass number is effective. It is impossible to form the all proton-rich nuclei by both reactions at the equal temperature. Therefore, we consider the two regions having different temperatures. When the (p, γ) and (γ, n) reactions took place in two regions, the light elements of the proton-rich nuclei can be produced by (p, γ) reactions in the region of the higher temperature and the very heavy proton-rich nuclei can be produced by (γ, n) reactions in the region of lower temperature. The abundances of the proton-rich heavy nuclei are shown in Fig. 6.



● The proton-rich heavy nuclei.
◐ Of the proton-rich heavy nuclei, the nuclei having a possibility produced in weak loops of the *s*-process chain.
○ The nuclei produced by neutron capture processes.

Fig. 1. A chart of the proton-rich heavy nuclei.

§ 2. Synthesis of the proton-rich heavy nuclei

(i) *Reaction rates*

Let us now consider the reaction rates for (p, γ), (γ, p), (γ, n) and (n, γ) reactions on the heavy nuclei. In general, for the (p, γ) reaction on a heavy nucleus (A, Z) a mean reaction time per one nucleus, $\tau_{p\gamma}$, is approximately given by using a statistical nonresonant formula as

$$1/\tau_{p\gamma}=\lambda_{p\gamma}=4.50\times10^{9}\rho_p\frac{Z^{5/6}}{A^{1/6}T_9^{2/3}}\exp\left[1.26\{Z(A^{1/3}+1)\}^{1/2}-4.25(Z^2/T_9)^{1/3}\right], \tag{1}$$

where $\lambda_{p\gamma}$ is a rate of reaction per one nucleus per unit time, ρ_p the density of protons in gcm^{-3} and T_9 the temperature in units of $10^9\,^\circ\mathrm{K}$.

A last proton binding energy Q_p of a fixed N nucleus becomes progressively smaller as more and more protons are added, and the (γ, p) reactions become operative. A mean reaction time for the (γ, p) reaction being in a state of detailed balance,

$$1/\tau_{\gamma p}=\lambda_{\gamma p}=2.00\times10^{10}\frac{T_9^{3/2}}{\rho_p\tau_{p\gamma}}\exp(-11.6Q_p/T_9) \tag{2}$$

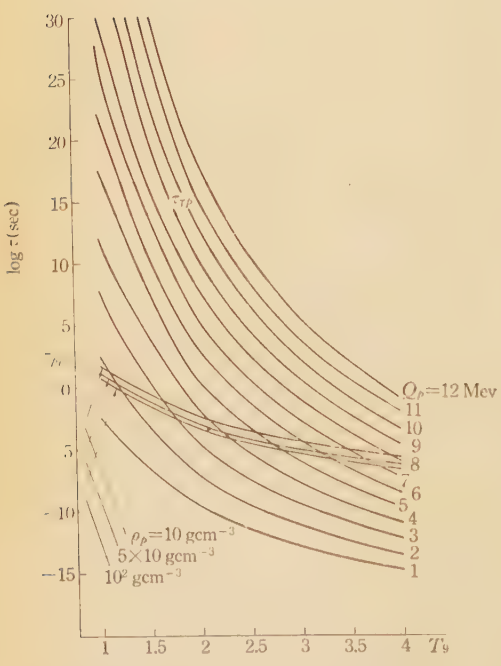


Fig. 2. (a) (p, γ) and (γ, p) reaction times for typical nuclei. $(Z, A)=(40, 90)$.

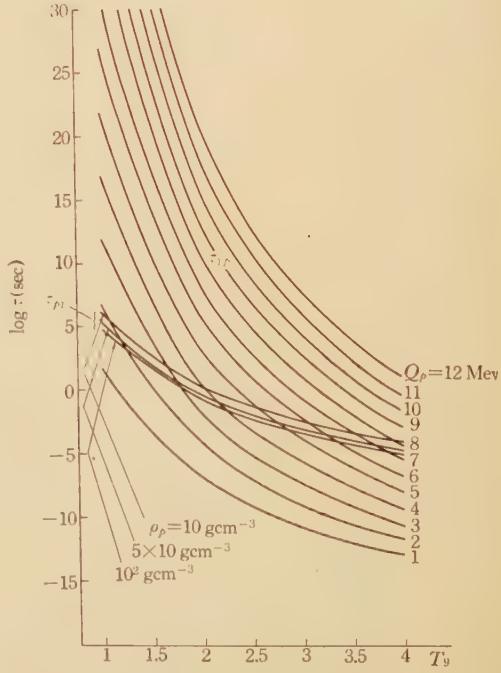


Fig. 2. (b) (p, γ) and (γ, p) reaction times for typical nuclei. $(Z, A)=(60, 142)$.

or

$$\log \tau_{\gamma p} = -10.30 - 1.5 \log T_9 + 5.04 \frac{Q_p}{T_9} + \log \tau_{\gamma p} \rho_p, \tag{3}$$

with all logarithms to the base 10; $\lambda_{\gamma p}$ a reaction rate of the (γ, p) reaction and Q_p in units of Mev. For $Z=40$ and 60, the values of $\tau_{p\gamma}$ and $\tau_{\gamma p}$ are shown in Fig. 2 assuming $\rho_p = 10,\ 5\times 10,\ 10^2\ \mathrm{gcm}^{-3}$.

On the other hand, a mean reaction time of the (γ, n) reaction, $\tau_{\gamma n}$, is also obtained from that of the (n, γ) reaction $\tau_{n\gamma}$. Write σ for the cross section of the (n, γ) reaction and let v and ρ_n be the velocity and density of neutrons responsible for its reaction, respectively, then

$$1/\tau_{n\gamma} = \lambda_{n\gamma} = 6.02\times 10^{23} \rho_n \sigma v, \tag{4}$$

where $\lambda_{n\gamma}$ is a reaction rate per one nucleus per unit time. For the value σ, we use the cross sections evaluated[4] from the reactivity measurement for a neutron energy of 25 kev and for the unknown cross sections we interpolate or extrapolate their values from the known experimental ones considering the difference between even A and odd A nuclei for which a ratio (odd A)/(even A) $=1\sim 4$ in the heavy element region. A mean reaction time of the (γ, n) reaction is also obtained from the equation for the equilibrium between (n, γ) and (γ, n) reactions.

$$1/\tau_{\gamma n} = \lambda_{\gamma n} = 2.00\times 10^{10} \frac{T_9^{3/2}}{\rho_n \tau_{n\gamma}} \times \exp(-11.6 Q_n / T_9) \tag{5}$$

or

$$\log \tau_{\gamma n} = -10.30 - 1.5 \log T_9 + 5.04 \frac{Q_n}{T_9} + \log \tau_{n\gamma} \rho_n, \tag{6}$$

where $\lambda_{\gamma n}$ is a reaction rate and Q_n a last neutron binding energy in a nucleus (A, Z) in units of Mev. The relation between these reaction times $\tau_{n\gamma}$ and $\tau_{\gamma n}$ is shown in Fig. 3.

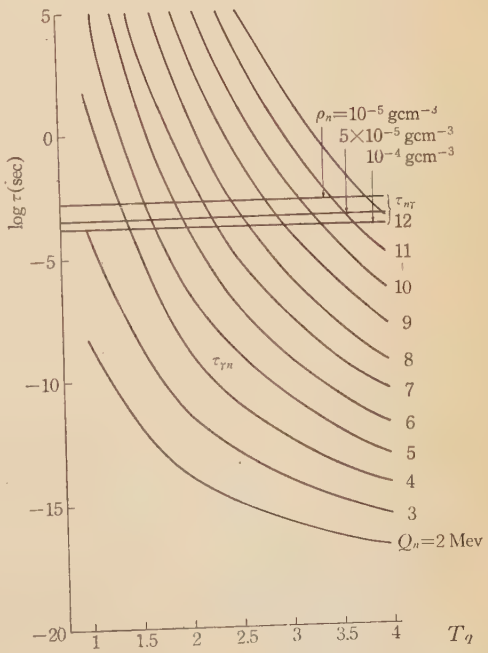


Fig. 3. (γ, n) and (n, γ) reaction times. (n, γ) reaction time is assumed to be constant for a given density ρ_n.

(ii) (p, γ) *reaction*

In this subsection we shall formulate and solve the general problem of the nuclear formation in which the proton-rich heavy nuclei are produced by the (p, γ) or (γ, n) reaction on a fast time scale. As was described in

§ 1, it is assumed that these processes occur at $2.5\sim3\times10^9$ degrees, and only about $1^o/_o$ of the heavy nuclei which have already been synthesized by the neutron capture process contribute to produce the proton-rich heavy nuclei.

Besides these processes, we must consider the reactions induced by α-particles, but their reactions occur much slower than those by protons, for example, a ratio of $\lambda_{\alpha\gamma}/\lambda_{p\gamma}$ being equal to $0.43\cdot\rho_\alpha/\rho_p\cdot\exp\{41.5-10(633/T_9)^{1/3}\}$ for $(A, Z)=(120, 50)$, in which $\lambda_{\alpha\gamma}$ is a reaction rate of (α, γ) reaction and ρ_α the α-particle density in units of gcm^{-3}. Therefore, it is able to neglect the effect of the reactions induced by α-particles hereafter.

Next, it should be noted whether the number of protons which can be added to the heavy nuclei is limited by the positron decay times. In the region of the proton-rich heavy nuclei, a mean lifetime of β^+-decay τ_{β^+} is agreeable to the calculated values as positron emission takes place by a forbidden transition to the ground state of their nuclei. Hence for an unknown mean lifetime of β^+-decay we estimate it using the expression as

$$\tau_{\beta^+}\simeq 10^6/W_{\beta^+}^5 \text{ sec} \quad \text{(forbidden transition)},$$
$$W_{\beta^+}=M(A, Z)-M(A, Z-1)-0.51, \tag{7}$$

where $M(A, Z)$ is the mass measured in units of Mev, for which we used the semi-empirical mass formula derived by Cameron.[5] Then the calculated value τ_{β^+} is of the order of 10 minutes at least for the nuclei of interest. Now the duration of a supernova explosion has been estimated to be $1\sim10^3$ sec, so we can neglect these β^+-decay processes in our rapid processes.

Thus, neglecting the α-induced reactions and β^+-decay reactions, we can write the time variation of abundances of nuclei in our process, in which is treated the (p, γ) reaction at first, as follows:

$$\begin{aligned}\frac{dn(0)}{dt}&=\lambda_{p\gamma}(-1)\,n(-1)-\{\lambda_{p\gamma}(0)+\lambda_{\gamma p}(0)\}\,n(0)+\lambda_{\gamma p}(1)\,n(1),\\ \frac{dn(1)}{dt}&=\lambda_{p\gamma}(0)\,n(0)-\{\lambda_{p\gamma}(1)+\lambda_{\gamma p}(1)\}\,n(1)+\lambda_{\gamma p}(2)\,n(2),\end{aligned} \tag{8}$$

etc.,

which $n(i)$ is the relative abundance of a nucleus i, $\lambda_{p\gamma}(i)$ and $\lambda_{\gamma p}(i)$ are the reaction rates of the (p, γ) and (γ, p) reactions on a nucleus i, respectively. As protons are very abundant in comparison with the other nuclei, the reaction time which the seed nuclei capture protons is much shorter than the duration of supernova outburst. Hence we need not consider the (p, γ) reaction time on the initial several seed nuclei until the last proton binding energy in a nucleus decreases and the inverse reaction (γ, p) becomes effective. That is, when the proton capture process proceeds and the (γ, p) reaction rate becomes comparable with the (p, γ) reaction rate, a quasi-statistical equilibrium would

be realized between (p, γ) and (γ, p) reactions for a short period. If the time during which the proton capture process takes place is long, the produced proton-rich nucleus is released from the system including such an equilibrium state.

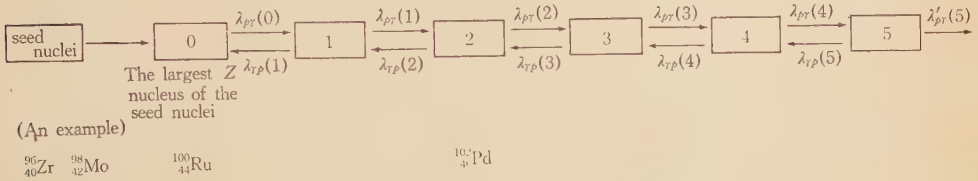


Fig. 4. Schematic diagram of the (p, γ) reactions.

When a nucleus reached one satisfying the condition such that $\lambda_{p\gamma}(0) \simeq \lambda_{\gamma p}(1)$, we call its nucleus a nucleus 0. We shall consider conveniently the system constitute of six nuclei. Then in most cases a nucleus 0 corresponds to an even-even nucleus which also is the largest Z nucleus of the seed nuclei having the same N (see Fig. 4). Thus we assume $\lambda_{\gamma p}(0) = 0$ because of $\lambda_{p\gamma}(0) \gg \lambda_{\gamma p}(0)$ and put the initial conditions such that $n(0) = 1$, $n(1) = n(2) = n(3) = \cdots = 0$ at $t = 0$. Then Eq. (8) are written in the following from a nucleus 0 to a nucleus 5 and their solutions can be obtained.

$$\begin{aligned}
&\frac{dn(0)}{dt} = -\lambda_{p\gamma}(0)\, n(0) + \lambda_{\gamma p}(1)\, n(1), \\
&\frac{dn(1)}{dt} = \lambda_{p\gamma}(0)\, n(0) - \{\lambda_{p\gamma}(1) + \lambda_{\gamma p}(1)\} n(1) + \lambda_{\gamma p}(2)\, n(2), \\
&\cdots\cdots\cdots

reason that it will be discussed later, $\lambda_{\gamma p}(5)$, $\lambda_{\gamma p}(3)$ and $\lambda_{\gamma p}(1)$ are all larger by the order two or more than the other λ, and it tells us $\lambda_{\gamma p}(5) \gg \lambda_{\gamma p}(3) \gg \lambda_{\gamma p}(1)$.

Taking account of the above relations among the reaction rates we can solve Eq. (9),

$$n(i) = A_i e^{-\delta_0 t} + \sum_{k=1}^{5} B_i e^{-\delta_k t} \quad (i=0\sim 5), \tag{11}$$

where δ_0, δ_1, δ_2, δ_3, δ_4 and δ_5 are the approximate solutions of the following equation:

$$\begin{aligned} &x^6+\lambda_{\gamma p}(5)\, x^5+\lambda_{\gamma p}(5)\,\lambda_{\gamma p}(3)\, x^4+\lambda_{\gamma p}(5)\,\lambda_{\gamma p}(3)\,\{\lambda_{\gamma p}(1) \\ &\quad+\lambda_{\gamma p}(4)\}\, x^3+\lambda_{\gamma p}(5)\,\lambda_{\gamma p}(4)\,\lambda_{\gamma p}(3)\,\lambda_{\gamma p}(1)\, x^2+\lambda_{\gamma p}(5) \\ &\quad\times\lambda_{\gamma p}(4)\,\lambda_{\gamma p}(3)\,\{\lambda_{p\gamma}(0)\,\lambda_{p\gamma}(1)+\lambda_{p\gamma}(0)\,\lambda_{\gamma p}(2)+\lambda_{\gamma p}(1)\,\lambda_{\gamma p}(2)\}\, x \\ &\quad+\lambda_{p\gamma}(0)\,\lambda_{p\gamma}(1)\,\lambda_{p\gamma}(2)\,\lambda_{p\gamma}(3)\,\lambda_{p\gamma}(4)\,\lambda'_{p\gamma}(5)=0. \end{aligned} \tag{12}$$

It may be allowed to neglect the higher terms in comparison with the first term of (11) considering $\delta_0 \ll \delta_1, \delta_2, \cdots \delta_5$. Then $n(i)$ have the following expression,

$$\begin{aligned} &n(i) \cong A_i e^{-\delta_0 t}, \\ &\delta_0 \cong \frac{\lambda_{p\gamma}(0)\,\lambda_{p\gamma}(1)\,\lambda_{p\gamma}(2)\,\lambda_{p\gamma}(3)\,\lambda_{p\gamma}(4)\,\lambda'_{p\gamma}(5)}{\lambda_{\gamma p}(3)\,\lambda_{\gamma p}(4)\,\lambda_{\gamma p}(5)\,\{\lambda_{p\gamma}(0)\,\lambda_{p\gamma}(1)+\lambda_{p\gamma}(0)\,\lambda_{\gamma p}(2)+\lambda_{\gamma p}(1)\,\lambda_{\gamma p}(2)\}}, \\ &A_0 : A_1 : A_2 \cong \lambda_{\gamma p}(1)\,\lambda_{\gamma p}(2) : \lambda_{p\gamma}(0)\,\lambda_{\gamma p}(2) : \lambda_{p\gamma}(0)\,\lambda_{p\gamma}(1), \end{aligned} \tag{13}$$

where A_i, etc., are the coefficients expressing the distribution of nuclei for $t \ll 1/\delta_0$ (i.e. $e^{-\delta_0 t} \simeq 1$), and A_3, A_4, A_5 are smaller by order two or more than A_0, A_2.

From (9) $\sim$ (13) we can evaluate the relative abundances of the proton-rich nuclei to their seed nuclei. Under the above conditions $T_9=3$, $\rho_p=10\sim 10^2$ gcm^{-3} the proton-rich nucleus corresponds to a nucleus 2 with the exception that $^{113}_{49}$In and $^{138}_{57}$La correspond to a nuclei 1 and $^{94}_{42}$Mo to a nucleus 0, in which $^{139}_{57}$La (only a seed nucleus of $^{138}_{57}$La) is exceptionally an odd-even nucleus. Thus $n(0)$, $n(1)$ and $n(2)$ are given by

$$\begin{aligned} &n(0) \cong A_0 e^{-\delta_0 t} \cong \frac{\lambda_{\gamma p}(1)\,\lambda_{\gamma p}(2)}{\lambda_{\gamma p}(1)\,\lambda_{\gamma p}(2)+\lambda_{p\gamma}(0)\,\lambda_{\gamma p}(2)+\lambda_{p\gamma}(0)\,\lambda_{p\gamma}(1)} e^{-\delta_0 t}, \\ &n(1) \cong A_1 e^{-\delta_0 t} \cong \frac{\lambda_{p\gamma}(0)\,\lambda_{\gamma p}(2)}{\lambda_{\gamma p}(1)\,\lambda_{\gamma p}(2)+\lambda_{p\gamma}(0)\,\lambda_{\gamma p}(2)+\lambda_{p\gamma}(0)\,\lambda_{p\gamma}(1)} e^{-\delta_0 t} \end{aligned}$$

and

$$n(2) \cong A_2 e^{-\delta_0 t} \cong \frac{\lambda_{p\gamma}(0)\,\lambda_{p\gamma}(1)}{\lambda_{\gamma p}(1)\,\lambda_{\gamma p}(2)+\lambda_{p\gamma}(0)\,\lambda_{\gamma p}(2)+\lambda_{p\gamma}(0)\,\lambda_{p\gamma}(1)} e^{-\delta_0 t}. \tag{14}$$

Since the nucleus $^{138}_{57}$La is a double shielded nucleus, which is not so proton-rich, it is unable to produce it by the β-decay processes considered as the freezing reaction in the cooling stage from the unstable nucleus, which is the product of the neutron capture process. Even if we also assume that $^{138}_{57}$La is the product of the proton-capture reaction from $^{137}_{56}$Ba, this nucleus cannot be produced at $T_9<3$ because $\lambda_{\gamma p}(^{139}_{58}\mathrm{Ce} \rightarrow ^{138}_{57}\mathrm{La}) \ll \lambda_{p\gamma}(^{138}_{57}\mathrm{La} \rightarrow ^{139}_{58}\mathrm{Ce})$, $^{138}_{57}$La being the Z odd-N odd nucleus. To produce $^{138}_{57}$La through the proton-capture process the temperature needs to be at least 3×10^9 °K. The astrophysical condition of $T_9>3$ would be implausible in the supernova envelope containing the excessive abundance of hydrogen. We therefore take $T_9=3$ for $\rho_p=10\sim10^2$ gcm^{-3} to produce $^{138}_{57}$La. When the temperature rises to about 3×10^9 °K, one must notice that the (γ, n) reaction time becomes shorter than the (p, γ) reaction time in the very heavy element region as indicated in Fig. 5. Namely, the (γ, n) reaction on their elements first occur and the seed nuclei for the (p, γ) process are almost brought off. Hence the formation of the proton-rich nuclei by the (p, γ) reaction is impossible there. Moreover in the envelope the neutron density is so low ($10^{-5}\sim10^{-4}$ gcm^{-3}) that the (γ, n) reaction rate is much larger than the (n, γ) reaction rate in the very heavy element region for this temperature. These facts tell us that the formation of the proton-rich nuclei in the very heavy element region is impossible in either process at $T_9=3$.

In order to produce the proton-rich nuclei by the (γ, n) reaction in the very heavy element region, we must lower the temperature for the densities $\rho_p=10\sim10^2$ gcm^{-3} and $\rho_n=10^{-5}\sim10^{-4}$ gcm^{-3}, these densities being the suitable values in the envelope of a supernova. As shown in Eqs. (4) and (5), the value of $\lambda_{n\gamma}$ is almost independent of temperature, while $\lambda_{\gamma n}$ changes sharply with the temperature. Therefore, if we adopt $T_9=2.5$ to increase the value of $\lambda_{\gamma n}$ for the same neutron density, the quasi-equilibrium state between (γ, n) and (n, γ) reactions realizes for a short period and the proton-rich nuclei can be produced through the (γ, n) process on the seed nuclei in the very heavy element region. In the light element region, at the temperature of $T_9=2.5$ the (p, γ) reaction time is shorter than the (γ, n) reaction time, so that the (p, γ) reaction is favorable as in the case of $T_9=3$ (see Fig. 5). In this case, however, the proton-rich nuclei cannot be produced by the (p, γ) reaction because of $\lambda_{p\gamma}>\lambda_{\gamma p}$ for $\rho_p=10\sim10^2$ gcm^{-3}. For $T_9=2.5$, therefore, the proton-rich nuclei in the light element region cannot be made in any case.

Here we assume that there exist two regions of $T_9=2.5$ and three for the densities of $\rho_p=10\sim10^2$ gcm^{-3} and $\rho_n=10^{-5}\sim10^{-4}$ gcm^{-3}. Then the light nuclei of the proton-rich nuclei would be produced by the (p, γ) reactions in the region of $T_9=3$, and the very heavy nuclei of the proton-rich nuclei would be produced by the (γ, n) reactions in the region of $T_9=2.5$.

The relation between the proton-rich heavy nuclei and their expected formation processes is shown in Table I.

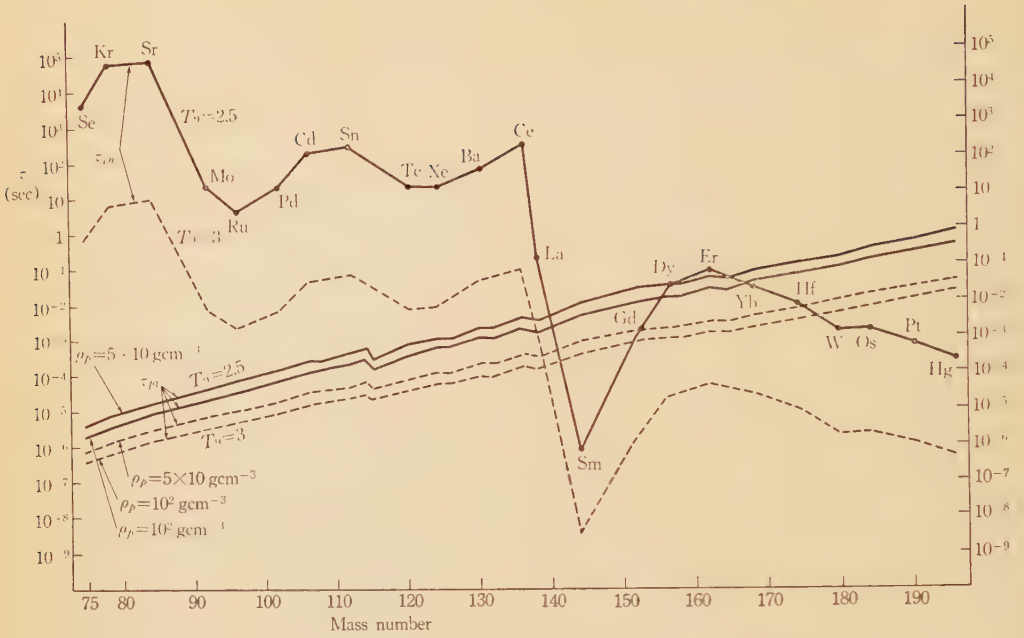


Fig. 5. The reaction times of (p, γ) and (γ, n) reactions for the production of proton-rich heavy nuclei.

(iii) (γ, n) *reaction*

The general formula forming the proton-rich nuclei through the (γ, n) process is treated with the same consideration as in the (p, γ) process. For $\lambda_{p\gamma}$ and $\lambda_{\gamma p}$ in Eqs. (8) and (9), we may substitute $\lambda_{\gamma n}$ and $\lambda_{n\gamma}$, respectively. Since $\lambda_{\gamma n}(0) \leq \lambda_{n\gamma}(0)$ for $\rho_n = 10^{-5} \sim 10^{-4}\,\mathrm{gcm}^{-3}$, $\lambda_{n\gamma}(0)$ being not negligible in comparison with $\lambda_{\gamma n}(0)$, $\lambda_{\gamma n}(0)$ is replaced by $\lambda_{\gamma n}(0) + \lambda_{n\gamma}(0)$. Then the following equations are obtained:

$$\begin{aligned}
\frac{dn(0)}{dt} &= -\{\lambda_{\gamma n}(0) + \lambda_{n\gamma}(0)\}\, n(0) + \lambda_{n\gamma}(1)\, n(1), \\
\frac{dn(1)}{dt} &= \lambda_{\gamma n}(0)\, n(0) - \{\lambda_{\gamma n}(1) + \lambda_{n\gamma}(1)\}\, n(1) + \lambda_{\gamma n}(2)\, n(2), \\
&\cdots\cdots\cdots

To explain the formation of the proton-rich nuclei through the quasi-equilibrium state between the (γ, n) and (n, γ) processes during the initial phases of the supernova explosion, we require the conditions such that $\rho_n = 10^{-5} \sim 10^{-4}$ gcm^{-3} for $T_9 = 2.5$ as discussed before. Taking here the value $\rho_n = 10^{-5} \sim 10^{-4}$ gcm^{-3}, $\lambda_{\gamma n}(1)$ and $\lambda_{\gamma n}(3)$ are larger by order one or more than the other all reaction rates $\lambda(\lambda_{\gamma n}(1) > \lambda_{\gamma n}(3))$, and $\lambda_{n\gamma}$'s values are almost constant $(\lambda_{n\gamma}(N \text{ odd})/\lambda_{n\gamma}(N \text{ even}) = 2 \sim 4)$ for fixed Z nuclei. Similarly to the case of (p, γ) reaction, we neglect the time until the seed nuclei capture the γ-rays and reach the nucleus 0, and take the initial conditions that $n(0) = 1$, $n(1) = n(2) = \cdots = n(5) = 0$ at $t = 0$, assuming the system from the nucleus 0 to the nucleus 5. The nucleus 2 generally corresponds the proton-rich heavy nucleus except $^{138}_{57}$La which corresponds to the nuclei 1. In the case that the proton-rich heavy nuclei have two isotopes, these nuclei correspond to the nuclei 0 and 2. Since the dependence of Q_n on the mass number is not so large as that of Q_p, the value $\lambda_{\gamma n}$ does not sharply increase with the mass number for fixed Z nuclei. The total number of seed nuclei will be distributed among comparatively many species of nuclei for each process, and A_i express the distribution coefficients of the produced nuclei for a short period.

$$\begin{aligned} &\cdots A_3 : A_2 : A_1 : A_0 : A_{-1} \cdots \\ &= \cdots \frac{\lambda_{\gamma n}(2)}{\lambda_{n\gamma}(3)} : 1 : \frac{\lambda_{n\gamma}(2)}{\lambda_{\gamma n}(1)} : \frac{\lambda_{n\gamma}(2)\,\lambda_{n\gamma}(1)}{\lambda_{\gamma n}(1)\,\lambda_{\gamma n}(0)} : \frac{\lambda_{n\gamma}(2)\,\lambda_{n\gamma}(1)\,\lambda_{n\gamma}(0)}{\lambda_{\gamma n}(1)\,\lambda_{\gamma n}(0)\,\lambda_{\gamma n}(-1)} \cdots \end{aligned} \tag{17}$$

where A_2 is normalized to 1, and the values A_4, $A_5 \cdots$ are smaller by order two or more than that of A_2.

Considering the above relations, the relative abundances of the objective nuclei are obtained as follows.

$$\begin{aligned} n(2) &\cong A_2 e^{-\delta_0' t}, \\ n(1) &\cong A_1 e^{-\delta_0' t}, \\ n(0) &\cong A_0 e^{-\delta_0' t}, \end{aligned} \tag{18}$$

$$\delta_0' \cong \frac{\{\lambda_{\gamma n}(0) + \lambda_{n\gamma}(0)\}\,\lambda_{\gamma n}(1)\,\lambda_{\gamma n}(2)\,\lambda_{\gamma n}(3)\,\lambda_{\gamma n}(4)\,\lambda_{\gamma n}'(5)}{\lambda_{n\gamma}(3)\,\lambda_{n\gamma}(4)\,\lambda_{n\gamma}'(5)\,\{(\lambda_{\gamma n}(0) + \lambda_{n\gamma}(0))\,(\lambda_{\gamma n}(1) + \lambda_{n\gamma}(2)) + \lambda_{n\gamma}(1)\,\lambda_{n\gamma}(2)\}}.$$

Using (14) and (18), the relative abundances of the proton-rich nuclei to the seed nuclei can be calculated for two processes, (p, γ) and (γ, n), the former is responsible for the large A nuclei at $T_9 = 3$, $\rho_p = 10 \sim 10^2$ gcm^{-3}, and the latter for the small A nuclei at $T_9 = 2.5$, $\rho_n = 10^{-5} \sim 10^{-4}$ gcm^{-3}. In Table I we show the relative abundances of all proton-rich nuclei to their seed nuclei, their ratios being about 1% or less. In either case, we assume that one percent of the seed nuclei contribute to our processes, namely the abundance of the nucleus 0 at $t = 0$, $n(0)$ is one percent of the abundance of all seed nuclei.

Then the abundances of the proton-rich heavy nuclei can be obtained as a function of time.

Table I. The relative abundances of proton-rich nuclei to the seed nuclei.[3] $^{113}_{49}$In is assumed to be produced by (p, γ) and β^+-decay processes.

Proton-rich nuclei	Seed nuclei in (p, γ) reactions		Proton-rich nuclei	Seed nuclei in (p, γ) reactions	
$^{74}_{34}$Se	$^{72}_{32}$Ge $^{71}_{31}$Ga $^{70}_{30}$Zn	0.030	$^{120}_{52}$Te	$^{118}_{50}$Sn $^{116}_{48}$Cd	0.011
$^{78}_{36}$Kr	$^{76}_{34}$Se $^{75}_{33}$As $^{74}_{32}$Ge	0.0061	$^{124}_{54}$Xe	$^{122}_{52}$Te $^{121}_{51}$Sb $^{120}_{50}$Sn	0.0055
$^{84}_{38}$Sr	$^{82}_{36}$Kr $^{81}_{35}$Br $^{80}_{34}$Se	0.0023	$^{126}_{54}$Xe	$^{124}_{52}$Te $^{123}_{51}$Sb $^{122}_{50}$Sn	0.0091
$^{92}_{42}$Mo	$^{90}_{40}$Zr $^{89}_{39}$Y $^{88}_{38}$Sr $^{87}_{37}$Rb $^{86}_{36}$Kr	0.0058	$^{130}_{56}$Ba	$^{128}_{54}$Xe $^{127}_{53}$I $^{126}_{52}$Te $^{124}_{50}$Sn	0.0020
$^{94}_{42}$Mo	$^{93}_{41}$Nb $^{92}_{40}$Zr	0.021	$^{132}_{56}$Ba	$^{130}_{54}$Xe $^{128}_{52}$Te	0.0022
$^{96}_{44}$Ru	$^{93}_{41}$Nb $^{92}_{40}$Zr	0.0082	$^{136}_{58}$Ce	$^{134}_{56}$Ba $^{133}_{55}$Cs $^{132}_{55}$Xe $^{130}_{52}$Te	0.0014
$^{98}_{44}$Ru	$^{96}_{42}$Mo $^{94}_{40}$Zr	0.0034	$^{138}_{58}$Ce	$^{136}_{56}$Ba $^{134}_{54}$Xe	0.0080
$^{102}_{46}$Pd	$^{100}_{44}$Ru $^{98}_{42}$Mo $^{96}_{40}$Zr	0.0024	$^{138}_{57}$La	$^{137}_{56}$Ba	0.0044
$^{106}_{48}$Cd	$^{104}_{46}$Pd $^{103}_{45}$Rh $^{102}_{44}$Ru $^{100}_{42}$Mo	0.011	$^{144}_{62}$Sm	$^{142}_{60}$Nd $^{141}_{59}$Pr $^{140}_{58}$Ce $^{139}_{57}$La $^{138}_{56}$Ba $^{136}_{54}$Xe	0.0014
$^{108}_{48}$Cd	$^{107}_{47}$Ag $^{106}_{46}$Pd $^{108}_{44}$Ru	0.013	$^{152}_{64}$Gd	$^{151}_{63}$Eu $^{150}_{62}$Sm $^{148}_{60}$Nd	0.0062
$^{112}_{50}$Sn	$^{110}_{48}$Cd $^{109}_{47}$Ag $^{108}_{46}$Pd	0.032	$^{156}_{66}$Dy	$^{154}_{64}$Gd $^{153}_{63}$Eu $^{152}_{62}$Sm $^{150}_{60}$Nd	0.00079
$^{113}_{49}$In	$^{111}_{48}$Cd	0.041	$^{158}_{66}$Dy	$^{156}_{64}$Gd $^{154}_{62}$Sm	0.0017
$^{114}_{50}$Sn	$^{112}_{48}$Cd $^{110}_{46}$Pd	0.030	$^{162}_{68}$Er	$^{160}_{66}$Dy $^{159}_{65}$Tb $^{158}_{64}$Gd	0.0011
$^{115}_{50}$Sn	$^{113}_{48}$Cd	0.042	$^{164}_{68}$Er	$^{162}_{66}$Dy $^{160}_{64}$Gd	0.016

Proton-rich nuclei	Seed nuclei in (γ, n) reactions		Proton-rich nuclei	Seed nuclei in (γ, n) reactions	
$^{196}_{80}$Hg	$^{198\ 200\ 202}_{199\ 201\ 201}$Hg	0.0016	$^{158}_{66}$Dy	$^{160\ 162\ 164}_{161\ 163\ \ \ }$Dy	0.00090
$^{190}_{78}$Pt	$^{192\ 195\ 198}_{194\ 196\ \ \ }$Pt	0.00062	$^{156}_{66}$Dy	$^{160\ 162\ 164}_{161\ 163\ \ \ }$Dy	0.00052
$^{184}_{76}$Os	$^{186\ 188\ 190}_{187\ 189\ 192}$Os	0.00018	$^{152}_{64}$Gd	$^{154\ 156\ 158}_{155\ 157\ 160}$Gd	0.0020
$^{180}_{74}$W	$^{182\ 184}_{183\ 186}$W	0.0012	$^{144}_{62}$Sm	$^{147\ 149\ 151}_{148\ 150\ 154}$Sm	0.016
$^{174}_{72}$Hf	$^{176\ 178\ 180}_{177\ 179\ \ \ }$Hf	0.0018	$^{138}_{57}$La	$^{139}_{57}$La	0.0009
$^{168}_{70}$Yb	$^{170\ 172\ 174}_{171\ 173\ 176}$Yb	0.0014	$^{138}_{58}$Ce	$^{140\ 142}_{58}$Ce	0.0025
$^{164}_{68}$Er	$^{166\ 168}_{167\ 170}$Er	0.015	$^{136}_{58}$Ce	$^{140\ 142}_{58}$Ce	0.0020
$^{162}_{68}$Er	$^{166\ 168}_{167\ 170}$Er	0.0010			

§ 3. Results and discussions

As shown in § 2, we divide the build-up process of the proton-rich heavy

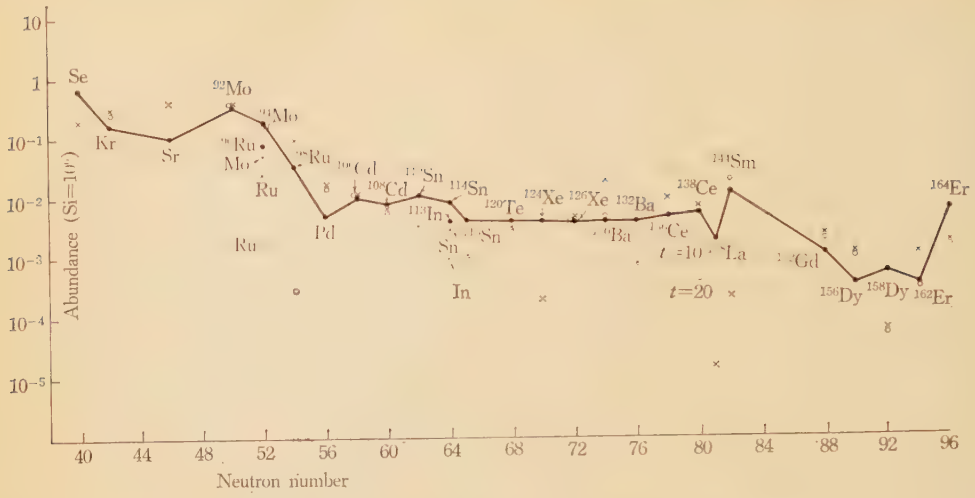
nuclei into two processes and evaluate their abundances. The time giving rise to these processes, t, may be of the same order of magnitude as or shorter than the duration of the supernova explosion ($1\sim10^3$ sec). Therefore we obtain the abundances of the objective nuclei for $1\leq t\leq10^3$ sec. Our calculated results are summarized in Fig. 6 and are shown in Table II. Since these processes are considered to occur in the supernova envelope, one may assume that the density of other elements is very low ($\rho\lesssim10^{-5}$).

Table II. Summary of our results.

Process	Temperature (in 10^9 °K)	Density (gcm^{-3})	Time scale (sec)	Our results (see Fig. 6)	Remarks
(p, γ)	3	$\rho_p=10$	$1\leq t\leq10^2$	Abund. of ^{124}Xe, ^{138}La and ^{144}Sm are too low $\frac{^{94}\text{Mo}}{^{96}\text{Ru}}=\frac{1}{25}$: small. (observed value : 2.7)	
		$\rho_p=5\times10$	$1\leq t\leq10^3$	Abund. of ^{74}Se, ^{84}Sr, and ^{98}Ru decrease sharply with t. At $t\geqq10$ these nuclei are very low-abundant. ^{138}La and ^{144}Sm are low-abundant.	Se, Sr and Ru can also be produced by p-capture process from ^{58}Ni.[6]
		$\rho_p=10^2$	$1\leq t\leq10^2$	$\frac{^{94}\text{Mo}}{^{96}\text{Ru}}=10^{-2}$: very small. ^{74}Se, ^{84}Sr and ^{138}La are hardly produced. Abund. of ^{138}Ce and ^{98}Ru decrease sharply with t.	In addition to synthesis by the p-capture, ^{94}Mo can also be made in weak loops of the s-process chain.[1]
(γ, n)	2.5	$\rho_n=10^{-5}$	$1\leq t\leq10^2$	^{196}Hg and ^{190}Pt are hardly produced. Abund. of ^{184}Os and ^{180}W decrease sharply with t.	In any case, ^{136}Ce, ^{138}Ce, La, Sm and Gd are hardly produced by (γ, n) process. ^{164}Er, ^{162}Er, ^{158}Dy and ^{156}Dy also be produced by (p, γ) process.
		$\rho_n=5\times10^{-5}$	$1\leq t\leq10^2$	$\frac{^{164}\text{Er}}{^{162}\text{Er}}$, $\frac{^{158}\text{Dy}}{^{156}\text{Dy}}$: large. Abund. of ^{196}Hg decreases sharply with t.	
		$\rho_n=10^{-4}$	$1\leq t\leq10^2$	Abund. of Dy is very low. $^{164}\text{Er}/^{162}\text{Er}\cong300$. (observed : 9/5) $^{158}\text{Dy}/^{156}\text{Dy}\cong600$. (observed : 10) Abund. of ^{196}Hg decreases sharply with t.	

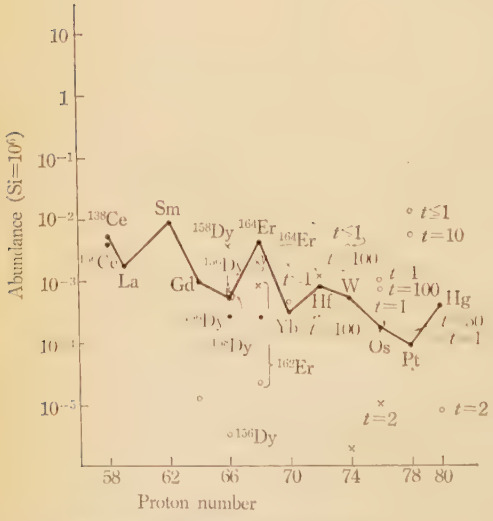
When $T_9=2.5$ or 3, $\rho_p=10\sim10^2$ gcm^{-3} and $\rho_n=10^{-5}\sim10^{-4}$ gcm^{-3}, there are the proton-rich nuclei having the possibilities produced by both reactions. These nuclei correspond to the ones such that the two reaction rates produced them are almost equal to each other ($\lambda_{p\gamma}\simeq\lambda_{\gamma n}$).

Considering the ambiguity of the values of the reaction rates, we take $^{136}_{58}$Ce,



● Abundances of proton-rich nuclei
× $\rho_p=10$ g cm^{-3}, $1\leq t$ (sec)$\leq 10^2$ (calculated abund.)
○ $\rho_p=10^2$ g cm^{-3}, $1\leq t$ (sec)$\leq 10^2$ (calculated abund.)
Se, Sr, La are hardly produced

Fig. 6 (a). The abundances of proton-rich heavy nuclei[3] and our results. The calculated abundances of proton-rich nuclei produced by (p, γ) reactions.



● Abundances of proton-rich nuclei
× $\rho_n=10^{-5}$gcm, $1\leq t$(sec)$\leq 10^2$ (calculated abund.)
○ $\rho_n=5\times10^{-5}$gcm^{-3}, $1\leq t$ (sec) $\leq 10^2$ (calculated abund.)

Fig. 6 (b). The abundances of proton-rich heavy nuclei[3] and our results. The calculated abundances of proton-rich nuclei produced by (γ, n) reactions.

$^{138}_{58}$Ce, $^{138}_{57}$La, $^{144}_{62}$Sm, $^{152}_{64}$Gd, $^{156}_{66}$Dy, $^{158}_{66}$Dy, $^{162}_{68}$Er and $^{164}_{68}$Er as the products through both processes, and evaluate their abundances.

In the case of the (p, γ) reaction, the abundances of the proton-rich nuclei are mostly decided by the magnitude of λ_i in (1·1), and their abundances are almost constant for time t. However, the abundances of $^{74}_{34}$Se, $^{84}_{38}$Sr and $^{98}_{44}$Ru sharply decrease with t at large proton density ($\rho_p \gtrsim 5\times10$ gcm^{-3}), and become low-abundant. In these nuclei, there is a probability that they may be formed through the successive proton-capture processes[6] from $^{58}_{28}$Ni for $T_9=0.5\sim2$ and $\rho_p=10^{-5}\sim10^2$ gcm^{-3} in the supernova envelope. As $^{113}_{49}$In is the Z odd-N even nucleus, the last proton binding energy of $^{114}_{50}$Sn is larger than

that of $^{113}_{49}$In ($\lambda_{p\gamma}(^{113}_{49}\mathrm{In}) \gg \lambda_{\gamma p}(^{114}_{50}\mathrm{Sn})$) ; hence $^{113}_{49}$In cannot be produced by the (p, γ) reactions. For this nucleus, it seems plausible to consider that the unstable nucleus $^{113}_{50}$Sn is produced by the (p, γ) reaction on $^{111}_{48}$Cd which has already been synthesized by neutron-capture and after the explosion of the supernova $^{113}_{50}$Sn is subjected to β^+-decay to produce $^{113}_{49}$In. At $T_9=3$ and $\rho_p=10\,\mathrm{gcm}^{-3}$ $^{138}_{57}$La is made from $^{137}_{56}$Ba, but its abundance is lower than the observed one by about order two.

In the case of the (γ, n) reaction, the abundances of the very heavy proton-rich nuclei sharply decrease with time t. At $\rho_n > 10^{-5}\,\mathrm{gcm}^{-3}$ the abundance ratios among two isotopes ($^{164}_{68}\mathrm{Er}/^{162}_{68}\mathrm{Er}$ and $^{158}_{66}\mathrm{Dy}/^{156}_{66}\mathrm{Dy}$) built by this process are in disagreement with the observed data. For $^{136}_{58}$Ce, $^{138}_{58}$Ce, $^{138}_{57}$La and $^{152}_{64}$Gd, it seems probable to consider that their nuclei cannot be produced by the (γ, n) reactions but can be produced only by the (p, γ) reactions. However, we have a number of uncertainties for the value of the reaction rates. For a last nucleon binding energy, there is some difference between the experimental values by Everling et al.[7] and the values using the semi-empirical mass formula by Cameron.[5] If the error between these values is 1 Mev, the difference of the two (γ, n) reaction rates using their values is of order two or three at $T_9=2.5\sim3$. Especially, the calculated abundances of the nuclei for which the time dependence is very large are not so reliable, these nuclei being $^{74}_{34}$Se, $^{84}_{38}$Sr, $^{98}_{44}$Ru, $^{138}_{58}$Ce, $^{138}_{57}$La, $^{184}_{76}$Os, $^{190}_{78}$Pt and $^{196}_{80}$Hg.

Finally we shall consider the foregoing astrophysical conditions in which the synthesis of the proton-rich heavy elements takes place. In general the temperature continuously changes in any region. However, the effective temperature by which the (p, γ) or (γ, n) reaction takes place is sharply determined, so that $T_9\simeq3$ for the (p, γ) reaction and $T_9\simeq2.5$ for the (γ, n) reaction. When we conveniently take the two independent regions of different temperatures in which the formation of the objective nuclei by the (p, γ) reaction is effective in the region of $T_9=3$ and the formation by the (γ, n) reaction is effective in the region of $T_9=2.5$, it would not give us so different results from true ones in our case. Therefore we assume that about one percent of abundances of the heavy-nuclei which have already been synthesized by neutron-capture are mixed into these two regions and give rise to our process. It will not be allowed that the temperature changes with time and the two effective reactions successively occur at different temperatures. Here, basing on the shock-theory of supernova explosion,[8] the mass fraction to the total mass of supernova in the stellar layer being heated to $T_9=3$ is $0.01\sim0.1$; meanwhile the ratio of the total mass in Galaxy to the ejected mass from the supernova is about 0.1.[1] Therefore it is considered, for example, that the heavy-nuclei existing in this layer would give rise to the effective (p, γ) reaction at $T_9=3$.

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The author would like to thank Prof. C. Hayashi who first pointed out the importance of this problem.

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Letters to the Editor

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Associated Production in Pion-Nucleon Collisions and Effects of Resonant States

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August 2, 1961

In this note we investigate the possibility of explaining the experimental features,[1] angular distribution, polarization and total cross section, of the reaction $\pi+P\to K^\circ+\Lambda$ at pion energy 910 Mev by taking account of N^* (π-*N third resonance*), Y^* (π-Λ *resonance*) and K^* (K-π *resonance*).[2]

S-matrix element for $\pi+N\to K+\Lambda$ is given by

$$S_{fi}=-i(2\pi)^4\delta^4(p_1+q_1-p_2-q_2)\times\left[\frac{m\Lambda}{4E_1E_2\omega_1\omega_2}\right]^{1/2}\bar{u}(p_2)\,T\tau_\alpha u(p_1), \tag{1}$$

$$T\equiv -A+\frac{i}{2}\gamma\cdot(q_1+q_2)B,$$

where $q_1(q_2)$ and $\mu(\kappa)$ are 4-momentum and mass of $\pi(K)$ and $p_1(p_2)$ and $m(\Lambda)$ are those of $N(\Lambda)$. ω_1, ω_2, E_1 and E_2 are the total energies of π, K, N and Λ, respectively. Here, we assume the same parity for N and Λ and K is pseudoscalar.

A and B are invariant functions of three variables $s=-(p_1+q_1)^2$, $s_c=-(p_1-q_2)^2$ and $t=-(p_1-p_2)^2$. In the c.m. system, $d\sigma/d\Omega$ is given by

$$\frac{d\sigma}{d\Omega}=\frac{1}{(2\pi)^2}\frac{m\Lambda}{4(\omega_1+E_1)}\frac{\bar{q}_2}{2\bar{q}_1}\times\sum|\bar{u}(p_2)\,Tu(p_1)|^2=\frac{\bar{q}_2}{2\bar{q}_1}\times\sum|\langle\Lambda|f_1+\frac{(\boldsymbol{\sigma}\boldsymbol{q}_2)(\boldsymbol{\sigma}\boldsymbol{q}_1)}{\bar{q}_1\bar{q}_2}f_2|p\rangle|^2, \tag{2}$$

where $\bar{q}_1$ and $\bar{q}_2$ are magnitudes of three-momenta of π and K.

We assume that A and B satisfy double dispersion relation.[3,4] Further, we assume that A and B can be approximated by

$$A=\frac{f_{\pi NN}g_{KN\Lambda}}{2}\frac{\Lambda-m}{m^2-s}\mp\frac{f_{\pi\Lambda\Sigma}g_{KN\Sigma}}{2}\times\frac{2\Sigma\mp(m+\Lambda)}{\Sigma'^2-s_c}+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}ds'\frac{A_1(s',t)}{s'-s}+\frac{1}{\pi}\int_{(\Lambda+\mu)^2}^{\infty}ds_c'\,\frac{A_2(s_c',t)}{s_c'-s_c}$$

$$+\frac{1}{\pi}\int_{(\kappa+\mu)^2}^{\infty} dt' \frac{A_3(s,t')}{t'-t},$$

$$B=\frac{f_{\pi NN}g_{KN\Lambda}}{m^2-s}-\frac{f_{\pi\Lambda\Sigma}g_{KN\Sigma}}{\Sigma^2-s_c}$$
$$+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds' \frac{B_1(s',t)}{s'-s}$$
$$+\frac{1}{\pi}\int_{(\Lambda+\mu)^2}^{\infty} ds_c' \frac{B_2(s_c',t)}{s_c'-s_c}$$
$$+\frac{1}{\pi}\int_{(\kappa+\mu)^2}^{\infty} dt' \frac{B_3(s,t')}{t'-t}, \qquad (3)$$

which have good correspondence to perturbational approach. Upper (lower) sign refers to even (odd) $P(\Sigma\Lambda)$, $\Sigma-\Lambda$ relative parity. Assumed resonant states are $N^*(l=3, J=5/2)$ in s-channel, $Y^*(l=0, J=1/2)$ in s_c-channel and $K^*(J=0$ *or* $1)$ in t-channel.

Imaginary parts of partial wave amplitudes corresponding to N^*, Y^* and K^* are written in Breit-Wigner formulas as

$$N^*: \quad \mathrm{Im}f_{l/2}^{I}=\frac{1}{4q}\frac{\Gamma_1\sqrt{\Gamma_1\Gamma_\lambda}}{(W-M^*)^2+\Gamma_1^2/4},$$

$$Y^*: \quad \mathrm{Im}f_{1/2}^{S}=\frac{1}{4q}\frac{\Gamma_2\sqrt{\Gamma_2\Gamma_\sigma}}{(W-M_Y)^2+\Gamma_2^2/4}$$
$$\approx\frac{\pi}{2q}\sqrt{\Gamma_2\Gamma_\sigma}\,\delta(W-M_Y),$$

$$K^*: \quad \mathrm{Im}f_{\pm}^{J}=\frac{N_{\pm}^{J}}{2}\frac{\Gamma}{(W-M_K)^2+\Gamma^2/4}$$
$$\approx\pi N_{\pm}^{J}\delta(W-M_K). \qquad (4)$$

For t-channel, we have used the method developed by Jacob and Wick.[5] $\Gamma_1(\Gamma_2)$ and $\Gamma_\lambda(\Gamma_\sigma)$ are half- and partial-width of $N^*(Y^*)$. M^*, M_Y and M_K are masses of N^*, Y^* and K^*, respectively. $N_{\pm}^{J}$ is the parameter of height of resonance in t-channel. Substituting these expressions in Eq. (3), we obtain A and B, and consequently $d\sigma/d\Omega$ and $P(\theta)$. The polarization is positive if the spin of Λ is parallel to $\boldsymbol{q}_2\times\boldsymbol{q}_1$ and negative if antiparallel.

Assuming reasonable values of usual coupling constants and widths of resonant states as $f_{\pi NN}^2/4\pi=f_{\pi\Lambda\Sigma}^2/4\pi=15$, $g_{KN\Lambda}^2/4\pi=5$, $g_{KN\Sigma}^2=/4\pi=1.5$, $\Gamma_1=100\Gamma_\lambda$ $=100$ Mev and $\Gamma_2\approx\Gamma_\sigma=20$ Mev, we can get angular distribution consistent with experiments but not the large $\overline{P(\theta)}$, polarization averaged over all angles. $d\sigma/d\Omega$ and $P(\theta)$ for this case are shown in Figs. 1 and 2. σ_{tot} and $\bar{P}$ are given in Table I.

Even if we take into account the nonresonant effects with low angular momentum, the situation is not so altered.[6] Nonresonant amplitudes are estimated by

$$\mathrm{Im}f_J^I=\frac{2}{2J+1}\frac{q}{4\pi}\sigma_J^I(tot)\sqrt{R},$$

where R is the branching ratio of $\pi^-+p\to K^\circ+\Lambda$ reaction and σ_J^I is partial wave π^-p total cross section. $d\sigma/d\Omega$, σ_{tot}, $P(\theta)$ and $\bar{P}$ with the effects of s- and p-wave nonresonant amplitudes are also shown in Table I and Figs. 1 and 2. With odd $P(\Sigma\Lambda)$, it is quite difficult to obtain a good fit with experiment. On account of the high angular momentum of N^*, very small values of $\bar{P}$ are obtained, which contradicts the experiments.*)

Finally, we can conclude that $d\sigma/d\Omega$

*) $(\alpha_\Lambda\overline{P})_{exp}\geqq 0.73\pm 0.14$.

Table I. Total cross section σ_{tot} and the polarization averaged over angles $\overline{P(\theta)}$ for the cases of N^* only and N^*+*nonresonant s- and p* 1/2-amplitude in *s*-channel.

	s-channel	J	N_+^1	N_-^1	σ_{tot}	$\overline{P}$
$\begin{pmatrix} f_{\pi NN}\, g_{KN\Lambda} > 0 \\ f_{\pi\Lambda\Sigma}\, g_{KN\Sigma} < 0 \end{pmatrix}$	N^*	1	0.08	0.0	0.59	0.04
			0.06	0.0	0.69	0.05
$P(\Sigma\Lambda) = +1$	N^*+nonres. (S+P $1/_2$)	1	0.13	0.0	0.63	−0.16

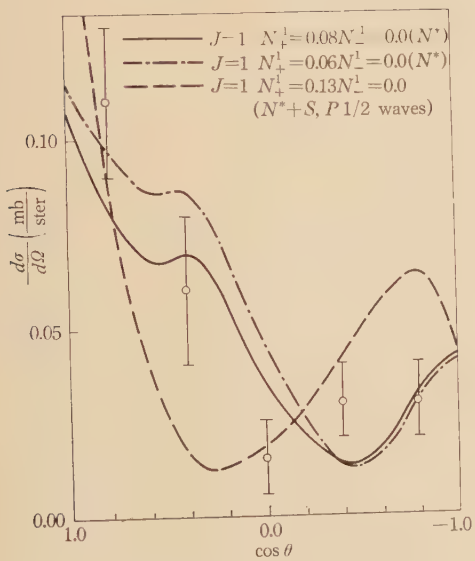

Fig. 1. The differential cross sections at the pion kinetic energy T_π=910 Mev. To avoid confusion, the sets in J=0 are not explicitly written. The sets (J=0; N_+^0=−1.0) and (J=0; N_+^0=−0.8) are quite similar to the sets (J=1; N_+^1=0.08, N_-^1=0.0) and (J=1; N_+^1=0.06, N_-^1=0.0), respectively.

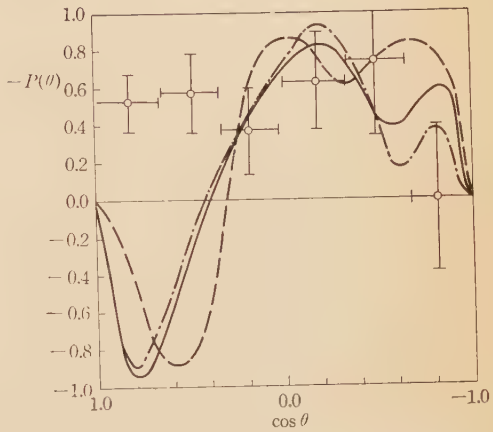

Fig. 2. The polarization of Λ in the pion kinetic energy range 910~990 Mev. The group of curves corresponds to T_π=910 Mev.

and σ_{tot} thus obtained are consistent with experiments but $\bar{P}$ is not by our treatment. This seems to suggest that the intermediate states might not be such simple ones that can be described in the conventional way.

In order to check up this point, we are now applying the same method to pion-nucleon scattering and photopionproduction near higher resonances and the results will appear in a later issue of this journal.

We wish to express our sincere thanks to Prof. D. Itô for his suggestion, to Dr. A. Kanazawa for his valuable comments and to the members of our Institute for their continuous interest.

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On the Interaction of the Pion and Cascade Particle*

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We first consider the old question of whether the coupling of the pion to the nucleon is direct or derivative by examining the π-Ξ interaction with the assumption that either direct coupling or derivative coupling is universal. The mass difference between the nucleon and the Ξ hyperon will be essential in this discussion and is included. We assume that in addition to the coupling type the (unrenormalized) coupling constant is universal and that the cutoff energy in the static theory will be more or less universal. Since the π-baryon interaction is the strongest interaction available, one might also hope that the renormalized π-baryon coupling constants are almost equal.

According to the static theory,[1] then, the resonance energy is approximately given by $\omega_{res}=(1/f^2)\omega_{max}$. Thus the location of the $(3/2, 3/2)$ resonance depends on whether $f^2=g^2(\mu/2m_B)^2$ is universal (derivative case), or if g^2 is universal (direct case).

If it should turn out that the Y_1^* is wholly or in part the resonance predicted by global symmetry,[2] then the calculation of Amati et al.[3] is at least consistent with our assumption (approximate universality of the renormalized coupling constant and cutoff), and implies that derivative coupling is universal by our criterion. However, we would like to argue that using the resonance predicted for the π-Ξ interaction would be a more effective way to examine this question of universal coupling. For the π-Ξ resonance, one expects a resonance energy of 2.1 times the pion mass for derivative coupling and about twice this value for direct coupling, using the same cutoff as in π-N theory. This difference in resonance energies should be large enough to be unambiguous even with the uncertainties of the model, for one would need a 100% change in the cutoff energy to cancel out the effect on the resonance energy of the difference in f^2 between the direct and derivative cases for the π-Ξ interaction; a 30% change in the cutoff will suffice for the π-Λ interaction.

The resonance reaction of the pion and cascade particle can be observed from high-energy kaon absorption by a proton with the production of a pion,

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cascade particle and a kaon:

$$K^-+p\rightarrow\Xi^{-*}+K^+\rightarrow\Xi^-+\pi^0+K^+ \rightarrow\Xi^0+\pi^-+K^+, \tag{1}$$

where Ξ^{-*} is the isobar state in which we are interested. The momentum spectrum of the positive kaon can be used to determine the excitation function of the π-Ξ interaction. If the Ξ^{-*} indeed exists, its isotopic spin can be determined from the branching ratio of the different decay modes shown in (1). Techniques for determining the spin state of the resonance will no doubt be established from the study of the Y_1^*.

The above remarks are, of course, based on the assumption of spin 1/2 for the Ξ hyperon. However, since this spin assignment is still not completely certain, it might be of interest to consider some predictions for a pion scattering from a spin 3/2 Ξ hyperon. We base these predictions on a model analogous to the static theory for pion-nucleon scattering. When we assume a p-wave interaction between π and Ξ, the Hamiltonian is written in the static theory as

$$H=H_0+H_1, \tag{2}$$

$$H_0=\sum_p a_p^+ a_p \omega_p, \tag{3}$$

$$H_1=\sum_p [V_p a_p+V_p^+ a_p^+], \tag{4}$$

where

$$V_p=(if_r^0/\mu)(\boldsymbol{S}\cdot/\boldsymbol{p}\sqrt{2\omega_p})(\tau_p v(p)), \tag{5}$$

$\boldsymbol{S}$ being the spin operator for spin 3/2 cascade particle, i.e.

$$S_i^+=S_i,\ [S_i, S_j]=i\epsilon_{ijk}S_k \quad \text{and} \quad \sum_i S_i^2=15/4. \tag{6}$$

The remainder of the notation is identical with that of Chew and Low.[1)]

The total isotopic spin T can take the values 1/2 and 3/2; the projection operators for these values of total isotopic spin are well known and are given in reference 1). The total angular momentum J can be either 1/2, 3/2, or 5/2; and the required angular momentum projection operators $B^J(p,q)$ are given by

$$B^{1/2}(p,q)=9/8(\boldsymbol{p}\cdot\boldsymbol{q}) +1/4(\boldsymbol{S}\cdot\boldsymbol{p})(\boldsymbol{S}\cdot\boldsymbol{q})-3/4(\boldsymbol{S}\cdot\boldsymbol{q})(\boldsymbol{S}\cdot\boldsymbol{p}),$$

$$B^{3/2}(p,q)=4/5(\boldsymbol{S}\cdot\boldsymbol{p})(\boldsymbol{S}\cdot\boldsymbol{q}),$$

$$B^{5/2}(p,q)=15/8(\boldsymbol{p}\cdot\boldsymbol{q})-21/20(\boldsymbol{S}\cdot\boldsymbol{p}) \times(\boldsymbol{S}\cdot\boldsymbol{q})+3/4(\boldsymbol{S}\cdot\boldsymbol{q})(\boldsymbol{S}\cdot\boldsymbol{p}). \tag{7}$$

Then, if we denote the six states $(T, J)=(1/2, 1/2)$, $(1/2, 3/2)$, $(1/2, 5/2)$, $(3/2, 1/2)$, $(3/2, 3/2)$ and $(3/2, 5/2)$ by the numbers $\alpha=1, 2, \cdots, 6$ in that order, we may write the residues of the pole terms for the $h_\alpha(z)$ as

$$\lambda=(1/6)f^2\begin{bmatrix} 5 & & & & & \\ & -28 & & & & \\ & & -3 & & & \\ & & & -10 & & \\ & & & & 11 & \\ & & & & & 6 \end{bmatrix}, \tag{8}$$

where f is some unrationalized, renormalized coupling constant. The crossing matrix is the direct product of the crossing matrix in isospace, C_T, and the crossing matrix in angular momentum space, C_J, where

$$C_T=1/3\begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$$

$$\text{and } C_J=\begin{bmatrix} 1/6 & -2/3 & 3/2 \\ -1/3 & 11/15 & 3/5 \\ 1/2 & 2/5 & 1/10 \end{bmatrix}. \tag{9}$$

In the first approximation, the effective ranges are given by (except for a positive constant which depends on the cutoff)

$$(I-B)\,\lambda \approx f^2 \begin{matrix} 6.0 \\ -4.1 \\ -2.0 \\ 4.0 \\ 8.0 \\ 8.0 \end{matrix}, \tag{10}$$

where $B_{\alpha\beta}=-(\lambda_\alpha/\lambda_\beta)C_{\alpha\beta}$.

Thus, if the first approximation can be trusted, one would expect resonances for the states $(T, J)=(3/2, 3/2)$, $(3/2, 5/2)$, $(1/2, 1/2)$ and $(3/2, 1/2)$ with resonance energies roughly in the ratio 3 : 4 : 6.

I would like to thank Dr. R. Arnowitt and Dr. A.O. Barut for a helpful discussion concerning this letter. I am especially indebted to Mr. J. Mazo for performing the calculations.

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Imaginary Part of the Optical Potential for Nuclei $A\sim60$ and $A\sim100$

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It was pointed out that the imaginary part of optical potential $W(r)$ will show shell effect[1] and that, for zero energy neutron, it will be large for nuclei with mass number $A\sim60$ and small for $A\sim100$.[2] We estimate this more quantitatively for ^{56}Fe and ^{112}Cd.

The incident $s_{1/2}$ neutron excites a nucleon in target in state j to j' and jumps to j''. For the delta function interaction the matrix elements are, for direct processes,

$$(-)^{j-j''}\sqrt{(2j'+1)(2j''+1)/2(2l''+1)}$$
$$\times(jj'1/2-1/2|J0)\,\delta_{Jl''}VI=D\cdot V\cdot I$$

and for exchange processes,

$$-(\text{above expression with exchange of prime and double prime, and without } \delta_{Jl''})$$
$$\times\sqrt{(2J+1)(2l'+1)}$$
$$\times W(j''Jl'j';1/2\,j)=E\cdot V\cdot I,$$

where J is vector sum of j and j'. V is the strength of nuclear force and I is the orbital overlap integral. For target neutrons, D and E are added and for protons, only D is needed. Each is squared and summed over J. These we call G^2.

The expectation value of $W(r)$, W, is

$$\sum_{jj'j''} G^2 V^2 I^2 \times \frac{W'}{(E_{1/2}+E_j-E_{j'}-E_{j''})^2+W'^2}^{n_j},$$

where E_j, etc., are the single particle energy (with pairing energy) of state j, etc., and n_j is the number of particles in state j. As there are unfilled shells, a further factor of c.f.p. is necessary if $j'=j$ or $j''=j$. Further, if $j'=j''$ for the same particles, coupling scheme is changed from the above one, that is, j' and j'' are first coupled to J' which should be even. These will be published later.

W' is W for the intermediate state. This may be smaller than W[3] but if the energy dependence of W is linear (corresponding to the energy inverse free two-body cross section)[4] W' is equal to W as W' is the sum of the imaginary part for each particle and a hole. We assume $W' \sim W \sim 3$ Mev.

We tentatively used Klinkenberg's level scheme[5] and assumed a pure configuration for a ground state. The sum of n_j (with c.f.p.) for which the energy difference is less than 4 Mev is for Fe 313.5 and for Cd 176.8. The smallness of the latter is due to the restriction through the parity conservation law and the fact that for target protons many processes in which j' and j'' are interchanged are forbidden as one of them is already occupied by neutrons. We calculated I with harmonic oscillator wave functions. G^2I^2 fluctuates very much but their weighted mean is (in 10^{26}cm^{-2})[3] for Fe $\sim 0.12\times10^{-4}$ and for Cd $\sim 0.065\times10^{-4}$. For Fe I^2 with $3s$ incident neutron was taken and for Cd the average of I^2 with $3s$ and $4s$ was taken. The smallness of the latter is due to the size effect. Thus we see that W for Cd is about 1/3 of that for Fe.

In the pairing correlation model, the wave functions are very different from those of the simple shell model but this can be treated in a similar way as in the calculation of inertial parameters.[6] The transitions of quasi-particles from states below the chemical potential to those above it are decreased compared with those for the pure shell model, but there appear new processes between states below (or above) the chemical potential and the net result will be an increase. However, the ratio mentioned above will not change very much.

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Effect of Pion-Pion Scattering Resonance on Nucleon Structure

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The effect of the π-π P-wave resonance on the electromagnetic structure of the nucleon has been investigated by Frazer and Fulco.[1] In their calculation, if the Legendre expansion is performed for the π-N scattering amplitudes over the left-hand cut, this cut produces divergent integrals which necessitate a cutoff procedure. The purpose of this note is to remedy this difficulty by a subtraction method.

In this note, the same notations as in F-F are used. With the use of the subtracted form, their functions $\Gamma_i(t)$ are expressed as follows:

$$\Gamma_i(t)=\Gamma_i(0)+\frac{t}{\pi}\int_{-\infty}^{a}dt'\frac{\mathrm{Im}\,\Gamma_i(t')}{t'(t'-t)}+\frac{t}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{\mathrm{Im}\,\Gamma_i(t')}{t'(t'-t)}. \quad (1)$$

In the approximation of neglecting all but two-pion intermediate state, $\Gamma_i(t)$ can be written as

$$\Gamma_i(t)=F_\pi(t)\left[\Gamma_i(0)+\frac{t}{\pi}\times\int_{-\infty}^{a}dt'\frac{\mathrm{Im}\,\Gamma_i(t')}{F_\pi(t')\,t'(t'-t)}\right], \quad (2)$$

where $\Gamma_i(0)$ are related to the forward π-N scattering amplitudes, and can be calculated directly from the experimental information of π-N scattering using the dispersion relations at forward. Unfortunately, $\Gamma_2(0)$ involves large errors coming from the cancellation of large terms, and Eq. (2) is not useful for $\Gamma_2(t)$, so we use the unsubtracted form as follows:

$$\Gamma_2(t)=\frac{F_\pi(t)}{\pi}\int_{-\infty}^{a}dt'\frac{\mathrm{Im}\,\Gamma_2(t')}{F_\pi(t')\,(t'-t)}. \quad (3)$$

On the other hand, $\Gamma_1(0)$ is well defined as shown by Ball and Wong,[2] and we use Eq. (2) for $\Gamma_1(t)$.

For the π-π P-wave amplitude, the one level formula assumed by Bowcock et al.[3] is used for simplicity, and the pion form factor is written as

$$F_\pi(t)=(t_r-\gamma)/(t_r-t-i\gamma q^3). \quad (4)$$

The unsubtracted spectral representation for $G_2^V(t)$ seems to be inconsistent with the experimental data given by Hofstadter et al.,[4] so we use a subtracted form,

$$G_2^V(t)=\frac{g_V e}{2m}+\frac{t}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{g_2^V(t')}{t'(t'-t)},$$

$$G_1^V(t)=\frac{e}{2}+\frac{t}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{g_1^V(t')}{t'(t'-t)}, \quad (5)$$

where $g_V=1.85$. Then with the use of the assumption of narrow resonance approximation for $F_\pi(t)$, $F_2^V(t)=(2m/g_V e)G_2^V(t)$ and $F_1^V(t)=(2/e)G_1^V(t)$ are written as

$$F_2^V(t)=(1-\alpha)+\alpha/(1-t/t_r),$$

$$F_1^V(t)=(1-\beta)+\beta/(1-t/t_r). \quad (6)$$

α and β are defined as follows:

$$\alpha \approx \frac{2m}{g_V}\frac{\sqrt{t_r}}{\gamma}\frac{1}{\pi}\int_{-\infty}^{a} dt' \frac{\mathrm{Im}\,\Gamma_2(t')}{t_r - t'}, \tag{7a}$$

$$\beta \approx \frac{2\sqrt{t_r}}{\gamma}\left[-\Gamma_1(0) + \frac{t_r}{\pi}\int_{-\infty}^{a} dt' \frac{\mathrm{Im}\,\Gamma_1(t')}{t'(t_r - t')}\right], \tag{7b}$$

approximating $F_\pi(t')$ to 1 in these integrands. For $t \leq a$, $\mathrm{Im}\,\Gamma_2(t)$ and $\mathrm{Im}\,\Gamma_1(t)$ are expressed by π-N scattering amplitudes which are evaluated by setting all partial cross sections equal to zero except for the (3, 3) state, and approximating the latter by a narrow resonance. Under these approximations, the integrals in Eqs. (7a) and (7b) are convergent and we shall cut off these integrals at $t = -26\mu^2$.

Comparing Eq. (6) with the experimental formula[4] of $F_2^V(t)$ and $F_1^V(t)$, the value of t_r is required to be $20 \sim 25\mu^2$. At $t_r = 22\mu^2$, using Eq. (7a), the value of γ calculated from the experimental value of α is $0.7 \sim 0.8\mu^{-1}$; on the other hand, using Eq. (7b), γ is $1.2 \sim 1.3\mu^{-1}$ from the value of β. The value of γ is sensitive to the integral value over the left-hand cut, so the disagreement of γ in $F_2^V(t)$ and $F_1^V(t)$ will be originated from using unsubtracted form for $\Gamma_2(t)$, and we shall not be able to assign the value γ definitely.

By using the fact that if the narrow resonance for the P-wave pion-pion scattering is assumed, the form factor should be of Clementel-Villi type,[5] we have obtained that the higher resonance energy is probable for the nucleon structure. The integral over the left-hand cut and the γ are included in α and β as the form of ratio, so we can always fit with the data by choosing the value of γ according to the integral value. Thus we should investigate the effect of the resonance on pion-nucleon scattering to analyse whether the integral value is consistent or not with the experimental information about pion-nucleon scattering.

We shall investigate the effect of π-π resonance on P-wave amplitudes of π-N scattering, $[(f_{11} - f_{31})/3k^2]_{k\to 0}$, and compare with the results derived by B. Their parameters C_1 and C_2 are given in our formalism as

$$C_1 = t_r\left[\Gamma_1(0) + \frac{t_r}{\pi}\int_{-\infty}^{a} dt' \frac{\mathrm{Im}\,\Gamma_1(t')}{t'(t' - t_r)}\right], \tag{8a}$$

$$C_2 = -\frac{t_r}{\pi}\int_{-\infty}^{a} dt' \frac{\mathrm{Im}\,\Gamma_2(t')}{t' - t_r}, \tag{8b}$$

and are estimated for $t_r = 22\mu^2$ to be -3.43 and -0.57 respectively which are compared with the values in B -1.0 and -0.27 respectively (hereafter we use $\mu = 1$ unit). If we write the π-π term in π-N scattering dispersion relations as

$$A_{\pi\pi}^{(-)} \equiv \frac{(s - \bar{s})}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{\rho(t')}{t' - t},$$

$$B_{\pi\pi}^{(-)} \equiv \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{\sigma(t')}{t' - t}, \tag{9}$$

the contribution of π-π term to $[(f_{11} - f_{31})/3k^2]_{k\to 0}$ is expressed by C_1 and C_2, and, in our estimation, turns out to be 0.10. According to the analy-

sis given by Ishida,[6] this quantity has the value 0.028 ± 0.013. This large effect 0.10 comes from that the values of $|C_1|$ and $|C_2|$ given by this note are larger than the values in B. However, if we use the subtracted form,

$$A_{\pi\pi}^{(-)} \equiv (s-\bar{s}) \left[C_A^{(-)} + \frac{t}{\pi} \int_{4\mu^2}^{\infty} dt' \frac{\rho(t')}{t'(t'-t)} \right],$$

$$B_{\pi\pi}^{(-)} \equiv C_B^{(-)} + \frac{t}{\pi} \int_{4\mu^2}^{\infty} dt' \frac{\sigma(t')}{t'(t'-t)}, \quad (10)$$

this contribution to $[(f_{11}-f_{31})/3k^2]_{k\to 0}$ reduces to 0.035 and does not contradict the data on π-N scattering. In Eq. (10) the new parameters $C_A^{(-)}$ and $C_B^{(-)}$ are calculated from the dispersion relations for π-N scattering at forward. Because we have not any information about the contribution coming from the high mass states at present, if the dispersion relations are applied to an analysis, the subtraction will be needed to get reliable results, thus we may conclude that the values of C_1 and C_2 given in this note are not inconsistent with the experimental data. Detailed formalism and discussions will be reported later.

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Proof of Partial-Wave Dispersion Relations for Pion-Pion and Nucleon-Nucleon Scatterings in Perturbation Theory

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Usual derivations[1] of partial-wave dispersion relations are always based on the Mandelstam representation.[2] However, up to now, for the Mandelstam representation no complete proof has been given even in perturbation theory[3] in spite of many efforts. In this note we shall completely prove the partial-wave dispersion relations for the equal-mass-particle, pion-pion and nucleon-nucleon scatterings in every order of perturbation theory. The results are identical with the ones derived from the Mandelstam representation.

As an example, we shall examine the nucleon-nucleon scattering. The scattering amplitude can be written as[4]

$$A(s,t) = \int_{-\infty}^{+\infty} d\alpha \int_0^1 dx \frac{f(\alpha, x)}{\alpha - sx - t(1-x)}$$

$$+\int_{-\infty}^{+\infty} d\beta \int_0^1 dy \frac{g(\beta, y)}{\beta - ty - u(1-y)}$$

$$+\int_{-\infty}^{+\infty} d\gamma \int_0^1 dz \frac{h(\gamma, z)}{\gamma - uz - s(1-z)}, \quad (1)$$

except for one-particle intermediate state. Here s and t are the invariant squares of energy and momentum transfer, and

$$u = 4m_N^2 - s - t. \quad (2)$$

The restrictions for the supports of the weight functions in Eq. (1) are[4]

$$\left.\begin{aligned} &\alpha \geq 4m_N^2 x + 4m_\pi^2(1-x), \\ &\beta \geq \mathrm{Min}[4m_\pi^2 y + 4m_N^2(1-y), \\ &\quad 4m_N^2 y + 4m_\pi^2(1-y)] \end{aligned}\right\} \quad (3)$$

and

$$\gamma \geq 4m_\pi^2 z + 4m_N^2(1-z).$$

The integral representation (1) and the supports (3) are valid in every order of perturbation theory.

The partial-wave amplitudes are defined by

$$A_l(s) = \int_{-1}^{1} d(\cos\theta)\, P_l(\cos\theta) \times A[s,\ (4m_N^2 - s)(1-\cos\theta)/2]. \quad (4)$$

From Eqs. (1) and (4), we see that $A_l(s)$ can have a singularity only when $A(s, t)$ has a singularity at the end point of the $\cos\theta$ integration, that is, $\cos^2\theta = 1$. Therefore, $A_l(s)$ is singular only when

$$\left.\begin{aligned} &\alpha = sx, \\ &\alpha = s(2x-1) + 4m_N^2(1-x), \\ &\beta = -s(1-y) + 4m_N^2(1-y), \\ &\beta = -sy + 4m_N^2 y, \\ &\gamma = -s(2z-1) + 4m_N^2 z \end{aligned}\right\} \quad (5)$$

and

$$\gamma = s(1-z).$$

Eqs. (3) and (5) are sufficient to assure that $A_l(s)$ is regular except for the cuts in the intervals $[-\infty, 4(m_N^2 - m_\pi^2)]$ and $[4m_N^2, +\infty]$. Using this analiticity, we can obtain the dispersion relation for $A_l(s)$.

For the equal-mass-particle, and pion-pion scatterings, the restrictions for the supports of the weight functions in Eq. (1) are

$$\alpha \geq 4\mu^2 \quad (6)$$

and

$$\alpha \geq \mathrm{Min}[4m_\pi^2 x + 16m_\pi^2(1-x), 16m_\pi^2 x + 4m_\pi^2(1-x)] \quad (7)$$

and likewise for β and γ. For these cases the similar discussions also hold. The partial-wave amplitudes are regular except for the cuts in the intervals $(-\infty, 0)$ and $(4\mu^2, +\infty)$ for the equal-mass-particle scattering, $(-\infty, 0)$ and $(4m_\pi^2, +\infty)$ for the pion-pion scattering.

After submitting this work, the author received Prog. Theor. Phys. **26**, No. 3 in which Nakanishi stated that the partial-wave dispersion relation can be derived from the integral representation (1) [see note added in proof of the reference 4)].

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Hypothesis of the Isotopic Spin Independence. II

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In the previous note[1] two speculations in the high energy limit[1] have been discussed:[2]

a) If a particle A with isotopic spin I_A collides on a target B with isotopic spin I_B, the total cross sections corresponding to the total isotopic spins $I=I_A+I_B$, I_A+I_B-1, $\cdots$, $|I_A-I_B|$ tend to a constant independent of I,

$$\sigma(I=I_A+I_B)=\sigma(I=I_A+I_B-1)=\cdots =\sigma(I=|I_A-I_B|)=\text{const.} \quad (1)$$

b) The S-matrices are independent of the total isotopic spin in the high energy limit. The assumption b) automatically guarantees the relation (1) except the last part.

The assumption b) obviously contradicts the argument that the peripheral collision is most important in the high energy region. It is pointed out in the present note that the assumption b) can be criticized by the process $\pi+N\to\pi+\pi+N$, in comparing with the results of the peripheral collision.

The cross sections are given in terms of the transition amplitudes $a_{1/2}(s)$, $a_{3/2}(s)$, $a_{1/2}(a)$, and $a_{3/2}(a)$, as[3]

$$\sigma(\pi^+ p\to\pi^+\pi^+ n)=\frac{4}{5}|a_{3/2}(s)|^2, \quad (2a)$$

$$\sigma(\pi^+ p\to\pi^+\pi^0 p)=\frac{1}{5}|a_{3/2}(s)|^2 +|a_{3/2}(a)|^2, \quad (2b)$$

$$\sigma(\pi^- p\to\pi^-\pi^0 p)=\frac{1}{5}|a_{3/2}(s)|^2 +\left|\frac{1}{3}a_{3/2}(a)-\frac{2}{3}a_{1/2}(a)\right|^2, \quad (2c)$$

$$\sigma(\pi^- p\to\pi^-\pi^+ n) =\left|\frac{\sqrt{2}}{3}a_{3/2}(a)+\frac{\sqrt{2}}{3}a_{1/2}(a)\right|^2 +\left|\frac{1}{3}\sqrt{\frac{2}{5}}a_{3/2}(s)+\frac{2}{3}a_{1/2}(s)\right|^2, \quad (2d)$$

$$\sigma(\pi^- p\to\pi^0\pi^0 n) =\left|-\frac{2}{3}\sqrt{\frac{1}{5}}a_{3/2}(s)+\frac{\sqrt{2}}{3}a_{1/2}(s)\right|^2, \quad (2e)$$

where the subscripts 1/2 and 3/2 denote the total isotopic spin, and s and a specify symmetric and antisymmetric with respect to the two pions in the final state. We have the following relation from Eqs. (2) under the assumption b),

$$\sigma(\pi^- p\to\pi\pi N)=\sigma(\pi^+ p\to\pi\pi N) =\sigma(\pi^- n\to\pi\pi N)=\sigma(\pi^+ n\to\pi\pi N), \quad (3)$$

where we define

$$\sigma(\pi^- p \to \pi\pi N) \equiv \sigma(\pi^- p \to \pi^- \pi^0 p) + \sigma(\pi^- p \to \pi^- \pi^+ n) + \sigma(\pi^- p \to \pi^0 \pi^0 n),$$

and so forth.

In order to obtain the relation (3), the assumption b) is interpreted as

$$|a_{1/2}(a)|^2 + |a_{1/2}(s)|^2 = |a_{3/2}(a)|^2 + |a_{3/2}(s)|^2. \quad (4)$$

If the assumption b) is interpreted more rigorously as

$$a_{1/2}(s) = a_{3/2}(s) \text{ and } a_{1/2}(a) = a_{3/2}(a), \quad (5)$$

one obtains, for instance,

$$\begin{aligned} 4\sigma(\pi^+ p \to \pi^+ \pi^0 p) &\geq \sigma(\pi^- p \to \pi^- \pi^+ n) \\ &\quad + \sigma(\pi^- p \to \pi^0 \pi^0 n) \\ &\geq 4\sigma(\pi^- p \to \pi^- \pi^0 p) \\ &\geq \sigma(\pi^+ p \to \pi^+ \pi^+ n), \end{aligned} \quad (6)$$

in addition to the relation (3). The relations (3) and (6) are criterions of the assumption b).

On the other hand, the peripheral process shown in Fig. 1 due to the pion-pion interaction does not give the relation (3). For the pion-pion interactions $\lambda(\varphi_\alpha \varphi_\alpha)^2$ and the $I=1$ state of $\lambda'(\varphi_\alpha \partial\varphi_\beta - \varphi_\beta \partial\varphi_\alpha)^2$ [4] the charge ratio is displayed in Table I, φ_α being the pion wave function.[5] Here the pion-pion rescattering correction is taken into account, while the pion-nucleon final state interaction is not.

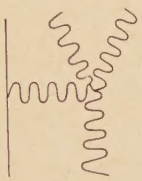

Fig. 1. Diagram for $\pi+N\to\pi+\pi+N$ due to the pion-pion interaction. Nucleon and meson are shown by solid and wavy lines, respectively.

Table I. Charge ratio of the $\pi+N\to\pi+\pi+N$

Process	$\lambda(\varphi_\alpha\varphi_\alpha)^2$	$I=1$ state of $\lambda'(\varphi_\alpha\partial\varphi_\beta-\varphi_\beta\partial\varphi_\alpha)^2$
$\sigma(\pi^+p\to\pi^+\pi^+n)$	4	0
$\sigma(\pi^+p\to\pi^+\pi^0p)$	1	1
$\sigma(\pi^-p\to\pi^-\pi^0p)$	1	1
$\sigma(\pi^-p\to\pi^-\pi^+n)$	8	2
$\sigma(\pi^-p\to\pi^0\pi^0n)$	1	0

If the peripheral collision gives the main contribution, we have

$$\sigma(\pi^- p \to \pi\pi N) = 2\sigma(\pi^+ p \to \pi\pi N) \text{ for } \lambda(\varphi_\alpha\varphi_\alpha)^2, \quad (7)$$

and

$$\sigma(\pi^- p \to \pi\pi N) = 3\sigma(\pi^+ p \to \pi\pi N), \text{ for } I=1 \text{ of } \lambda'(\varphi_\alpha\partial\varphi_\beta - \varphi_\beta\partial\varphi_\alpha)^2. \quad (8)$$

If the result (3) contradicts experiment, the assumption b) is wrong. On the other hand, even if Eq. (7) or (8) disagrees with experiment, it does not always mean the peripheral collision to be unimportant.

For $N+N\to\pi+N+N$ we have similar discussions. The cross sections are written as

$$\sigma(pp\to\pi^+pn) = |a_1(a)|^2 + \frac{1}{2}|a_1(s)|^2, \quad (9a)$$

$$\sigma(pp\to\pi^0pp) = \frac{1}{2}|a_1(s)|^2, \quad (9b)$$

$$\sigma(pn\to\pi^+nn) = \left|\frac{1}{\sqrt{6}}a_0(s) - \frac{1}{2}a_1(s)\right|^2, \quad (9c)$$

$$\sigma(pn\to\pi^- pp)=\left|\frac{1}{\sqrt{6}}a_0(s)+\frac{1}{2}a_1(s)\right|^2, \tag{9d}$$

$$\sigma(pn\to\pi^0 pn)=\frac{1}{6}|a_0(s)|^2+\frac{1}{2}|a_1(a)|^2. \tag{9e}$$

The notations are understood to be similar to Eqs. (2).

If we assume

$$|a_0(s)|^2=|a_1(s)|^2+|a_1(a)|^2, \tag{10}$$

$$\sigma(pp\to\pi NN)=\sigma(pn\to\pi NN) \tag{11}$$

is obtained. In parallel with Eqs. (5), one may interpret the assumption b) as

$$a_0(s)=a_1(s) \text{ and } a_1(a)=0. \tag{12}$$

From Eqs. (9) and (12) we have, in addition to Eq. (11),

$$\begin{aligned}\sigma(pp\to\pi^+ np)&=\sigma(pp\to\pi^0 pp)\\&=3\sigma(pn\to\pi^0 pn)\\&=\frac{3}{5}\{\sigma(pn\to\pi^- pp)\\&\quad+\sigma(pn\to\pi^+ nn)\}.\end{aligned} \tag{13}$$

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